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**Wave operators on the singular spectrum.
The case of rank-two commutators**

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ABSTRACT:

A problem of existence of a wave operator is considered in the case of rank-two commutators, the spectral measure of the unitary operators being allowed to be singular. An application is obtained for the boundary behaviour of pseudocontinuable functions in the unit disk.

Key words: wave operator, summation methods, singular spectral measure.

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Let $U_1 : H_1 \rightarrow H_1$, $U_2 : H_2 \rightarrow H_2$ be two unitary operators, $X : H_1 \rightarrow H_2$ a bounded linear operator. Set $K = XU_1 - U_2X$. Define

$$A_r = (1 - r) \sum_{n=0}^{\infty} r^n U_2^n X U_1^{-n},$$

i.e., A_r are the Abel means of the sequence $U_2^n X U_1^{-n}$. The general conjecture is the following.

Conjecture 0.1. *Let U_1, U_2 be unitary operators, and assume that X is such that $K = XU_1 - U_2X$ is a finite rank operator (or from the trace class, etc.) Then the Abel limit of the sequence $U_2^n X U_1^{-n}$ exists in the weak operator topology, i.e., the operators A_r weakly converge as $r \nearrow 1$.*

It is shown in [1] that Conjecture 0.1 for the case of rank $K = 2$ reduces to the case, where $H_1 = H_2 = L^2(\mu)$ with singular probability measure μ on the unit circle having no atomic masses, $U := U_1 = U_2$ is the operator of multiplication by the independent variable on $L^2(\mu)$, and

$$(1) \quad K = (\cdot, \bar{f})1 - (\cdot, 1)f$$

for some $f \in L^2(\mu)$.

In this article we discuss this case. It is connected with the problem of constructing the Hilbert transform on a singular measure and with a certain boundary convergence of functions from K_θ , see Proposition 2.1. Our Theorem 3.1 is a reformulation of the problem in terms of truncated Toeplitz operators. It immediately implies a result (Theorem 4.1) about the boundary behaviour of functions from a wide subclass of K_θ . A sufficient condition for three equivalent properties from Proposition 2.1 is given in Theorem 5.6.

1. THE HILBERT TRANSFORM AND CAUCHY TYPE INTEGRALS

A wave operator is an operator with prescribed commutator with a given unitary operator. This construction turns out to be closely connected with the Hilbert transform. For a singular measure μ it is natural to define the Hilbert transform by the formula

$$Hf = \int \frac{f(\xi) - f(z)}{\xi - z} d\mu(\xi).$$

However, this map (initially defined on smooth functions) is not continuous in $L^2(\mu)$.

Proposition 1.1. *Let μ be a singular probability measure on the unit circle without atomic masses. The operator H defined on sufficiently smooth functions f cannot be extended to a continuous map on $L^2(\mu)$.*

A proof of this fact will be presented in the next section.

A natural question appears about the class of functions to which the Hilbert transform can be applied and how it acts on them. One of possible ways consists in constructing a limit of $\int \frac{f(\xi) - f(z)}{\xi - rz} d\mu(\xi)$ as $r \nearrow 1$.

If a function f on the unit circle is sufficiently smooth, define a [regular] integral operator on $L^2(\mu)$ with kernel $\frac{f(\xi)-f(z)}{\xi-z}$,

$$h \mapsto \int \frac{f(\xi)-f(z)}{\xi-z} h(\xi) d\mu(\xi).$$

It is easily seen that its commutator with the operator of multiplication by the independent variable coincides with (1).

If $K = XU - UX$, it is easy to check the relation

$$(2) \quad X - U^{n+1} X U^{-(n+1)} = \sum_{m=0}^n U^m K U^{-(m+1)}, \quad n \geq 0.$$

Denote by B_r the Abel means of the sequence (2):

$$\begin{aligned} B_r &= (1-r) \sum_{n=0}^{\infty} r^n (X - U^{n+1} X U^{-(n+1)}) \\ &= (1-r) \sum_{n=0}^{\infty} r^n \sum_{m=0}^n U^m K U^{-(m+1)} = \sum_{m=0}^{\infty} r^m U^m K U^{-(m+1)}. \end{aligned}$$

By construction, $\|B_r\| \leq 2\|X\|$, and convergence of B_r is equivalent to that of the Abel means of $U^n X U^{-n}$.

If $K = \sum_k (\cdot, \bar{u}_k) v_k$, for $h \in L^2(\mu)$ and for z with $|z| = 1$ we have

$$\begin{aligned} (B_r h)(z) &= \left(\sum_{m=0}^{\infty} r^m U^m K U^{-(m+1)} h \right) (z) \\ &= \sum_{m=0}^{\infty} r^m z^m \sum_k \int \bar{\xi}^{-(m+1)} h(\xi) u_k(\xi) d\mu(\xi) \cdot v_k(z) = \sum_k v_k(z) \cdot \int \frac{\bar{\xi} h(\xi) u_k(\xi) d\mu(\xi)}{1 - r \bar{\xi} z}. \end{aligned}$$

Apply this formula to the operator K given by (1):

$$(3) \quad (B_r h)(z) = \int \frac{\bar{\xi} h(\xi) f(\xi) d\mu(\xi)}{1 - r \bar{\xi} z} - f(z) \cdot \int \frac{\bar{\xi} h(\xi) d\mu(\xi)}{1 - r \bar{\xi} z} = \int \frac{f(\xi) - f(z)}{\xi - rz} h(\xi) d\mu(\xi).$$

Thus, the limit of the functions $B_r 1$ (whenever it exists) gives us a definition of the Hilbert transform of f .

From now on, instead of (1), we work with K defined on $L^2(\mu)$ by

$$(4) \quad K = (\cdot, \bar{z}) f - (\cdot, \bar{z} f) 1.$$

To write it in the form $K = XU - UX$, take the operator $-XU$ in place of X that corresponds to K from formula (1). For the operators B_r we then have

$$(5) \quad (B_r h)(z) = \int \frac{f(z) - f(\xi)}{1 - r \bar{\xi} z} h(\xi) d\mu(\xi).$$

To obtain the Hilbert transform of f , one should take the limit of $B_r h$ with $h(\xi) = -\bar{\xi}$.

2. THE CLARK MEASURES

Fix an inner function θ with $\theta(0) = 0$, set $K_\theta = H^2 \ominus \theta H^2$, P_θ denotes the orthogonal projection onto K_θ .

For a complex number α with $|\alpha| = 1$ consider the singular probability measure σ_α on the unit circle determined by

$$\frac{1 + \bar{\alpha}\theta(z)}{1 - \bar{\alpha}\theta(z)} = \int \frac{1 + \bar{\xi}z}{1 - \xi z} d\sigma_\alpha(\xi).$$

It is well known [3] that the unitary operator U_α , $U_\alpha h = P_\theta z h + \alpha(h, \bar{z}\theta)1$, is unitarily equivalent to the operator of multiplication by z on $L^2(\sigma_\alpha)$. The unitary identification between K_θ and $L^2(\sigma_\alpha)$ takes a function from K_θ to its boundary values σ_α -almost everywhere [6], the inverse map sends a function $s \in L^2(\sigma_\alpha)$ to the function from K_θ whose value at a point z of the unit disk is $(1 - \theta(z)) \int \frac{s(\xi)d\sigma_\alpha(\xi)}{1 - \xi z}$.

For $\alpha = 1$ we usually write $\sigma_1 = \mu$, $U_1 = U$.

Let $f \in L^2(\mu)$, consider the function $\varphi \in K_\theta$ such that $\varphi = f$ μ -almost everywhere. Define K by (4), construct the operators B_r (5), denote by g_r the function $B_r h$ with $h \equiv 1$. We have

$$g_r(z) = \int \frac{f(z) - f(\xi)}{1 - r\bar{\xi}z} d\mu(\xi) = \frac{\varphi(z) - \varphi(rz)}{1 - \theta(rz)}.$$

Proposition 2.1. *Assume that the commutator $K = XU - UX$ has the form (4) for some $f \in L^2(\mu)$. Construct the inner function θ for which $\mu = \sigma_1$ and $\varphi \in K_\theta$ such that $\varphi = f$ μ -almost everywhere, define the functions g_r . The following are equivalent:*

- 1) *the Abel means of $U^n XU^{-n}$ have a strong (resp., weak) limit;*
- 2) *the strong (resp., weak) limit of $\int \frac{f(\xi) - f(z)}{\xi - rz} d\mu(\xi)$ exists as $r \nearrow 1$ (and thus the Hilbert transform Hf can be defined);*
- 3) *the functions g_r have a strong (resp., weak) limit as $r \nearrow 1$.*

Proof. Convergence of the Abel means of $U^n XU^{-n}$ is equivalent to that of B_r . If we apply B_r to the vector $h \equiv 1$, we obtain g_r , for $h(\xi) = -\bar{x}i$ we obtain $(B_r h)(z) = \int \frac{f(\xi) - f(z)}{\xi - rz} d\mu(\xi)$. It remains to notice that the subset of $L^2(\mu)$ on which the convergence holds is a closed subspace reducing U , and convergence for $h \equiv 1$ or for $h = \bar{z}$ implies the same convergence on the whole of $L^2(\mu)$. \square

The limit g of g_r and Hf satisfy the relation

$$(6) \quad g(z) + z(Hf)(z) = \lim \left(\int \frac{f(z) - f(\xi)}{1 - r\bar{\xi}z} d\mu(\xi) + z \cdot \int \frac{f(\xi) - f(z)}{\xi - rz} d\mu(\xi) \right) = f(z) - \int f d\mu.$$

Now we present a proof of the fact that the Hilbert transform H is not a continuous operator on $L^2(\mu)$.

Proof of Proposition 1.1. Let Q denote the transplantation of the map $f \mapsto f - Hzf$ from $L^2(\mu)$ to K_θ by the standard identification of K_θ and $L^2(\mu)$. We have $H1 = 0$, $Hz = 1$; if $\int f d\mu = 0$ then $Hzf = zHf$. Then, accordingly, Q possesses the

following properties:

$$\text{a) } Q1 = 0; \quad \text{b) } Q(\bar{z}\theta) = \bar{z}\theta; \quad \text{c) } h, zh \in K_\theta \implies Qzh = zQh.$$

Continuity of Q on $L^2(\mu)$ and properties a) and c) would imply $Q = 0$, cf. [4]. Indeed, for $h \in K_\theta$ we have

$$Qh = Q(h - h(0)) + h(0)Q1 = zQ\frac{h - h(0)}{z}.$$

Iterations of the operator $h \mapsto \frac{h-h(0)}{z}$ form a sequence in K_θ with norms tending to 0, hence $Qh = 0$. At the same time, $Q \neq 0$ by property b). \square

3. TRUNCATED TOEPLITZ OPERATORS

Basic properties of truncated Toeplitz operators were studied in [2].

For $\psi \in L^2$ the truncated Toeplitz operator A_ψ is defined by

$$A_\psi h = P_\theta \psi h, \quad h \in K_\theta;$$

we assume that the symbol ψ is such that A_ψ is bounded.

Obviously, functions from $\theta H^2 + \overline{\theta H^2}$ determine zero operator, thus we may restrict ourselves by the assumption that the symbol belongs to $K_\theta + \overline{K_\theta}$. For a truncated Toeplitz operator A take a symbol $\psi \in K_\theta + \overline{K_\theta}$. We write

$$\psi_+ = P_+ \psi = A1 \in K_\theta, \quad \psi_- = P_- \psi \in \overline{K_\theta}, \quad \psi_0 = (\psi, 1) = (A1, 1) = (A(\bar{z}\theta), \bar{z}\theta).$$

We also have

$$(\psi_-, 1) = 0, \quad A(\bar{z}\theta) = (\psi_- + \psi_0)\bar{z}\theta.$$

For the symbol we then have

$$(7) \quad \psi = \psi_+ + \psi_- = A1 + z\bar{\theta} \cdot A(\bar{z}\theta) - (A1, 1)1.$$

The following theorem shows that Conjecture 0.1 with $\text{rank } K = 2$ reduces to a problem about truncated Toeplitz operators.

Theorem 3.1. 1) *An operator A on K_θ is a truncated Toeplitz operator if and only if*

$$(8) \quad K = AU - UA = (\cdot, \bar{z}\theta)\varphi - (\cdot, \bar{z}\theta\bar{\varphi})1$$

for some $\varphi \in K_\theta$.

2) *If A is a truncated Toeplitz operator with $(A1, 1) = 0$, then*

$$(9) \quad \varphi = A1 - zA(\bar{z}\theta)$$

up to an additive constant.

3) *If $A = A_\psi$, then*

$$(10) \quad \varphi = \psi_+ - \theta\psi_-,$$

hence $\varphi = \psi$ σ_{-1} -almost everywhere. A truncated Toeplitz operator A_ψ commutes with U if and only if $\varphi = \psi_+ - \theta\psi_- \equiv \text{const}$.

For a constant function φ we obtain $K = 0$ in (8), therefore, the function φ is determined up to a constant. The standard choice will be determined by the condition $\varphi(0) = 0$.

Proof. 1) Theorem 8.1 from [2] may be rewritten as follows: *X is a truncated Toeplitz operator if and only if $(f, \bar{z}\theta) = 0$, $(g, 1) = 0$ imply $(Kf, g) = 0$.* This is equivalent to a representation of K in the form $K = (\cdot, \bar{z}\theta)\varphi + (\cdot, \gamma)1$ for some $\varphi, \gamma \in K_\theta$, and thus (8) implies that A is a truncated Toeplitz operator. Conversely, by Theorem 6.1 of [1] we obtain $z\varphi + \bar{\gamma} = 0$ μ -almost everywhere, hence γ is the function from K_θ whose values equal $\bar{z}\varphi$ μ -almost everywhere. This means that $\gamma = \bar{z}\theta\bar{\varphi}$, which yields (8).

2) Apply relation (8) to the vector $\bar{z}\theta$:

$$\begin{aligned} K(\bar{z}\theta) &= (AU - UA)(\bar{z}\theta) = A1 - UA(\bar{z}\theta) = A1 - zA(\bar{z}\theta); \\ &= (\bar{z}\theta, \bar{z}\theta)\varphi - (\bar{z}\theta, \bar{z}\theta\bar{\varphi})1 = \varphi - \varphi(0). \end{aligned}$$

3) For a constant function ψ the fact is trivial. Now without loss of generality we may think that $\psi_0 = 0$. By the preceding formula, for $A = A_\psi$ we get

$$\varphi - \varphi(0) = A_\psi 1 - zA_\psi(\bar{z}\theta) = \psi_+ - z(\psi_- + \psi_0)\bar{z}\theta = \psi_+ - \theta\psi_-.$$

The claim now easily follows. \square

A complex measure \varkappa on the unit circle will be called a *quasisymbol* of a truncated Toeplitz operator A acting on K_θ if $(Af, g) = \int f\bar{g} d\varkappa$ for all (continuous) $f, g \in K_\theta$.

Proposition 3.2. *Take α with $|\alpha| = 1$ and $q \in L^\infty(\sigma_\alpha)$. Let ω be the function from K_θ whose values coincide with q σ_α -everywhere. Then $q(U_\alpha)$ is a truncated Toeplitz operator with symbol $(1 + \alpha\bar{\theta})\omega$, and $q\sigma_\alpha$ is a quasisymbol of $q(U_\alpha)$. For the function φ that determines the commutator (8) we have $\varphi = (1 - \alpha)\omega$.*

Proof. Sarason observed that all operators commuting with U_α are truncated Toeplitz operators, see [2], Section 12.

The fact is obvious if q is constant. Therefore, we may assume that $\psi_0 = \int q d\sigma_\alpha = \omega(0) = 0$. For the symbol of $A = q(U_\alpha)$ use formula (7):

$$\psi = A1 + z\bar{\theta} \cdot A(\bar{z}\theta) - (A1, 1)1 = \omega + (z\bar{\theta}) \cdot \bar{z}\alpha\omega = (1 + \alpha\bar{\theta})\omega.$$

By (10), from the formula for ψ for $A = q(U_\alpha)$ we obtain $\psi_+ = \omega$, $\psi_- = \alpha\bar{\theta}\omega$, $\varphi = \psi_+ - \theta\psi_- = (1 - \alpha)\omega$. \square

4. BOUNDARY BEHAVIOUR OF FUNCTIONS FROM K_θ

Denote by \mathfrak{K} the class of functions $\varphi \in K_\theta$, for which there exists a bounded truncated Toeplitz operator A such that formula (8) is fulfilled. Conjecture 0.1 for the case rank $K = 2$ is equivalent to the weak convergence of functions (5) as $r \nearrow 1$ for all $\varphi \in \mathfrak{K}$.

It follows from Proposition 3.2 that for any unimodular α , $\alpha \neq 1$, a function from K_θ belongs to \mathfrak{K} if its boundary values belong to $L^\infty(\sigma_\alpha)$ for some unimodular $\alpha \neq 1$. (A natural question is whether or not the same is true for $\alpha = 1$.) The example from [5] implies that there exist functions from \mathfrak{K} whose boundary values σ_α -almost everywhere

do not constitute a function from $L^\infty(\sigma_\alpha)$. There are other sufficient conditions for $\varphi \in K_\theta$ to belong to \mathfrak{K} . In particular, all bounded functions from K_θ belong to \mathfrak{K} ; also all functions of the form $P_\theta u$ with $u \in H^\infty$ are in \mathfrak{K} , they are realized by the truncated Toeplitz operator with symbol u .

Poltoratski [6] established convergence $\varphi(rz) \rightarrow \varphi(z)$, $\varphi \in K_\theta$, as $r \nearrow 1$ in $L^2(\sigma_\alpha)$, and, moreover, for σ_α -almost all z . We obtain another stronger version of the former convergence for functions from \mathfrak{K} .

Theorem 4.1. *Let $\varphi \in \mathfrak{K}$. Then $\varphi(z) - \varphi(rz) = (1 - \theta(rz))g_r(z)$, where the norms $\|g_r\|_{L^2(\mu)}$ are bounded uniformly in r .*

Proof. If $\varphi \in \mathfrak{K}$, then formula (8) is fulfilled for some truncated Toeplitz operator A . Hence g_r have the form $B_r h$ with $\|h\| = 1$ and this implies that their norms in $L^2(\mu)$ are bounded by $2 \cdot \|A\|$. \square

In Theorem 5.6 below we establish a sufficient condition for convergence of g_r as $r \nearrow 1$. As a partial case of Theorems 4.1 and 5.6, we obtain the following result.

Theorem 4.2. *Take $\varphi \in K_\theta$, let g_r be defined by $\varphi(z) - \varphi(rz) = (1 - \theta(rz))g_r(z)$. If φ coincides with a bounded (resp., continuous) function σ_α -almost everywhere for some unimodular $\alpha \neq 1$, then the norms $\|g_r\|_{L^2(\mu)}$ are bounded uniformly in r (resp., the limit of g_r exists in $L^2(\mu)$).*

5. CONVERGENCE FOR CONTINUOUS FUNCTIONS

For a sequence (x_0, x_1, x_2, \dots) the Cesaro means are its arithmetical means $\frac{1}{n+1} \sum_{k=0}^n x_k$.

Theorem 5.1. *Suppose that U is a unitary operator with singular spectrum having no eigenvectors, X is a compact operator. Then the Cesaro means of the sequence $U^n X U^{-n}$ tend to zero in the strong operator topology.*

Proof. Since every compact operator can be approximated in norm by finite-rank operators, it suffices to consider the case of a rank-one operator X . One may think that X acts on $L^2(\mu)$, where, as usual, U is the operator of multiplication by the independent variable. Let $X = (\cdot, \bar{u})v$. Then $U^n X U^{-n} h = (z^{-n} h, \bar{u}) z^n v$, and for the norms we have $\|U^n X U^{-n} h\| \leq \|v\| \cdot \left| \int z^{-n} h(z) u(z) d\mu(z) \right|$. We obtain Fourier coefficients for the complex measure $h u \mu$ on the unit circle, the Cesaro means of their absolute values tend to zero by the classic Wiener theorem. \square

Lemma 5.2. *If q is a continuous function on the unit circle, α_1, α_2 are two distinct numbers on the unit circle, then $q(U_{\alpha_1}) - q(U_{\alpha_2})$ is a compact operator.*

Proof. Since any continuous function may be uniformly approximated by [trigonometrical] polynomials, it suffices to prove the lemma for the polynomials. We have $\text{rank}(U_{\alpha_1} - U_{\alpha_2}) = 1$, and this easily implies that $\text{rank}(q(U_{\alpha_1}) - q(U_{\alpha_2})) < \infty$ if q is a polynomial. \square

Corollary 5.3. *For any continuous function q on the unit circle and for every unimodular constant α the Cesaro means of the sequence $U^n q(U_\alpha) U^{-n}$ strongly tend to $q(U)$.*

Proof. Apply Theorem 5.1 to the operator $X = q(U_\alpha) - q(U)$, which is compact by the Lemma. \square

Theorem 5.4. *Let A be a truncated Toeplitz operator such that formula (8) is fulfilled for $\varphi \in K_\theta$. If φ coincides with a continuous function q σ_α -almost everywhere for some $\alpha \neq 1$, then the strong limit of the sequence of Cesaro means of $U^n A U^{-n}$ exists and equals $A - \frac{1}{1-\alpha}(q(U_\alpha) - q(U))$.*

Proof. From Proposition 3.2 it follows that the operator $A - \frac{1}{1-\alpha}q(U_\alpha)$ commutes with U . Therefore,

$$\begin{aligned} U^n A U^{-n} &= U^n \left(A - \frac{1}{1-\alpha}q(U_\alpha) \right) U^{-n} + U^n \cdot \frac{1}{1-\alpha}q(U_\alpha) U^{-n} \\ &= \left(A - \frac{1}{1-\alpha}q(U_\alpha) \right) + \frac{1}{1-\alpha}U^n q(U_\alpha) U^{-n}, \end{aligned}$$

it remains to apply Corollary 5.3. \square

Corollary 5.5. *The Cesaro means of the sequence $U^n A_\psi U^{-n}$ tend to a limit for a truncated Toeplitz operator A_ψ with symbol $\psi \in K_\theta + \overline{K_\theta}$ if ψ coincides with a continuous function σ_{-1} -almost everywhere.*

Theorem 5.6. *In the conditions of Proposition 2.1 assume that for some unimodular complex number $\alpha \neq 1$ the function $\varphi \in K_\theta$ coincides σ_α -almost everywhere with a continuous function q . Then the equivalent properties 1), 2), 3) of the Proposition are fulfilled. Moreover, if $\int f d\mu = 0$, then*

$$Hf = \bar{z} \frac{q - \alpha f}{1 - \alpha};$$

for a constant function f we have $Hf = 0$.

This formula may be compared with formula (2.1) in [7]. The latter also connects the Hilbert transform, the ‘‘Clark transform’’ ($f \mapsto q$), and extension by continuity, but on the real line and under stronger smoothness assumptions.

In the particular case $\alpha = -1$ we have

$$(Hf)(z) = \frac{f(z) + q(z)}{2z}.$$

A simpler formula is true for the limit g of g_r , even without the assumption $\int f d\mu = 0$. From (6) we obtain

$$g = \frac{f - q}{1 - \alpha}.$$

(This relation is obvious if f is constant.)

Once again, it is interesting to know if the theorem holds true under the same assumptions for $\alpha = 1$.

Proof. The convergence of B_r follows from Theorem 5.4.

To obtain a formula for the Hilbert transform, we use formula (5) with $h(\xi) = \bar{\xi}$, in which we need to pass to the limit as $r \nearrow 1$. Also recall that the operators B_r were defined as the Abel means of the sequence (2). To fulfill formula (4) in $L^2(\mu)$, take the truncated Toeplitz operator $A = \frac{1}{1-\alpha}q(U_\alpha)$ on K_θ . The operators B_r are the Abel means of the sequence $A - U^nAU^{-n}$, by Corollary 5.3 they tend to $\frac{1}{1-\alpha}(q(U_\alpha) - q(U))$. As was shown above, to find the Hilbert transform Hf , we must apply this formula to the function from K_θ corresponding to the vector $h \in L^2(\mu)$, $h(\xi) = -\bar{\xi}$, that is, to $-\bar{z}\theta$, and consider the values of the resulting function $\frac{-1}{1-\alpha}(q(U_\alpha) - q(U))(\bar{z}\theta)$ μ -almost everywhere. If q is a constant, we trivially obtain 0. Now take q with $\int q d\sigma_\alpha = 0$; equivalently, $\int f d\mu = 0$; this also yields $\bar{z}\omega \in K_\theta$. We have $q(U_\alpha)\bar{z}\theta = q\bar{z}\alpha = \alpha\bar{z}\omega$ σ_α -almost everywhere, hence $q(U_\alpha)(\bar{z}\theta) = \alpha\bar{z}\omega = \alpha\bar{z}f$ μ -almost everywhere. Furthermore, $q(U)(\bar{z}\theta) = \bar{z}q$ μ -almost everywhere. We obtain $Hf = \frac{-1}{1-\alpha}(\alpha\bar{z}f - \bar{z}q) = \bar{z}\frac{q-\alpha f}{1-\alpha}$. \square

The expressions obtained essentially depend on a possibility to extend functions defined σ_α -almost everywhere to μ -almost all points by continuity. It is natural to ask if the continuity property may be weakened. A natural generalization may hold if μ -almost all points are Lebesgue points for σ_{-1} . It would also be important to describe the class of $f \in L^2(\mu)$ to which these arguments may be applied.

Conjecture 5.7. *Set $\mu = \sigma_1$, $\nu = \sigma_{-1}$, take $q \in L^2(\nu)$ that coincides with a symbol of a bounded truncated Toeplitz operator A ν -almost everywhere. Denote by $I_{z,\epsilon}$ the arc of length ϵ centered at z . If the limit*

$$\lim \frac{\int_{I_{z,\epsilon}} q(\xi) d\nu(\xi)}{\int_{I_{z,\epsilon}} d\nu(\xi)}$$

exists for μ -almost all z , then the Abel means of the sequence U^nAU^{-n} tend to a limit as $r \nearrow 1$.

We considered the sequence $U_2^n XU_1^{-n}$ as $n \rightarrow +\infty$, the same arguments work for this sequence as $n \rightarrow +\infty$. It is an interesting question if the limits as $n \rightarrow +\infty$ and as $n \rightarrow -\infty$ must coincide (if they exist) in the case of rank-two commutator for singular spectral measures. In general, one may consider their half-sum, which would also give us a natural definition of the Hilbert transform as soon as existence of a limit will be established.

Now suppose that Conjecture 0.1 is true, hence a wave operator and the Hilbert transform are constructed. One can see from our results that this means the existence of “natural” maps $L^\infty(\sigma_\alpha) \rightarrow L^\infty(\mu)$ sending continuous functions to “themselves”, but these maps cannot be extended to continuous maps $L^2(\sigma_\alpha) \rightarrow L^2(\mu)$. This gives a reason to think that Conjecture 0.1 must be disproved.

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