ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

РЕДКОЛЛЕГИЯ

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On the linear problem arising in the study of a free boundary problem for the Navier–Stokes equations

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Dedicated to Professor V. M. Babich on the occasion of his 80th birthday

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ABSTRACT:

We prove the solvability in the Sobolev–Slobodetskii spaces and obtain coercive estimates of the solution of a basic linear problem arising in the study of evolution of an isolated mass of the capillary viscous incomressible liquid. The results obtained in the paper can be used in the proof of local solvability of the above-mentioned free boundary problem without any specific restrictions on the shape of initial configuration of the liquid.

Key words: Stokes and Navier–Stokes equations, coercive estimates.

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1 Introduction

The paper is devoted to the linear problem

$$\begin{cases} \boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\ \nabla\cdot\boldsymbol{v} = f(x,t) = \nabla\cdot\boldsymbol{F}(x,t), \quad x\in\mathcal{F}, \quad t>0, \\ T(\boldsymbol{v},p)\boldsymbol{N}(x) + \sigma\boldsymbol{N}(x)L\rho = \boldsymbol{d}(x,t), \\ \rho_t(x,t) + \boldsymbol{V}(x)\cdot\nabla_\tau\rho - \boldsymbol{v}(x,t)\cdot\boldsymbol{N}(x) = g(x,t), \quad x\in\mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x\in\mathcal{F}, \quad \rho(x,0) = \rho_0(x), \quad x\in\mathcal{G}, \end{cases} \tag{1.1}$$

in a bounded domain $\mathcal{F}\subset\mathbb{R}^3$ with a smooth boundary $\mathcal{G}$. Unknown are the vector field $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$ and functions $p(x,t)$, $x\in\mathcal{F}$, and $\rho(x,t)$, $x\in\mathcal{G}$. By $T(\boldsymbol{v},p) = -pI+\nu S(\boldsymbol{v})$ we mean the stress tensor, $S(\boldsymbol{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, $\boldsymbol{N}$ is the exterior normal to $\mathcal{G}$, ν and σ are positive constants, and

$$L\rho = -\Delta_{\mathcal{G}}\rho + b(x)\rho,$$

where $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$ and $b(x)$ is a smooth function. Finally, $\boldsymbol{V}(x)$ is a vector field defined on $\mathcal{G}$ and ∇_τ is the tangential part of the gradient.

The problem (1.1) arises as a result of linearization of a free boundary problem for the Navier-Stokes equations, governing the evolution of an isolated mass of a viscous incompressible capillary liquid. The latter has been studied in the papers [1,2] and others, where the method of the Lagrangian coordinates was used. It turned out to be especially fruitful in the case when the surface tension is not taken into account [3]. The problem (1.1) is obtained when so called Hanzawa coordinate transformation is applied to the free boundary problem in order to write it a fixed domain (see formula (5.2)). This transformation gives some technical advantages in the case of a capillary liquid, when the coefficient of the surface tension σ is positive. It may be useful also in the analysis of some more complicated problems (e.g., problems of magnetohydrodynamics).

In [4] the problem (1.1) has been studied in the Hölder spaces of functions.

The main result of the paper is the coercive estimate of the solution of the problem (1.1) in anisotropic Sobolev-Slobodetskii spaces $W_2^{l,l/2}(Q_T)$ in a cylindrical domain $Q_T = \mathcal{F}\times(0,T)$. We recall the definition of these spaces. Let Ω be a domain in $\mathbb{R}^n$. The (isotropic) Sobolev space $W_2^l(\Omega)$ with $l>0$ is the space of functions $u(x)$, $x\in\Omega$, with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{0\le|j|\le l}\|D^j u\|^2_{L_2(\Omega)} \equiv \sum_{0\le|j|\le l}\int_\Omega |D^j u(x)|^2 dx,$$

if $l=[l]$, i.e. l is an integral number, and

$$\|u\|^2_{W_2^l(\Omega)} = \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|j|=[l]}\int_\Omega\int_\Omega |D^j u(x) - D^j u(y)|^2 \frac{dxdy}{|x-y|^{n+2\lambda}},$$

if $l=[l]+\lambda$, $\lambda\in(0,1)$. As usual, $D^j u$ denotes a (generalized) partial derivative $\frac{\partial^{|j|}u}{\partial x_1^{j_1}\dots\partial x_n^{j_n}}$ where $j=(j_1,j_2,\dots j_n)$ and $|j| = j_1+\dots+j_n$. The anisotropic space $W_2^{l,l/2}(Q_T)$, $Q_T = \Omega\times(0,T)$, can be defined as

$$L_2((0,T),W_2^l(\Omega)) \cap W_2^{l/2}((0,T),L_2(\Omega))$$

and supplied with the norm

$$\|u\|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{l/2}(0,T)}dx. \tag{1.2}$$

There exist many other equivalent norms in $W_2^{l,l/2}(Q_T)$; some of them will be used below. Sobolev spaces of functions given on smooth surfaces, in particular, on $\mathcal{G}$ and on $G_T = \mathcal{G}\times(0,T)$, are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to introduce the spaces $W_2^{l,0}(Q_T) = L_2((0,T),W_2^l(\Omega))$ and $W_2^{0,l/2}(Q_T) = W_2^{l/2}((0,T),L_2(\Omega))$; the squares of norms in these spaces coincide, respectively, with the first and the second term in (1.2). Finally, by $|u|_{l/2,r,Q_T}$ and $|u|_{l/2,r,G_T}$ we mean the norms of u in $W_2^{l/2}(0,T;W_2^r(\Omega))$ and $W_2^{l/2}(0,T;W_2^r(\mathcal{G}))$, respectively.

Theorem 1.1. *Assume that $l\in[0,5/2)$, $l\neq 1/2,1,3/2$ and that the data of the problem (1.1) possess the following regularity properties: $\boldsymbol{f}\in W_2^{l,l/2}(Q_T)$, $f\in W_2^{l+1,0}(Q_T)$, $f(x,t)=\nabla\cdot\boldsymbol{F}(x,t)$, $\boldsymbol{F}\in W_2^{0,1+l/2}(Q_T)$, $\boldsymbol{d}\cdot\boldsymbol{N}\in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}\mathcal{G}))$, $\boldsymbol{d}-\boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\in W_2^{l+1/2,l/2+1/4}(G_T)$, $g\in W_2^{l+3/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{3/2}(\mathcal{G}))$, $\boldsymbol{v}_0\in W_2^{l+1}(\mathcal{F})$, $\rho_0\in W_2^{l+2}(\mathcal{G})$ where $T<\infty$, $Q_T=\mathcal{F}_1\times(0,T)$, $G_T=\mathcal{G}\times(0,T)$. Assume also that $\boldsymbol{V}\in W_2^{l+3/2}(\mathcal{G})$. Finally, let the compatibility conditions*

$$\begin{aligned}&\nabla\cdot\boldsymbol{v}_0(x)=f(x,0),\quad x\in\mathcal{F},\quad \text{if}\quad l<1/2,\\ &\nabla\cdot\boldsymbol{v}_0(x)=f(x,0),\quad x\in\mathcal{F},\quad \nu\Pi_{\mathcal{G}}S(\boldsymbol{v}_0)\boldsymbol{N}=\Pi_{\mathcal{G}}\boldsymbol{d}(x,0),\quad x\in\mathcal{G},\quad \text{if}\quad l>1/2\end{aligned} \tag{1.3}$$

be satisfied, where $\Pi_{\mathcal{G}}\boldsymbol{d}=\boldsymbol{d}-\boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})$ is the projection of $\boldsymbol{d}$ on the tangent plane to $\mathcal{G}$. Then the problem (1.1) has a unique solution $\boldsymbol{v},p,\rho$ such that $\boldsymbol{v}\in W_2^{l+2,l/2+1}(Q_T)$, $\nabla p\in W_2^{l,l/2}(Q_T)$, $p\in W_2^{l+1/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(\mathcal{G}))$, $\rho\in W_2^{l+5/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))$, $\rho_t\in W_2^{l+3/2,0}(G_T)\cap W_2^{l/2}(0,T;W_2^{3/2}(\mathcal{G}))$, $\rho(\cdot,t)\in W_2^{l+2}(\mathcal{G})$, $\forall t\in(0,T)$, and the solution satisfies the inequality

$$\begin{aligned}Y_T(\boldsymbol{v},p,\rho)\equiv&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T)}+\|\nabla p\|_{W_2^{l,l/2}(Q_T)}+\|p\|_{W_2^{l+1/2,0}(G_T)}+|p|_{l/2,1/2,G_T}\\ &+\|\rho\|_{W_2^{l+5/2,0}(G_T)}+|\rho|_{l/2,5/2,G_T}+\|\rho_t\|_{W_2^{l+3/2,0}(G_T)}+|\rho_t|_{l/2,3/2,G_T}\\ &\le c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}+\|f\|_{W_2^{l+1,0}(Q_T)}+\|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)}\\ &+\|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}+\|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)}+|\boldsymbol{d}\cdot\boldsymbol{N}|_{l/2,1/2,G_T}\\ &+\|g\|_{W_2^{l+3/2,0}(G_T)}+|g|_{l/2,3/2,G_T}+\|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)}+\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big)\equiv N_T.\end{aligned} \tag{1.4}$$

Moreover, if $g\in W_2^{l+3/2,l/2+3/4}(G_T)$, then $\rho_t\in W_2^{l+3/2,l/2+3/4}(G_T)$, and

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T)}+\|\nabla p\|_{W_2^{l,l/2}(Q_T)}+\|p\|_{W_2^{l+1/2,0}(G_T)}+|p|_{l/2,1/2,G_T}\\ &+\|\rho\|_{W_2^{l+5/2,0}(G_T)}+|\rho|_{l/2,5/2,G_T}+\|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\\ &\le c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)}+\|f\|_{W_2^{l+1,0}(Q_T)}+\|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)}\\ &+\|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)}+\|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)}+|\boldsymbol{d}\cdot\boldsymbol{N}|_{l/2,1/2,G_T}\\ &+\|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)}+\|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)}+\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big).\end{aligned} \tag{1.5}$$

The restriction $l < 5/2$ minimizes the order of compatility of the initial and boundary data expressed by (1.3). The requirement $l \neq 1/2, 1, 3/2$ is technical; it is made to avoid the cases in which the compatibility conditions (1.3) should be strongly modified (in this connection see [5,6]).

It follows from the imbedding theorems that in the case $f = 0$, $\boldsymbol{F} = 0$ the estimate (1.4) is coercive, i.e.,

$$N_T \leq cY_T(\boldsymbol{v}, p, \rho);$$

the same is true for (1.5). Hence Theorem 1.1 guarantees the existence of the solution of the problem (1.1) with maximal regularity properties.

By the trace theorem for the space $W_2^{l+2,l/2+1}(Q_T)$, $\boldsymbol{v}(\cdot, t) \in W_2^{l+1}(\mathcal{F})$, i.e., $\boldsymbol{v}$ is as smooth as $\boldsymbol{v}_0$. It follows from Proposition 4.1 that the same is true for ρ.

The proof of Theorem 1 is given in Sec. 2-4. Sec. 5 contains a short discussion (without detailed proofs) of the application of Theorem 1.1 to the free boundary problem with the initial domain of arbitrary shape and with the vector field $\boldsymbol{v}_0(x)$ of the initial velocity that needs not be small.

2 Parameter-dependent problem

Following [7], we consider the problem with a complex parameter s:

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \\ \nabla \cdot \boldsymbol{v}(x) = 0, \quad x \in \mathcal{F}, \\ T(\boldsymbol{v}, p)\boldsymbol{N} + \sigma\boldsymbol{N}L_0\rho = \boldsymbol{d}(x), \\ s\rho + \boldsymbol{V} \cdot \nabla_\tau\rho - \boldsymbol{v}(x) \cdot \boldsymbol{N}(x) = g(x), \quad x \in \mathcal{G}. \end{cases} \tag{2.1}$$

The solution of (2.1) is also sought in the space of complex-valued functions.

Theorem 2.2. *Let* $\operatorname{Re} s \geq a \gg 1$, $\boldsymbol{f} \in W_2^l(\mathcal{F})$, $\boldsymbol{d} \in W_2^{l+1/2}(\mathcal{G})$, $g \in W_2^{l+3/2}(\mathcal{G})$ *with* $l \in [0, 5/2)$. *Then the problem* (2.1) *has a unique solution* $\boldsymbol{v} \in W_2^{2+l}(\mathcal{F})$, $p \in W_2^{1+l}(\mathcal{F})$, $\rho \in W_2^{l+5/2}(\mathcal{G})$, *and*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{2+l}(\mathcal{F})} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|p\|_{W_2^{l+1}(\mathcal{F})} + |s|^{l/2}\|p\|_{W_2^1(\mathcal{F})} + |s|^{1+l/2}\|\rho\|_{W_2^{3/2}(\mathcal{G})} \\ &+ |s|\|\rho\|_{W_2^{l+3/2}(\mathcal{G})} + |s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathcal{G})} + \|\rho\|_{W_2^{l+5/2}(\mathcal{G})} \leq c\big(\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F})} \\ &+ |s|^{1/4+l/2}\|\boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\|_{L_2(\mathcal{G})} + \|\boldsymbol{d}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathcal{G})} \\ &+ |s|^{l/2}\|g\|_{W_2^{3/2}(\mathcal{G})} + \|g\|_{W_2^{l+3/2}(\mathcal{G})}\big). \end{aligned} \tag{2.2}$$

with the constant independent of $|s|$ *(but it may depend on* a*).*

Proof. We start with the proof of the estimate (2.2). Without loss of generality we may assume that $\boldsymbol{f}$ is divergence free, because arbitrary $\boldsymbol{f} \in L_2(\mathcal{F})$ can be decomposed in the orthogonal sum

$$\boldsymbol{v} = \boldsymbol{f}' + \nabla\varphi,$$

where $\boldsymbol{f}'$ is divergence free and φ is a solution of the Dirichlet problem

$$\nabla^2\varphi = 0, \quad x \in \mathcal{F}, \quad \varphi|_{\mathcal{G}} = 0.$$

Since

$$c_1\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} \le \|\nabla\varphi\|_{W_2^l(\mathcal{F})} + \|\boldsymbol{f}'\|_{W_2^l(\mathcal{F})} \le c_2\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})},$$

the problem (2.1) is equivalent to a similar problem with $\boldsymbol{f}$ and p replaced by $\boldsymbol{f}'$ and $p' = p - \nabla\varphi$, respectively.

STEP 1. We consider a model problem in the half-space $\mathbb{R}^3_+ = \{x_3 > 0\}$

$$\begin{cases} s\boldsymbol{v}(x) + \boldsymbol{V}'\cdot\nabla'\boldsymbol{v}(x) - \nu\nabla^2\boldsymbol{v}(x) + \nabla p(x) = 0, \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = b_j(x'), \quad j = 1,2, \\ -p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = b_3(x'), \\ s\rho + \boldsymbol{V}'\cdot\nabla'\rho + v_3(x) = g(x), \quad x_3 = 0, \end{cases} \tag{2.3}$$

where $\boldsymbol{V}'$ is a constant vector of the form $\boldsymbol{V}' = (V_1, V_2)$, $x' = (x_1, x_2)$ and $\nabla' = \Big(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\Big)$. Using the Fourier transform in x_1, x_2, we reduce (2.3) to the boundary value problem on the half-axis $\mathbb{R}_+ = \{x_3 > 0\}$:

$$\begin{cases} \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + i\xi_j\widetilde{p} = 0 \quad j = 1,2, \\ \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_3 + \dfrac{d\widetilde{p}}{dx_3} = 0, \quad i\xi_1\widetilde{v}_1 + i\xi_2\widetilde{v}_2 + \dfrac{d\widetilde{v}_3}{dx_3} = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{dv_j}{dx_3} + i\xi_j\widetilde{v}_3\Big) = \widetilde{b}_j, \quad j = 1,2, \\ -\widetilde{p} + 2\nu\dfrac{d\widetilde{v}_3}{dx_3} + \sigma|\xi|^2\widetilde{\rho} = \widetilde{b}_3, \\ s_1\widetilde{\rho} + \widetilde{v}_3 = \widetilde{g}, \quad x_3 = 0, \\ \widetilde{\boldsymbol{v}} \to 0, \quad \widetilde{p} \to 0, \quad (x_3 \to \infty), \end{cases} \tag{2.4}$$

where $\xi = (\xi_1, \xi_2)$, $r_1 = r_1(s,\xi) = \sqrt{s_1\nu^{-1} + |\xi|^2}$, $-\pi \le arg\ r_1 < \pi$, $s_1 = s + i\boldsymbol{V}'\cdot\xi$.

It is convenient to exclude the function $\widetilde{\rho}$ from (2.4) and write this problem in the form

$$\begin{cases} \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + i\xi_j\widetilde{p} = 0 \quad j = 1,2, \\ \nu\Big(r_1^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + \dfrac{d\widetilde{p}}{dx_3} = 0, \quad i\xi_1\widetilde{v}_1 + i\xi_2\widetilde{v}_2 + \dfrac{d\widetilde{v}_3}{dx_3} = 0, \quad x_3 > 0, \\ \nu\Big(\dfrac{d\widetilde{v}_j}{dx_3} + i\xi_j\widetilde{v}_3\Big) = \widetilde{b}_j, \quad j = 1,2, \\ -\widetilde{p} + 2\nu\dfrac{d\widetilde{v}_3}{dx_3} - \dfrac{\sigma}{s_1}|\xi|^2\widetilde{v}_3 = \widetilde{b}_3 - \dfrac{\sigma}{s_1}|\xi|^2\widetilde{g}, \quad x_3 = 0, \\ \widetilde{\boldsymbol{v}} \to 0, \quad \widetilde{p} \to 0, \quad (x_3 \to \infty). \end{cases} \tag{2.5}$$

In the paper [2] the explicit formula for the solution of (2.5) is obtained, in particular, it

is shown that in the case $\mathrm{Re}s_1 > 0$

$$\widetilde{v}_i = -\frac{1-\delta_{i3}}{\nu r_1} e_0(x_3)\widetilde{b}_i + \frac{e_0(x_3)}{\nu^2 r_1(r_1+|\xi|)P_1}\sum_{j=1}^{3} U_{ij}\widetilde{b}_j + \frac{e_1(x_3)}{\nu^2(r_1+|\xi|)P_1}\sum_{j=1}^{3} V_{ij}\widetilde{b}_j$$
$$- \frac{\sigma|\xi|^2 e_0(x_3)}{\nu^2 s_1 r_1(r_1+|\xi|)P_1} U_{i3}\widetilde{g} - \frac{\sigma|\xi|^2 e_1(x_3)}{\nu^2 s_1(r_1+|\xi|)P_1} V_{i3}\widetilde{g}, \quad i = 1,2,3, \tag{2.6}$$

$$\widetilde{p} = \frac{r_1 s_1}{\nu P_1}\Big[\Big(2\nu + \frac{\sigma\xi^2}{s_1 r_1}\Big)(i\xi_1\widetilde{b}_1 + i\xi_2\widetilde{b}_2) - \nu\Big(r_1 + \frac{\xi^2}{r_1}\Big)(\widetilde{b}_3 - \frac{\sigma}{s_1}|\xi|^2\widetilde{g})\Big]e^{-|\xi|x_3}, \tag{2.7}$$

where

$$e_0(x_3) = e^{-r_1 x_3}, \quad e_1(x_3) = \frac{e^{-r_1 x_3} - e^{-|\xi|x_3}}{r_1 - |\xi|}, \tag{2.8}$$

$$P_1 = (r_1^2 + |\xi|^2)^2 - 4r_1|\xi|^2 + \frac{\sigma}{\nu^2}|\xi|^3 = \frac{s_1}{\nu}\Big(\frac{s_1}{\nu} + 4|\xi|^2\Big(1 - \frac{|\xi|}{r_1+|\xi|}\Big) + \frac{\sigma|\xi|^3}{\nu s_1}\Big), \tag{2.9}$$

and U_{ij}, V_{ij} are the elements of the matrices

$$\mathcal{U} = \begin{pmatrix} \xi_1^2((3r_1-|\xi|)s_1 + \frac{\sigma}{\nu}|\xi|^2) & \xi_1\xi_2((3r_1-|\xi|)s_1 + \frac{\sigma}{\nu}|\xi|^2) & i\xi_1 r_1 s_1(r_1-|\xi|) \\ \xi_1\xi_2((3r_1-|\xi|)s_1 + \frac{\sigma}{\nu}|\xi|^2) & \xi_2^2((3r_1-|\xi|)s_1 + \frac{\sigma}{\nu}|\xi|^2) & i\xi_1 r_1 s_1(r_1-|\xi|) \\ -i\xi_1 r_1 s_1(r_1-|\xi|) & -i\xi_2 r_1 s_1(r_1-|\xi|) & -|\xi| r_1 s_1(r_1+|\xi|) \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} -\xi_1^2(2r_1 s_1 + \frac{\sigma}{\nu}|\xi|^2) & -\xi_1\xi_2(2r_1 s_1 + \frac{\sigma}{\nu}|\xi|^2) & -i\xi_1 s_1(r_1^2+|\xi|^2) \\ -\xi_1\xi_2(2r_1 s_1 + \frac{\sigma}{\nu}|\xi|^2) & -\xi_2^2(2r_1 s_1 + \frac{\sigma}{\nu}|\xi|^2) & -i\xi_2 s_1(r_1^2+|\xi|^2) \\ -i\xi_1|\xi|(2r_1 s_1 + \frac{\sigma}{\nu}|\xi|^2) & -i\xi_2\xi|(2r_1 s_1 + \frac{\sigma}{\nu}|\xi|^2) & |\xi| s_1(r_1^2+|\xi|^2) \end{pmatrix}.$$

It is shown in [2] that in the case $\mathrm{Re}s_1 \geq \gamma > 0$

$$\frac{\gamma^2}{\nu^2} + |s_1||\xi|^2 + |s_1|^2 + \sigma|\xi|^3 \leq c(\gamma)|P_1|, \tag{2.10}$$

$$\int_0^\infty \Big|\frac{d^j e_0(x_3)}{dx_3^j}\Big|^2 dx_3 \leq \frac{1}{\sqrt{2}}|r_1|^{2j-1},$$

$$\int_0^\infty \Big|\frac{d^j e_1(x_3)}{dx_3^j}\Big|^2 dx_3 \leq c\frac{|r_1|^{2j-1} + |\xi|^{2j-1}}{|r_1|^2},$$

$$\int_0^\infty\int_0^\infty \Big|\frac{d^j e_0(x_3+z)}{dx_3^j} - \frac{d^j e_0(x_3)}{dx_3^j}\Big|^2 \frac{dx_3 dz}{|z|^{1+2\kappa}} \leq c|r_1|^{2(j+\kappa)-1},$$

$$\int_0^\infty\int_0^\infty \Big|\frac{d^j e_1(x_3+z)}{dx_3^j} - \frac{d^j e_1(x_3)}{dx_3^j}\Big|^2 \frac{dx_3 dz}{|z|^{1+2\kappa}} \leq c\frac{|r_1|^{2(j+\kappa)j-1} + |\xi|^{2(j+\kappa)-1}}{|r_1|^2},$$

where $j \geq 0$, $\kappa \in (0,1)$. Moreover, if $\mathrm{Re}s \geq \gamma > 0$, and $\gamma > (4\nu)^{-1}|\boldsymbol{V}'|^2$, then

$$c|r_1(s_1,\xi)| \leq \sqrt{|s| + |\xi|^2} \leq c'|r_1(s_1,\xi)|.$$

Using the above inequalities and repeating the calculations in the proof of Theorem 3.1.in [2] (carried out in the case $\boldsymbol{V}' = 0$), we obtain

$$
\begin{aligned}
&\|\widetilde{\boldsymbol{v}}\|^2_{\dot{W}_2^{l+2}(\mathbb{R}_+)} + |r(s,\xi)|^{2(l+2)}\|\widetilde{\boldsymbol{v}}\|^2_{L_2(\mathbb{R}_+)} + \|\widetilde{p}_{x_3}\|^2_{\dot{W}_2^{l}(\mathbb{R}_+)} + |r|^{2l}|\xi|^2\|\widetilde{p}\|^2_{L_2(\mathbb{R}_+)}\\
&\le c(\|\widetilde{\boldsymbol{v}}\|^2_{\dot{W}_2^{l+2}(\mathbb{R}_+)} + |r_1(s_1,\xi)|^{2(l+2)}\|\widetilde{\boldsymbol{v}}\|^2_{L_2(\mathbb{R}_+)} + \|\widetilde{p}_{x_3}\|_{L_2(\mathbb{R}_+)} + |r_1|^{2l}|\xi|^2\|\widetilde{p}\|^2_{L_2(\mathbb{R}_+)})\\
&\le c(|r_1|^{2l+1}(|\widetilde{b}_1|^2 + |\widetilde{b}_2|^2) + |\xi||r_1|^{2l}|\widetilde{b}_3|^2 + |r_1|^{2l}|\xi|^3|\widetilde{g}|^2)\\
&\le c(|r|^{2l+1}(|\widetilde{b}_1|^2 + |\widetilde{b}_2|^2) + |\xi||r|^{2l}|\widetilde{b}_3|^2 + |r|^{2l}|\xi|^3|\widetilde{g}|^2),
\end{aligned}
$$

where $\|\cdot\|_{\dot{W}_2^l(\mathbb{R}^n)}$ is the principal part of the norm in $W_2^l(\mathbb{R}^n)$:

$$
\|u\|^2_{\dot{W}_2^l(\mathbb{R}^n)} = \sum_{|j|=[l]}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|D^j u(x) - D^j u(y)|^2\frac{dxdy}{|x-y|^{n+2\lambda}}, \quad \lambda = l - [l] \in (0,1).
$$

Now we integrate this inequality with respect to $\xi \in \mathbb{R}^2$ and use the Parceval formula. This leads to

$$
\begin{aligned}
&\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{2+l}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)}\\
&\le c\Big(\|\boldsymbol{b}\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l+1/2}\|\boldsymbol{b}'\|^2_{L_2(\mathbb{R}^2)} + |s|^l\|b_3\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^l\|g\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).
\end{aligned}
\tag{2.11}
$$

We supplement (2.11) with the estimates of $p|_{x_3=0} \equiv p(0)$ and ρ. By (2.7),

$$
|\widetilde{p}(0)| \le c(|\widetilde{\boldsymbol{b}}| + (1+|\xi|)|\widetilde{g}|), \tag{2.12}
$$

which implies

$$
\begin{aligned}
&\|p(0)\|_{W_2^{1/2}(\mathbb{R}^2)} \le c\Big(\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{3/2}(\mathbb{R}^2)}\Big),\\
&\|p(0)\|_{W_2^{l+1/2}(\mathbb{R}^2)} \le c\Big(\|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\Big).
\end{aligned}
\tag{2.13}
$$

To estimate the norms of ρ, we use the equations

$$
\begin{aligned}
&s_1\widetilde{\rho} = \widetilde{g} - \widetilde{v}_3(0),\\
&\sigma|\xi|^2\widetilde{\rho} = \widetilde{b}_3 + (\widetilde{p} - 2\nu\frac{d\widetilde{v}_3}{dx_3})\Big|_{x_3=0} = \widetilde{b}_3 + (\widetilde{p}(0) + 2\nu\sum_{j=1}^2 i\xi_j\widetilde{v}_j(0)).
\end{aligned}
\tag{2.14}
$$

Since

$$
\widetilde{v}_3(0) = \frac{\sum_{j=1}^3 U_{3j}\widetilde{b}_j}{\nu^2 r_1(r_1+|\xi|)P_1} - \frac{\sigma\xi^2 U_{33}\widetilde{g}}{\nu^2 s_1 r_1(r_1+|\xi|)P_1},
$$

and, as a consequence,

$$
|\widetilde{v}_3(0)| \le c(|\xi||r|^{-2}|\widetilde{\boldsymbol{b}}| + |\widetilde{g}|),
$$

equations (2.14) imply

$$
\begin{aligned}
&\gamma|\widetilde{\rho}| \le c(|\xi||r|^{-2}|\widetilde{\boldsymbol{b}}| + |\widetilde{g}|),\\
&\sigma|\xi|^2|\widetilde{\rho}| \le c(|\widetilde{\boldsymbol{b}}| + (1+|\xi|)|\widetilde{g}|).
\end{aligned}
\tag{2.15}
$$

Hence

$$\begin{aligned}
&\|\rho\|_{W_2^{5/2}(\mathbb{R}^2)} \le c\Big(\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{3/2}(\mathbb{R}^2)}\Big),\\
&\|\rho\|_{W_2^{l+5/2}(\mathbb{R}^2)} \le c\Big(\|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\Big).
\end{aligned} \tag{2.16}$$

Now we pass to the estimate of $s\widetilde{\rho}$. By (2.14),

$$|s||\widetilde{\rho}| \le |\boldsymbol{V}'||\xi||\widetilde{\rho}| + c(|\xi||r|^{-2}|\widetilde{\boldsymbol{b}}| + |\widetilde{g}|),$$

which implies

$$\begin{aligned}
&|s|\|\rho\|_{W_2^{l+3/2}(\mathbb{R}^2)} \le |\boldsymbol{V}'|\|\rho\|_{W_2^{l+5/2}(\mathbb{R}^2)} + c(\|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}),\\
&|s|^{1+l/2}\|\rho\|_{W_2^{3/2}(\mathbb{R}^2)} \le |\boldsymbol{V}'||s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathbb{R}^2)} + c|s|^{l/2}(\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{3/2}(\mathbb{R}^2)}).
\end{aligned} \tag{2.17}$$

Estimates (2.16),(2.17) yield

$$\begin{aligned}
&|s|^{l/2}\|p(0)\|_{W_2^{1/2}(\mathbb{R}^2)} + \|p(0)\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathbb{R}^2)} + \|\rho\|_{W_2^{l+5/2}(\mathbb{R}^2)}\\
&+ |s|\|\rho\|_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{1+l/2}\|\rho\|_{W_2^{3/2}(\mathbb{R}^2)}\\
&\le c\Big(|s|^{l/2}\|\boldsymbol{b}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|\boldsymbol{b}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|g\|_{W_2^{3/2}(\mathbb{R}^2)} + \|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\Big).
\end{aligned} \tag{2.18}$$

Step 2. We consider the problem

$$\begin{cases}
s\boldsymbol{v} + (\boldsymbol{V}'\cdot\nabla')\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x),\\
\nabla\cdot\boldsymbol{v}(x) = 0, \quad x_3 > 0,\\
\nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = 0, \quad j = 1,2,\\
-p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = 0,\\
s\rho + \boldsymbol{V}'\cdot\nabla'\rho + v_3 = 0, \quad x_3 = 0.
\end{cases} \tag{2.19}$$

Our goal is to construct the solution of (2.19) and to obtain the estimate similar to (2.11), (2.18). Without loss of generality we can assume that $\nabla\cdot\boldsymbol{f} = 0$, otherwise we could decompose $\boldsymbol{f}$ in the sum of a divergence free and a potential vector fields:

$$\boldsymbol{f} = \boldsymbol{f}' + \nabla\phi,$$

where ϕ is a solution of the Dirichlet problem

$$\nabla^2\phi(x) = \nabla\cdot\boldsymbol{f}(x), \quad x_3 > 0, \quad \phi|_{x_3=0} = 0.$$

The problem (2.19) is equivalent to a similar problem with $\boldsymbol{f}'$ instead of $\boldsymbol{f}$ and $p' = p - \phi$ instead of p.

Thus, we assume that $\boldsymbol{f}$ is divergence free and we extend $\boldsymbol{f}$ in $\mathbb{R}^3$ with preservation of this property and of the regularity properties, namely, we require that $\nabla\cdot\boldsymbol{f}^* = 0$,

$$\|\boldsymbol{f}^*\|_{L_2(\mathbb{R}^3)} \le c\|\boldsymbol{f}\|_{L_2(\mathbb{R}^3_+)}, \quad \|\boldsymbol{f}^*\|_{W_2^l(\mathbb{R}^3)} \le c\|\boldsymbol{f}\|_{W_2^l(\mathbb{R}^3_+)},$$

where $\boldsymbol{f}^*$ is the extension of $\boldsymbol{f}$.

We define $\boldsymbol{u}$ as the solution of the system

$$s\boldsymbol{u} + (\boldsymbol{V}' \cdot \nabla)\boldsymbol{u} - \nu\nabla^2\boldsymbol{u} = \boldsymbol{f}^*(x), \quad x \in \mathbb{R}^3.$$

Making the Fourier transform with respect to x_1, x_2, x_3, we obtain the solution in the form

$$\widetilde{\boldsymbol{u}}(\xi) = \frac{\widetilde{\boldsymbol{f}}^*}{s_1 + \nu\xi^2},$$

where $\xi = (\xi_1, \xi_2, \xi_3)$ is the dual variable and $s_1 = s + i\boldsymbol{V}' \cdot \xi$. It is clear that $\nabla \cdot \boldsymbol{u} = 0$. The corresponding pressure p vanishes. The vector field $\boldsymbol{u}$ satisfies the inequalities

$$|s|\|\boldsymbol{u}\|_{L_2(\mathbb{R}^3)} \le c\|\boldsymbol{f}^*\|_{L_2(\mathbb{R}^3)}, \quad \|\boldsymbol{u}\|_{W_2^{l+2}(\mathbb{R}^3)} \le c\|\boldsymbol{f}^*\|_{W_2^l(\mathbb{R}^3)},$$

and, as a consequence,

$$|s|^{1+l/2}\|\boldsymbol{u}\|_{L_2(\mathbb{R}^3)} + \|\boldsymbol{u}\|_{W_2^{l+2}(\mathbb{R}^3)} \le c(|s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathbb{R}^3)} + \|\boldsymbol{f}\|_{W_2^l(\mathbb{R}^3)}).$$

The difference $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}$ is a solution of (2.8) with

$$b_j(x) = -\nu\Big(\frac{\partial u_3}{\partial x_j} + \frac{\partial u_j}{\partial x_3}\Big), \quad j = 1, 2, 3, \quad g = -u_3.$$

Hence $\boldsymbol{w}, p, \rho$ satisfy (2.11), (2.18). It follows that

$$\begin{aligned}
&\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{l+2}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)} \\
&+ |s|^l\|p(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|p(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|\rho\|^2_{W_2^{5/2}(\mathbb{R}^2)} + \|\rho\|^2_{W_2^{l+5/2}(\mathbb{R}^2)} \\
&+ |s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathbb{R}^2)} \le c\Big(\|\boldsymbol{f}\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\boldsymbol{f}\|^2_{L_2(\mathbb{R}^3_+)}\Big).
\end{aligned} \tag{2.20}$$

Step 3. We consider the problem

$$\begin{cases}
s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \\
\nabla \cdot \boldsymbol{v}(x) = 0, \quad x_3 > 0, \\
\nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = b_j, \quad j = 1, 2, \\
-p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = b_3(x'), \\
s\rho + \boldsymbol{V}' \cdot \nabla'\rho + v_3 = g(x'), \quad x_3 = 0.
\end{cases} \tag{2.21}$$

The first equation can be written in the form

$$s\boldsymbol{v} + (\boldsymbol{V}' \cdot \nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x) + (\boldsymbol{V}' \cdot \nabla)\boldsymbol{v},$$

and the term $(\boldsymbol{V}' \cdot \nabla)\boldsymbol{v}$ can be estimated by the interpolation inequality as follows:

$$\begin{aligned}
&\|(\boldsymbol{V}' \cdot \nabla)\boldsymbol{v}\|_{W_2^l(\mathbb{R}^3_+)} \le c|s|^{-1/2}(\|\boldsymbol{v}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)}), \\
&|s|^{l/2}\|(\boldsymbol{V}' \cdot \nabla)\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)} \le c|s|^{-1/2}(\|\boldsymbol{v}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)}),
\end{aligned}$$

hence

$$\|(\boldsymbol{V}'\cdot\nabla)\boldsymbol{v}\|_{W_2^l(\mathbb{R}^3_+)}+|s|^{l/2}\|(\boldsymbol{V}'\cdot\nabla)\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)}\le c|s|^{-1/2}(\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)}+|s|^{1+l/2}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)}).$$

If $|s|$ is large enough, then (2.11), (2.18), (2.20) yield

$$\begin{aligned}&\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)}+|s|^l\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)}+\|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)}+|s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)}\\&+|s|^l\|p(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)}+\|p(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}+|s|^l\|\rho\|^2_{W_2^{5/2}(\mathbb{R}^2)}+\|\rho\|^2_{W_2^{l+5/2}(\mathbb{R}^2)}\\&+|s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathbb{R}^2)}\\&\le c\Big(\|\boldsymbol{f}\|^2_{W_2^l(\mathbb{R}^3_+)}+|s|^l\|\boldsymbol{f}\|^2_{L_2(\mathbb{R}^3_+)}+\|\boldsymbol{b}\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}+|s|^{l+1/2}\|\boldsymbol{b}'\|^2_{L_2(\mathbb{R}^2)}\\&+|s|^l\|b_3\|^2_{W_2^{1/2}(\mathbb{R}^2)}+\|g\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).\end{aligned}\tag{2.22}$$

Step 4. We consider the problem

$$\begin{cases}s\boldsymbol{v}(x)-\nu\nabla^2\boldsymbol{v}(x)+\nabla p(x)=0,\\ \nabla\cdot\boldsymbol{v}(x)=h(x),\quad x\in\mathbb{R}^3_+,\\ \nu\Big(\dfrac{\partial v_3}{\partial x_j}+\dfrac{\partial v_j}{\partial x_3}\Big)=0,\quad j=1,2,\\ -p+2\nu\dfrac{\partial v_3}{\partial x_3}-\sigma\Delta'\rho=0,\\ s\rho+(\boldsymbol{V}'\cdot\nabla)\rho+v_3=0,\quad x_3=0,\end{cases}\tag{2.23}$$

under the assumption that h decays at infinity sufficiently rapidly, and

$$h=\nabla\cdot\boldsymbol{H}(x)+h'(x)\tag{2.24}$$

with compactly supported h'. We reduce (2.23) to (2.21). To this end, we introduce $\boldsymbol{w}=\nabla\Phi(x)$, where Φ is a solution of the Dirichlet problem

$$\nabla^2\Phi(x)=h(x),\quad x\in\mathbb{R}^3_+,\quad \Phi(x)|_{x_3=0}=0.\tag{2.25}$$

By the Green identity,

$$\begin{aligned}&\int_{\mathbb{R}^3_+}|\nabla\Phi(x)|^2dx=-\int_{\mathbb{R}^3_+}\Phi(x)\nabla^2\Phi(x)dx=\int_{\mathbb{R}^3_+}(\nabla\Phi(x)\cdot\boldsymbol{H}-h'(x)\Phi(x))dx\\&\le c\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)}\|\nabla\Phi\|_{L_2(\mathbb{R}^3_+)}+\|h'\|_{L_{6/5}(supp\ h')}\|\Phi\|_{L_6(\mathbb{R}^3_+)}\Big)\\&\le c\|\nabla\Phi\|_{L_2(\mathbb{R}^3_+)}\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)}+\|h'\|_{L_2(\mathbb{R}^3_+)}\Big).\end{aligned}\tag{2.26}$$

Moreover, coercive estimate for the problem (2.25) yields

$$\|\nabla\Phi\|_{\dot W_2^{2+l}(\mathbb{R}^3_+)}\le c\|h\|_{\dot W_2^{1+l}(\mathbb{R}^3_+)},$$

hence

$$\begin{aligned}&\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)}+|s|^{1+l/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}\\&\le c|s|^{1+l/2}\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)}+\|h'\|_{L_2(\mathbb{R}^3_+)}\Big)+c\|h\|_{W_2^{1+l}(\mathbb{R}^3_+)}.\end{aligned}\tag{2.27}$$

The functions $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{w}$, p, ρ represent the solution of the problem (2.21) with the data

$$\begin{aligned}
&\boldsymbol{f} = -s\boldsymbol{w} + \nu\nabla^2\boldsymbol{w},\\
&b_j = -\nu\Big(\frac{\partial w_j}{\partial x_3} + \frac{\partial w_3}{\partial x_j}\Big), \quad j = 1,2, \quad b_3 = -2\nu\frac{\partial w_3}{\partial x_3}, \quad g = -w_3,
\end{aligned}$$

and they can be estimated by (2.22). This estimate, together with (2.27), yields

$$\begin{aligned}
&\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{2+l}\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla p\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla p\|^2_{L_2(\mathbb{R}^3_+)}\\
&+ |s|^l\|p(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|p(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|\rho\|^2_{W_2^{5/2}(\mathbb{R}^2)} + \|\rho\|^2_{W_2^{l+5/2}(\mathbb{R}^2)}\\
&+ |s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathbb{R}^2)}\\
&\le c|s|^{2+l}\Big(\|\boldsymbol{H}\|^2_{L_2(\mathbb{R}^3_+)} + \|h'\|^2_{L_2(\mathbb{R}^3_+)}\Big) + c\|h\|^2_{W_2^{1+l}(\mathbb{R}^3_+)}.
\end{aligned} \tag{2.28}$$

Step 5. We estimate the solution of (2.1) in the neighborhood of an arbitrary fixed point $x_0 \in \mathcal{G}$ by Schauder's localization method. Without loss of generality we may assume that $x_0 = 0$ and that the interior normal $-\boldsymbol{N}(0)$ is parallel to $\boldsymbol{e}_3$. Let $\zeta(x)$ be a smooth cut-off function equal to 1 for $|x| \le \delta/2$ and to zero in the domain $|x| \ge \delta$. The functions $\boldsymbol{w} = \zeta(x)\boldsymbol{v}(x)$, $\boldsymbol{q} = \zeta\boldsymbol{p}$, $r = \zeta\rho$ satisfy the equations

$$\begin{cases}
s\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = \boldsymbol{f}(x)\zeta(x) + \boldsymbol{m}_1(\boldsymbol{v}, p),\\
\nabla\cdot\boldsymbol{w}(x) = \nabla\zeta\cdot\boldsymbol{v}(x), \quad x \in \mathcal{F},\\
T(\boldsymbol{w}, q)\boldsymbol{N} - \sigma\boldsymbol{N}\Delta_{\mathcal{G}} r = \zeta(x)\boldsymbol{d}(x) + \boldsymbol{m}_2(\boldsymbol{v}, \rho),\\
sr(x) + \boldsymbol{V}'\cdot\nabla r - \boldsymbol{w}(x)\cdot\boldsymbol{N}(x) = \rho\boldsymbol{V}'\cdot\nabla\zeta + g(x)\zeta(x), \quad x \in \mathcal{G},
\end{cases} \tag{2.29}$$

where

$$\begin{aligned}
&\boldsymbol{m}_1(\boldsymbol{v}, p) = -2\nu\nabla\zeta(x)\nabla\boldsymbol{v} - \nu\boldsymbol{v}\nabla^2\zeta + p\nabla\zeta,\\
&\boldsymbol{m}_2(\boldsymbol{v}, \rho) = \nu\Big(\boldsymbol{v}(x)\frac{\partial\zeta}{\partial N} + \nabla\zeta(x)(\boldsymbol{v}\cdot\boldsymbol{N})\Big) + \boldsymbol{N}\sigma(\zeta(x)\Delta_{\mathcal{G}}\rho - \Delta_{\mathcal{G}}(\zeta\rho) - b(x)\zeta(x)\rho(x)).
\end{aligned}$$

We assume that in the d-neighborhood of the origin ($d \ge 2\delta$) the surface $\mathcal{G}$ is given by the equation

$$x_3 = \phi(x'), \quad x' = (x_1, x_2).$$

The function ϕ is smooth and $\phi(0) = 0$, $\nabla\phi(0) = 0$, which implies

$$|\nabla\phi(x')| \le c|x'|, \quad |\phi(x')| \le c|x'|^2 \tag{2.30}$$

for $|x'| \le d$. The components of $\boldsymbol{N}$ and the Laplace-Beltrami operator $\Delta_{\mathcal{G}}$ are expressed in terms of ϕ as follows:

$$N_\alpha = \frac{\phi_{y_\alpha}}{\sqrt{1 + |\nabla\phi|^2}}, \quad \alpha = 1,2, \quad N_3 = -\frac{1}{\sqrt{1 + |\nabla\phi|^2}},$$

$$\Delta_{\mathcal{G}} = \frac{1}{\sqrt{1 + |\nabla\phi|^2}}\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial y_\alpha}\Big(\delta_{\alpha\beta}\sqrt{1 + |\nabla\phi|^2} - \frac{\phi_{y_\alpha}\phi_{y_\beta}}{\sqrt{1 + |\nabla\phi|^2}}\Big)\frac{\partial}{\partial y_\beta}.$$

We make the change of variables in (2.29):

$$y = F(x): \quad y' = x', \quad y_3 = x_3 - \phi(x').$$

If d is small enough, then the transformation F is invertible, and it establishes one-to one correspondence between the domain $K_d = \{|x| \le d,\ x \in \mathcal{F}\}$ and a certain subdomain D of $\mathbb{R}^3_+$. The operators ∇_x and $S(\boldsymbol{v})$ are transformed into $\widehat{\nabla} = \nabla_y - \frac{\partial}{\partial y_3}\nabla\phi(y')$ and $\widehat{S}(\boldsymbol{v}) = \widehat{\nabla}\boldsymbol{v} + (\widehat{\nabla}\boldsymbol{v})^T$, respectively, and it holds

$$\nabla_x \cdot \boldsymbol{f}(x) = \widehat{\nabla} \cdot \boldsymbol{f}(x(y)) = \nabla_y \cdot \widehat{\boldsymbol{f}}(y),$$

where $\widehat{f}_i = f_i - \delta_{i3}\sum_{\alpha=1}^2 \phi_{y_\alpha} f_\alpha$.

We write the equations (2.29) in the variables $\{y\}$, keeping the old notation for all the transformed functions. We have

$$\begin{cases} s\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = \boldsymbol{M}_1(\boldsymbol{w}, q) + \boldsymbol{m}_1(\boldsymbol{v}, p) + \zeta\boldsymbol{f}, \\ \nabla \cdot \boldsymbol{w} = (\nabla - \widehat{\nabla}) \cdot \boldsymbol{w} + \widehat{\nabla}\zeta \cdot \boldsymbol{v}, \end{cases} \tag{2.31}$$

where $\nabla = \nabla_y$,

$$\boldsymbol{M}_1(\boldsymbol{w}, q) = \nu(\widehat{\nabla}^2 - \nabla^2)\boldsymbol{w} + (\nabla - \widehat{\nabla})q. \tag{2.32}$$

We note that the function $\nabla\zeta \cdot \boldsymbol{v}$ can be written in the form

$$\begin{aligned} \nabla\zeta \cdot \boldsymbol{v} &= \frac{1}{s}\nabla\zeta \cdot (\nu\nabla^2\boldsymbol{v} - \nabla p + \boldsymbol{f}) \\ &= \nabla \cdot \boldsymbol{A}_s(\boldsymbol{v}, p) + a_s(\boldsymbol{v}, p) + \frac{1}{s}\nabla\zeta \cdot \boldsymbol{f}, \end{aligned} \tag{2.33}$$

where

$$\begin{aligned} \boldsymbol{A}_s(\boldsymbol{v}, p) &= \frac{1}{s}(\nu\nabla\zeta\nabla\boldsymbol{v} - p\nabla\zeta), \\ a_s(\boldsymbol{v}, p) &= \frac{1}{s}\Big(-\nu D^2\zeta : \nabla\boldsymbol{v} + p\nabla^2\zeta\Big), \end{aligned} \tag{2.34}$$

$D^2\zeta = \left(\frac{\partial^2\zeta}{\partial x_i \partial x_j}\right)_{i,j=1,2,3}$, $\nabla\boldsymbol{v} = \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2,3}$. Hence $h \equiv (\nabla - \widehat{\nabla}) \cdot \boldsymbol{w} + \widehat{\nabla}\zeta \cdot \boldsymbol{v}$ satisfies (2.24) with

$$\boldsymbol{H} = \boldsymbol{e}_3 \sum_{\alpha=1}^{2} \phi_{y_\alpha} w_\alpha + \widehat{A}_s(\boldsymbol{v}, p), \quad h' = a_s(\boldsymbol{v}, p) + \frac{1}{s}\widehat{\nabla}\zeta \cdot \boldsymbol{f}. \tag{2.35}$$

We write the boundary condition $T\boldsymbol{N} - \sigma\boldsymbol{N}\Delta_{\mathcal{G}} r = \zeta\boldsymbol{d} + \boldsymbol{m}_2$ for the tangential and normal components separately, moreover, we can take only the first two components of the tangential part. This gives the system of three equations

$$\nu\Big(\sum_{i=1}^{3} \widehat{S}_{\alpha i}(\boldsymbol{w})N_i - N_\alpha(\boldsymbol{N} \cdot \widehat{S}(\boldsymbol{w})\boldsymbol{N})\Big) = \zeta(d_\alpha - N_\alpha(\boldsymbol{d} \cdot \boldsymbol{N})) + m_{2\alpha} - N_\alpha(\boldsymbol{N} \cdot \boldsymbol{m}_2), \quad \alpha = 1, 2,$$

$$-q + \nu\boldsymbol{N} \cdot \widehat{S}(\boldsymbol{w})\boldsymbol{N} + \sigma\Delta_{\mathcal{G}} r = \zeta\boldsymbol{d} \cdot \boldsymbol{N} + \boldsymbol{m}_2 \cdot \boldsymbol{N},$$

i.e.,

$$\begin{cases} \nu S_{\alpha 3}(\boldsymbol{w}) = L_\alpha(\boldsymbol{w}) + l_\alpha(\boldsymbol{v}) + \zeta d'_\alpha(y), \quad \alpha = 1, 2, \\ -q + \nu S_{33}(\boldsymbol{w}) - \sigma\Delta' r = L_3(\boldsymbol{w}) + B' r + l_3(\boldsymbol{v}) + \zeta\boldsymbol{d} \cdot \boldsymbol{N}, \end{cases} \tag{2.36}$$

where $d'_\alpha = d_\alpha - N_\alpha(\boldsymbol{d}\cdot\boldsymbol{N})$,

$$
\begin{aligned}
&L_\alpha(\boldsymbol{w}) = \nu\Big(S_{\alpha 3} - \sum_{j=1}^{3}\widehat{S}_{\alpha j}N_j + N_\alpha(\boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N})\Big),\\
&L_3(\boldsymbol{w}) = \nu\Big(S_{33}(\boldsymbol{w}) - \boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N}\Big),\\
&B'r = -\sigma(\Delta' + \Delta_{\mathcal{G}})r,\\
&l_\alpha(\boldsymbol{v}) = m_{2\alpha}(\boldsymbol{v}) - N_\alpha(\boldsymbol{m}_2(\boldsymbol{v})\cdot\boldsymbol{N}),\\
&l_3(\boldsymbol{v}) = \boldsymbol{m}_2\cdot\boldsymbol{N}.
\end{aligned}
$$

Finally, we have

$$
sr + \boldsymbol{V}'\cdot\nabla' r + w_3 = (w_3 + \boldsymbol{w}\cdot\boldsymbol{N}) + \rho\boldsymbol{V}'\cdot\nabla'\zeta + \zeta g. \tag{2.37}
$$

Now, we extend $\boldsymbol{w}$, q, r by zero into $\mathbb{R}^3_+$ and $\mathbb{R}^2$ and consider (2.31), (2.36), (2.37) as the problem of the type (2.21) in the half-space. We estimate $\boldsymbol{w}, q, r$ by (2.22), (2.28). We note that in view of (2.30) the leading coefficients of the operators $\boldsymbol{M}_1$, $\nabla - \widehat{\nabla}$, L_i, B' are small in the case of a small δ. By Lemma 4.1 in [2],

$$
\|\boldsymbol{M}_1\|_{W_2^l(\mathbb{R}^3_+)} \le c\delta^\theta\Big(\|\boldsymbol{w}\|_{W_2^{2+l}(\mathbb{R}^3_+)} + \|\nabla q\|_{W_2^l(\mathbb{R}^3_+)}\Big) + c(\theta)(\|\boldsymbol{w}\|_{W_2^{l+1}(\mathbb{R}^3_+)} + \|\nabla q\|_{W_2^{l-1}(\mathbb{R}^3_+)}),
$$

$$
|s|^{l/2}\|\boldsymbol{M}_1\|_{L_2(\mathbb{R}^3_+)} \le c|s|^{l/2}\Big(\delta(\|\boldsymbol{w}\|_{W_2^l(\mathbb{R}^3_+)} + \|\nabla q\|_{L_2(\mathbb{R}^3_+)}) + \|\boldsymbol{w}\|_{W_2^1(\mathbb{R}^3_+)}\Big),
$$

where $\theta\in(0,1)$. By interpolation inequalities,

$$
\|\boldsymbol{w}\|_{W_2^{l+1}(\mathbb{R}^3_+)} \le c(|s|^{-1/2}\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1/2+l/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}),
$$

$$
\|\nabla q\|_{W_2^{l-1}(\mathbb{R}^3_+)} \le c(|s|^{-1/2}\|\nabla q\|_{W_2^l(\mathbb{R}^3_+)} + |s|^{l/2-1/2}\|\nabla q\|_{L_2(\mathbb{R}^3_+)}),
$$

$$
\|\boldsymbol{w}\|_{W_2^1(\mathbb{R}^3_+)} \le c(|s|^{-l/2-1/2}\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{1/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}),
$$

whence

$$
\begin{aligned}
&\|\boldsymbol{M}_1\|_{W_2^l(\mathbb{R}^3_+)} + |s|^{l/2}\|\boldsymbol{M}_1\|_{L_2(\mathbb{R}^3_+)}\\
&\le c(\delta^\theta + c(\delta)|s|^{-1/2})\Big(\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + \|\nabla q\|_{W_2^l(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)} + |s|^{l/2}\|\nabla q\|_{L_2(\mathbb{R}^3_+)}\Big).
\end{aligned}
$$

In a similar way we obtain

$$
\begin{aligned}
&\sum_{j=1}^{2}(\|L_j(\boldsymbol{w})\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2+1/4}\|L_j(\boldsymbol{w})\|_{L_2(\mathbb{R}^2)})\\
&+ \|L_3(\boldsymbol{w})\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|L_3(\boldsymbol{w})\|_{W_2^{1/2}(\mathbb{R}^2)}\\
&+ \|w_3 + \boldsymbol{w}\cdot\boldsymbol{N}\|_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{l/2+3/4}\|w_3 + \boldsymbol{w}\cdot\boldsymbol{N}\|_{L_2(\mathbb{R}^2)}\\
&+ \|(\nabla - \widehat{\nabla})\cdot\boldsymbol{w}\|_{W_2^{l+1}(\mathbb{R}^3_+)} + |s|^{1+l/2}\|\boldsymbol{e}_3\sum_{\alpha=1}^{2}\phi_{y_\alpha}w_\alpha\|_{L_2(\mathbb{R}^3_+)}\\
&\le c(\delta^\theta + c(\delta)|s|^{-1/2})\Big(\|\boldsymbol{w}\|_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{l/2+1}\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\|B'r\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|B'r\|_{W_2^{1/2}(\mathbb{R}^2)} \\
&\le c\delta^\theta \|r\|_{W_2^{l+5/2}(\mathbb{R}^2)} + c(\theta)\|r\|_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{l/2}(c\delta^\theta \|r\|_{W_2^{5/2}(\mathbb{R}^2)} + c(\theta)\|r\|_{W_2^{3/2}(\mathbb{R}^2)}) \\
&\le c(\delta^\theta + c(\delta)|s|^{-1/2})(\|r\|_{W_2^{l+5/2}(\mathbb{R}^2)} + |s|^{l/2}\|r\|_{W_2^{5/2}(\mathbb{R}^2)} + |s|^{1+l/2}\|r\|_{W_2^{3/2}(\mathbb{R}^2)}).
\end{aligned}
$$

Now we pass to the estimate of $\widehat{\nabla}\zeta \cdot \boldsymbol{v}$, $\boldsymbol{m}_1$ and $\boldsymbol{m}_2$. We have

$$
\begin{aligned}
&\|\widehat{\nabla}\zeta \cdot \boldsymbol{v}\|_{W_2^{l+1}(\mathbb{R}^2)} + \|\boldsymbol{m}_1\|_{W_2^l(\mathbb{R}^3_+)} + |s|^{l/2}\|\boldsymbol{m}_1\|_{L_2(\mathbb{R}^3_+)} + \sum_{\alpha=1}^{2}\Big(\|m_{2\alpha} - N_\alpha(\boldsymbol{m}_2\cdot\boldsymbol{N})\|_{W_2^{l+1/2}(\mathbb{R}^2)} \\
&+ |s|^{l/2+1/4}\|m_{2\alpha} - N_\alpha(\boldsymbol{m}_2\cdot\boldsymbol{N})\|_{L_2(\mathbb{R}^2)}\Big) + \|\boldsymbol{m}_2\cdot\boldsymbol{N}\|_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l/2}\|\boldsymbol{m}_2\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathbb{R}^2)} \\
&\le c(\delta)\Big(\|\boldsymbol{v}\|_{W_2^{l+1}(K_{2\delta})} + |s|^{l/2}\|\boldsymbol{v}\|_{W_2^1(K_{2\delta})} + \|\boldsymbol{v}\|_{W_2^{l+1/2}(S_{2\delta})} + |s|^{l/2+1/4}\|\boldsymbol{v}\|_{L_2(S_{2\delta})} \\
&+ |s|^{l/2}\|\boldsymbol{v}\|_{W_2^{1/2}(S_{2\delta})} + \|\rho\|_{W_2^{l+3/2}(S_{2\delta})} + |s|^{l/2}\|\rho\|_{W_2^{3/2}(S_{2\delta})} \\
&+ \|p\|_{W_2^l(K_{2\delta})} + |s|^{l/2}\|p\|_{L_2(K_{2\delta})}\Big),
\end{aligned}
$$

moreover,

$$
|s|^{1+l/2}\Big(\|\boldsymbol{A}_s\|_{L_2(\mathbb{R}^3_+)} + \|a_s\|_{L_2(\mathbb{R}^3_+)}\Big) \le c|s|^{l/2}\Big(\|\boldsymbol{v}\|_{W_2^1(K_{21\delta})} + \|p\|_{L_2(K_{2\delta})}\Big).
$$

The coefficient $\delta^\theta + c(\delta)|s|^{-1/2}$ can be made arbitrarily small by the choice of a small δ and large $|s|$. In this case, it is not hard to verify that the application of (2.22), to our problem leads to

$$
\begin{aligned}
&\|\boldsymbol{w}\|^2_{W_2^{l+2}(\mathbb{R}^3_+)} + |s|^{2+l}\|\boldsymbol{w}\|^2_{L_2(\mathbb{R}^3_+)} + \|\nabla q\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\nabla q\|^2_{L_2(\mathbb{R}^3_+)} \\
&+ |s|^l\|q(0)\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|q(0)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|r\|^2_{W_2^{5/2}(\mathbb{R}^2)} + \|r\|^2_{W_2^{l+5/2}(\mathbb{R}^2)} \\
&+ |s|^2\|r\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^{2+l}\|r\|^2_{W_2^{3/2}(\mathbb{R}^2)} \\
&\le c\Big(\|\zeta\boldsymbol{f}\|^2_{W_2^l(\mathbb{R}^3_+)} + |s|^l\|\zeta\boldsymbol{f}\|^2_{L_2(\mathbb{R}^3_+)} + \|\zeta\boldsymbol{d}'\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^{l+1/2}\|\zeta\boldsymbol{d}'\|^2_{L_2(\mathbb{R}^2)} \\
&+ \|\zeta\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + |s|^l\|\zeta\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathbb{R}^2)} + \|\zeta g\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^l\|\zeta g\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big) \\
&+ c\Big(\|p\|^2_{W_2^l(K_{2\delta})} + |s|^l\|p\|^2_{L_2(K_{2\delta})} + \|\boldsymbol{v}\|^2_{W_2^{l+1}(K_{2\delta})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^1(K_{2\delta})} + \|\boldsymbol{v}\|^2_{W_2^{l+1/2}(S_{2\delta})} \\
&+ |s|^{l+1/2}\|\boldsymbol{v}\|^2_{L_2(S_{2\delta})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^{1/2}(S_{2\delta})} + \|\rho\|^2_{W_2^{l+3/2}(S_{2\delta})} + |s|^l\|\rho\|^2_{W_2^{3/2}(S_{2\delta})}\Big).
\end{aligned}
\tag{2.38}
$$

Inequalities of this type can be obtained in a neighborhood of any point of $\mathcal{G}$ and of any interior point of $\mathcal{F}$ as well, if the distance of this point to $\mathcal{G}$ is larger that $\delta_1 > 0$ (in this case the norms of g and $\boldsymbol{d}$ do not occur in the estimate). If we cover $\mathcal{F}$ by a finite number of such

neighborhoods and add estimates (2.38) together, we obtain

$$
\begin{aligned}
&\|\boldsymbol{v}\|^2_{W_2^{2+l}(\mathcal{F})} + |s|^l\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})} + \|\nabla p\|^2_{W_2^l(\mathcal{F})} + |s|^l\|\nabla p\|^2_{L_2(\mathcal{F})} \\
&+ \|p\|^2_{W_2^{l+1/2}(\mathcal{G})} + |s|^l\|p\|^2_{W_2^{1/2}(\mathcal{G})} + \|\rho\|^2_{W_2^{l+5/2}(\mathcal{G})} + |s|^l\|\rho\|^2_{W_2^{5/2}(\mathcal{G})} \\
&+ |s|^2\|\rho\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^{2+l}\|\rho\|^2_{W_2^{3/2}(\mathcal{G})} \\
&\le c\Big(\|\boldsymbol{f}\|^2_{L_2(\mathcal{F})} + |s|^l\|\boldsymbol{f}\|^2_{L_2(\mathcal{F})} + \|\Pi_{\mathcal{G}}\boldsymbol{d}\|^2_{W_2^{l+1/2}\mathcal{G}} + |s|^{l+1/2}\|\Pi_{\mathcal{G}}\boldsymbol{d}\|^2_{L_2(\mathcal{G})} \\
&+ \|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathcal{G})} + |s|^l\|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathcal{G})} + \|g\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l\|g\|^2_{W_2^{3/2}(\mathcal{G})}\Big) \\
&+ c\Big(\|p\|^2_{W_2^l(\mathcal{F})} + |s|^l\|p\|^2_{L_2(\mathcal{F})} + N_1^2(\boldsymbol{v}) + N_2^2(\rho)\Big),
\end{aligned}
\tag{2.39}
$$

where

$$
N_1^2(\boldsymbol{v}) = \|\boldsymbol{v}\|^2_{W_2^{l+1}(\mathcal{F})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^1(\mathcal{F})} + \|\boldsymbol{v}\|^2_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l+1/2}\|\boldsymbol{v}\|^2_{L_2(\mathcal{G})} + |s|^l\|\boldsymbol{v}\|^2_{W_2^{1/2}(\mathcal{G})},
$$

$$
N_2^2(\rho) = \|\rho\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l\|\rho\|^2_{W_2^{3/2}(\mathcal{G})}.
$$

The next step is the estimate of p.

Step 6. We have assumed that $\boldsymbol{f}$ is divergence free. Hence p can be regarded as a solution of the problem

$$
\nabla^2 p = 0, \quad x \in \mathcal{F}, \quad p = \nu\boldsymbol{N}\cdot\boldsymbol{S}(\boldsymbol{v})\boldsymbol{N} + \sigma L\rho - \boldsymbol{d}\cdot\boldsymbol{N}, \quad x \in \mathcal{G}.
$$

It is well known that

$$
\begin{aligned}
\|p\|_{L_2(\mathcal{F})} &\le c\|\nu\boldsymbol{N}\cdot\boldsymbol{S}(\boldsymbol{v})\boldsymbol{N} + \sigma L\rho - \boldsymbol{d}\cdot\boldsymbol{N}\|_{L_2(\mathcal{G})} \\
&\le c\Big(\|\nabla\boldsymbol{v}\|_{L_2(\mathcal{G})} + \|\rho\|_{W_2^2(\mathcal{G})} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{L_2(\mathcal{G})}\Big).
\end{aligned}
$$

In view of the interpolation inequality

$$
\|p\|^2_{W_2^l(\mathcal{F})} \le \epsilon\|\nabla p\|^2_{W_2^l(\mathcal{F})} + c(\epsilon)\|p\|^2_{L_2(\mathcal{F})},
$$

we have

$$
\begin{aligned}
&\|p\|^2_{W_2^l(\mathcal{F})} + |s|^l\|p\|^2_{L_2(\mathcal{F})} \le \epsilon\|\nabla p\|^2_{W_2^l(\mathcal{F})} \\
&+ c(\epsilon)|s|^l\Big(\|\nabla\boldsymbol{v}\|^2_{L_2(\mathcal{G})} + \|\rho\|^2_{W_2^2(\mathcal{G})} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{L_2(\mathcal{G})}\Big),
\end{aligned}
\tag{2.40}
$$

moreover,

$$
\begin{aligned}
|s|^l\|\rho\|^2_{W_2^2(\mathcal{G})} &\le c|s|^l\Big(|s|^{-1/2}\|\rho\|^2_{W_2^{5/2}(\mathcal{G})} + |s|^{1/2}\|\rho\|^2_{W_2^{3/2}(\mathcal{G})}\Big), \\
|s|^l\|\nabla\boldsymbol{v}\|^2_{L_2(\mathcal{G})} &\le c|s|^l\Big(|s|^{-l-1/2}\|\boldsymbol{v}\|^2_{W_2^{l+2}(\mathcal{F})} + |s|^{3/2}\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})}\Big).
\end{aligned}
$$

We estimate the expressions $N_1^2(\boldsymbol{v})$ and $N_2^2(\rho)$ in a similar way and show that (2.39) and (2.40) imply (2.2) in the case of large $|s|$.

The solvability of the problem (2.1) is established in Sec. 3.

3 End of the proof of Theorem 2.1.

We continue the proof of Theorem 2.1 and establish the solvability of the problem (2.1). We use the method applied in [7] for the analysis of parabolic initial-boundary value problems and in [2] to the evolution Stokes problem similar to (1.1).

We need the following auxiliary proposition.

Proposition 3.1. *For arbitrary* $\boldsymbol{f} \in W_2^l(\mathcal{F})$, $\boldsymbol{d} \in W_2^{l+1/2}(\mathcal{G})$ *the problem*

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x \in \mathcal{F}, \\ \nu\Pi S(\boldsymbol{v})\boldsymbol{N} = \Pi\boldsymbol{d}(x), \\ -p + \nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} = \boldsymbol{d}\cdot\boldsymbol{N}, \quad x \in \mathcal{G}, \end{cases} \tag{3.1}$$

with $Res \gg 1$ *has a unique solution* $\boldsymbol{v} \in W_2^{2+l}(\mathcal{F})$, $p \in W_2^{l+1}(\mathcal{F})$, *and the solution satisfies the inequality*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{l+2}(\mathcal{F})} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|\nabla p\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\nabla p\|_{L_2(\mathcal{F})} \\ &+ \|p\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|p\|_{W_2^{l/2}(\mathcal{G})} \le c\Big(\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F})} \\ &+ \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2+1/4}\|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{L_2(\mathcal{G})} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathcal{G})}\Big). \end{aligned} \tag{3.2}$$

The theorem is proved in [8], see also [9].

We consider the problem (2.1) with $\boldsymbol{f} = 0$ and $\boldsymbol{d} = 0$. Let $\{\varphi_k\}$, $k = 1, 2, ...$ be sufficiently "fine" smooth partition of unity: $\sum_k \varphi_k(x) = 1$ defined on $\mathcal{G}$ and in a certain neighborhood of $\mathcal{G}$. We may assume that $supp\varphi_i \subset K_\delta^{(i)}$, where $K_\delta^{(i)}$ is a ball $|x - x_i| \le \delta$, $x_i \in \mathcal{G}$. We also assume that there exist smooth functions $\psi_i(x)$ with $supp\psi_i \subset K_\delta^i$ such that $\psi_i(x)\varphi_i(x) = \varphi_i(x)$. We suppose that

$$|D^j\varphi_i(x)| + |D^j\psi_i(x)| \le c\delta^{-|j|}$$

and that each point x can belong no more than M_0 balls $K_\delta^{(i)}$ with M_0 independent of δ.

Let $y_3 = \phi_i(y')$, $y' = (y_1, y_2)$ be the equation of $\mathcal{G}$ in a neighborhood of the point x_i in a local Cartesian coordinate system $y = (y_1, y_2, y_3)$ with the center at x_i and with y_3-axis directed along the vector $-\boldsymbol{N}(x_i)$. It is clear that $y = C_i(x - x_i)$ where C_i is an orthogonal matrix. Without restriction of generality it may be assumed that ϕ_i is defined on the entire plane $y_3 = 0$ (i.e., on the tangent plane to $\mathcal{G}$ at the point x_i) and satisfying (2.30) near the origin. The transformation $z_1 = y_1, z_2 = y_2, z_3 = y_3 - \phi_i(y')$ "rectifies" $\mathcal{G}$ near x_i. We denote by $Z_j(x)$ the composition of this transformation with $y = C_i(x - x_i)$.

Now we describe briefly the scheme of the proof of the solvability of the problem

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x \in \mathcal{F}, \\ T(\boldsymbol{v}, p)\boldsymbol{N} + \sigma\boldsymbol{N}L\rho = 0, \\ s\rho + \boldsymbol{V}\cdot\nabla_\tau\rho - \boldsymbol{v}(x)\cdot\boldsymbol{N}(x) = g(x), \quad x \in \mathcal{G}. \end{cases} \tag{3.3}$$

We construct a linear operator R that makes correspond to every function $g \in W_2^{l+3/2}(\mathcal{G})$ the element $U = (\boldsymbol{v}, p, \rho)$, where $\boldsymbol{v}$ is a divergence free vector field belonging to $W_2^{l+2}(\mathcal{F})$,

$p \in W_2^{l+1}(\mathcal{F})$ and $\rho \in W_2^{l+4/3}(\mathcal{G})$ such that

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v}(x) = 0, \quad x \in \mathcal{F}, \\ T(\boldsymbol{v},p)\boldsymbol{N} + \sigma\boldsymbol{N}L\rho = 0, \\ s\rho + \boldsymbol{V}\cdot\nabla_\tau\rho - \boldsymbol{v}(x)\cdot\boldsymbol{N}(x) = g(x) + Ag(x), \quad x \in \mathcal{G}, \end{cases} \tag{3.4}$$

where A is a linear continuous operator in $W_2^{l+3/2}(\mathcal{G})$, and the operator $I + A$ is invertible. Then $U = R(I + A)^{-1}g$ is a solution of (3.4), which is required to prove.

We define Rg as the sum of three terms:

$$Rg = R_1g + R_2g + R_3g = U_1 + U_2 + U_3, \quad U_j = (\boldsymbol{v}^{(j)}, p^{(j)}, \rho^{(j)}).$$

We set

$$R_1g = \sum_k \psi_k(x)(\boldsymbol{v}_k(x), p_k(x), \rho_k(x)),$$

where

$$\boldsymbol{v}_k(x) = C_k^{-1}\boldsymbol{u}_k(Z_kx), \quad p(x) = q_k(Z_kx), \quad \rho_k(x) = r_k(Z_kx),$$

and $(\boldsymbol{u}_k, q_k, r_k)$ is a solution of the half-space problem

$$\begin{cases} s\boldsymbol{u}_k(z) - \nu\nabla_z^2\boldsymbol{u}_k(z) + \nabla_z q_k(z) = 0, \\ \nabla\cdot\boldsymbol{u}_k(z) = 0, \quad z \in \mathbb{R}^3_+ = \{z_3 > 0\}, \\ \dfrac{\partial u_{k3}}{\partial z_j} + \dfrac{\partial u_{kj}}{\partial z_3} = 0, \quad j = 1, 2, \\ -q_k + 2\nu\dfrac{\partial u_{k3}}{\partial z_3} - \sigma\Delta_z' r_k = 0, \\ sr_k(z) + \boldsymbol{V}_k\cdot\nabla_z' r_k(z) + \boldsymbol{u}_{k3} = g(z)\varphi_k(z), \quad z_3 = 0, \end{cases} \tag{3.5}$$

with $\boldsymbol{V}_k = C_k\boldsymbol{V}(x_k)$, $z = Z_j(x)$. It is clear that $\boldsymbol{v}^{(1)} \in W_2^{l+2}(\mathcal{F})$, $p^{(1)} \in W_2^{l+1}(\mathcal{F})$, $\rho^{(1)} \in W_2^{l+5/2}(\mathcal{G})$. We set

$$\begin{cases} s\boldsymbol{v}^{(1)} - \nu\nabla^2\boldsymbol{v}^{(1)} + \nabla p^{(1)} \equiv \boldsymbol{f}_1(x), \\ \nabla\cdot\boldsymbol{v}^{(1)}(x) \equiv f_1(x), \quad x \in \mathcal{F}, \\ T(\boldsymbol{v}^{(1)}, p^{(1)})\boldsymbol{N} + \sigma\boldsymbol{N}L\rho^{(1)}(x) \equiv \boldsymbol{d}_1(x), \\ s\rho^{(1)} + \boldsymbol{V}(x)\cdot\nabla_\tau\rho^{(1)} - \boldsymbol{v}^{(1)}(x)\cdot\boldsymbol{N}(x) \equiv g(x) + g_1(x), \quad x \in \mathcal{G}, \end{cases} \tag{3.6}$$

and we define R_2g as $R_2g = (\boldsymbol{v}^{(2)}, 0, 0)$, $\boldsymbol{v}^{(2)} = \nabla\Phi(x)$, where Φ is a solution of the Dirichlet problem

$$\nabla^2\Phi(x) = -f_1(x), \quad x \in \mathcal{F}, \quad \Phi|_{\mathcal{G}} = 0.$$

Finally, $R_3g = (\boldsymbol{v}^{(3)}, p^{(3)}, 0)$ is a solution of

$$\begin{cases} s\boldsymbol{v}^{(3)} - \nu\nabla^2\boldsymbol{v}^{(3)} + \nabla p^{(3)} = \boldsymbol{f}_2(x), \\ \nabla\cdot\boldsymbol{v}^{(3)}(x) = 0, \quad x \in \mathcal{F}, \\ T(\boldsymbol{v}^{(3)}, p^{(3)})\boldsymbol{N} = \boldsymbol{d}_2(x), \end{cases}$$

where

$$\boldsymbol{f}_2 = -\boldsymbol{f}_1 - (s\boldsymbol{v}^{(2)} - \nu\nabla^2\boldsymbol{v}^{(2)}),$$
$$\boldsymbol{d}_2 = -\boldsymbol{d}_1 - S(\boldsymbol{v}^{(2)})\boldsymbol{N}.$$

Then $Ag = g_1 + \boldsymbol{v}^{(2)}\cdot\boldsymbol{N} + \boldsymbol{v}^{(3)}\cdot\boldsymbol{N}$. We have $\boldsymbol{v}^{(2)} \in W_2^{2+l}(\mathcal{F})$, $\boldsymbol{f}_2 \in W_2^l(\mathcal{F})$, $\boldsymbol{d}_2 \in W_2^{l+1/2}(\mathcal{G})$, $\boldsymbol{v}^{(3)} \in W_2^{l+2}(\mathcal{F})$, $p^{(3)} \in W_2^{l+1}(\mathcal{F})$, $Ag \in W_2^{l+3/2}(\mathcal{G})$.

It remains to prove the invertibility of $I + A$.

We compute the functions $\boldsymbol{f}_1$, f_1, $\boldsymbol{d}_1$, g_1 in (3.6). Since

$$\boldsymbol{v}^{(1)}(x) = \sum_k \psi_k(x)\boldsymbol{v}_k(x), \quad \boldsymbol{p}^{(1)}(x) = \sum_k \psi_k(x)p_k(x), \quad x \in \mathcal{F},$$
$$\rho^{(1)}(x) = \sum_k \psi_k(x)\rho_k(x), \quad x \in \mathcal{G},$$

we have

$$\boldsymbol{f}_1(x) = \sum_k \psi_k(x)(s\boldsymbol{v}_k - \nu\nabla^2\boldsymbol{v}_k + \nabla p_k) - \nu\sum_k(\nabla^2(\psi_k\boldsymbol{v}_k) - \psi_k\nabla^2\boldsymbol{v}_k)$$
$$+ \sum_k(\nabla(\psi_k p_k) - \psi_k\nabla p_k),$$
$$f_1(x) = \sum_k(\nabla\psi_k\cdot\boldsymbol{v}_k + \psi_k\nabla\cdot\boldsymbol{v}_k(x)),$$
$$\Pi_{\mathcal{G}}\boldsymbol{d}_1 = \sum_k \psi_k\Pi_{\mathcal{G}}S(\boldsymbol{v}_k)\boldsymbol{N} + \sum_k \Pi_{\mathcal{G}}(S(\psi_k\boldsymbol{v}_k) - \psi_k S(\boldsymbol{v}_k))\boldsymbol{N},$$
$$\boldsymbol{d}_1\cdot\boldsymbol{N} = \sum_k \psi_k(-p_k + \nu\boldsymbol{N}\cdot S(\boldsymbol{v}_k)\boldsymbol{N} + \sigma L\rho_k) + \nu\sum_k \boldsymbol{N}\cdot(S(\psi_k\boldsymbol{v}_k) - \psi_k S(\boldsymbol{v}_k))\boldsymbol{N}$$
$$+ \sigma\sum_k(L(\psi_k\rho_k) - \psi_k L\rho_k).$$

We consider the main terms in the above formulas. In view of (3.5),

$$\begin{aligned} &s\boldsymbol{v}_k - \nu\nabla_x^2\boldsymbol{v}_k + \nabla_x p_k = C_k^{-1}(\nu(\nabla_y^2 - \nabla_z^2)\boldsymbol{u}_k + (\nabla_y - \nabla_z)q_k), \\ &\nabla\cdot\boldsymbol{v}_k = (\nabla_y - \nabla_z)\cdot\boldsymbol{u}_k. \end{aligned} \tag{3.7}$$

Moreover, we have $\Pi_{\mathcal{G}}S_x(\boldsymbol{v}_k)\boldsymbol{N} = C_k^{-1}\Pi_k S_y(\boldsymbol{u}_k)\boldsymbol{N}_k$, where $\Pi_k\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{N}_k(\boldsymbol{N}_k\cdot\boldsymbol{f})$ and $\boldsymbol{N}_k = C_k\boldsymbol{N}$ is the vector whose components are given by

$$N_{kj}(y) = \frac{\phi_{ky_j}}{\sqrt{1 + |\nabla\phi_k|^2}}, \quad j = 1, 2, \quad N_{k3} = -\frac{1}{\sqrt{1 + |\nabla\phi_k|^2}}$$

in a neighborhood of the origin. Hence

$$\begin{aligned} &\Pi_{\mathcal{G}}S_x(\boldsymbol{v}_k)\boldsymbol{N} = \Pi_{\mathcal{G}}\Big(\Pi_{\mathcal{G}}C_k^{-1}(S_y(\boldsymbol{u}_k)\boldsymbol{N}_k - \sum_{j=1}^{2}\boldsymbol{e}_j S_{zj3}(\boldsymbol{u}_k))\Big), \\ &-p_k + \nu\boldsymbol{N}\cdot S_x(\boldsymbol{v}_k)\boldsymbol{N} + \sigma L\rho_k = \nu(\boldsymbol{N}_k\cdot S_y(\boldsymbol{u}_k)\boldsymbol{N}_k - 2\frac{\partial u_{k3}}{\partial z_3}) + \sigma(L + \Delta_z')r_k, \end{aligned} \tag{3.8}$$

where $S_y(\boldsymbol{u}) = (\nabla_y \boldsymbol{u}) + (\nabla_y \boldsymbol{u})^T$. Finally, from

$$s\rho^{(1)} + \boldsymbol{V}(x)\cdot\nabla_\tau\rho^{(1)} - \boldsymbol{v}^{(1)}\cdot\boldsymbol{N} = \sum_k \psi_k(s\rho_k + \boldsymbol{V}(x)\cdot\nabla_\tau\rho_k - \boldsymbol{v}_k\cdot\boldsymbol{N}) + \sum_k(\boldsymbol{V}(x)\cdot\nabla_\tau\psi_k)\rho_k$$

it follows that

$$g_1 = \sum_k \psi_k((\boldsymbol{V}_k(y)\cdot\nabla_{y\tau} - \boldsymbol{V}_k(0)\cdot\nabla'_z)r_k - (\boldsymbol{u}_k\cdot\boldsymbol{N} + \boldsymbol{u}_{k3})) + \sum_k(\boldsymbol{V}(x)\cdot\nabla_\tau\psi_k)\rho_k \quad (3.9)$$

with $\boldsymbol{V}_k = C_k\boldsymbol{V}$.

A simple calculation shows that

$$\frac{\partial}{\partial z_j} = \frac{\partial}{\partial y_j} - \frac{\partial}{\partial y_3}\phi_{k,j}(y'), \quad \phi_{k,j} = (1-\delta_{3j})\frac{\partial\phi_k}{\partial y_j}$$

and, as a consequence,

$$\nabla_y\cdot\boldsymbol{u}_k - \nabla_z\cdot\boldsymbol{u}_k = \sum_{j=1}^{2}\phi_{k,j}\frac{\partial u_{kj}}{\partial y_3}.$$

Hence

$$f_1(x) = \sum_k(\nabla\psi_k\cdot\boldsymbol{v}_k + \psi_k(\nabla_y - \nabla_z)\cdot\boldsymbol{u}_k) = \sum_k \nabla\psi_k\cdot\boldsymbol{v}_k$$
$$+\sum_k\sum_{m,j=1}^{3}\psi_k C_{3m}\frac{\partial}{\partial x_m}\phi_{k,j}u_{kj} = \sum_{m,j=1}^{3}\frac{\partial}{\partial x_m}\sum_k\psi_k C_{3m}\phi_{k,j}u_{kj} + \sum_k\boldsymbol{\chi}_k\cdot\boldsymbol{u}_k,$$

where $\boldsymbol{\chi}_k$ is a vector field with the components

$$\chi_{kj} = \sum_{m=1}^{3}C_{jm}\frac{\partial\psi_k}{\partial x_m} - \sum_{m=1}^{3}\frac{\partial}{\partial x_m}(\sum_k\psi_k C_{3m}\phi_{k,j}).$$

Since

$$\boldsymbol{u}_k(z,t) = \frac{1}{s}(\nu\nabla_z^2\boldsymbol{u}_k - \nabla_z q_k) = \frac{1}{s}\sum_{j,m=1}^{3}\frac{\partial}{\partial x_m}(C_{jm} - C_{3m}\phi_{k,j})\Big(\nu\frac{\partial\boldsymbol{u}_k}{\partial z_j} - \boldsymbol{e}_j q_k\Big),$$

we have

$$f_1 = \nabla\cdot\boldsymbol{F} + F'$$

with

$$F_m(x) = \sum_k\boldsymbol{\chi}_k\cdot\sum_{j=1}^{3}\frac{1}{s}(C_{jm} - C_{3m}\phi_{k,j})\Big(\nu\frac{\partial\boldsymbol{u}_k}{\partial z_j} - \boldsymbol{e}_j q_k\Big)$$
$$+\sum_k\psi_k(x)C_{3m}\sum_{j=1}^{3}\sum_{\alpha=1}^{2}\phi_{k,\alpha}u_{k\alpha}, \quad (3.10)$$
$$F'(x) = -\sum_k\sum_{m=1}^{3}\frac{\partial\boldsymbol{\chi}_k(x)}{\partial x_m}\cdot\sum_{j=1}^{3}\frac{1}{s}(C_{jm} - C_{3m}\phi_{k,j})\Big(\nu\frac{\partial\boldsymbol{u}_k}{\partial z_j} - \boldsymbol{e}_j q_k\Big).$$

Now we pass to the estimates. Since every point of $\mathcal{F}\cap\mathcal{G}$ belongs to no more that M_0 domains $K_\delta^{(i)}$, the functions of the form $f(x)=\sum_j f_j(x)$, $supp f_j\subset K_\delta^{(j)}$, satisfy the inequality

$$\|f\|^2_{W_2^r(\mathcal{F})}\le c_0\sum_j\|f_j\|^2_{W_2^r(\mathcal{F})}$$

with c_0 independent of δ. By (2.16) and (2.28),

$$\begin{aligned}&\|\boldsymbol{u}_k\|^2_{W_2^{2+l}(\mathbb{R}^3_+)}+|s|^{2+l}\|\boldsymbol{u}_k\|^2_{L_2(\mathbb{R}^3_+)}+\|\nabla q_k\|^2_{W_2^l(\mathbb{R}^3_+)}+|s|^l\|\nabla q_k\|^2_{L_2(\mathbb{R}^3_+)}\\&+\|q_k\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}+|s|^l\|q_k\|^2_{W_2^{1/2}(\mathbb{R}^2)}+\|r_k\|^2_{W_2^{l+5/2}(\mathbb{R}^2)}+|s|^l\|r_k\|^2_{W_2^{5/2}(\mathbb{R}^2)}\\&+|s|^2\|r_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^{2+l}\|r_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\le c\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}.\end{aligned}$$

We note that (3.7)-(3.10) are linear differential expressions with respect to $\boldsymbol{u}_k$, q_k, r_k, whose leading coefficients are small in $K_\delta^{(k)}$. Hence we may use Lemma 4.1 in [2] and obtain the inequality

$$\begin{aligned}&\|\boldsymbol{f}_1\|^2_{W_2^l(\mathcal{F})}+|s|^l\|\boldsymbol{f}_1\|^2_{L_2(\mathcal{F})}+\|f_1\|^2_{W_2^{l+1}(\mathcal{F})}\\&+|s|^{2+l}(\|\boldsymbol{F}\|^2_{L_2(\mathcal{F})}+\|F'\|^2_{L_2(\mathcal{F})})+\|\Pi_{\mathcal{G}}\boldsymbol{d}_1\|^2_{W_2^{l+1/2}(\mathcal{G})}\\&+|s|^{l+1/2}\|\Pi_{\mathcal{G}}\boldsymbol{d}_1\|^2_{L_2(\mathcal{G})}+\|\boldsymbol{d}_1\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathcal{G})}+|s|^l\|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathcal{G})}\\&+\|g_1\|^2_{W_2^{l+3/2}(\mathcal{G})}+|s|^{l+3/2}\|g_1\|^2_{L_2(\mathcal{G})}\\&\le c(\delta^\theta+c(\delta)|s|^{-1/2})^2\sum_k\Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).\end{aligned}$$

In addition, we have

$$\begin{aligned}&\|\boldsymbol{v}^{(2)}\|^2_{W_2^{2+l}(\mathcal{F})}+|s|^{2+l}\|\boldsymbol{v}^{(2)}\|^2_{L_2(\mathcal{F})}\\&\le c\Big(\|f_1\|^2_{W_2^{1+l}(\mathcal{F})}+|s|^{2+l}(\|\boldsymbol{F}\|^2_{L_2(\mathcal{F})}+\|F'\|^2_{L_2(\mathcal{F})})\Big)\\&\le c(\delta^\theta+c(\delta)|s|^{-1/2})^2\sum_k\Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big)\end{aligned}$$

and, by Proposition 3.1,

$$\begin{aligned}&\|\boldsymbol{v}^{(3)}\|^2_{W_2^{2+l}(\mathcal{F})}+|s|^{2+l}\|\boldsymbol{v}^{(3)}\|^2_{L_2(\mathcal{F})}+\|\nabla p^{(3)}\|^2_{W_2^l(\mathcal{F})}\\&+|s|^l\|\nabla p^{(3)}\|^2_{L_2(\mathcal{F})}+\|p^{(3)}\|^2_{W_2^{l+1/2}(\mathcal{G})}+|s|^l\|p^{(3)}\|^2_{W_2^{1/2}(\mathcal{G})}\\&\le c\Big(\|\boldsymbol{f}_2\|^2_{W_2^l(\mathcal{F})}+|s|^l\|\boldsymbol{f}_2\|^2_{L_2(\mathcal{F})}+\|\Pi_{\mathcal{G}}\boldsymbol{d}_2\|^2_{W_2^{l+1/2}(\mathcal{G})}\\&+|s|^{l+1/2}\|\Pi_{\mathcal{G}}\boldsymbol{d}_1\|^2_{L_2(\mathcal{G})}+\|\boldsymbol{d}\cdot\boldsymbol{N}\|^2_{W_2^{l+1/2}(\mathcal{G})}+|s|^l\|\boldsymbol{d}_2\cdot\boldsymbol{N}\|^2_{W_2^{1/2}(\mathcal{G})}\\&\le c(\delta^\theta+c(\delta)|s|^{-1/2})^2\sum_k\Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}+|s|^l\|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)}\Big).\end{aligned}$$

Hence

$$\begin{aligned}
&\|Ag\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l \|Ag\|^2_{W_2^{3/2}(\mathcal{G})} \\
&\le c\Big(\|g_1 + \boldsymbol{v}^{(2)} \cdot \boldsymbol{N} + \boldsymbol{v}^{(3)} \cdot \boldsymbol{N}\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l \|g_1 + \boldsymbol{v}^{(2)} \cdot \boldsymbol{N} + \boldsymbol{v}^{(3)} \cdot \boldsymbol{N}\|^2_{W_2^{3/2}(\mathcal{G})} \Big) \\
&\le c(\delta^\theta + c(\delta)|s|^{-1/2})^2 \sum_k \Big(\|g\varphi_k\|^2_{W_2^{l+3/2}(\mathbb{R}^2)} + |s|^l \|g\varphi_k\|^2_{W_2^{3/2}(\mathbb{R}^2)} \Big).
\end{aligned}$$

It can be verified that the expression on the right does not exceed

$$c_1(\delta^\theta + c_1(\delta)|s|^{-1/2})^2 (\|g\|^2_{W_2^{l+3/2}(\mathcal{G})} + |s|^l \|g\|^2_{W_2^{3/2}(\mathcal{G})}),$$

which shows that A is a contraction operator in the case of small δ and large $|s|$. This completes the proof of the solvability of the problem (3.3).

The solution of (2.1) can be constructed as the sum

$$\boldsymbol{v} = \boldsymbol{w}_1 + \boldsymbol{w}_2, \quad p = \pi_1 + \pi_2, \quad \rho = r_1 + r_2,$$

where $(\boldsymbol{w}_1, \pi_1, r_1)$ is a solution of (3.1) and $(\boldsymbol{w}_2, \pi_2, r_2)$ is a solution of (3.3) with g replaced by $g + \boldsymbol{w}_1 \cdot \boldsymbol{N} + \boldsymbol{w}_2 \cdot \boldsymbol{N}$. Theorem 2.1 is proved.

Remark. We have assumed that Res is sufficiently large positive number. In fact, the statement of Theorem 2.2 is true for Re$s > a$, where a is determined by the spectrum of the problem (2.6). It is well known (see [10,4]) that in the case $\boldsymbol{d} = 0$ this problem can be written in the form

$$(sI - \mathcal{A})U = G, \tag{3.11}$$

where $U = (\boldsymbol{v}, \rho)^T$, $G = (\boldsymbol{f}, g)^T$, I is 2×2 unit matrix and $\mathcal{A}$ is the 2×2 matrix operator

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

with the elements defined by

$$\begin{aligned}
A_{11}\boldsymbol{v} &= \nu \nabla^2 \boldsymbol{v} - p_1(v), & A_{12}\rho &= -p_2(\rho), \\
A_{21}\boldsymbol{v} &= \boldsymbol{v} \cdot \boldsymbol{N}|_{\mathcal{G}}, & A_{22}\rho &= -\boldsymbol{V} \cdot \nabla_\tau \rho|_{\mathcal{G}}.
\end{aligned}$$

By $p_1(\boldsymbol{v})$ and $p_2(\rho)$ we mean harmonic functions in $\mathcal{F}$ satisfying the boundary conditions

$$p_1(v) = \nu \boldsymbol{N} \cdot S(\boldsymbol{v})\boldsymbol{N}, \quad p_2(\rho) = -L\rho$$

on $\mathcal{G}$ (hence the pressure as an independent function is excluded). The domain of $\mathcal{A}$ is a subspace of $W_2^2(\mathcal{F}) \times W_2^{5/2}(\mathcal{G})$ defined by the conditions $\nabla \cdot \boldsymbol{v} = 0$, $\Pi_{\mathcal{G}} S(\boldsymbol{v})\boldsymbol{N}|_{\mathcal{G}} = 0$. According to Theorem 2.2, for large positive Res the equation (3.11) is solvable, and the inequality (3.7) with $l = 0$ yields

$$\|(sI - \mathcal{A})^{-1} G\|_D \le c\|G\|_X,$$

where $D = W_2^2(\mathcal{F}) \times W_2^{5/2}(\mathcal{G})$, $X = L_2(\mathcal{F}) \times W_2^{3/2}(\mathcal{G})$. Hence $(sI - \mathcal{A})^{-1}$ is completely continuous and the spectrum of $\mathcal{A}$ consists of eigenvalues with the only accumulation point

at infinity (in the left complex half-plane). There can exist no more than a finite number of points of the spectrum in the right half-plane. Let a_0 is the upper bound of the real parts of these points. Proposition 3.1 holds for $\mathrm{Re}\, s > 0$, hence we can require in Theorem 2.1 that $\mathrm{Re}\, s > max(0, a_0) = a$.

Remark. From the interpolation inequality

$$|s|^{l/2+3/4}\|\nabla_\tau \rho\|_{L_2(\mathcal{G})} \le \delta \|\rho\|_{W_2^{l+5/2}(\mathcal{G})} + c(\delta)|s|^{1/2}\|\rho\|_{L_2(\mathcal{G})}, \quad \delta \ll 1,$$

and the equation $s\rho + \boldsymbol{V}\cdot\nabla_\tau\rho - \boldsymbol{v}\cdot\boldsymbol{N} = g$ it follows that the solution of the problem (2.1) satisfies, along with (2.2), the inequality

$$\begin{aligned}
&\|\boldsymbol{v}\|_{W_2^{2+l}(\mathcal{F})} + |s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|p\|_{W_2^{l+1}(\mathcal{F})} + |s|^{l/2}\|p\|_{W_2^1(\mathcal{F})} + |s|^{1+l/2+3/4}\|\rho\|_{L_2(\mathcal{G})} \\
&+ |s|\|\rho\|_{W_2^{l+3/2}(\mathcal{G})} + |s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathcal{G})} + \|\rho\|_{W_2^{l+5/2}(\mathcal{G})} \le c\big(\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F})} \\
&+ |s|^{1/4+l/2}\|\boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d}\cdot\boldsymbol{N})\|_{L_2(\mathcal{G})} + \|\boldsymbol{d}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathcal{G})} \\
&+ |s|^{l/2+3/4}\|g\|_{L_2(\mathcal{G})} + \|g\|_{W_2^{l+3/2}(\mathcal{G})}\big).
\end{aligned} \tag{3.12}$$

4 Proof of Theorem 1.1.

We start with the following auxiliary proposition.

Proposition 4.1 *Let l be as in Theorem 1.1. If $\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))$, $\rho_t \in W_2^{l+3/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{3/2}(\mathcal{G}))$, then*

$$\|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} \le c\big(\|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_t\|_{W_2^{l+3/2,0}(G_T)}\big), \tag{4.1}$$

and in the case $l > 1$

$$\|\rho_t(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})} \le c\big(\|\rho_t\|_{W_2^{l+3/2,0}(G_T)} + |\rho_t|_{l/2,3/2,G_T)}\big). \tag{4.2}$$

If $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, $l > 1/2$, then

$$\|\rho_t(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})} \le \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)}. \tag{4.3}$$

For arbitrary $\rho_0 \in W_2^{l+2}(\mathcal{G})$, $\rho_1 \in W_2^{l+1/2}(\mathcal{G})$ there exists $\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T;W_2^{5/2}(\mathcal{G}))$ with $\rho_t \in W_2^{l+3/2,0l/2+3/4}(G_T)$ such that

$$\rho(x,0) = \rho_0(x), \quad \rho_t(x,0) = \rho_1(x)$$

and

$$\begin{aligned}
&\|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + |\rho|_{l/2,5/2,G_T} \\
&\le c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_1\|_{W_2^{l+2}(\mathcal{G})}\Big)
\end{aligned} \tag{4.4}$$

Proof. Inequalities (4.1)-(4.3) are consequences of the trace theorems for isotropic and anisotropic Sobolev-Slobodetskii spaces. Let us turn to the second statement of the proposition. By the inverse trace theorem of L.N.Slobodetskii [11], we can construct $r_1 \in W_2^{l+5/2}(G_T)$ such that $r_1(x,0) = \rho_0(x)$, $r_{1t}(x,0) = 0$ and

$$\|r_1\|_{W_2^{l+5/2}(G_T)} \le c\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}.$$

By a similar theorem in the anisotropic case, there exist $r_2 \in W_2^{7/2+l,7/4+l/2}(\Sigma_T)$ such that $r_2(x,0)=0$, $r_{2t}(x,0)=\rho_1(x)$ and

$$\|r_2\|_{W_2^{l+7/2,l/2+7/4}(G_T)} \le c\|\rho_1\|_{W_2^{l+1/2}(\mathcal{G})}.$$

It is easily verified that $\rho = r_1 + r_2$ possesses all the necessary properties. The proposition is proved.

Proof of Theorem 1.1. We reduce (2.1) to a similar problem with zero divergence by construction of an auxiliary vector field $\boldsymbol{u}_1(x,t) = \nabla\Phi(x,t)$, where Φ is a solution of the Dirichlet problem

$$\nabla^2\Phi(x,t) = f(x,t), \quad x \in \mathcal{F}, \quad \Phi(x,t)|_{x\in\mathcal{G}} = 0.$$

This function satisfies the inequality

$$\|\Phi\|_{W_2^{l+3,0}(Q_T)} \le c\|f\|_{W_2^{l+1}(Q_T)}; \tag{4.5}$$

in addition, since

$$\nabla^2\Phi_t(x,t) = f_t(x,t) = \nabla\cdot\boldsymbol{F}_t(x,t), \quad x\in\mathcal{F}, \quad \Phi_t(x,t)=0, \quad x\in\mathcal{G},$$

we have

$$\|\nabla\Phi_t\|_{W_2^{0,l/2}(Q_T)} \le c\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(Q_T)}, \tag{4.6}$$

hence

$$\|\boldsymbol{u}_1\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\Big(\|f\|_{W_2^{l+1}(Q_T)} + \|\boldsymbol{F}_t\|_{W_2^{0,l/2}(Q_T)}\Big). \tag{4.7}$$

The functions $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{u}_1$, p, ρ satisfy the relations

$$\begin{cases} \boldsymbol{v}_{1t} - \nu\nabla^2\boldsymbol{v}_1 + \nabla p = \boldsymbol{f}_1(x,t), \\ \nabla\cdot\boldsymbol{v}_1(x,t) = 0, \qquad x\in\mathcal{F}, \quad t>0, \\ T(\boldsymbol{v}_1,p)\boldsymbol{N} + \boldsymbol{N}L\rho = \boldsymbol{d}_1(x,t), \\ \rho_t + \boldsymbol{V}\cdot\nabla_\tau\rho - \boldsymbol{v}_1(x,t)\cdot\boldsymbol{N}(x) = g_1(x,t), \quad x\in\mathcal{G}, \\ \boldsymbol{v}_1(x,0) = \boldsymbol{v}_0 - \boldsymbol{u}_1(x,0) \equiv \boldsymbol{w}_0(x) \quad x\in\mathcal{F}, \quad \rho(x,0)=\rho_0(x), \quad x\in\mathcal{G}, \end{cases} \tag{4.8}$$

where

$$\begin{cases} \boldsymbol{f}_1 = \boldsymbol{f} - \boldsymbol{u}_{1t} + \nu\nabla^2\boldsymbol{u}_1, \\ \boldsymbol{d}_1 = \boldsymbol{d} - \nu S(\boldsymbol{u}_1)\boldsymbol{N}, \quad g_1 = g + \boldsymbol{u}_1\cdot\boldsymbol{N}. \end{cases} \tag{4.9}$$

In particular,

$$\boldsymbol{d}_1\cdot\boldsymbol{N} = \boldsymbol{d}\cdot\boldsymbol{N} - \nu\boldsymbol{N}\cdot S(\boldsymbol{u}_1)\boldsymbol{N}|_{x\in\mathcal{G}}.$$

Now we reduce (4.8) to a similar problem with zero initial data. If $l<1$, we introduce a solenoidal vector field $\boldsymbol{u}_2(x,t)$ such that $\boldsymbol{u}_2(x,0) = \boldsymbol{w}_0(x)$ and

$$\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})}.$$

In the case $l>1$ we also compute

$$\boldsymbol{w}_t(x,0)|_{t=0} = \nu\nabla^2\boldsymbol{w}_0 - \nabla p_0(x) + \boldsymbol{f}_1(x,0) \equiv \boldsymbol{w}_1(x),$$

where p_0 is a solution of the problem

$$\begin{cases} \nabla^2 p_0(x) = \nabla\cdot\big(\boldsymbol{f}_1(x,0)\big), \quad x\in\mathcal{F},\\ p_0(x) = \nu\boldsymbol{N}\cdot S(\boldsymbol{w}_0)\boldsymbol{N} + L\rho_0 - \boldsymbol{d}_1\cdot\boldsymbol{N}, \quad x\in\mathcal{G}. \end{cases}$$

It satisfies the inequality

$$\|p_0\|_{W_2^l(\mathcal{F})} \le c\Big(\|\boldsymbol{f}_1(\cdot,0)\|_{W_2^{l-1}(\mathcal{F})} + \|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \|\boldsymbol{d}_1(\cdot,0)\cdot\boldsymbol{N}\|_{W_2^{l-1/2}(\mathcal{G})}\Big),$$

whence

$$\|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathcal{F})} \le c\Big(\|\boldsymbol{f}_1\|_{W_2^{l-1}(\mathcal{F})} + \|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \|\boldsymbol{d}_1\cdot\boldsymbol{N}\|_{W_2^{l-1/2}(\mathcal{G})}\Big). \quad (4.10)$$

We find a solenoidal vector field $\boldsymbol{u}_2(x,t)$ such that

$$\boldsymbol{u}_2(x,0) = \boldsymbol{w}_0(x), \quad \boldsymbol{u}_{2t}(x,0) = \boldsymbol{w}_1(x)$$

and

$$\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{w}_1\|_{W_2^{l+1}(\mathcal{F})}\Big). \quad (4.11)$$

Moreover, we construct $p_1(x,t)$ and $\rho_1(x,t)$ such that $p_1(x,0) = p_0(x)$,

$$\rho_1(x,0) = \rho_0(x), \quad \rho_{1t}(x,0) = -\boldsymbol{V}(x)\cdot\nabla_\tau\rho_0 - \boldsymbol{w}_0(x)\cdot\boldsymbol{N} + g_1(x,0) \equiv \rho_1'(x) \quad (4.12)$$

and

$$\begin{aligned} &\|p_1\|_{W_2^{l+1,l/2+1/2}(Q_T)} \le c\|p_0\|_{W_2^l(\mathcal{F})},\\ &\|\rho_1\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_{1t}\|_{W_2^{l+3/2,l/2+1/4}(G_T)} + |\rho|_{l/2,5/2,G_T} \le c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_1'\|_{W_2^{l+1/2}(\mathcal{G})}\Big). \end{aligned} \quad (4.13)$$

The construction of ρ_1 is described in Proposition 4.1. The construction of $\boldsymbol{u}_2$ is carried out in the following way. We find $\boldsymbol{w}_0(x,0)$ and $\boldsymbol{w}_1(x,0)$, $x\in\mathbb{R}^3$, in the form $\boldsymbol{w}_i(x,0) = \boldsymbol{\xi}_i + \boldsymbol{\eta}_i$, where $\boldsymbol{\xi}_i$ is the extension of $\boldsymbol{w}_i(x)$ into $\mathbb{R}^3$ with preservation of class; we assume that $\boldsymbol{\xi}_i$ has a compact support. Then, using the result of Bogovskii [12], we can find $\boldsymbol{\eta}_i$, also with a compact support, satisfying the equation $\nabla\cdot\boldsymbol{\eta}_i = -\nabla\cdot\boldsymbol{\xi}_i$ and the inequalities

$$\begin{aligned} &\|\boldsymbol{\eta}_1\|_{W_2^{l+1}(\mathbb{R}^3)} \le c\|\boldsymbol{\xi}_1\|_{W_2^{l+1}(\mathbb{R}^3)} \le c\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega)},\\ &\|\boldsymbol{\eta}_2\|_{W_2^{l-1}(\mathbb{R}^3)} \le c\|\boldsymbol{\xi}_2\|_{W_2^{l-1}(\mathbb{R}^3)} \le c\|\boldsymbol{w}_1\|_{W_2^{l-1}(\Omega)}. \end{aligned}$$

Finally, we define the vector field $\boldsymbol{u}_2(x,t)$ satisfying

$$\begin{aligned} &\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(\mathbb{R}^3_+\times\mathbb{R}_+)} \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathbb{R}^3)} + \|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathbb{R}^3)}\Big)\\ &\le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathcal{F})}\Big). \end{aligned}$$

Usually $\boldsymbol{u}_2$ is expressed in terms of $\boldsymbol{w}_0$ and $\boldsymbol{w}_1$ as a sum of convolution integrals (with respect to x_i); then it is divergence free.

For $\boldsymbol{v}_2 = \boldsymbol{v}_1 - \boldsymbol{u}_2,$, $p_2 = p - p_1$, $\rho_2 = \rho - \rho_1$ we have

$$\begin{cases} \boldsymbol{v}_{2t} - \nu\nabla^2\boldsymbol{v}_2 + \nabla p_2 = \boldsymbol{f}_2(x,t), \\ \nabla\cdot\boldsymbol{v}_2(x,t) = 0, \qquad x\in\mathcal{F}, \quad t>0, \\ T(\boldsymbol{v}_2,p_2)\boldsymbol{N} + \boldsymbol{N}L\rho_2 = \boldsymbol{d}_2(x,t), \\ \rho_{2t} + \boldsymbol{V}\cdot\nabla_\tau\rho_2 = \boldsymbol{v}_2(x,t)\cdot\boldsymbol{N}(x) + g_2(x,t), \quad x\in\mathcal{G}, \\ \boldsymbol{v}_2(x,0) = 0, \quad x\in\mathcal{F}, \quad r_2(x,0) = 0, \quad x\in\mathcal{G}, \end{cases} \tag{4.14}$$

where

$$\begin{aligned} \boldsymbol{f}_2 &= \boldsymbol{f}_1 - \Big(\boldsymbol{w}_{2t} - \nu\nabla^2\boldsymbol{w}_2 + \nabla p_1\Big), \\ \boldsymbol{d}_2 &= \boldsymbol{d}_1 - (T(\boldsymbol{w}_2,p_1)\boldsymbol{N} + B_0(\rho_1)\boldsymbol{N}), \\ g_2 &= g_1 - \boldsymbol{V}\cdot\nabla\rho_1 + \boldsymbol{w}_1\cdot\boldsymbol{N} - \rho_{1t}. \end{aligned}$$

Since $\boldsymbol{d}_2$, g_2 and in the case $l>1$ also $\boldsymbol{f}_2$ vanish for $t=0$ and $l<5/2$, we can extend these functions by zero to the domain $t<0$ with preservation of class, after which we extend them, also with preservation of class, into the domain $t>T$. Then we apply the Laplace transform, as in [7], assuming that $\mathrm{Re}\, s \equiv a$ is sufficiently large (s is the dual variable). The problem (4.15) is then converted in (3.1), whose solvability is proved in Theorem 2.1. The inverse Laplace transform yields the solution of (4.15) defined in an infinite time interval $(-\infty,+\infty)$. Using the estimate (2.7) and the Parceval equality, we obtain the estimate of this solution in weighted Sobolev spaces with the weight e^{-at}. From this estimate it follows that

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_{-\infty,T})} + \|\nabla p\|_{W_2^{l,l/2}(Q_{-\infty,T})} + \|p\|_{W_2^{l+1/2,0}(G_{-\infty,T})} + |p|_{l/2,1/2,G_{-\infty,T}} \\ &+ \|\rho\|_{W_2^{l+5/2,0}(G_{-\infty,T})} + |\rho|_{l/2,5/2,G_{-\infty,T}} + \|\rho_t\|_{W_2^{l+3/2,0}(G_{-\infty,T})} + |\rho_t|_{l/2,3/2,G_{-\infty,T}} \\ &\le c\Big(\|\boldsymbol{f}_2\|_{W_2^{l,l/2}(Q_{-\infty,T})} + \|\boldsymbol{d}_2 - \boldsymbol{N}(\boldsymbol{d}_2\cdot\boldsymbol{N})\|_{W_2^{l+1/2,l/2+1/4}(G_{-\infty,T})} \\ &+ \|\boldsymbol{d}_2\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_{-\infty,T})} + |d_2\cdot\boldsymbol{N}|_{l/2,1/2,G_{-\infty,T}} \\ &+ \|g_2\|_{W_2^{l+3/2,0}(G_{-\infty,T})} + |g_2|_{l/2,3/2,G_{-\infty,T}}\Big), \end{aligned} \tag{4.15}$$

where $Q_{-\infty,T} = \mathcal{F}\times(-\infty,T)$, $G_{-\infty,T} = \mathcal{G}\times(-\infty,T)$. All the functions in (4.15) vanish for $t<0$, and the constant c is bounded for finite T. From (4.15) it is easy to deduce the estimate

$$\begin{aligned} &Y_T(\boldsymbol{v}_2,q_2,\rho_2) \le c(T)\Big(\|\boldsymbol{f}_2\|_{W_2^{l,l/2}(Q_T)} + \|\boldsymbol{d}_2 - \boldsymbol{N}(\boldsymbol{d}_2\cdot\boldsymbol{N})\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \\ &+ \|\boldsymbol{d}_2\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + |d_2\cdot\boldsymbol{N}|_{l/2,1/2,G_T} + \|g_2\|_{W_2^{l+3/2,0}(G_T)} + |g_2|_{l/2,3/2,G_T}\Big), \end{aligned} \tag{4.16}$$

and from (4.16), (4.3), (4.7) inequality (2.4) follows. This completes the proof of Theorem 1.1.

For the application of this theorem to the proof of the local solvability of a nonlinear problem (see Sec. 5) it is important to be sure that the constant in the basic inequality (2.4) remains bounded for small T. In fact it is not always the case, because the norm $\|u\|_{W_2^l(-\infty,T)}$, $l=[l]+\lambda$, $0<\lambda<1$ of the function $u(t)$ vanishing for $t<0$ is equivalent to

$$\Big(\|u\|^2_{W_2^l(0,T)} + \int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}dt\Big)^{1/2}$$

(in this connection see [13] and [6], Ch.4). If $\lambda > 1/2$ and $D_t^{[l]}u|_{t=0} = 0$, then

$$\int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}dt \le c\int_0^T \frac{dh}{h^{1+2\lambda}}\int_h^T |u(t-h)-u(t)|^2 dt$$

with the constant independent of T. In the case $\lambda < 1/2$ we have

$$\begin{aligned} c_1\Big(\|u\|^2_{W_2^l(0,T)} + \int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}dt\Big) &\le \|u\|^2_{W_2^l(0,T)} + \frac{1}{T^{2\lambda}}\int_0^T |D_t^{[l]}u(t)|^2 dt \\ &\le c_2\Big(\|u\|^2_{W_2^l(0,T)} + \int_0^T \frac{|D_t^{[l]}u(t)|^2}{t^{2\lambda}}dt\Big), \end{aligned}$$

where the constants are also independent of T. Hence the constant $c(T)$ in (4.16) becomes uniformly bounded for finite T, if all the $W_2^{l/2}(0,T)$-norms in this inequality are replaced with the $\widehat{W}_2^{l/2}(0,T)$-norms defined by

$$\begin{aligned} &\|u\|_{\widehat{W}_2^{l/2}(0,T)} = \|u\|_{W_2^{l/2}(0,T)}, \quad \text{if} \quad \lambda > 1/2, \\ &\|u\|_{\widehat{W}_2^{l/2}(0,T)} = \Big(\|u\|^2_{W_2^{l/2}(0,T)} + \frac{1}{T^{2\lambda}}\int_0^T |D_t^{[l/2]}u(t)|^2 dt\Big)^{1/2}, \quad \text{if} \quad \lambda > 1/2 \end{aligned} \tag{4.17}$$

(here λ is a fractional part of $l/2$). As a consequence of (4.15), we have

$$\begin{aligned} &\|\boldsymbol{v}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \langle p\rangle_{l/2,1/2,G_T} \\ &+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \langle\rho\rangle_{l/2,5/2,G_T} + \|\rho_t\|_{W_2^{l+3/2,0}(G_T)} + \langle\rho_t\rangle_{l/2,3/2,G_T} + \langle\rho\rangle_{l/2,5/2,G_T} \\ &\le c\Big(\|\boldsymbol{f}_2\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|\boldsymbol{d}_2 - \boldsymbol{N}(\boldsymbol{d}_2\cdot\boldsymbol{N})\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_T)} \\ &+ \|\boldsymbol{d}_2\cdot\boldsymbol{N}\|_{\widehat{W}_2^{l+1/2,0}(G_T)} + \langle d_2\cdot\boldsymbol{N}\rangle_{l/2,1/2,G_T} + \|g_2\|_{\widehat{W}_2^{l+3/2,0}(G_T)} + \langle g_2\rangle_{l/2,3/2,G_T}\Big), \end{aligned} \tag{4.18}$$

where $\langle\cdot\rangle_{l/2,r,G_T}$ is the norm in $\widehat{W}^{l/2}(0,T;W_2^r(\mathcal{G}))$. By $\widehat{W}_2^{l,l/2}(Q_T)$ we mean the space with the modified norm (1.2):

$$\|u\|^2_{\widehat{W}_2^{l,l/2}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_\Omega \|u(x,\cdot)\|^2_{\widehat{W}_2^{l/2}(0,T)}dx. \tag{4.19}$$

It is clear that the norms (1.2) and (4.19) are equivalent.

Now we turn to the inequality (4.11). We can set there $T = \infty$; it is possible also to assume that $\boldsymbol{u}_2$ vanishes for $t > t_0$. We use the inequality (see [13], Lemma 2)

$$\int_0^\infty \frac{|v(t)|^2}{t^{2\lambda}}dt \le c\int_0^\infty \frac{dh}{h^{1+2\lambda}}\int_h^\infty |v(t-h)-v(y)|^2 dt \tag{4.20}$$

valid for $\lambda \in (0,1/2)$ and for $\lambda \in (1/2,1)$, $v(0) = 0$. It follows that the norm $\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(Q_T)}$ in (4.11) can be replaced with $\|\boldsymbol{u}_2\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)}$. The same is true for the inequalities (4.6),

(4.7). Hence we have, along with (1.4), (1.5):

$$\begin{aligned}\widehat{Y}_T(\boldsymbol{v},p,\rho) \equiv &\|\boldsymbol{v}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \langle p\rangle_{l/2,1/2,G_T} \\ &+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \langle\rho\rangle_{l/2,5/2,G_T} + \|\rho_t\|_{W_2^{l+3/2,0}(G_T)} + \langle\rho_t\rangle_{l/2,3/2,G_T} + \langle\rho\rangle_{l/2,5/2,G_T} \\ &\le c\Big(\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widehat{W}_2^{0,1+l/2}(Q_T)} \\ &+ \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + \langle\boldsymbol{d}\cdot\boldsymbol{N}\rangle_{l/2,1/2,G_T} \\ &+ \|g\|_{W_2^{l+3/2,0}(G_T)} + \langle g\rangle_{l/2,3/2,G_T} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big),\end{aligned} \tag{4.21}$$

$$\begin{aligned}&\|\boldsymbol{v}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + \langle p\rangle_{l/2,1/2,G_T} \\ &+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \langle\rho\rangle_{l/2,5/2,G_T} + \|\rho_t\|_{\widehat{W}_2^{l+3/2,l/2+3/4}(G_T)} + \langle\rho\rangle_{l/2,5/2,G_T} \\ &\le c\Big(\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widehat{W}_2^{0,1+l/2}(Q_T)} \\ &+ \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + \langle\boldsymbol{d}\cdot\boldsymbol{N}\rangle_{l/2,1/2,G_T} \\ &+ \|g\|_{\widehat{W}_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F}_1)} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big)\end{aligned} \tag{4.22}$$

with the constants independent of T.

5 On the free boundary problem

Theorem 2.1 provides an analytical basis for the proof of the solvability of the free boundary problem governing the motion of an isolated liquid mass:

$$\begin{cases}\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v} = 0, \quad x\in\Omega_t, \quad t>0, \\ T(\boldsymbol{v},p)\boldsymbol{n}(x) = \sigma H, \\ V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x\in\Gamma_t, \\ \boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y\in\Omega_0.\end{cases} \tag{5.1}$$

Unknown are the domain Ω_t with the boundary Γ_t for $t>0$, $\boldsymbol{v}(x,t)$ and $p(x,t)$, $x\in\Omega_t$. The domain Ω_0 is given. By $\boldsymbol{n}$ we mean the exterior normal to Γ_t, and V_n is the velocity of the evolution of Γ_t in the normal direction.

We assume that Γ_0 is close to a smooth closed surface $\mathcal{G}$, so that Γ_0 can be regarded as a normal perturbation of $\mathcal{G}$:

$$x = y + \boldsymbol{N}(y)\rho_0(y), \quad y\in\mathcal{G},$$

where $\boldsymbol{N}(y)$ is the unit exterior normal to $\mathcal{G}$ and ρ_0 is a given small function. We denote by $\mathcal{F}$ the domain bounded by $\mathcal{G}$. And we assume that at least for small t Γ_t is also close to $\mathcal{G}$ and can be given by the equation

$$x = y + \boldsymbol{N}(y)\rho(y,t), \quad y\in\mathcal{G}$$

with an unknown function $\rho(y,t)$.

As usual, the free boundary problem (5.1) is written as a nonlinear problem in a given domain, which is achieved by the mapping of Ω_t onto this domain. We use the transformation

$$x = y + \boldsymbol{N}^*(y)\rho^*(y) \equiv e_\rho(y) : \mathcal{F} \to \Omega_t, \tag{5.2}$$

where ρ^* and $\boldsymbol{N}^*$ are extensions of ρ and $\boldsymbol{N}$ from $\mathcal{G}$ in $\mathcal{F}$ such that $\boldsymbol{N}^*$ is a sufficiently regular vector field and ρ^* is a function with a small C^1- norm. This guarantees the invertibility of e_ρ.

We denote by $\mathcal{L} = \mathcal{L}(y,\rho^*)$ the Jacobi matrix of the transformation $x = e_\rho(y)$ and we set $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$. By $l_{ij}(y,\rho^*)$, $l^{ij}(y,\rho^*)$ $\widehat{L}_{ij}(y,\rho^*)$ we denote the elements of $\mathcal{L}$, $\mathcal{L}^{-1}$, $\widehat{\mathcal{L}}$. The transformation (5.2) converts the operator of the gradient with respect to x, ∇_x, in $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla$, $\nabla = \nabla_y$. Equations (5.1) take the form

$$\begin{cases} \boldsymbol{u}_t(y,t) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u},q,\rho), \\ \nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u},\rho), \quad y\in\mathcal{F}, t>0, \\ \Pi_{\mathcal{G}} S(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{l}_2(\boldsymbol{u},\rho), \\ -q + \nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}(y) + \sigma\boldsymbol{N}(y)L(\rho) = l_4(\boldsymbol{u},\rho) + l_5(\rho) + \sigma\mathcal{H}(y), \\ \rho_t + \boldsymbol{V}(y)\cdot\nabla_\tau\rho - \boldsymbol{u}\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u},\rho), \quad y\in\mathcal{G}, \quad y\in\mathcal{G}, \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}, \quad \rho(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \end{cases} \tag{5.3}$$

where $\boldsymbol{u}(y,t) = \boldsymbol{v}(e_\rho(y),t)$, $q(y,t) = p(e_\rho(y),t)$,

$$\begin{cases} \boldsymbol{l}_1(\boldsymbol{u},q,\rho) = \nu(\widetilde{\nabla}^2 - \nabla^2)\boldsymbol{u} + (\nabla - \widetilde{\nabla})q + \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*(y)\cdot\nabla)\boldsymbol{u} - (\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u}, \\ l_2(\boldsymbol{u},\rho) = (I - \widetilde{\mathcal{L}}^T)\nabla\cdot\boldsymbol{u} = \nabla\cdot(I - \widehat{\mathcal{L}})\boldsymbol{u}, \\ \boldsymbol{l}_3(\boldsymbol{u},\rho) = \Pi_{\mathcal{G}}(\Pi_{\mathcal{G}} S(\boldsymbol{u})\boldsymbol{N})(y) - \Pi\widetilde{S}(\boldsymbol{u})\boldsymbol{n}(e_\rho(y)), \\ l_4(\boldsymbol{u},\rho) = \nu(\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{n}\cdot\widetilde{S}(\boldsymbol{u})\boldsymbol{n}), \\ l_5(\rho) = \displaystyle\int_0^1 (1-s)\frac{d^2}{ds^2}\frac{\mathcal{L}^T(y,s\rho)\boldsymbol{N}}{|\mathcal{L}^T(y,s\rho)\boldsymbol{N}|}ds, \\ l_6(\boldsymbol{u},\rho) = (\dfrac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(y,\rho)|} - \boldsymbol{N} + \nabla_\tau\rho)\cdot\boldsymbol{u} + (\boldsymbol{V}(y) - \boldsymbol{u}(y,t))\cdot\nabla_\tau\rho, \quad y\in\mathcal{G}. \end{cases} \tag{5.4}$$

By $\widetilde{S}$ we mean the transformed rate-of-strain tensor: $\widetilde{S} = (\widetilde{\nabla}\boldsymbol{u}) + (\widetilde{\nabla}\boldsymbol{u})^T$; the normals $\boldsymbol{n}(e_\rho)$ and $N(y)$ are connected by

$$\boldsymbol{n}(e_\rho(y)) = \frac{\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|},$$

$\Pi\boldsymbol{f} = \boldsymbol{f} - \boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{f})$. The expression $L\rho = -\Delta_{\mathcal{G}}\rho + (\mathcal{H}^2 - 2\mathcal{K})\rho$ is computed as the first variation of

$$H - \mathcal{H} = -\nabla_x\cdot\boldsymbol{n}(x)|_{x=e_\rho} + \nabla_y\cdot\boldsymbol{N}(y)$$

with respect to ρ.

Theorem 5.1. *If $\boldsymbol{u}_0 \in W_2^{l+1}(\mathcal{F})$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$ and the compatibility and the smallness conditions*

$$\widetilde{\nabla}\cdot\boldsymbol{u}_0 = 0, \quad \Pi\widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n} = 0, \quad t = 0,$$
$$\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \le \epsilon \ll 1$$

are satisfied, then the problem (5.1) *has a unique solution with finite norm* $\widehat{Y}_T(\boldsymbol{u}, q, \rho)$ *(see* (4.21)*) defined on a certain (small) time interval* $(0, T)$.

The solvability of the problem (5.3) can be established by the method of successive approximations, according to a usual scheme:

$$\begin{cases} \boldsymbol{u}_{m+1,t}(y,t) - \nu\nabla^2\boldsymbol{u}_{m+1} - \nabla q_{m+1} = \boldsymbol{l}_1(\boldsymbol{u}_m, q_m, \rho_m), \\ \nabla\cdot\boldsymbol{u}_{m+1} = l_2(\boldsymbol{u}_m, \rho_m), \quad y\in\mathcal{F}, t>0, \\ \Pi_{\mathcal{G}} S(\boldsymbol{u}_{m+1})\boldsymbol{N} = \boldsymbol{l}_2(\boldsymbol{u}_m, \rho_m), \\ -q_{m+1} + \nu\boldsymbol{N}\cdot S(\boldsymbol{u}_{m+1})\boldsymbol{N}(y) + \sigma\boldsymbol{N}(y)L_0(\rho_{m+1}) = l_4(\boldsymbol{u}_m,\rho_m) + l_5(\rho_m) + \sigma\mathcal{H}(y), \\ \rho_{m+1,t} + \boldsymbol{V}(y)\cdot\nabla_\tau\rho_{m+1} - \boldsymbol{u}_{m+1}\cdot\boldsymbol{N}(y) = l_6(\boldsymbol{u}_m,\rho), \quad y\in\mathcal{G}, \quad y\in\mathcal{G}, \\ \boldsymbol{u}_{m+1}(y,0) = \boldsymbol{u}_0(y), \quad y\in\mathcal{F}, \quad \rho_{m+1}(y,0) = \rho_0(y), \quad y\in\mathcal{G}, \end{cases} \tag{5.5}$$

$m = 1, 2, \ldots$. As the first approximation, we take the functions $(\boldsymbol{u}_1, \rho_1)$ satisfying the conditions

$$\nabla\cdot\boldsymbol{u}_1 = 0, \quad \boldsymbol{u}_1(y,0) = \boldsymbol{u}_0(y), \quad \rho_1(y,0) = \rho_0(y),$$

and we set $q_1 = 0$. We require that

$$\begin{aligned} &\|\boldsymbol{u}_1\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F})}, \\ &\|\rho_1\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_{1,t}\|_{W_2^{l+3/2,l/2+3/4}(G_T)} \le c\|\rho_0\|_{W_2^{l+1}(\mathcal{G})} \end{aligned} \tag{5.6}$$

Then the compatibility conditions (1.3) in the linear problems (5.5) are satisfied for all $m \ge 1$. Moreover, the estimates of nonlinear terms (that are omitted) enable us to show, using Theorem 2.1, that

$$\begin{aligned} &\widehat{Y}_T(\boldsymbol{u}_{m+1}, q_{m+1}, \rho_{m+1}) \le \delta_3\widehat{Y}_T(\boldsymbol{u}_m, q_m, \rho_m) + \delta_4\widehat{Y}_T^2(\boldsymbol{u}_m, q_m, \rho_m) \\ &+ c\Big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{q}_0\|_{W_2^l(\mathcal{F})} + \|\rho_0\|_{W_2^{l+2}(\mathcal{F})} + \|\mathcal{H}\|_{W_2^l(\mathcal{G})}\Big), \end{aligned} \tag{5.7}$$

where δ_3 and δ_4 are numbers depending on T and $\|\rho_0\|_{W_2^{l+3/2}(\mathcal{F})}$ and going to zero as T and the $W_2^{l+3/2}(\mathcal{F})$- norm of ρ_0 tend to zero. When these numbers are sufficiently small, then inequalities (5.7) guarantee a uniform estimate for $Y_T(\boldsymbol{u}_{m+1}, q_{m+1}, \rho_{m+1})$. The convergence of $(\boldsymbol{u}_m, q_m, \rho_m)$ to the solution of (2.2) is proved by similar arguments.

Estimates (5.7) hold, when the vector field $\boldsymbol{V}(x)$ is chosen properly. According to our calculations, it should belong to $W_2^{l+3/2}(\mathcal{G})$ and satisfy the condition

$$\sup_{\mathcal{G}}|\boldsymbol{V}(x) - \boldsymbol{u}_0| + \|\boldsymbol{V} - \boldsymbol{u}_0\|_{W_2^1(\mathcal{G})} \le \delta_5 \ll 1. \tag{5.8}$$

Detailed proof of Theorem 5.1 will be given in subsequent publications.

References.

1. V.A.Solonnikov, *Solvability of the problem on the evolution of an isolated volume of viscous incompresible capillary fluid*, Zapiski nauchn. semin. LOMI, **140** (1984), 179-186.

2. V.A.Solonnikov, *On initial-boundary value problem for the Stokes system arising in the study of a free boundary problem*, Trudy Mat. Inst. Steklov, **188** (1990), 150-188.

3. V.A.Solonnikov, *On the stability of uniformly rotating viscous incompressible self-gravitating liquid.* J.Math.Sci. **152**, No 5 (2008), 4343-4370.

4. V.A.Solonnikov, *On the problem of stability of equilibrium figures of uniformly rotating viscous incompressible liquid*, in: Instability in models connected with fluid flows II, 189-254, Int. Math. Ser. (N.Y.) 7, Springer, New York, 2008.

5. V.A.Solonnikov, *On the linear problem related to the stability of uniformly rotating liquid*, J.Math.Sci., **144**, No 6 (2007), 4671-4695.

6. O.A. Ladyzhenskaya, V.A.Solonnikov, N.N.Uraltseva, *Linear and quasilinear equations of parabolic type*, AMS, Providence, Rhode Island, 1968.

7. M.S.Agranovich, M.I.Vishik, Elliptic problems with a parameter and parabolic problems of general type, Uspehi Mat. Nauk, **19** No 3 (1964), 53-161.

8. G.Grubb, V.A.Solonnikov, *Boundary value problems for the nonstationary Navier-Stokes equations treated by pseudo-differential methods*, Math. Scandinavica, **69** (1991), 217-290.

9. V.A.Solonnikov, *On the stability of uniformly rotating viscous incompressible self-gravitating liquid*, PDMI Preprint 03/2010, 1-101.

10. J.T.Beale, T.Nishida, *Large-time behavior of viscous surface waves*, Lect. Notes in Num. Appl. Anal. **8** (1985), 1-14.

11. L.N.Slobodetskii, *Generalized S.L.Sobolev spaces and their applications to the boundary value problems for partial differential equations.* Uchen. Zap. Leningr. Herzen Ped. Inst. 197 (1958), 54-112.

12. M.E.Bogovskii, *Solution of some problems of vector analysis connected with operators div and grad.* Trudy Semin. S.L.Soboleva, No 1 (1980), Acad. Nauk SSSR, Sib. Otdel, Inst. Math., Novosibirsk (1980), 5-40.

13. V.A.Solonnikov, *A-priori estimates for second order parabolic equations*, Trudy Mat. Inst. Steklov **70**, (1964), 133-212.