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ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

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SYMMETRIC REPRESENTATIONS OF DISTRIBUTIONS OVER $\mathbb{R}^2$ BY DISTRIBUTIONS WITH NOT MORE THAN THREE POINT SUPPORT

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Abstract

We construct symmetric representations of distributions over $\mathbb{R}^2$ with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values. These representations are two-dimensional analogs of the following easy verified formula for distributions $\mathbf{p}$ over $\mathbb{R}^1$ with a mean value u:

$$\mathbf{p} = \int_{x=u^-}^{\infty} \mathbf{p}(dx) \int_{y=-\infty}^{u^+} \frac{x-y}{\int_{t=u}^{\infty}(t-u)\cdot\mathbf{p}(dt)} \cdot \mathbf{p}^u_{x,y} \cdot \mathbf{p}(dy),$$

where, for $y < u < x$, distributions $\mathbf{p}^u_{x,y} = ((x-u)\cdot\delta^y + (u-y)\cdot\delta^x)/(x-y)$, δ^x is the degenerate distribution with the single-point support x, and $\mathbf{p}^u_{x,u} = \mathbf{p}^u_{u,y} = \delta^u/2$.

Key words: probability distributions over the plane, mean values, extreme points of convex sets, convex combinations of distributions.

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1. Introduction. Setting of problem.

We consider the set $\mathbf{P}(\mathbb{R}^2)$ of probability distributions $\mathbf{p}$ over the plane $\mathbb{R}^2 = \{z = (x, y)\}$ with finite first absolute moments

$$\int_{\mathbb{R}^2} |x| \cdot \mathbf{p}(dz) < \infty, \quad \int_{\mathbb{R}^2} |y| \cdot \mathbf{p}(dz) < \infty.$$

We denote by $\mathbf{E_p}[x]$ and $\mathbf{E_p}[y]$ the mean values of distribution $\mathbf{p}$:

$$\mathbf{E_p}[x] = \int_{\mathbb{R}^2} x \cdot \mathbf{p}(dz) < \infty, \quad \mathbf{E_p}[y] = \int_{\mathbb{R}^2} y \cdot \mathbf{p}(dz) < \infty.$$

We construct symmetric representations of the convex set of distributions with given mean values

$$\Theta(u, v) = \{\mathbf{p} \in \mathbf{P}(\mathbb{R}^2) : \mathbf{E_p}[x] = u, \mathbf{E_p}[y] = v\},$$

as a convex hull of its extreme points.

This is sufficient to give the representation for the set $\Theta(0, 0)$. The extreme points of the set $\Theta(0, 0)$ are the degenerate distribution δ^0 with the single-point support $0 = (0, 0)$, distributions $\mathbf{p}^0_{z_1,z_2} \in \Theta(0, 0)$ with two-point supports (z_1, z_2), and distributions $\mathbf{p}^0_{z_1,z_2,z_3} \in \Theta(0, 0)$ with three-point supports (z_1, z_2, z_3).

This problem arose from investigating multistage bidding models where two types of risky assets are traded [1]. As the example for imitation we take the symmetric representation of one-dimensional probability distributions over the integer lattice that was exploited in [2] for analysis of bidding models with single-type asset. Let $\mathbf{p}$ be a probability distribution over the set of integers $\mathbb{Z}^1$ with zero mean value. Then

$$\mathbf{p} = p(0) \cdot \delta^0 + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \frac{k + l}{\sum_{t=1}^{\infty} t \cdot p(t)} \mathbf{p}(-l)\mathbf{p}(k) \cdot \mathbf{p}^0_{k,-l}, \tag{1}$$

where $\mathbf{p}^0_{k,-l}$ is the probability distribution with the support $\{-l, k\}$ and with zero mean value. Formula (1) can be written as

$$\mathbf{p} = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{k + l}{\sum_{t=1}^{\infty} t \cdot \mathbf{p}(t)} \mathbf{p}(-l)\mathbf{p}(k) \cdot \mathbf{p}^0_{k,-l},$$

if we put $\mathbf{p}^0_{k,0} = \mathbf{p}^0_{0,-l} = \delta^0/2$.

Observe that the coefficients $\mathbf{P_p}(\mathbf{p}^0_{k,-l})$ of decomposition (1), that may be treated as probabilities of corresponding distributions $\mathbf{p}^0_{k,-l}$ in the two-step lottery realizing distribution $\mathbf{p}$, have the form

$$\mathbf{P_p}(\mathbf{p}^0_{k,-l}) = \alpha(k, -l)\beta(\mathbf{p})\mathbf{p}(k)\mathbf{p}(-l),$$

where $\alpha(k, l) = k + l$ and $\beta(\mathbf{p}) = 1/\sum_{t=1}^{\infty} t \cdot \mathbf{p}(t) = 1/\sum_{t=1}^{\infty} t \cdot \mathbf{p}(-t)$, the last equality playing the crucial role. We mean just this form of coefficients saying that the representation (1) is symmetric. We aim for constructing the representation of two-dimensional probability distributions with the analogous characteristics.

Formula (1) can be easily generalized for probability distributions over the set of real numbers $\mathbb{R}^1$ with zero mean value. Namely

$$\mathbf{p} = \int_{x=0^-}^{\infty} \mathbf{p}(dx) \int_{y=-\infty}^{0^+} \frac{x - y}{\int_{t=0}^{\infty} t \cdot \mathbf{p}(dt)} \cdot \mathbf{p}^0_{x,-y} \cdot \mathbf{p}(dy),$$

where, for $y < 0 < x$, the distributions $\mathbf{p}^0_{x,-y} = (x \cdot \delta(y) - y \cdot \delta(x))/(x - y)$, and $\mathbf{p}^0_{x,0} = \mathbf{p}^0_{0,-y} = \delta(0)/2$.

Consider the set of three-point sets that form triangles containing the point $(0,0)$:

$$\Delta^0 = \{(z_1, z_2, z_3), z_i \neq (0,0) : (0,0) \in \triangle(z_1, z_2, z_3)\}.$$

The set Δ^0 is a manifold with boundary. Its interior IntΔ^0 is the set of three-point sets $(z_1, z_2, z_3) \in \Delta^0$ such that $(0,0)$ belongs to the interior of the $\triangle(z_1, z_2, z_3)$. Its boundary $\partial\Delta^0$ is the set of three-point sets $(z_1, z_2, z_3) \in \Delta^0$ such that $(0,0)$ belongs to the boundary of the $\triangle(z_1, z_2, z_3)$.

The distribution $\mathbf{p}^0_{z_1,z_2,z_3} \in \Theta(0,0)$ with the support $\{z_1, z_2, z_3\} \in \Delta^0$ is given by

$$\mathbf{p}^0_{z_1,z_2,z_3} = \frac{\sum_{j=1}^3 \det[z_{i+1}, z_{i+2}] \cdot \delta(z_i)}{\sum_{j=1}^3 \det[z_j, z_{j+1}]}, \tag{2}$$

where $\det[z_i, z_{i+1}] = x_i \cdot y_{i+1} - y_i \cdot x_{i+1}$. All arithmetical operations with subscripts are fulfilled modulo 3. If the points $(z_1, z_2, z_3) \in \Delta^0$ are indexed counterclockwise, then $\det[z_i, z_{i+1}] \geq 0$.

If $(z_1, z_2, z_3) \in \partial\Delta^0$, then there is an index i such that $\det[z_i, z_{i+1}] = 0$. In this case $\arg z_{i+1} = \arg z_i + \pi (\text{mod } 2\pi)$, the point $(0,0) \in [z_i, z_{i+1}]$ and the distribution $\mathbf{p}^0_{z_1,z_2,z_3}$ degenerates into the distribution $\mathbf{p}^0_{z_i,z_{i+1}}$ with the support $\{z_i, z_{i+1}\}$.

2. Key invariant of distributions $\mathbf{p} \in \Theta(0,0)$.

For $\psi \in [0, 2\pi)$, let R_ψ be the half-line

$$R_\psi = \{z : \arg z = \psi (\text{mod } 2\pi)\}.$$

With each $\psi \in [0, 2\pi)$ we associate the set of two-point sets

$$\Delta^0(\psi) = \{(z_1, z_2), z_i \neq (0,0) : \forall z \in R_\psi \quad (0,0) \in \triangle(z_1, z_2, z)\}.$$

Denote by Int$\Delta^0(\psi)$ and $\partial\Delta^0(\psi)$ the sets of two-point sets (z_1, z_2) such that, for $z \in R_\psi$, the set (z_1, z_2, z) belongs to IntΔ^0 and to $\partial\Delta^0$ respectively. We take, that the points (z_1, z_2) are indexed counterclockwise.

Consider the quantity

$$\Phi(\mathbf{p}, \psi) = \int_{\mathrm{Int}\Delta^0(\psi)} \det[z_1, z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2) + 1/2 \int_{\partial\Delta^0(\psi)} \det[z_1, z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2). \tag{3}$$

Using polar coordinates $z_1 = (r_1, \varphi_1)$, $z_2 = (r_2, \varphi_2)$ we get

$$\Phi(\mathbf{p}, \psi) = \int_{\varphi_1=\psi^+}^{\pi+\psi^+} \int_{r_1=0^+}^{\infty} \mathbf{p}(dr_1 d\varphi_1) \int_{\varphi_2=\pi+\psi^+}^{\pi+\varphi_1^+} \int_{r_2=0^+}^{\infty} r_1 \cdot r_2 \cdot \sin(\varphi_2 - \varphi_1)\mathbf{p}(dr_2 d\varphi_2).$$

Remark 1. The quantity

$$\partial\Phi(\mathbf{p}, \psi) = 1/2 \int_{\partial\Delta^0(\psi)} \det[z_1, z_2]\mathbf{p}(dz_1)\mathbf{p}(dz_2)$$

differs from zero only if the measure $\mathbf{p}(R_{\psi+\pi})$ is more than zero. In this case

$$\partial\Phi(\mathbf{p}, \psi) = \int_{R_{\psi+\pi}} r_2 \mathbf{p}(dr_2) \cdot \int_{Hp_\psi} \det[e_\psi, z_1]\mathbf{p}(dz_1)$$

$$= \int_{R_{\psi+\pi}} r_1 \mathbf{p}(dr_1) \cdot \int_{Hp_{\psi+\pi}} \det[z_2, e_\psi] \mathbf{p}(dz_2), \tag{4}$$

where $e_\psi = (1, \psi)$ and Hp_φ is the half-plane

$$Hp_\varphi = \{z : \arg z \in (\varphi, \varphi + \pi)(\mathrm{mod}\ 2\pi)\}.$$

The next fact produces the base for constructing symmetric representations of distributions over $\mathbb{R}^2$ with given mean values as convex combinations of distributions with supports containing not more than three points and with the same mean values.

Theorem 1. *For any distribution $\mathbf{p} \in \Theta(0,0)$ the quantity $\Phi(\mathbf{p}, \psi)$ does not depend on ψ, i.e. this is an invariant $\Phi(\mathbf{p})$ of distribution $\mathbf{p} \in \Theta(0,0)$.*

Proof. We begin with proving Theorem 1 for distributions $\mathbf{p} \in \Theta^f(0,0)$ with finite supports. Let $\psi_1, \psi_2 \in [0, 2\pi)$, $\psi_1 < \psi_2$, be such two values of argument that the support of the distribution $\mathbf{p} \in \Theta^f(0,0)$ does not contain points z with $\psi_1 < \arg z < \psi_2$.

Set

$$U(\psi_1 + \pi, \psi_2 + \pi) = \{z \in \mathbb{R}^2 : \psi_1 + \pi < \arg z \leq \psi_2 + \pi\}.$$

We have

$$\Phi(\mathbf{p}, \psi_1) - \Phi(\mathbf{p}, \psi_2) = \sum_{z_3 \in U(\psi_1+\pi, \psi_2+\pi)} \sum_{z_2 \in \mathbb{R}^2} p(z_2) p(z_3) \det[z_2, z_3].$$

Since, for distributions $\mathbf{p} \in \Theta(0,0)$,

$$\sum_{z_2 \in \mathbb{R}^2} p(z_2) \det[z_2, z_3] = 0,$$

we obtain

$$\Phi(\mathbf{p}, \psi_1) - \Phi(\mathbf{p}, \psi_2) = 0.$$

Iterating this argument the relevant number of times we obtain the statement of Theorem 1 for any distribution $\mathbf{p} \in \Theta^f(0,0)$.

As the set $\Theta^f(0,0)$ is weakly* everywhere dense in $\Theta(0,0)$ we obtain the statement of Theorem 1 for arbitrary distributions $\mathbf{p} \in \Theta(0,0)$.

□

Remark 2. This theorem is a two-dimensional analog of the fact that, for $\mathbf{p} \in \Theta(0) \subset \mathbf{P}(\mathbb{R}^1)$, the equality

$$\int_{t=0}^{\infty} t \cdot \mathbf{p}(dt) = \int_{t=0}^{\infty} t \cdot \mathbf{p}(-dt)$$

holds.

Example 1. Consider the distribution $\mathbf{p}^0_{z_1,z_2,z_3}$ with $(z_1, z_2, z_3) \in \mathrm{Int}\Delta^0$. For this distribution, if $\arg z_i = \varphi_i$ and

$$\varphi_i + \pi < \psi < \varphi_{i+1} + \pi (\mathrm{mod}\ 2\pi),$$

then the support of the measure induced by $\mathbf{p}^0_{z_1,z_2,z_3}$ over the set $\Delta^0(\psi)$ is the set $\{(z_i, z_{i+1})\} \subset \mathrm{Int}\Delta^0(\psi)$. Thus

$$\Phi(\mathbf{p}^0_{z_1,z_2,z_3}, \psi) = \det[z_i, z_{i+1}] \cdot \mathbf{p}^0_{z_1,z_2,z_3}(z_i) \mathbf{p}^0_{z_1,z_2,z_3}(z_{i+1}) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}.$$

If $\varphi_i + \pi = \psi (\mathrm{mod}\ 2\pi)$, then the support of the induced measure is the set $\{(z_{i-1}, z_i), (z_i, z_{i+1})\} \subset \partial\Delta^0(\psi)$. Thus

$$\Phi(\mathbf{p}^0_{z_1,z_2,z_3}, \psi) = 1/2 \cdot (\det[z_{i-1}, z_i] \cdot \mathbf{p}^0_{z_1,z_2,z_3}(z_{i-1})\mathbf{p}^0_{z_1,z_2,z_3}(z_i)$$

$$+ \det[z_i, z_{i+1}] \cdot \mathbf{p}^0_{z_1,z_2,z_3}(z_i)\mathbf{p}^0_{z_1,z_2,z_3}(z_{i+1})) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}.$$

Thus, in accordance with Theorem 1, $\Phi(\mathbf{p}^0_{z_1,z_2,z_3}, \psi)$ has the same value $\Phi(\mathbf{p}^0_{z_1,z_2,z_3})$ for all values of ψ.

3. Decomposition of distributions $\mathbf{p} \in \Theta(0,0)$.

The invariance of the quantity $\Phi(\mathbf{p})$ proved in the previous section allows us to formulate the following preliminary variant of decomposition theorem for two-dimensional distributions. This variant demonstrate a perfect analogy with the decomposition of one-dimensional distributions.

Proposition 2. *Any distribution $\mathbf{p} \in \Theta(0,0)$ has the following symmetric decomposition into a convex combination of distributions with not more than three-point supports:*

$$\mathbf{p} = \mathbf{p}(0,0) \cdot \delta^0 + \int_{Int\Delta^0} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3} \mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3)$$

$$+1/2 \int_{\partial\Delta^0} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3} \mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3), \tag{5}$$

where $\Phi(\mathbf{p})$ is given by (3).

Proof. We begin with proving Proposition 2 for distributions $\mathbf{p} \in \Theta^f(0,0)$ with finite supports. Take a point $z_1 = (r_1, \varphi_1) \in \mathrm{supp}\ \mathbf{p}$. This point occurs in three point set (z_1, z_2, z_3) if $(z_2, z_3) \in \Delta^0(\varphi_1)$. The probability $\mathbf{p}'(z_1)$ calculated according to formula (5) is

$$\mathbf{p}'(z_1) = \sum_{(z_2,z_3)\in \mathrm{Int}\Delta^0(\varphi_1)} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3}(z_1)\mathbf{p}(z_1)\mathbf{p}(z_2)\mathbf{p}(z_3)$$

$$+1/2 \sum_{(z_2,z_3)\in \partial\Delta^0(\varphi_1)} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3}(z_1)\mathbf{p}(z_1)\mathbf{p}(z_2)\mathbf{p}(z_3).$$

Substituting the values $\mathbf{p}^0_{z_1,z_2,z_3}(z_1)$ given by (2) we get

$$\mathbf{p}'(z_1) = \frac{\mathbf{p}(z_1)}{\Phi(\mathbf{p})} \Big(\sum_{\mathrm{Int}\Delta^0(\varphi_1)} +1/2 \sum_{\partial\Delta^0(\varphi_1)} \Big) \det[z_2, z_3] \cdot \mathbf{p}(z_2)\mathbf{p}(z_3) = \mathbf{p}(z_1).$$

This proves Proposition 2 for any distribution $\mathbf{p} \in \Theta^f(0,0)$.

As the set $\Theta^f(0,0)$ is weakly* everywhere dense in $\Theta(0,0)$ we obtain the statement of Proposition 2 for arbitrary distributions $\mathbf{p} \in \Theta(0,0)$.

□

The term

$$\partial\mathbf{p} = 1/2 \int_{\partial\Delta^0} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3} \mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3)$$

of decomposition (5) contains all distributions $\mathbf{p}^0_{z_i,z_{i+1}}$ with two-point supports (z_i, z_{i+1}), where $z_i \in R_\psi$ and $z_{i+1} \in R_{\psi+\pi}$. In order that such combination of points could appear with nonzero probability, it is necessary that the measure $\mathbf{p}(R_\psi)$ and the measure $\mathbf{p}(R_{\psi+\pi})$ are more than zero. This is possible for a not more than countable set $\Psi(\mathbf{p})$ of values ψ.

These considerations make possible the final formulation of the principal Theorem:

Theorem 3. *Any probability distribution $\mathbf{p} \in \Theta(0,0)$ has the following symmetric representation as a convex combination of distributions with one-, two-, and three-point supports:*

$$\mathbf{p} = \mathbf{p}(0,0)\cdot\delta^0 + \int_{Int\Delta^0} \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}^0_{z_1,z_2,z_3}\mathbf{p}(dz_1)\mathbf{p}(dz_2)\mathbf{p}(dz_3)$$

$$+\sum_{\Psi(\mathbf{p})} \frac{\partial\Phi(\mathbf{p},\psi)}{\Phi(\mathbf{p})} \int_{R_\psi}\int_{R_{\psi+\pi}} \frac{r_1 + r_2}{\int_{R_{\psi+\pi}} t\mathbf{p}(dt)} \mathbf{p}^0_{(r_1,\psi),(r_2,\psi+\pi)}\mathbf{p}(dr_2)\mathbf{p}(dr_1). \qquad (6)$$

Proof. For a pair of points $z_1 = (r_1, \psi)$, $z_2 = (r_2, \psi+\pi)$, their combination with any point z from Hp_ψ or from $Hp_{\psi+\pi}$ reduces to the distribution $\mathbf{p}^0_{z_1,z_2}$. Since

$$\int_{Hp_\psi} \det[e_\psi, z]\mathbf{p}(dz) = \int_{Hp_{\psi+\pi}} \det[z, e_\psi]\mathbf{p}(dz),$$

where $e_\psi = (1,\psi)$, we get

$$\partial\mathbf{p} = \sum_{\Psi(\mathbf{p})} \int_{Hp_\psi} \det[e_\psi, z]\mathbf{p}(dz) \int_{R_\psi}\int_{R_{\psi+\pi}} \frac{r_1 + r_2}{\Phi(\mathbf{p})} \mathbf{p}^0_{(r_1,\psi),(r_2,\psi+\pi)}\mathbf{p}(dr_2)\mathbf{p}(dr_1).$$

It follows from (4) that

$$\int_{Hp_\psi} \det[e_\psi, z_1]\mathbf{p}(dz_1) = \frac{\partial\Phi(\mathbf{p},\psi)}{\int_{R_{\psi+\pi}} r_2\mathbf{p}(dr_2)}.$$

Substituting this expression in place of this integral we obtain

$$\partial\mathbf{p} = \sum_{\Psi(\mathbf{p})} \frac{\partial\Phi(\mathbf{p},\psi)}{\Phi(\mathbf{p})} \int_{R_\psi}\int_{R_{\psi+\pi}} \frac{r_1 + r_2}{\int_{R_{\psi+\pi}} t\mathbf{p}(dt)} \mathbf{p}^0_{(r_1,\psi),(r_2,\psi+\pi)}\mathbf{p}(dr_2)\mathbf{p}(dr_1).$$

Substituting this into formula (5) we obtain (6). This proves Theorem 3.

□

Remark 3. For distributions $\mathbf{p} \in \Theta(0,0)$ with discrete supports this theorem indicates probabilities $\mathbf{P}_\mathbf{p}(\mathbf{p}^0_{z_1,z_2,z_3})$ and $\mathbf{P}_\mathbf{p}(\mathbf{p}^0_{z_1,z_2})$ of appearance of distributions with two-, and three-point supports in their symmetric representations:

$$\mathbf{P}_\mathbf{p}(\mathbf{p}^0_{z_1,z_2,z_3}) = \frac{\sum_{j=1}^3 \det[z_j, z_{j+1}]}{\Phi(\mathbf{p})} \mathbf{p}(z_1)\mathbf{p}(z_2)\mathbf{p}(z_3);$$

$$\mathbf{P}_\mathbf{p}(\mathbf{p}^0_{(r_1,\varphi),(r_2,\varphi+\pi)}) = \frac{\partial\Phi(\mathbf{p},\varphi)}{\Phi(\mathbf{p})} \frac{r_1 + r_2}{\sum_{R_{\psi+\pi}} t\mathbf{p}(t)} \mathbf{p}(r_1,\varphi)\mathbf{p}(r_2,\varphi+\pi).$$

4. Examples.

Here we give several elementary examples concerning calculation of invariants $\Phi(\mathbf{p})$ and constructing symmetric representations as a convex combinations of distributions with one-, two-, and three-point supports, for simple distributions with finite supports.

Example 1'. We return to the distribution $\mathbf{p}^0_{z_1,z_2,z_3}$ with $(z_1,z_2,z_3)\in \mathrm{Int}\Delta^0$. For this distribution, as it is shown in Example 1,

$$\Phi(\mathbf{p}^0_{z_1,z_2,z_3}) = \det[z_i,z_{i+1}]\cdot \mathbf{p}^0_{z_1,z_2,z_3}(z_i)\mathbf{p}^0_{z_1,z_2,z_3}(z_{i+1}) = \frac{\prod_{j=1}^3 \det[z_j,z_{j+1}]}{(\sum_{j=1}^3 \det[z_j,z_{j+1}])^2}.$$

As the distribution $\mathbf{p}^0_{z_1,z_2,z_3}$ is an extreme point of the set $\Theta(0,0)$ its symmetric representation is trivial. To check it formally put

$$\mathbf{P}_{\mathbf{p}^0_{z_1,z_2,z_3}}(\mathbf{p}^0_{z_1,z_2,z_3}) = \frac{\sum_{i=1}^3 \det[z_i,z_{i+1}]}{\Phi(\mathbf{p}^0_{z_1,z_2,z_3})}\mathbf{p}^0_{z_1,z_2,z_3}(z_1)\mathbf{p}^0_{z_1,z_2,z_3}(z_2)\mathbf{p}^0_{z_1,z_2,z_3}(z_3) = 1.$$

Example 2. For $\mathbf{z} = (z_1,z_2,z_3)\in \mathrm{Int}\Delta^0$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)>0$, $\sum_{i=1}^3\alpha_i = 1$, consider the distribution

$$\mathbf{p}_{\mathbf{z},\alpha} = \sum_{j=1}^3 \alpha_i \mathbf{p}^0_{z_i,-z_i}\in\Theta(0,0).$$

For this distribution,

$$\mathrm{supp}\ \mathbf{p}_{\mathbf{z},\alpha} = \{z_1,z_2,z_3,-z_1,-z_2,-z_3\},\quad \mathbf{p}_{\mathbf{z},\alpha}(z_i) = \mathbf{p}_{\mathbf{z},\alpha}(-z_i) = \alpha_i/2.$$

Let $\arg z_i = \varphi_i$. If

$$\varphi_i < \psi < \varphi_{i-1} + \pi (\mathrm{mod}\ 2\pi),$$

then the support of the measure induced by $\mathbf{p}_{\mathbf{z},\alpha}$ over the set $\Delta^0(\psi)$ is the set

$$\{(-z_i,z_{i-1}),(-z_i,-z_{i+1}),(z_{i+1},z_{i-1})\}\subset \mathrm{Int}\Delta^0(\psi).$$

Since $\det[-z_i,z_{i-1}] = \det[z_{i-1},z_i]$, $\det[-z_i,-z_{i+1}] = \det[z_i,z_{i+1}]$, we get

$$\Phi(\mathbf{p}_{\mathbf{z},\alpha}) = \sum_{i=1}^3 \det[z_i,z_{i+1}]\cdot \mathbf{p}_{\mathbf{z},\alpha}(z_i)\mathbf{p}_{\mathbf{z},\alpha}(z_{i+1}) = 1/4\sum_{i=1}^3 \det[z_i,z_{i+1}]\cdot\alpha_i\cdot\alpha_{i+1}.$$

The symmetric representation of the distribution $\mathbf{p}_{\mathbf{z},\alpha}$ includes five extreme distributions: two three-point distributions $\mathbf{p}^0_{z_1,z_2,z_3}$ and $\mathbf{p}^0_{-z_1,-z_2,-z_3}$, and three two-point distributions $\mathbf{p}^0_{z_1,-z_1}$, $\mathbf{p}^0_{z_2,-z_2}$, and $\mathbf{p}^0_{z_3,-z_3}$. These distributions occur with probabilities

$$\mathbf{P}_{\mathbf{p}_{\mathbf{z},\alpha}}(\mathbf{p}^0_{z_1,z_2,z_3}) = \mathbf{P}_{\mathbf{p}_{\mathbf{z},\alpha}}(\mathbf{p}^0_{-z_1,-z_2,-z_3}) = \frac{\sum_{i=1}^3\det[z_i,z_{i+1}]}{\Phi(\mathbf{p}_{\mathbf{z},\alpha})}\mathbf{p}_{\mathbf{z},\alpha}(z_1)\mathbf{p}_{\mathbf{z},\alpha}(z_2)\mathbf{p}_{\mathbf{z},\alpha}(z_3)$$

$$= 1/2\frac{\sum_{i=1}^3\det[z_i,z_{i+1}]}{\sum_{i=1}^3\det[z_i,z_{i+1}]\cdot\alpha_i\cdot\alpha_{i+1}}\alpha_1\alpha_2\alpha_3;$$

$$\mathbf{P}_{\mathbf{p}_{\mathbf{z},\alpha}}(\mathbf{p}^0_{z_i,-z_i}) = \frac{2\det[z_i,z_{i+1}]\mathbf{p}_{\mathbf{z},\alpha}(z_{i+1}) + 2\det[z_{i+2},z_i]\mathbf{p}_{\mathbf{z},\alpha}(z_{i+2})}{\Phi(\mathbf{p}_{\mathbf{z},\alpha})}\mathbf{p}_{\mathbf{z},\alpha}(z_i)^2$$

$$= \frac{\det[z_i, z_{i+1}]\alpha_{i+1} + \det[z_{i+2}, z_i]\alpha_{i+2}}{\sum_{i=1}^3 \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}} \alpha_i^2.$$

Observe that

$$\mathbf{P}_{\mathbf{p}_{\mathbf{z},\alpha}}(\mathbf{p}^0_{z_1,z_2,z_3}) + \mathbf{P}_{\mathbf{p}_{\mathbf{z},\alpha}}(\mathbf{p}^0_{-z_1,-z_2,-z_3}) + \sum_{i=1}^3 \mathbf{P}_{\mathbf{p}_{\mathbf{z},\alpha}}(\mathbf{p}^0_{z_i,-z_i})$$

$$= \frac{\sum_{i=1}^3 \det[z_i, z_{i+1}]\alpha_1\alpha_2\alpha_3 + \sum_{i=1}^3(\det[z_i, z_{i+1}]\alpha_i^2\alpha_{i+1} + \det[z_{i+2}, z_i]\alpha_i^2\alpha_{i+2})}{\sum_{i=1}^3 \det[z_i, z_{i+1}] \cdot \alpha_i \cdot \alpha_{i+1}}$$

$$= \frac{(\sum_{i=1}^3 \det[z_i, z_{i+1}]\alpha_i\alpha_{i+1})(\sum_{i=1}^3 \alpha_i)}{\sum_{i=1}^3 \det[z_i, z_{i+1}]\alpha_i\alpha_{i+1}} = 1$$

Example 3. For $\mathbf{z} = (z_1, z_2, z_3) \in \mathrm{Int}\Delta^0$, $\beta \in (0,1)$, consider the distribution

$$\mathbf{p}_{\beta,\mathbf{z}} = \beta\mathbf{p}^0_{z_1,z_2,z_3} + (1-\beta)\mathbf{p}^0_{-z_1,-z_2,-z_3} \in \Theta(0,0).$$

This distribution has the same support as the distribution $\mathbf{p}_{\mathbf{z},\alpha}$ of the previous example:

$$\text{supp } \mathbf{p}_{\beta,\mathbf{z}} = \{z_1, z_2, z_3, -z_1, -z_2, -z_3\},$$

The probabilities of these points are

$$\mathbf{p}_{\beta,\mathbf{z}}(z_i) = \beta\frac{\det[z_{i+1}, z_{i+2}]}{\sum_{j=1}^3 \det[z_j, z_{j+1}]} \quad \mathbf{p}_{\beta,\mathbf{z}}(-z_i) = (1-\beta)\frac{\det[z_{i+1}, z_{i+2}]}{\sum_{j=1}^3 \det[z_j, z_{j+1}]}.$$

For this distribution, if

$$\varphi_i < \psi < \varphi_{i-1} + \pi (\mathrm{mod}\ 2\pi),$$

then the support of the measure induced by $\mathbf{p}_{\beta,\mathbf{z}}$ over the set $\Delta^0(\psi)$ is the set

$$\{(-z_i, z_{i-1}), (-z_i, -z_{i+1}), (z_{i+1}, z_{i-1})\} \subset \mathrm{Int}\Delta^0(\psi).$$

Since

$$\det[-z_i, z_{i-1}]\mathbf{p}_{\beta,\mathbf{z}}(-z_i)\mathbf{p}_{\beta,\mathbf{z}}(z_{i-1}) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}\beta(1-\beta),$$

$$\det[-z_i, -z_{i+1}]\mathbf{p}_{\beta,\mathbf{z}}(-z_i)\mathbf{p}_{\beta,\mathbf{z}}(-z_{i+1}) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}(1-\beta)^2,$$

we get

$$\Phi(\mathbf{p}_{\beta,\mathbf{z}}) = \frac{\prod_{j=1}^3 \det[z_j, z_{j+1}]}{(\sum_{j=1}^3 \det[z_j, z_{j+1}])^2}(\beta^2 + \beta(1-\beta) + (1-\beta)^2).$$

The symmetric representation of the distribution $\mathbf{p}_{\beta,\mathbf{z}}$ includes the same five extreme distributions as in the previous example: two three-point distributions $\mathbf{p}^0_{z_1,z_2,z_3}$ and $\mathbf{p}^0_{-z_1,-z_2,-z_3}$, and three two-point distributions $\mathbf{p}^0_{z_1,-z_1}$, $\mathbf{p}^0_{z_2,-z_2}$, and $\mathbf{p}^0_{z_3,-z_3}$. These distributions occur with probabilities

$$\mathbf{P}_{\mathbf{p}_{\beta,\mathbf{z}}}(\mathbf{p}^0_{z_1,z_2,z_3}) = \frac{\sum_{i=1}^3 \det[z_i, z_{i+1}]}{\Phi(\mathbf{p}_{\beta,\mathbf{z}})}\mathbf{p}_{\beta,\mathbf{z}}(z_1)\mathbf{p}_{\beta,\mathbf{z}}(z_2)\mathbf{p}_{\beta,\mathbf{z}}(z_3)$$

$$= \frac{\beta^3}{\beta^2 + \beta(1-\beta) + (1-\beta)^2};$$

$$\mathbf{P}_{\mathbf{p}_{\beta,\mathbf{z}}}(\mathbf{p}^0_{-z_1,-z_2,-z_3}) = \frac{\sum_{i=1}^3 \det[z_i, z_{i+1}]}{\Phi(\mathbf{p}_{\beta,\mathbf{z}})} \mathbf{p}_{\beta,\mathbf{z}}(-z_1)\mathbf{p}_{\beta,\mathbf{z}}(-z_2)\mathbf{p}_{\beta,\mathbf{z}}(-z_3)$$

$$= \frac{(1-\beta)^3}{\beta^2 + \beta(1-\beta) + (1-\beta)^2};$$

$$\mathbf{P}_{\mathbf{p}_{\beta,\mathbf{z}}}(\mathbf{p}^0_{z_i,-z_i}) = \frac{2\det[z_i, z_{i+1}]\mathbf{p}_{\beta,\mathbf{z}}(z_{i+1}) + 2\det[z_{i+2}, z_i]\mathbf{p}_{\beta,\mathbf{z}}(-z_{i+2})}{\Phi(\mathbf{p}_{\beta,\mathbf{z}})} \mathbf{p}_{\beta,\mathbf{z}}(z_i)\mathbf{p}_{\beta,\mathbf{z}}(-z_i)$$

$$= \frac{\det[z_{i+1}, z_{i+2}]}{\sum_{j=1}^3 \det[z_j, z_{j+1}]} \frac{2\beta(1-\beta)}{\beta^2 + \beta(1-\beta) + (1-\beta)^2}.$$

Observe that

$$\mathbf{P}_{\mathbf{p}_{\beta,\mathbf{z}}}(\mathbf{p}^0_{z_1,z_2,z_3}) + \mathbf{P}_{\mathbf{p}_{\beta,\mathbf{z}}}(\mathbf{p}^0_{-z_1,-z_2,-z_3}) + \sum_{i=1}^3 \mathbf{P}_{\mathbf{p}_{\beta,\mathbf{z}}}(\mathbf{p}^0_{z_i,-z_i})$$

$$= \frac{\beta^3 + (1-\beta)^3 + 2\beta(1-\beta)}{\beta^2 + \beta(1-\beta) + (1-\beta)^2} = \frac{\beta^2 - \beta(1-\beta) + (1-\beta)^2 + 2\beta(1-\beta)}{\beta^2 + \beta(1-\beta) + (1-\beta)^2} = 1.$$

References

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