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On the stability of uniformly rotating viscous incompressible self-gravitating liquid

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ABSTRACT:

The present communication contains lectures given by the author at the Institute of Mathematics of the Academy of Sciences of the People Republic of China in April 2009. They are devoted to the free boundary problem for the Navier–Stokes equations governing the evolution of an isolated liquid mass. The main attention is given tothe problem of stability (and instability) of a finite mass subjected to the forces of self-gravitation and, possibly, of the surface tension on a freeboundary and rotating uniformly about a fixed axis. In addition, the lectures contain an auxiliary material used in the analysis of problems of fluid mechanics. In particular, basic properties of the Sobolev–Slobodetskii spaces are analyzed and some auxiliary relations and inequalities (e.g. Korn's inequality) are proved.

Key words: rotating fluid, stability, free boundary

1 Formulation of main results

The lectures are devoted to the free boundary problem governing the evolution of an isolated mass of a viscous incompressible fluid bounded only by a free surface. It is assumed that the liquid is subject to the forces of self-gravitation and to the capillary forces on the boundary. The problem consists of determination of a bounded domain $\Omega_t \subset \mathbb{R}^3$, $t > 0$, as well as of the vector field of velocities $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$ and the pressure function $p(x,t)$, $x \in \Omega_t$, $t > 0$, satisfying the equations

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \nabla^2 \boldsymbol{v} + \nabla p = \kappa \nabla U(x,t), \\ \nabla \cdot \boldsymbol{v} = 0, \qquad x \in \Omega_t, \ \ t > 0, \\ T(\boldsymbol{v}, p)\boldsymbol{n} = \sigma H \boldsymbol{n}, \quad V_n = \boldsymbol{v} \cdot \boldsymbol{n}, \qquad x \in \Gamma_t \equiv \partial \Omega_t, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad x \in \Omega_0. \end{cases}$$

Here $\nu = const > 0$ is the viscosity coefficient, κ and σ are non-negative gravitational constant and the coefficient of the surface tension, respectively, non vanishing simultaneously (i.e., $\kappa + \sigma > 0$),

$$U(x,t) = \int_{\Omega_t} \frac{dz}{|x - z|}$$

is the Newtonian potential depending on an unknown domain Ω_t, Γ_t is the boundary of Ω_t, $H(x)$ is the doubled mean curvature of Γ_t, negative for convex domains, $T(\boldsymbol{v}, p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor, $S(\boldsymbol{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j} \right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, $\boldsymbol{n}$ is the exterior normal to Γ_t, V_n is the velocity of evolution of Γ_t in the normal direction. The density of the liquid is assumed to be equal to one. The domain Ω_0 is given.

By introducing a new pressure $p - \kappa U$ instead of p, we can write the above problem in the form

$$\begin{cases} \boldsymbol{v}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nu \nabla^2 \boldsymbol{v} + \nabla p = 0, \quad \nabla \cdot \boldsymbol{v} = 0, \qquad x \in \Omega_t, \ \ t > 0, \\ T(\boldsymbol{v}, p)\boldsymbol{n} = (\kappa U(x,t) + \sigma H)\boldsymbol{n}, \quad V_n = \boldsymbol{v} \cdot \boldsymbol{n}, \qquad x \in \Gamma_t \equiv \partial \Omega_t, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad x \in \Omega_0. \end{cases} \tag{1.1}$$

From the conservation of mass and of momenta it follows that

$$|\Omega_t| = |\Omega_0|,$$

$$\int_{\Omega_t} \boldsymbol{v}(x,t)dx = \int_{\Omega_0} \boldsymbol{v}_0(x)dx, \quad \int_{\Omega_t} \boldsymbol{v} \times x dx = \int_{\Omega_0} \boldsymbol{v}_0(x) \times x dx.$$

The problem (1.1) is well posed, in the sense that for arbitrary initial data $\boldsymbol{v}_0(x)$ and Ω_0 possessing some regularity properties and satisfying natural compatibility conditions it has a unique solution defined at least in a finite time interval.

There exist solutions of the problem (1.1) corresponding to the uniform rotation of the liquid as a rigid body about a fixed axis (x_3-axis). Then the velocity and the pressure are given by

$$\boldsymbol{V}(x) = \omega(\boldsymbol{e}_3 \times \boldsymbol{x}) = \omega(-x_2, x_1, 0), \qquad P(x) = \frac{\omega^2}{2}|x'|^2 + p_0, \tag{1.2}$$

where $x' = (x_1, x_2, 0)$, $p_0 = const$, $\boldsymbol{e}_3$ is a unit vector directed along the x_3-axis and ω is the constant angular velocity of rotation. The boundary conditions furnish the equation that determines the domain $\mathcal{F}$, filled with the rotating liquid:

$$\sigma\mathcal{H}(x) + \frac{\omega^2}{2}|x'|^2 + \kappa\mathcal{U} + p_0 = 0, \quad x \in \mathcal{G} = \partial\mathcal{F}, \tag{1.3}$$

where $\mathcal{H}$ is the doubled mean curvature of $\mathcal{G}$, negative for convex domains,

$$\mathcal{U}(x) = \int_{\mathcal{F}} \frac{dz}{|x-z|},$$

and ω is a constant angular velocity of rotation. The domain $\mathcal{F}$ is known as the equilibrium figure of uniformly rotating liquid. In the case $\sigma = 0$ equilibrium figures have been studied in classical papers of Maclaurin, Jacobi, Riemann, Poincaré, Lyapunov and many other great mathematicians (see, for instance, the book of P.Appell [1] where the case $\sigma > 0$ is also considered). It was found that $\mathcal{F}$ may be axially symmetric with respect to the x_3-axis (as Maclaurin ellipsoids in the case $\sigma = 0$) or not (Jacobi ellipsoids, pear-formed figures of Poincaré etc.) In the first case the functions (1.2) given in the domain $\mathcal{F}$ represent the stationary solution of (1.1). In the second case the same functions defined in the variable domain $\mathcal{F}_{\omega t+\varphi}$, where $\mathcal{F}_\theta$ is obtained from a fixed equilibrium figure $\mathcal{F} \equiv \mathcal{F}_0$ by rotation of an angle θ about the x_3-axis, represent a periodic solution of (1.1).

The question of stability of the solution (1.2) is of a special interest. As in classical mechanics, following Lagrange, one used to decide about stability according to the properties of the second variation of the energy functional. This is a quadratic form

$$\begin{aligned}\delta^2\mathcal{R}(\rho) = \sigma\int_{\mathcal{G}} |\nabla_{\mathcal{G}}\rho|^2 dS + \int_{\mathcal{G}} b(x)\rho^2(x)dS + \frac{\omega^2}{\int_{\mathcal{F}}|z'|^2dz}\Big(\int_{\mathcal{G}} |y'|^2\rho(y)dS\Big)^2 \\ - \kappa\int_{\mathcal{G}}\int_{\mathcal{G}} \frac{\rho(y)\rho(z)}{|y-z|}dS_y dS_z\end{aligned} \tag{1.4}$$

where $\nabla_{\mathcal{G}}$ is the surface gradient on $\mathcal{G}$,

$$b(x) = -\sigma(\mathcal{H}^2 - 2\mathcal{K}) - \omega^2\boldsymbol{x}'\cdot\boldsymbol{N}(x) - \kappa\frac{\partial\mathcal{U}(x)}{\partial N} = -\sigma(k_1^2 + k_2^2) - \omega^2\boldsymbol{x}'\cdot\boldsymbol{N}(x) - \kappa\frac{\partial\mathcal{U}(x)}{\partial N}, \tag{1.5}$$

$\boldsymbol{N}(x)$ is an exterior normal to $\mathcal{G}$ and $\mathcal{K}$ is the Gaussian curvature, k_1 and k_2 are principal curvatures of $\mathcal{G}$ (the functional $\mathcal{R}$ is defined in (6.30) and (1.4) is justified in Sec.8).

If the form (1.4) is positive definite for arbitrary function $\rho(x)$ given on $\mathcal{G}$ and satisfying the conditions

$$\int_{\mathcal{G}} \rho(x)dS = 0, \quad \int_{\mathcal{G}} \rho(x)x_i dS = 0, \quad i = 1, 2, 3, \tag{1.6}$$

$$\int_{\mathcal{G}} \rho(x)h(x)dS_x = 0, \tag{1.7}$$

where $h(x) = \boldsymbol{N}(x)\cdot(\boldsymbol{e}_3 \times x)$, $x \in \mathcal{G}$, then the solution (1.2) of (1.1) was considered stable, and in the opposite case, i.e., when this form can take negative values, the solution was considered as unstable. (The function $h(x)$ equals zero, if $\mathcal{F}$ is axially symmetric, and it satisfies the equation $\delta^2\mathcal{R}(h) = 0$ that is a consequence of (7.3)).

Here these statements are justified by analyzing the evolution free boundary problem for the perturbations of the velocity and the pressure. We assume that $\mathcal{F}$ is a given bounded domain with a smooth surface $\mathcal{G}$ defined by the equation (1.3) whose barycenter is located at the origin, i.e., the equations

$$\int_{\mathcal{F}} x_i dx = 0, \quad i = 1,2,3$$

hold. The initial data, $\boldsymbol{v}_0$ and Ω_0, are close to $\boldsymbol{V}$ and $\mathcal{F}$, respectively, and they satisfy the natural conditions

$$|\Omega_0| = |\mathcal{F}|, \quad \int_{\Omega_0} x_i dx = 0, \quad i = 1,2,3 \tag{1.8}$$

and

$$\begin{gathered}\int_{\Omega_0} \boldsymbol{v}_0(x)dx = 0,\\ \int_{\Omega_0} (x \times \boldsymbol{v}_0)dx = \int_{\mathcal{F}} (x \times \boldsymbol{V}(x))dx = \beta \boldsymbol{e}_3,\end{gathered} \tag{1.9}$$

where $\beta = \omega \int_{\mathcal{F}} |x'|^2 dx$ is the magnitude of the angular momentum of the rotating liquid. Then, if the form (1.4) is positive definite, we prove that the problem (1.1) has a unique solution defined in the infinite time interval $t > 0$, and $\boldsymbol{v} \to \boldsymbol{V}$, $p \to P$, $\Omega_t \to \mathcal{F}_\varphi$, as $t \to \infty$. Moreover, we establish that it is not the case when the form (1.4) can take negative values for some ρ satisfying (1.6), (1.7).

For simplicity we assume that $\mathcal{F}$ is rotationally symmetric with respect to the x_3-axis. The case of non-symmetric $\mathcal{F}$ is treated in [2,3].

It is convenient to work with the free boundary problem for the perturbations of the velocity and the pressure $\boldsymbol{v}-\boldsymbol{V}$, $p-P$ written in the coordinate system that rotates uniformly with the same angular velocity ω. It has the form

$$\begin{cases} \boldsymbol{w}_t + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu\nabla^2\boldsymbol{w} + \nabla s = 0, \\ \nabla\cdot\boldsymbol{w}(y,t) = 0, \qquad y \in \Omega'_t, \quad t > 0, \\ T(\boldsymbol{w}, s)\boldsymbol{n}' = (\sigma H' + \dfrac{\omega^2}{2}|y'|^2 + \kappa U'(y,t) + p_0)\boldsymbol{n}', \\ V'_n = \boldsymbol{w}\cdot\boldsymbol{n}', \quad y \in \Gamma'_t, \\ \boldsymbol{w}(y,0) = \boldsymbol{v}_0(y) - \boldsymbol{V}(y) \equiv \boldsymbol{w}_0(y), \quad y \in \Omega_0, \end{cases} \tag{1.10}$$

where $\Omega'_t = \mathcal{Z}(\omega t)\Omega_t$, $\boldsymbol{w}$ and s are the perturbations of the velocity and pressure written as functions of $y \in \Omega'_t$,

$$\mathcal{Z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$\boldsymbol{n}'$ is the exterior normal to Γ'_t, H' is the doubled mean curvature of Γ'_t and $U' = \int_{\Omega'_t} |y - z|^{-1} dz$. The conditions (1.9) are transformed into

$$\begin{gathered}\int_{\Omega_0} \boldsymbol{w}_0(x)dx = 0,\\ \int_{\Omega_0} \boldsymbol{w}_0(x)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_0} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx = \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx,\end{gathered} \tag{1.11}$$

where $\boldsymbol{\eta}_i(x) = \boldsymbol{e}_i \times x$, $\boldsymbol{e}_i = (\delta_{ij})_{j=1,2,3}$. It is easily verified that (1.11) hold for arbitrary $t \geq 0$, i.e.,

$$\begin{aligned}
&\int_{\Omega'_t} \boldsymbol{w}(x,t)dx = 0, \\
&\int_{\Omega'_t} \boldsymbol{w}(x,t) \cdot \boldsymbol{\eta}_i(x)dx + \omega \int_{\Omega'_t} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x)dx \quad = \omega \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x)dx,
\end{aligned} \tag{1.12}$$

$i = 1, 2, 3$. We also have

$$|\Omega'_t| = |\mathcal{F}|, \quad \int_{\Omega'_t} x_i dx = 0, \quad i = 1, 2, 3. \tag{1.13}$$

To the solution (1.2) of (1.1) corresponds the zero solution of (1.10).

If the surface Γ'_t is close to $\mathcal{G}$, then it can be considered as a normal perturbation of $\mathcal{G}$, i.e., it can be given by the equation

$$x = y + \boldsymbol{N}(y)\rho(y,t), \quad y \in \mathcal{G}, \tag{1.14}$$

where ρ is a small function. In particular, Γ_0 is defined by the equation (1.14) with a given function $\rho_0(y)$.

We can extend $\boldsymbol{N}(y)$ and $\rho(y,t)$ from $\mathcal{G}$ into $\mathcal{F}$ so that $\boldsymbol{N}$ remains sufficiently regular and ρ remains small together with its gradient (more detailed assumptions concerning the extensions $\boldsymbol{N}^*$ and ρ^* of $\boldsymbol{N}$ and ρ will be formulated in Sec. 5). Then the relation

$$x = y + \boldsymbol{N}^*(y)\rho^*(y,t) \equiv e_\rho(y), \quad y \in \mathcal{F}, \tag{1.15}$$

defines an invertible mapping of $\mathcal{F}$ onto Ω'_t. We observe that the condition $V_n = \boldsymbol{w} \cdot \boldsymbol{n}'$ is equivalent to

$$\rho_t(y,t) = \frac{\boldsymbol{w}(x,t) \cdot \boldsymbol{n}'(x)}{\boldsymbol{n}'(x) \cdot \boldsymbol{N}(y)}, \tag{1.16}$$

where x and y are connected with each other according to (1.14), and (1.13) can be written in terms of ρ in the form

$$\int_{\mathcal{G}} \varphi(z,\rho)dS = 0, \quad \int_{\mathcal{G}} \psi_i(z,\rho)dS = 0 \quad i = 1, 2, 3, \tag{1.17}$$

where

$$\varphi(z,\rho) = \rho - \frac{\rho^2}{2}\mathcal{H}(z) + \frac{\rho^3}{3}\mathcal{K}(z),$$

$$\psi_i(z,\rho) = \varphi(z,\rho)z_i + N_i(z)\Big(\frac{\rho^2}{2} - \frac{\rho^3}{3}\mathcal{H}(z) + \frac{\rho^4}{4}\mathcal{K}(z)\Big).$$

Now we present our main results concerning stability of the zero solution of the problem (1.10), (1.12). This will be done for the cases $\sigma > 0$ and $\sigma = 0$ separately.

1. $\sigma > 0$.

First of all, we write (1.10), (1.12) as a nonlinear problem in a fixed domain $\mathcal{F}$. We map $\mathcal{F}$ on Ω'_t by the transformation (1.15). We denote by $\mathcal{L} = \mathcal{L}(y,\rho^*)$ the Jacobi matrix of the transformation (1.15) and we set $L = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$. By $l_{ij}(y,\rho^*)$, $l^{ij}(y,\rho^*)$ $\widehat{L}_{ij}(y,\rho^*)$ we denote the elements of $\mathcal{L}$, $\mathcal{L}^{-1}$, $\widehat{\mathcal{L}}$. Under the transformation (1.15), the equations (1.10) take the form

$$\begin{cases} \boldsymbol{u}_t - \rho_t^*(\mathcal{L}^{-1}\boldsymbol{N}^*\cdot\nabla)\boldsymbol{u} + (\mathcal{L}^{-1}\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + 2\omega(e_3\times\boldsymbol{u}) - \nu\widetilde{\nabla}\cdot\widetilde{\nabla}\boldsymbol{u} + \widetilde{\nabla}q = 0, \\ \nabla\cdot\widehat{\mathcal{L}}\boldsymbol{u} = 0, \quad y\in\mathcal{F}, \\ \widetilde{T}(\boldsymbol{u},q)\boldsymbol{n} \\ \quad = \boldsymbol{n}\Big(\sigma(H(x)-\mathcal{H}(y)) + \dfrac{\omega^2}{2}(|x'|^2-|y'|^2) + \kappa(U(x,t)-\mathcal{U}(y))\Big)\Big|_{x=e_\rho(y)} \equiv M\boldsymbol{n}, \quad y\in\mathcal{G}, \\ \rho_t(y,t) = \dfrac{\boldsymbol{u}(y,t)\cdot\boldsymbol{n}(e_\rho)}{\boldsymbol{N}(y)\cdot\boldsymbol{n}(e_\rho)}, \quad y\in\mathcal{G}, \\ \rho(y,0)=\rho_0(y), \quad y\in\mathcal{G}, \quad \boldsymbol{u}(y,0)=\boldsymbol{u}_0(y), \quad y\in\mathcal{F}, \end{cases} \tag{1.18}$$

where $\boldsymbol{u}(y,t) = \boldsymbol{w}(e_\rho(y).t)$, $q(y,t) = s(e_\rho(y),t)$, $\widetilde{\nabla} = \mathcal{L}^{-T}\nabla$, $\mathcal{L}^{-T}$ is the transposed matrix $\mathcal{L}^{-1}$, $\nabla = \nabla_y$, $\widetilde{T}$ is the transformed stress tensor: $\widetilde{T}(\boldsymbol{u},q) = -qI + \nu\widetilde{S}(\boldsymbol{u})$, $\widetilde{S}(\boldsymbol{u}) = \widetilde{\nabla}\boldsymbol{u} + (\widetilde{\nabla}\boldsymbol{u})^T$ is the transformed rate-of-strain tensor (we have omitted the primes). The conditions (1.12) are converted into

$$\begin{aligned} &\int_{\mathcal{F}} \boldsymbol{u}(y,t)L\,dy = 0, \\ &\int_{\mathcal{F}} L\boldsymbol{u}(y,t)\cdot\boldsymbol{\eta}_i(e_\rho(y))dy = -\omega\int_{\mathcal{F}} L\boldsymbol{\eta}_3(e_\rho(y),t)\cdot\boldsymbol{\eta}_i(e_\rho(y))dy + \omega\int_{\mathcal{F}}\boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dy, \end{aligned} \tag{1.19}$$

The normals $\boldsymbol{n}(e_\rho(y))$ and $\boldsymbol{N}(y)$ are connected with each other as follows:

$$\boldsymbol{n}(e_\rho(y)) = \frac{\widehat{\mathcal{L}}^T\boldsymbol{N}(y)}{|\widehat{\mathcal{L}}^T\boldsymbol{N}(y)|}. \tag{1.20}$$

Let $W^{l,l/2}_{2,\gamma}(\Omega\times(0,\infty))$ be the Sobolev space with an exponential weight $e^{\gamma t}$, $\gamma<0$, equipped with the norm

$$\|u\|_{W^{l,l/2}_{2,\gamma}(\Omega\times(0,\infty))} = \|e^{\gamma t}u\|_{W^{l,l/2}_{2}(\Omega\times(0,\infty))}.$$

The spaces $W^{l,0}_{2,\gamma}(\Omega\times(0,\infty))$, $W^{0,l/2}_{2,\gamma}(\Omega\times(0,\infty))$ are defined in a similar manner.

Theorem 1.1. *Let $l\in(1,3/2)$, $Q_\infty = \mathcal{F}\times(0,\infty)$, $G_\infty = \mathcal{G}\times(0,\infty)$. Assume that $\boldsymbol{u}_0\in W^{l+1}_2(\mathcal{F})$ and $\rho_0\in W^{l+2}_2(\mathcal{G})$ satisfy the compatibility conditions*

$$\begin{aligned} &\widetilde{\nabla}\cdot\boldsymbol{u}_0(x) = 0, \quad x\in\mathcal{F}, \\ &\widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0 - \boldsymbol{n}_0(\boldsymbol{n}_0\cdot\widetilde{S}(\boldsymbol{u}_0)\boldsymbol{n}_0) = 0, \end{aligned}$$

where $\boldsymbol{n}_0(e_{\rho_0})$ is the normal to Γ_0, and the smallness condition

$$\|\boldsymbol{u}_0\|_{W^{l+1}_2(\mathcal{F})} + \|\boldsymbol{\rho}_0\|_{W^{l+2}_2(\mathcal{G})} \le \varepsilon \ll 1. \tag{1.21}$$

Assume finally that

$$\delta^2\mathcal{R}(\rho) \ge c\|\rho\|^2_{W^1_2(\mathcal{G})} \tag{1.22}$$

for arbitrary $\rho \in W_2^1(\mathcal{G})$ *satisfying* (1.6). *Then the problem* (1.18) *has a unique solution* $\boldsymbol{u}, q, \rho$ *such that* $\boldsymbol{u} \in W_{2,\gamma}^{l+2,l/2+1}(Q_\infty)$, $\nabla q \in W_{2,\gamma}^{l,l/2}(Q_\infty)$, $q|_{G_\infty} \in W_{2,\gamma}^{l+l/2,0}(G_\infty)$, $e^{-\gamma t}q \in W_2^{l/2}(0,\infty; W_2^{1/2}(\mathcal{G}))$, $\rho \in W_{2,\gamma}^{l+5/2,0}(G_\infty)$, $\rho_t \in W_{2,\gamma}^{l+3/2,l/2+3/4}(G_\infty)$, $\rho(\cdot,t) \in W_2^{l+2}(\mathcal{G})$, $\quad \forall t > 0$, $\gamma < 0$. *The solution satisfies the inequality*

$$\begin{aligned}
&\|\boldsymbol{u}\|_{W_{2,\gamma}^{l+2,l/2+1}(Q_\infty)} + \|\nabla q\|_{W_{2,\gamma}^{l,l/2}(Q_\infty)} + \|q\|_{W_{2,\gamma}^{l+1/2,0}(G_\infty)} + |e^{-\gamma t}q|_{l/2,1/2,G_\infty} \\
&+ \|\rho\|_{W_{2,\gamma}^{l+5/2,0}(G_\infty)} + \|\rho_t\|_{W_{2,\gamma}^{l+3/2,l/2+3/4}(G_\infty)} + \sup_{t>0} e^{-\gamma t}\|\rho\|_{W_2^{l+2}(\mathcal{G}\infty)} \\
&\leq c\Big(\|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big),
\end{aligned} \tag{1.23}$$

where $|\cdot|_{l/2,r,\mathcal{G}_\infty}$ *is the norm in* $W_2^{l/2}(0,\infty; W_2^r(\mathcal{G}))$.

The estimate (1.23) shows that the zero solution of the problem (1.18) is exponentially stable. In [2,4] similar estimate has been obtained for the Hölder norms of the solution of (1.10).

When we omit all the nonlinear terms with respect to $\boldsymbol{w}$, p, ρ in (1.18), we arrive at the linear problem in $\mathcal{F}$

$$\begin{cases}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\
\nabla \cdot \boldsymbol{v}(y,t) = 0, \qquad y \in \mathcal{F}, \quad t > 0, \\
T(\boldsymbol{v}, p)\boldsymbol{N} + B_0\rho = 0, \\
\rho_t = \boldsymbol{v}(y,t) \cdot \boldsymbol{N}(y), \quad y \in \mathcal{G}, \\
\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y) \quad y \in \mathcal{F}, \quad \rho(y,0) = \rho_0(y), \quad y \in \mathcal{G},
\end{cases} \tag{1.24}$$

where

$$B_0\rho(x,t) = -\sigma\Delta_{\mathcal{G}}\rho - b(x)\rho - \kappa\int_{\mathcal{G}} \frac{\rho(y,t)dS}{|x-y|} \tag{1.25}$$

is the first variation of the expression M in (1.18) with respect to ρ, and $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$. The function $b(x)$ is defined in (1.5). Linearization of (1.17), (1.12) leads to

$$\int_{\mathcal{G}} \rho(y,t)dS = 0, \quad \int_{\mathcal{G}} y_i\rho(y,t)dS = 0, \tag{1.26}$$

$$\begin{aligned}
&\int_{\mathcal{F}} \boldsymbol{v}(x,t)dx = 0, \\
&\int_{\mathcal{F}} \boldsymbol{v}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\mathcal{G}} \rho(x,t)\boldsymbol{\eta}_i(x)\cdot\boldsymbol{\eta}_3(x)dS = 0, \quad i = 1,2,3.
\end{aligned} \tag{1.27}$$

2. $\sigma = 0$**.**

We pass to the Lagrangian coordinates $\xi \in \Omega_0$ connected with the Eulerian coordinates $x \in \Omega_t$ by

$$x = \xi + \int_0^t \boldsymbol{u}(\xi,\tau)d\tau \equiv X(\xi,t), \tag{1.28}$$

where

$$\boldsymbol{u}(\xi,t) \equiv \boldsymbol{w}(X(\xi,t),t).$$

By the transformation (1.28) the relations (1.10) are converted into

$$\begin{cases} \boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla_u^2\boldsymbol{u} + \nabla_u q = 0, \\ \nabla_u \cdot \boldsymbol{u} = 0, \qquad \xi \in \Omega_0, \ \ t > 0, \\ T_u(\boldsymbol{u}, q)\boldsymbol{n} = (\kappa(U(X,t) - \mathcal{U}(\overline{X})) + \dfrac{\omega^2}{2}(|X'|^2 - |\overline{X}'|^2)\boldsymbol{n}, \quad \xi \in \Gamma_0, \\ \boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \qquad \xi \in \Omega_0, \end{cases} \tag{1.29}$$

where $q(\xi,t) = s(X(\xi,t),t))$, $\boldsymbol{n} = \boldsymbol{n}(X)$, $\bar{X}$ is the closest point of $\mathcal{G}$ to X, ∇_u is the transformed gradient with respect to x, T_u is the transformed stress tensor, i.e.,

$$\nabla_u = A\nabla, \quad T_u(\boldsymbol{u}, q) = -qI + \nu S_u(\boldsymbol{u}), \quad S_u(\boldsymbol{u}) = (\nabla_u\boldsymbol{u}) + (\nabla_u\boldsymbol{u})^T,$$

$\nabla = \left(\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_3}\right) \equiv \nabla_\xi$, $A = (A_{ij})_{i,j=1,2,3}$, A_{ij} is a co-factor of the element $a_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i(\xi,\tau)}{\partial\xi_j}d\tau$ of the Jacobi matrix of the transformation (1.28) (the determinant of this matrix equals one). The normal $\boldsymbol{n}_0(\xi)$ to Γ_0 is related to $\boldsymbol{n}(X(\xi,t))$ by

$$\boldsymbol{n}(X) = \frac{A(\xi,t)\boldsymbol{n}_0(\xi)}{|A(\xi,t)\boldsymbol{n}_0(\xi)|}. \tag{1.30}$$

The equations (1.12) take the form

$$\begin{aligned} &\int_{\Omega_0} \boldsymbol{u}(\xi,t)d\xi = 0, \\ &\int_{\Omega_0} \boldsymbol{u}(\xi,t)\cdot\boldsymbol{\eta}_i(X)d\xi = -\omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(X)\cdot\boldsymbol{\eta}_i(X)d\xi + \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_i(y)dy. \end{aligned} \tag{1.31}$$

The function ρ in (1.14) can be written as a function of $\xi \in \Gamma_0$:

$$\rho(z,t) = R(X(\xi,t)) \equiv r(\xi,t), \tag{1.32}$$

where $z = \overline{X(\xi,t)}$,

$$R(x) = \pm dist(x, \mathcal{G}),$$

the sign " $-$ " corresponds to the case $x \in \mathcal{F}$ and the sign " $+$ " to the case $x \in \mathbb{R}^3 \setminus \mathcal{F}$. The function $R(x)$ is smooth in a certain neighborhood of $\mathcal{G}$, and

$$\nabla R(x) = \boldsymbol{N}(\bar{x}).$$

Hence

$$R_t(X(\xi,t)) = \nabla_X R(X)\cdot X_t = \boldsymbol{N}(\bar{X})\cdot\boldsymbol{u}(\xi,t), \tag{1.33}$$

and $\boldsymbol{u}, q, r$ can be regarded as a solution to the problem

$$\begin{cases} \boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla_u^2\boldsymbol{u} + \nabla_u q = 0, \\ \nabla_u \cdot \boldsymbol{u} = 0, \qquad \xi \in \Omega_0, \ \ t > 0, \\ T_u(\boldsymbol{u}, q)\boldsymbol{n} = (\kappa(U(X,t) - \mathcal{U}(\overline{X})) + \dfrac{\omega^2}{2}(|X'|^2 - |\overline{X}'|^2)\boldsymbol{n}, \quad \xi \in \Gamma_0, \\ r_t = \boldsymbol{N}(\overline{X})\cdot\boldsymbol{u}, \\ \boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \qquad \xi \in \Omega_0, \quad r(\xi,0) = \rho(\bar{\xi},0) \equiv r_0(\xi). \end{cases} \tag{1.34}$$

This problem is studied in the weighted Sobolev spaces $\widetilde{W}_2^{l,l/2}(Q_T^0)$, $l>1$, with the norm

$$\|u\|^2_{\widetilde{W}_2^{l,l/2}(Q_T^0)} = \|u\|^2_{W_2^{l,l/2}(Q_T^0)} + \|tu\|^2_{W_2^{l-1,(l-1)/2}(Q_T^0)}, \quad Q_T^0 = \Omega_0 \times (0,T)$$

(the weight improves the behavior of the elements of these spaces for large t). We also find it convenient to introduce the spaces $\widetilde{W}_2^{l,0}(Q_T^0)$ and $\widetilde{W}_2^{0,l/2}(Q_T^0)$ with the norms

$$\|u\|^2_{\widetilde{W}_2^{l,0}(Q_T^0)} = \|u\|^2_{W_2^{l,0}(Q_T^0)} + \|tu\|^2_{W_2^{l-1,0}(Q_T^0)},$$

$$\|u\|^2_{\widetilde{W}_2^{0,l/2}(Q_T^0)} = \|u\|^2_{W_2^{0,l/2}(Q_T^0)} + \|tu\|^2_{W_2^{0,(l-1)/2}(Q_T^0)}.$$

The main result concerning the solvability of the problem (1.34) is as follows.

Theorem 1.2 [5]. *Let $\boldsymbol{w}_0 \in W_2^{l+1}(\Omega_0)$ with $l\in(1,3/2)$ and let the surface Γ_0 be given by (1.8) with $\rho_0 \in W_2^{l+3/2}(\mathcal{G})$. Assume also that $\boldsymbol{w}_0$ satisfies the compatibility conditions*

$$\begin{aligned} &\nabla\cdot\boldsymbol{w}_0(\xi)=0, \quad \xi\in\Omega_0,\\ &S(\boldsymbol{w}_0)\boldsymbol{n}_0 - \boldsymbol{n}_0(\boldsymbol{n}_0\cdot S(\boldsymbol{w}_0)\boldsymbol{n}_0)=0, \quad \xi\in\Gamma_0 \end{aligned} \tag{1.35}$$

and that

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \le \epsilon \tag{1.36}$$

where ϵ is a sufficiently small positive number. Moreover, let the condition

$$\delta^2\mathcal{R} \ge c\|\rho\|^2_{L_2(\mathcal{G})} \tag{1.37}$$

hold for arbitrary $\rho\in L_2(\mathcal{G})$ satisfying (1.6)*. Then the problem* (1.34) *has a unique solution $\boldsymbol{u},q,r$ such that $\boldsymbol{u}\in\widetilde{W}_2^{l+2,l/2+1}(Q_\infty^0)$, $\nabla q\in\widetilde{W}_2^{l,l/2}(Q_\infty^0)$, $r\in\widetilde{W}_2^{l+1/2,0}(G_\infty^0)$, $r_t\in\widetilde{W}_2^{l+3/2,l/2+3/4}(G_\infty^0)$, where $Q_\infty^0=\Omega_0\times\mathbb{R}_+$, $G_\infty^0=\Gamma_0\times\mathbb{R}_+$, $p|_{G_\infty^0}\in\widetilde{W}_2^{l+1/2,l/2+1/4}(G_\infty^0)$, $r\in W_2^{l+1}(\Gamma_0)$, $\forall t>0$. The surface Γ_t is representable in the form* (1.14) *where ρ is connected with $r=R(X)$ by* (1.32)*. The solution satisfies the inequality*

$$\begin{aligned} &\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_\infty^0)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_\infty^0)} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_\infty^0)}\\ &+\|r\|_{\widetilde{W}_2^{l+1/2,0}(G_\infty^0)} + \|r_t\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_\infty^0)} + \sup_{t>0}\|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)}\\ &+\sup_{t>0}\|tr(\cdot,t)\|_{W_2^{l}(\Gamma_0)} \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)}\Big). \end{aligned} \tag{1.38}$$

Estimate (1.38) shows that the solution of the problem (1.34) tends to zero as $t\to\infty$ like a power function.

The difference in the treatment of the problem (1.1) with $\sigma>0$ and $\sigma=0$ can be explained as follows: since the elements of the matrix $\mathcal{L}$ depend on the derivatives of $\rho(y,t)$, these derivatives appear in the equation for ρ_t, in view of (1.20), and the corresponding nonlinearity in this equation turns out to be too strong, if $\sigma=0$. This is not the case in the Lagrangian coordinates (see (1.33)). On the other hand, since $\boldsymbol{u}\in W_2^{l+2,l/2+1}(Q_T^0)$, the transformation (1.28) does not seem to be regular enough in the case $\sigma>0$.

The proof of the solvability of the problems (1.18) and (1.34) relies on the estimates for the solution of the non-homogeneous problem (1.24) and of the second (Neumann) evolution initial-boundary value problem for the Stokes system

$$
\begin{cases}
\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(y,t), \\
\nabla\cdot\boldsymbol{v}(y,t) = f(y,t), \qquad y\in\mathcal{F}, \quad t>0, \\
T(\boldsymbol{v},p)\boldsymbol{N} = \boldsymbol{d}(y,t), \qquad y\in\mathcal{G}, \\
\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y\in\mathcal{F}.
\end{cases}
\tag{1.39}
$$

These linear problems are studied in Sec. 2-4. In Sec. 5 and 6 basic ideas of the proof of Theorems 1.1 and 1.2 are presented; the main attention is given to the estimates (1.23) and (1.38). Sec. 7 is concerned with the case when $\delta^2\mathcal{R}(\rho)$ can take negative values for some ρ satisfying (1.6). It is proved that in this case the regime of rigid rotation is not stable in a linear approximation, i.e., that the problem (1.24)-(1.27) has solutions growing exponentially as $t\to\infty$ for appropriate initial data. Finally, in Sec. 8 auxiliary material is presented. It includes the Sobolev-Slobodetskii spaces, some auxiliary inequalities, in particular, Korn's inequality, and the calculation of variations of some functions and functionals under the normal perturbation of the domain where the functionals are defined.

2 On the second boundary- and initial-boundary value problems for the Stokes equations

1. Stationary problem

Let Ω be a bounded domain in $\mathbb{R}^3$ with the boundary $S \in C^2$. We consider the boundary value problem

$$\begin{cases} -\nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x) \quad \nabla\cdot\boldsymbol{v} = 0, \quad x \in \Omega, \\ T(\boldsymbol{v}, p)\boldsymbol{n} = \boldsymbol{d}(x), \quad x \in S, \end{cases} \tag{2.1}$$

where $\boldsymbol{n}(x)$ is the exterior normal to S and $T(\boldsymbol{v}, p)$ is the stress tensor. We observe that the homogeneous problem (2.1) (with $\boldsymbol{f} = 0$, $\boldsymbol{d} = 0$) has a non-trivial solution $\boldsymbol{v} = \boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{x}$, $p = 0$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors. We define a weak solution of the problem (2.1). We multiply the first equation in (2.1) by a test vector field $\boldsymbol{\varphi}$ and integrate over Ω. Since

$$-\nu\nabla^2\boldsymbol{v} + \nabla p = -\nabla\cdot T \stackrel{\text{def}}{=} -\nu\nabla\cdot S(\boldsymbol{v}) + \nabla p,$$

integration by parts leads to

$$\frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi})dx - \int_\Omega p\nabla\cdot\boldsymbol{\varphi}dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx + \int_S \boldsymbol{d}\cdot\boldsymbol{\varphi}dS. \tag{2.2}$$

It is natural to define a weak solution of (2.1) as a vector field $\boldsymbol{v} \in W_2^1(\Omega)$ and the function $p \in L_2(\Omega)$ satisfying (2.2) for arbitrary $\boldsymbol{\varphi} \in W_2^1(\Omega)$ (cf. [6]). The vector fields $\boldsymbol{f}$ and $\boldsymbol{d}$ should satisfy the necessary compatibility conditions

$$\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\eta}dx + \int_S \boldsymbol{d}\cdot\boldsymbol{\eta}dS = 0, \tag{2.3}$$

because the left hand side of (2.2) vanishes when $\boldsymbol{\varphi} = \boldsymbol{\eta}$.

If $\boldsymbol{\varphi}$ is divergence free, then the term with the pressure drops out, and (2.2) takes the form

$$\frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx + \int_S \boldsymbol{d}\cdot\boldsymbol{\varphi}dS. \tag{2.4}$$

We introduce the following spaces:

$\boldsymbol{V}$: the space of (real valued) vector fields $\boldsymbol{v} \in W_2^1(\Omega)$ orthogonal to all vectors of rigid motion $\boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b}\times\boldsymbol{x}$, i.e,

$$\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\eta}dx = 0; \tag{2.5}$$

$\boldsymbol{J}$: the subspace of divergence free vector fields in $\boldsymbol{V}$.

$\boldsymbol{V}_S \subset W_2^{1/2}(S)$: the space of traces of the elements $\boldsymbol{\varphi} \in \boldsymbol{V}$ on S.

In $\boldsymbol{V}$, we introduce the scalar product

$$[\boldsymbol{v}, \boldsymbol{\varphi}] = \int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi})dx.$$

By the Korn inequality (see Proposition 8.9) this bilinear form possesses all the properties of the scalar product, and the form $[\boldsymbol{v}, \boldsymbol{v}]^{1/2} = \|\boldsymbol{v}\|_{\boldsymbol{V}}$ is equivalent to $\|\boldsymbol{v}\|_{W_2^1(\Omega)}$ for all $\boldsymbol{v} \in \boldsymbol{V}$.

Let $\boldsymbol{V}'$ and $\boldsymbol{V}'_s$ be dual spaces to $\boldsymbol{V}$ and $\boldsymbol{V}_S$. For arbitrary $\boldsymbol{f} \in \boldsymbol{V}'$, $\boldsymbol{d} \in \boldsymbol{V}'_S$,$\boldsymbol{\varphi} \in \boldsymbol{V}$ we have

$$\begin{aligned} &|\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} dx| \leq \|\boldsymbol{f}\|_{\boldsymbol{V}'} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}}, \\ &|\int_S \boldsymbol{d} \cdot \boldsymbol{\varphi} dS| \leq \|\boldsymbol{d}\|_{\boldsymbol{V}'_S} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}_S} \leq C \|\boldsymbol{d}\|_{\boldsymbol{V}'_S} \|\boldsymbol{\varphi}\|_{\boldsymbol{V}}. \end{aligned} \tag{2.6}$$

We also set $\boldsymbol{W} = W_2^1(\Omega)$, $\boldsymbol{W}_S = W_2^{1/2}(S)$, and we introduce the dual spaces $\boldsymbol{W}'$ and $\boldsymbol{W}'_S$. It is clear that $L_2(\Omega) \subset \boldsymbol{W}' \subset \boldsymbol{V}'$, $L_2(S) \subset \boldsymbol{W}'_S \subset \boldsymbol{V}'_S$, and inequalities (2.6) hold with $\|\boldsymbol{f}\|_{L_2(\Omega)}$ and $\|\boldsymbol{f}\|_{\boldsymbol{W}'}$ instead of $\|\boldsymbol{f}\|_{\boldsymbol{V}'}$ and $\|\boldsymbol{d}\|_{L_2(S)}$, $\|\boldsymbol{d}\|_{\boldsymbol{W}'_S}$ instead of $\|\boldsymbol{d}\|_{\boldsymbol{V}'_S}$.

Our first objective is to prove

Proposition 2.1 *For arbitrary $\boldsymbol{f} \in \boldsymbol{W}'$, $\boldsymbol{d} \in \boldsymbol{W}'_S$ satisfying the compatibility condition* (2.3)*, the problem* (2.1) *has a unique weak solution $\boldsymbol{v} \in \boldsymbol{J}$, $p \in L_2(\Omega)$, and*

$$\|\boldsymbol{v}\|_{W_2^1(\Omega)} + \|p\|_{L_2(\Omega)} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}). \tag{2.7}$$

Proof. The proof is carried out in several steps.

Step 1. Determination of $\boldsymbol{v}$.

We find $\boldsymbol{v}$ as a vector field $\boldsymbol{v} \in \boldsymbol{J}$ satisfying (2.4) for arbitrary $\boldsymbol{\varphi} \in \boldsymbol{J}$. By the Riesz representation theorem, a linear functional

$$\frac{2}{\nu}\Big(\int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} dx + \int_S \boldsymbol{d} \cdot \boldsymbol{\varphi} dS\Big)$$

can be represented in a unique way in the form of the scalar product $[\boldsymbol{v}, \boldsymbol{\varphi}]$, which gives (2.4). Taking $\boldsymbol{\varphi} = \boldsymbol{v}$ in (2.4) we easily obtain

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} \leq \frac{2}{\nu}(\|\boldsymbol{f}\|_{\boldsymbol{V}'} + \|\boldsymbol{d}\|_{\boldsymbol{V}'_S}). \tag{2.8}$$

Step 2. Construction of p.

We introduce the space

$$\boldsymbol{K} = \boldsymbol{V} \ominus \boldsymbol{J} \quad (\boldsymbol{V} = \boldsymbol{J} \oplus \boldsymbol{K})$$

of the elements $\boldsymbol{w} \in \boldsymbol{V}$ satisfying $[\boldsymbol{w}, \boldsymbol{\varphi}] = 0$, $\forall \boldsymbol{\varphi} \in \boldsymbol{J}$, and we prove the following lemma:

Proposition 2.2 *For arbitrary $f \in L_2(\Omega)$ there exists a unique element $\boldsymbol{w} \in \boldsymbol{K}$ such that*

$$\nabla \cdot \boldsymbol{w} = f. \tag{2.9}$$

It satisfies the inequality

$$\|\boldsymbol{w}\|_{W_2^1(\Omega)} \leq C\|f\|_{L_2(\Omega)}. \tag{2.10}$$

The correspondence between f and $\boldsymbol{w}$ is linear.

Proof. The vector field

$$\boldsymbol{w}_1(x) = -\frac{1}{4\pi}\nabla \int_\Omega \frac{f(y)}{|x-y|} dy$$

belongs to $W_2^1(\Omega)$ and satisfies (2.9) and the inequality

$$\|\boldsymbol{w}_1\|_{W_2^1(\Omega)} \leq C\|f\|_{L_2(\Omega)}$$

(due to the theorem of Calderon-Zygmund). But it needs not satisfy (2.5) and belong to $\boldsymbol{V}$. Assuming for simplicity that $\int_\Omega x_k dx = 0 \quad k = 1,2,3$ (which can be achieved by a simple coordinate transformation), we define $\boldsymbol{w}_2 \in \boldsymbol{V}$ by

$$\boldsymbol{w}_2(x) = \boldsymbol{w}_1(x) + \boldsymbol{a} + \boldsymbol{b} \times x = \boldsymbol{w}_1(x) + \sum_{i=1}^{3}(a_i\boldsymbol{e}_i + b_i\boldsymbol{\eta}_i).$$

The condition $\int_\Omega \boldsymbol{w}_2 \cdot \boldsymbol{\eta} dx = 0$ yields the following simple equations for a_i and b_i:

$$\begin{aligned} a_i &= -\frac{1}{|\Omega|}\int_\Omega w_{1i}(x)dx, \\ \sum_{j=1}^{3} S_{ij}b_j &= -\int_\Omega \boldsymbol{w}_1 \cdot \boldsymbol{\eta}_i dx \quad i = 1,2,3, \end{aligned} \tag{2.11}$$

where $S_{ij} = \int_\Omega \boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_j dx$. Since the matrix $(S_{ij})_{i,j=1,2,3}$ is positive definite, the algebraic system (2.11) for b_j is uniquely solvable, and we have

$$\|\boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}\|_{W_2^1(\Omega)} \leq C\|\boldsymbol{w}_1\|_{L_2(\Omega)} \leq C\|f\|_{L_2(\Omega)},$$

which implies

$$\|\boldsymbol{w}_2\|_{W_2^1(\Omega)} \leq C\|f\|_{L_2(\Omega)}.$$

Finally, we define $\boldsymbol{w}$ as a projection of $\boldsymbol{w}_2 \in \boldsymbol{V}$ on $\boldsymbol{K}$. It is clear that $\boldsymbol{w}$ satisfies (2.9) and (2.10). The uniqueness of $\boldsymbol{w}$ follows from the fact that $\nabla \cdot \boldsymbol{w} = 0$, $\boldsymbol{w} \in \boldsymbol{K}$ implies $\boldsymbol{w} \in \boldsymbol{J} \cap \boldsymbol{K}$, i.e, $\boldsymbol{w} = 0$. The proposition is proved. ■

The inequality (2.10) shows that

$$\|\boldsymbol{w}\|_{W_2^1(\Omega)} \leq C\|\nabla \cdot \boldsymbol{w}\|_{L_2(\Omega)}, \quad \forall \boldsymbol{w} \in \boldsymbol{K}.$$

The estimate $\|\mathrm{div}\boldsymbol{w}\|_{L_2(\Omega)} \leq C\|\boldsymbol{w}\|_{\boldsymbol{V}}$ is obvious, and we see that the norms $\|\boldsymbol{w}\|_{W_2^1(\Omega}$ and $\|\nabla \cdot \boldsymbol{w}\|_{L_2(\Omega)}$ are equivalent in $\boldsymbol{K}$. Hence we can introduce a new scalar product

$$\langle \boldsymbol{w}, \boldsymbol{\varphi} \rangle = \int_\Omega (\nabla \cdot \boldsymbol{w})(\nabla \cdot \boldsymbol{\varphi})dx \tag{2.12}$$

in this space.

To define the pressure $p(x)$, we consider the expression

$$L(\boldsymbol{\varphi}) = \frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi})dx - \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} dx - \int_S \boldsymbol{d} \cdot \boldsymbol{\varphi} dS$$

with $\boldsymbol{v}(x)$ that has been just found. $L(\boldsymbol{\varphi})$ is a linear functional in $\boldsymbol{\varphi} \in \boldsymbol{W}$, and

$$|L(\boldsymbol{\varphi})| \leq \left(c\|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}\right)\|\boldsymbol{\varphi}\|_{W_2^1(\Omega)}.$$

This yields the estimate for the norm of the functional L:

$$\|L\| \leq c\|\boldsymbol{v}\|_{\boldsymbol{V}} + \|\boldsymbol{f}\|_{\boldsymbol{V}'} + \|\boldsymbol{d}\|_{\boldsymbol{V}'_S} \leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}).$$

For $\boldsymbol{\varphi} \in \boldsymbol{J}$, $L(\boldsymbol{\varphi}) = 0$. By the Riesz representation theorem, for arbitrary $\boldsymbol{\varphi} \in \boldsymbol{K}$, $L(\boldsymbol{\varphi})$ can be represented as a scalar product (2.12), i.e,

$$L(\boldsymbol{\varphi}) = \int_\Omega (\nabla \cdot \boldsymbol{w})(\nabla \cdot \boldsymbol{\varphi}) dx, \quad \forall \boldsymbol{\varphi} \in \boldsymbol{K}, \tag{2.13}$$

where $\boldsymbol{w} \in \boldsymbol{K}$ is determined in a unique way and

$$\|\nabla \cdot \boldsymbol{w}\|_{L_2(\Omega)} \leq \|L\| \leq C(\|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}). \tag{2.14}$$

It is clear that equation (2.13) holds also for $\boldsymbol{\varphi} \in \boldsymbol{J}$, i.e, it is verified for arbitrary $\boldsymbol{\varphi} \in \boldsymbol{V}$; moreover, in view of (2.3), (2.2) holds for arbitrary $\boldsymbol{\varphi} \in W_2^1(\Omega)$.

It coincides with (2.2), if we set $p = \nabla \cdot \boldsymbol{w}$. The estimate (2.7) is a consequence of (2.8) and (2.14). The proposition 2.1 is proved. ■

The problem (2.1) is elliptic, and the solution satisfies well known coercive estimates.

Proposition 2.3 *If* $\boldsymbol{f} \in W_2^l(\Omega)$, $\boldsymbol{d} \in W_2^{l+1/2}(S)$, *then* $\boldsymbol{v} \in W_2^{2+l}(\Omega)$, $p \in W_2^{1+l}(\Omega)$ *and*

$$\|\boldsymbol{v}\|_{W_2^{2+l}(\Omega)} + \|p\|_{W_2^{1+l}(\Omega)} \leq C(\|\boldsymbol{f}\|_{W_2^l(\Omega)} + \|\boldsymbol{d}\|_{W_2^{l+1/2}(S)}). \tag{2.15}$$

Estimate (2.15) requires a certain regularity of the boundary S; we assume that it is sufficiently smooth.

The proof is carried out by standard methods of the theory of elliptic systems. At first, assuming that $\boldsymbol{f} \in L_2(\Omega)$, $\boldsymbol{d} \in W_2^{1/2}(S)$, we can obtain the estimate (2.15) for $l = 0$. This is done via local estimates of the same kind as in the case of the Dirichlet problem for the Laplacean and for the Stokes operator (see for instance [7]). Further improvement of smoothness of the solution is obtained by using the theorem of regularity for general elliptic boundary value problems.

2. Problem with a parameter

Now, we consider the problem with a complex parameter s

$$\begin{cases} s\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \quad \nabla \cdot \boldsymbol{v} = 0, \quad x \in \Omega, \\ T(\boldsymbol{v}, p)\boldsymbol{n} = \boldsymbol{d}(x), \quad x \in S. \end{cases} \tag{2.16}$$

The solution is sought also in the space of complex-valued functions.

A weak solution of (2.16) is defined as a divergence free vector field $\boldsymbol{v} \in W_2^1(\Omega)$ and a function $p \in L_2(\Omega)$ satisfying the integral identity

$$s\int_\Omega \boldsymbol{v} \cdot \boldsymbol{\varphi} dx + \frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi}) dx - \int_\Omega p\nabla \cdot \boldsymbol{\varphi} dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} dx + \int_S \boldsymbol{d} \cdot \boldsymbol{\varphi} dS, \tag{2.17}$$

where $\boldsymbol{\varphi} \in W_2^1(\Omega)$, $\boldsymbol{v} \cdot \boldsymbol{\varphi} = \sum_{i=1}^3 v_i\varphi_i$, etc.

Proposition 2.4 *If* $\operatorname{Re} s \geq a > 0$, $\boldsymbol{f} \in \boldsymbol{W}'$, $\boldsymbol{d} \in \boldsymbol{W}'_S$, *then the problem* (2.16) *has a unique weak solution, and*

$$|s|^{1/2}\|\boldsymbol{v}\|_{L_2(\Omega)} + \|\boldsymbol{v}\|_{W_2^1(\Omega)} \leq c(\|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}), \tag{2.18}$$

$$\|p\|_{L_2(\Omega)} \leq c(|s|\|\boldsymbol{v}\|_{\boldsymbol{W}'} + \|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}) \tag{2.19}$$

with the constant independent of s *(but it may depend on* a*).*

Proof. We denote by $\boldsymbol{H}$ the space of divergence free vector fields from $W_2^1(\Omega)$ and, as above, we find $\boldsymbol{v}$ as the element of $\boldsymbol{H}$ satisfying the equation

$$s\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\varphi}dx + \frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi})dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx + \int_S \boldsymbol{d}\cdot\boldsymbol{\varphi}dS \tag{2.20}$$

for arbitrary $\boldsymbol{\varphi} \in \boldsymbol{H}$. Since s is a complex number, the form

$$Q_s(\boldsymbol{v},\boldsymbol{\varphi}) = s\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\varphi}dx + \frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi})dx$$

can not be taken as a scalar product in $W_2^1(\Omega)$, and we use the Lax-Milgram theorem. It is possible, because

$$\begin{aligned}
|Q_s(\boldsymbol{v},\bar{\boldsymbol{v}})|^2 &= \big(\operatorname{Re} s\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\Omega)}\big)^2 + (\operatorname{Im} s)^2\|\boldsymbol{v}\|^4_{L_2(\Omega)} \\
&\geq C\big(|s|^2\|\boldsymbol{v}\|^4_{L_2(\Omega)} + \|S(\boldsymbol{v})\|^4_{L_2(\Omega)}\big) \\
&\geq C\big(|s|\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\Omega)}\big)^2 \\
&\geq C\big(|a|\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\Omega)}\big)^2 \geq C\|\boldsymbol{v}\|^4_{W_2^1(\Omega)}.
\end{aligned} \tag{2.21}$$

By the Lax-Milgram theorem, there exists a unique $\boldsymbol{v} \in \boldsymbol{H}$ satisfying (2.20) for arbitrary $\boldsymbol{\varphi} \in \boldsymbol{H}$. Setting $\boldsymbol{\varphi} = \bar{\boldsymbol{v}}$, we obtain, in view of (2.21),

$$|s|\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\Omega)} \leq c\|\boldsymbol{v}\|_{W_2^1(\Omega)}(\|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}),$$

which implies (2.18).

The construction of p is carried out essentially in the same way as in Proposition 2.1: $p = \nabla\cdot\boldsymbol{w}$, where $\boldsymbol{w} \in \boldsymbol{H}^\perp = W_2^1(\Omega) \ominus \boldsymbol{H}$ is determined by

$$\begin{aligned}
L_s(\varphi) = s\int_\Omega \boldsymbol{v}\cdot\boldsymbol{\varphi}dx + \frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{\varphi})dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx - \int_S \boldsymbol{d}\cdot\boldsymbol{\varphi}dS \\
= \int_\Omega \nabla\cdot\boldsymbol{w}\nabla\cdot\boldsymbol{\varphi}dx.
\end{aligned}$$

It is clear that

$$\|p\|_{L_2(\Omega)} = \|\nabla\cdot\boldsymbol{w}\|_{W_2^1(\Omega)} \leq c(|s|\|\boldsymbol{v}\|_{\boldsymbol{W}'} + \|\boldsymbol{f}\|_{\boldsymbol{W}'} + \|\boldsymbol{d}\|_{\boldsymbol{W}'_S}).$$

The proposition is proved. ■

Proposition 2.5 *If $Res \geq a > 0$, $\boldsymbol{f} \in W_2^l(\Omega)$, $\boldsymbol{d} \in W_2^{l+1/2}(S)$, then the problem* (2.16) *has a unique solution $\boldsymbol{v} \in W_2^{2+l}(\Omega)$, $p \in W_2^{1+l}(\Omega)$ and*

$$\begin{aligned}
&|s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\Omega)} + \|\boldsymbol{v}\|_{W_2^{2+l}(\Omega)} + \|p\|_{W_2^{1+l}(\Omega)} + |s|^{l/2}\|p\|_{W_2^1(\Omega)} \\
&\quad \leq C\big(\|\boldsymbol{f}\|_{W_2^l(\Omega)} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\Omega)} + \|\boldsymbol{d}\|_{W_2^{l+1/2}(S)} \\
&+ |s|^{1/4+l/2}\|\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n})\|_{L_2(\Omega)} + |s|^{l/2}\|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{1/2}(S)}\big), \\
&|s|^{l/2+1}\|p\|_{L_2(S)} \leq C\Big(\|\boldsymbol{f}\|_{W_2^l(\Omega)} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\Omega)} + \|\boldsymbol{d}\|_{W_2^{l+1/2}(S)} + |s|^{1/4+l/2}\|\boldsymbol{d}\|_{L_2(\Omega)}\Big).
\end{aligned} \tag{2.22}$$

with the constant independent of $|s|$.

Proof. We go back to (2.20) and write the estimate of $\boldsymbol{v}$, assuming that $\boldsymbol{f} \in L_2(\Omega)$ and $\boldsymbol{d}\cdot\boldsymbol{n} \in W_2^{1/2}(S)$, $\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n}) \equiv \boldsymbol{d}_{tan} \in L_2(S)$. We obtain

$$\begin{aligned}
&|\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}dx| + |\int_S \boldsymbol{d}\cdot\boldsymbol{v}dS| \leq \|\boldsymbol{f}\|_{L_2(\Omega)}\|\boldsymbol{v}\|_{L_2(\Omega)} + \|\boldsymbol{d}_{tan}\|_{L_2(S)}\|\boldsymbol{v}_{tan}\|_{L_2(S)} \\
&+ \|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{1/2}(S)}\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W'}_S)} \\
&\leq c\Big(\|\boldsymbol{f}\|_{L_2(\Omega)} + c\|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{1/2}(S)}\Big)\|\boldsymbol{v}\|_{L_2(\Omega)} + \|\boldsymbol{d}_{tan}\|_{L_2(S)}\|\boldsymbol{v}_{tan}\|_{L_2(S)},
\end{aligned} \tag{2.23}$$

because

$$\|\boldsymbol{v}\cdot\boldsymbol{n}\|_{\boldsymbol{W'}_S} \leq c\|\boldsymbol{v}\|_{L_2(\Omega)}$$

for divergence free vector fields. From (2.20) and (2.23) it follows that

$$|s|\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \|\boldsymbol{v}\|^2_{W_2^1(\Omega)} \leq C\Big((\|\boldsymbol{f}\|_{L_2(\Omega)} + \|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{1/2}(S)})\|\boldsymbol{v}\|_{L_2(\Omega)} + \|\boldsymbol{d}_{tan}\|_{L_2(S)}\|\boldsymbol{v}_{tan}\|_{L_2(S)}\Big) \tag{2.24}$$

We multiply (2.24) by $|s|$ and apply the inequality $ab \leq \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon}$, $a, b, \epsilon > 0$. This gives

$$\begin{aligned}
&|s|^2\|\boldsymbol{v}\|^2_{L_2(\Omega)} + |s|\|\boldsymbol{v}\|^2_{W_2^1(\Omega)} \\
&\leq C(\epsilon)(\|\boldsymbol{f}\|^2_{L_2(\Omega)} + |s|^{1/2}\|\boldsymbol{d}_{tan}\|^2_{L_2(S)} + \|\boldsymbol{d}\cdot\boldsymbol{n}\|^2_{W_2^{1/2}(S)} + \epsilon|s|^2\|\boldsymbol{v}\|^2_{L_2(\Omega)} + \epsilon|s|^{3/2}\|\boldsymbol{v}\|^2_{L_2(S)}.
\end{aligned}$$

The last term we estimate by the interpolation inequality

$$|s|^{3/2}\|\boldsymbol{v}\|^2_{L_2(S)} \leq C(|s|^2\|\boldsymbol{v}\|^2_{L_2(\Omega)} + |s|\|\boldsymbol{v}\|^2_{W_2^1(\Omega)}).$$

Taking ϵ sufficiently small, we arrive at

$$|s|^2\|\boldsymbol{v}\|^2_{L_2(\Omega)} + |s|\|\boldsymbol{v}\|^2_{W_2^1(\Omega)} \leq C(\|\boldsymbol{f}\|^2_{L_2(\Omega)} + |s|^{1/2}\|\boldsymbol{d}_{tan}\|^2_{L_2(S)} + \|\boldsymbol{d}\cdot\boldsymbol{n}\|^2_{W_2^{1/2}(S)}) \tag{2.25}$$

Now we consider $\boldsymbol{v}$ as a weak solution of the problem (2.1) with $\boldsymbol{f}$ replaced by $\boldsymbol{f} - s\boldsymbol{v}$. By (2.19) and (2.25),

$$\begin{aligned}
\|p\|_{L_2(\Omega)} &\leq C(\|\boldsymbol{f}\|_{L_2(\Omega)} + |s|\|\boldsymbol{v}\|_{L_2(\Omega)} + \|\boldsymbol{d}\|_{L_2(S)}) \\
&\leq C(\|\boldsymbol{f}\|_{L_2(\Omega)} + |s|^{1/4}\|\boldsymbol{d}\|_{L_2(S)} + \|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{1/2}(S)})
\end{aligned}$$

In addition, from the regularity theorem for solutions of elliptic boundary value problems it follows that $\boldsymbol{v} \in W_2^{l+2}(\Omega)$, $p \in W_2^l(\Omega)$ and

$$\|\boldsymbol{v}\|_{W_2^{l+2}(\Omega)} + \|p\|_{W_2^{l+1}(\Omega)} \le C(\|\boldsymbol{f} - s\boldsymbol{v}\|_{W_2^l(\Omega)} + \|\boldsymbol{d}\|_{W_2^{l+1/2}(S)}).$$

Applying again the interpolation inequality

$$|s|\|\boldsymbol{v}\|_{W_2^l(\Omega)} \le \epsilon_1\|\boldsymbol{v}\|_{W_2^{2+l}(\Omega)} + C(\epsilon_1)|s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\Omega)} \tag{2.26}$$

with a small ϵ_1 and taking (2.25) into account, we obtain the first estimate (2.22). Now we easily deduce the second estimate from the boundary condition

$$p(x,t) = \nu\boldsymbol{n}\cdot S(\boldsymbol{v})\boldsymbol{n} - \boldsymbol{d}\cdot\boldsymbol{n}(x), \quad x \in S.$$

The theorem is proved.

3. Evolution second initial-boundary problem

We pass to the analysis of the evolution problem

$$\begin{cases} \boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \quad \nabla\cdot\boldsymbol{v} = f(x,t), \quad x \in \Omega, \quad t \le T, \quad T < +\infty \\ T(\boldsymbol{v},p)\boldsymbol{n} = \boldsymbol{d}(x,t), \quad x \in S \\ \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0(x), \quad x \in \Omega. \end{cases} \tag{2.27}$$

We shall look for the solution of (2.27) such that $\boldsymbol{v} \in W_2^{2+l,1+l/2}(D_T)$, $l \in [0,5/2)$, $\nabla p \in W_2^{l,l/2}(D_T)$, where $l \in [0,5/2)$, $D_T = \Omega\times(0,T)$. The restriction $l < 5/2$ minimizes the order of compatibility of initial and boundary conditions. As above, we assume for simplicity that the surface S is sufficiently regular.

We start with the case of zero initial data and zero divergence.

Proposition 2.6 *Assume that $\boldsymbol{v}_0(x) = 0$, $f = 0$, $\boldsymbol{f} \in W_2^{l,l/2}(D_T)$, $\boldsymbol{d}-\boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n}) \in W_2^{l+1/2,l/2+1/4}(\Sigma_T)$, $\boldsymbol{d}\cdot\boldsymbol{n} \in W_2^{l+1/2,0}(\Sigma_T) \cap W_2^{l/2}(0,T;W_2^{1/2}(S))$, where $l \in [0,5/2)$, $D_T = \Omega\times(0,T)$, $\Sigma_T = S\times(0,T)$, and that the zero extension of $\boldsymbol{f}$ and $\boldsymbol{d}$ in $\Omega\times(-\infty,0)$ and $S\times(-\infty,0)$ is the extension with preservation of class, i.e., $\boldsymbol{f}^0 \in W_2^{l,l/2}(D_T^\infty)$, $\boldsymbol{d}^0 \in W_2^{l+1/2,l/2+1/4}(\Sigma_T^\infty)$, where $\boldsymbol{f}^0 = \boldsymbol{f}$, $\boldsymbol{d}^0 = \boldsymbol{d}$ for $t > 0$ $\boldsymbol{f}^0 = 0$, $\boldsymbol{d}^0 = 0$ for $t < 0$; $D_T^\infty = \Omega\times(-\infty,T)$, $\Sigma_T^\infty = S\times(-\infty,T)$. Then the problem (2.27) has a unique solution $\boldsymbol{v} \in W_2^{2+l,1+l/2}(D_T)$, $\nabla p \in W_2^{l,l/2}(D_T)$, $p \in W_2^{l+1/2,0}(\Sigma_T) \cap W_2^{l/2}(0,T;W_2^{1/2}(S))$ and*

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(D_T)} + \|\nabla p\|_{W_2^{l,l/2}(D_T)} + \|p\|_{W_2^{l+1/2,0}(\Sigma_T)} + |p|_{l/2,1/2,\Sigma_T} \\ &\le C(T)(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)} + \|\boldsymbol{d}-\boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n})\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)} + \|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{l+1/2,0}(\Sigma_T)} + |\boldsymbol{d}\cdot\boldsymbol{n}|_{l/2,1/2,\Sigma_T}), \end{aligned} \tag{2.28}$$

where $|\cdot|_{l/2,r,\Sigma_T}$ is the norm in $W_2^{l/2}(0,T;W_2^r(S))$.

If, in addition, $\boldsymbol{d}\cdot\boldsymbol{n} \in W_2^{0,l/2+1/4}(\Sigma_T)$, then

$$\begin{aligned} &\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(D_T)} + \|\nabla p\|_{W_2^{l,l/2}(D_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)} \\ &\le c(T)\Big(\|\boldsymbol{f}\|_{W_2^l(D_T)} + \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/2}(\Sigma_T)}\Big). \end{aligned} \tag{2.29}$$

Moreover,

$$\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(D_T)} + \|\nabla p\|_{W_2^{l,l/2}(D_T)} + \|p\|_{W_2^{l+1/2,0}(\Sigma_T)} + |p|_{l/2,1/2,\Sigma_T}$$
$$\le c_1(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)} + \|\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n})\|_{W_2^{l+1/2,l/2+1/2}(\Sigma_T)} + \|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{l+1/2,0}(\Sigma_T)} \tag{2.30}$$
$$+ |\boldsymbol{d}\cdot\boldsymbol{n}|_{l/2,1/2,\Sigma_T}) + c_2\|\boldsymbol{v}\|_{L_2(D_T)},$$

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(D_T)} + \|\nabla p\|_{W_2^{l,l/2}(D_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}\\ &\le c_3(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)} + \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}) + c_4\|\boldsymbol{v}\|_{L_2(D_T)}\end{aligned} \tag{2.31}$$

with the constants independent of T.

Remark 2.1 $\boldsymbol{f}^0$ *is the extension of* $\boldsymbol{f}$ *with preservation of class, if* $l/2 < 1/2$ (*i.e,* $l < 1$) *or if* $l > 1$, $\boldsymbol{f}(x,0) = 0$ *(according to the trace theorem,* $\boldsymbol{f}(x,0) \in W_2^{l-1}(\Omega)$*). If* $l = 1$, *then* $\boldsymbol{f}(x,t)$ *should have a finite norm* $\big(\int_0^T \|\boldsymbol{f}(\cdot,t)\|^2_{L_2(\Omega)}\frac{dt}{t}\big)^{1/2}$ *that should be added to the right hand side of* (2.28) - (2.31).

$\boldsymbol{d}^0$ *is the extension of* $\boldsymbol{d}$ *with preservation of class, if* $l/2+1/4 < 1/2$, (*i.e,* $l < 1/2$) *or* $l > 1/2$, $\boldsymbol{d}^0(x,0) = 0$. *For* $l = 1/2$, *there should appear an additional term* $\big(\int_0^T \|\boldsymbol{d}(\cdot,t)\|^2\frac{dt}{t}\big)^{1/2}$ *in* (2.28) - (2.31).

Proof. For simplicity, we assume that $l \ne 1$, $l \ne 1/2$. We extend $\boldsymbol{f}^0$ and $\boldsymbol{d}^0$ with preservation of class in the domains $\Omega\times(T,+\infty)$, $S\times(T,+\infty)$, respectively, and consider the problem

$$\begin{cases}\boldsymbol{u}_t - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{f}^*(x,t), \quad \nabla\cdot\boldsymbol{u} = 0, \quad x\in\Omega, \quad -\infty < t < \infty,\\ T(\boldsymbol{u},q)\boldsymbol{n} = \boldsymbol{d}^*(x,t), \quad x\in S,\end{cases} \tag{2.32}$$

where $\boldsymbol{f}^*$ and $\boldsymbol{d}^*$ are the extended $\boldsymbol{f}$ and $\boldsymbol{d}$. They satisfy the inequalities

$$\|\boldsymbol{f}^*\|_{W_2^l(D_\infty^\infty)} \le c\|\boldsymbol{f}^0\|_{W_2^l(D_T^\infty)} \le c\|\boldsymbol{f}\|_{W_2^l(D_T)},$$

$$\|\boldsymbol{d}^*\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_\infty^\infty)} \le c\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}.$$

Following Agranovich and Vishik [8], we reduce (2.32) to the corresponding parameter-dependent problem, using the Laplace transformation. We recall that it is defined by the formula

$$\tilde f(s) = \int_0^{+\infty} e^{-st}f(t)dt \overset{\text{def}}{=} Lf, \quad s = a+i\xi, \quad a > 0,$$

and the inverse transformation has the form

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+\infty} e^{st}\tilde f(s)ds \overset{\text{def}}{=} L^{-1}\tilde f.$$

It is easily seen that $\tilde f(a+i\xi)$ is the Fourier transform of the function $e^{-at}f(t)$ extended by zero in the half-axis $t < 0$, hence the following Parseval formula holds (see Theorem 8.1):

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}|\tilde f(a+i\xi)|^2d\xi = \int_0^\infty |f(t)|^2e^{-2at}dt.$$

Since $Lf_t = sLf$, the application of the Laplace transform converts (2.32) into (2.16). Estimate (2.22) and the Parseval equality yield

$$\|\boldsymbol{u}\|_{W_{2,a}^{l+2,l/2+1}(D_\infty^\infty)} + \|\nabla q\|_{W_{2,a}^{l,l/2}(D_\infty^\infty)} \le c(a)(\|\boldsymbol{f}^*\|_{W_{2,a}^{l,l/2}(D_\infty^\infty)}$$
$$+ \|(\boldsymbol{d}^* - \boldsymbol{n}(\boldsymbol{d}^* \cdot \boldsymbol{n}))\|_{W_{2,a}^{l+1/2,l/2+1/4}(\Sigma_\infty^\infty)} + \|\boldsymbol{d}^* \cdot \boldsymbol{n}\|_{W_{2,a}^{l+1/2,0}(\Sigma_\infty^\infty)} + |e^{-at}\boldsymbol{d}^* \cdot \boldsymbol{n}|_{l/2,1/2,\Sigma_\infty^\infty}), \tag{2.33}$$

where $Q_\infty^\infty = \Omega \times (-\infty, \infty)$, $\Sigma_\infty^\infty = S \times (-\infty, \infty)$. Since $a > 0$ and $\boldsymbol{f}^*(x,t) = 0$, $\boldsymbol{d}^*(x,t) = 0$ for $t < 0$, we have

$$\begin{aligned}
&\|\boldsymbol{f}^*\|_{W_{2,a}^{l,l/2}(D_\infty^\infty)} + \|(\boldsymbol{d}^* - \boldsymbol{n}(\boldsymbol{d}^* \cdot \boldsymbol{n}))\|_{W_{2,a}^{l+1/2,l/2+1/4}(\Sigma_\infty^\infty)} \\
&+ \|\boldsymbol{d}^* \cdot \boldsymbol{n}\|_{W_{2,a}^{l+1/2,0}(\Sigma_\infty^\infty)} + |e^{-at}\boldsymbol{d}^* \cdot \boldsymbol{n}|_{l/1,1/2,\Sigma_\infty^\infty} \\
&\le C(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)} + \|\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d} \cdot \boldsymbol{n})\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)} \\
&+ \|\boldsymbol{d} \cdot \boldsymbol{n}\|_{W_2^{l+1/2,0}(\Sigma_T)} + |\boldsymbol{d} \cdot \boldsymbol{n}|_{l/2,1/2,\Sigma_T})
\end{aligned} \tag{2.34}$$

We define the solution of our problem (2.27) with $\boldsymbol{v}_0 = 0$ as the restriction of $\boldsymbol{u}(x,t)$, $q(x,t)$ to the time interval $(0,T)$. It is clear that $\boldsymbol{u} = 0$ and $q = 0$ for $t \le 0$. Inequalities (2.33), (2.34) imply

$$\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(D_T)} + \|\nabla p\|_{W_2^{l,l/2}(D_T)} \le C(T)(\|\boldsymbol{f}\|_{W_2^l(D_T)}$$
$$+ \|\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d} \cdot \boldsymbol{n})\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)} + \|\boldsymbol{d} \cdot \boldsymbol{n}\|_{W_2^{l+1/2,0}(\Sigma_T)} + |\boldsymbol{d} \cdot \boldsymbol{n}|_{l/2,1/2,\Sigma_T}\Big).$$

Using the boundary condition $p = \nu \boldsymbol{n} \cdot S(\boldsymbol{v})\boldsymbol{n} - \boldsymbol{d} \cdot \boldsymbol{n}$, we estimate $\|p\|_{W_2^{l+1/2,0}(\Sigma_T)} + |p|_{l/2,1/2,\Sigma_T}$ or $\|p\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}$ and obtain (2.28), (2.29).

The solution defined in this way is unique. For parabolic problems, this has been proved in [8]. Indeed, let $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $p = p_1 - p_2$ be the difference of two solutions $(\boldsymbol{v}_1, p_1)$ and $(\boldsymbol{v}_2, p_2)$. We can extend $(\boldsymbol{v}, p)$ in D_∞^∞ so that $\boldsymbol{v} \in W_{2,a}^{l+2,l/2+1}(D_\infty^\infty)$, $\nabla p \in W_{2,a}^{l,l/2}(D_\infty^\infty)$, $p_{x \in S} \in W_{2,a}^{l+1/2,l/2+1/4}(\Sigma_\infty^\infty)$, $\boldsymbol{v}(x,t) = 0$, $p(x,t) = 0$ for $t < 0$ and $\nabla \cdot \boldsymbol{v} = 0$. Let

$$\boldsymbol{f}(x,t) = \boldsymbol{v}_t - \nu \nabla^2 \boldsymbol{v} + \nabla p, \quad \boldsymbol{d} = T(\boldsymbol{v}, p)\boldsymbol{n}|_S, \quad t \in (-\infty, +\infty).$$

It is clear that $\boldsymbol{f} \in W_{2,a}^{l,l/2}(D_\infty^\infty)$, $\boldsymbol{d} \in W_{2,a}^{l+1/2,l/2+1/4}(\Sigma_\infty^\infty)$. Since $\boldsymbol{f}(x,t)$ and $\boldsymbol{d}(x,t)$ vanish for $t < T$, the functions $\boldsymbol{f}'(x,t) = \boldsymbol{f}(x, t+T)$, $\boldsymbol{d}'(x,t) = \boldsymbol{d}(x, t+T)$ vanish for $t < 0$. In addition,

$$\tilde{\boldsymbol{f}}'(x,s) = e^{sT}\tilde{\boldsymbol{f}}(x,s), \quad \tilde{\boldsymbol{d}}'(x,s) = e^{sT}\tilde{\boldsymbol{d}}(x,s).$$

Let $\boldsymbol{v}' \in W_{2,a}^{l+2,l/2+1}(D_\infty^\infty)$, $\nabla p' \in W_{2,a}^{l,l/2}(D_\infty^\infty)$ be a solution of the problem (2.29) in the infinite time interval, corresponding to the functions $(\boldsymbol{f}', \boldsymbol{d}')$. It vanishes for $t < 0$, moreover,

$$\tilde{\boldsymbol{v}}'(x,s) = e^{sT}\tilde{\boldsymbol{v}}(x,s), \quad \nabla \tilde{p}'(x,s) = e^{sT}\nabla \tilde{p}(x,s).$$

But these relations mean that

$$\boldsymbol{v}'(x,t) = \boldsymbol{v}(x, t+T), \quad \nabla p'(x,t) = \nabla p(x, t+T),$$

hence

$$\boldsymbol{v}(x,t+T)=0,\quad \text{for}\quad t<0,\quad \nabla p(x,t+T)=0,\quad \text{for}\quad t<0,\quad \text{q.e.d.}$$

In order to prove (2.30), (2.31), we apply (2.33) to $\boldsymbol{v}_a=e^{at}\boldsymbol{v}$, $p_a=e^{at}p$. These functions satisfy the relations

$$\begin{cases} \boldsymbol{v}_{at}-\nu\nabla^2\boldsymbol{v}_a+\nabla p_a=a\boldsymbol{v}_a+\boldsymbol{f}_a, \quad \nabla\cdot\boldsymbol{v}_a=0, \quad x\in\Omega,\\ T(\boldsymbol{v}_a,p_a)\boldsymbol{n}=\boldsymbol{d}_a, \quad x\in S,\\ \boldsymbol{v}_a|_{t=0}=0. \end{cases}$$

Inequality (2.33) applied to $\boldsymbol{v}_a$, p_a gives (2.30) and (2.31) is obtained in a similar way. The proposition is proved. ■

Theorem 2.1. *Assume that $l\in[0,5/2)$, $l\neq 1/2$, $l\neq 1$, and that the data of the problem (2.27) possess the following properties:*

$\boldsymbol{f}\in W_2^{l,l/2}(D_T)$, $f\in W_2^{l+1,0}(D_T)$, $f=\nabla\cdot\boldsymbol{F}$, $\boldsymbol{F}_t\in W_2^{0,l/2}(D_T)$, $\boldsymbol{d}-\boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n})\in W_2^{l+1/2,l/2+1/4}(\Sigma_T)$, $\boldsymbol{d}\cdot\boldsymbol{n}\in W_2^{l+1/2,0}(\Sigma_T)\cap W_2^{l/2}(0,T;W_2^{1/2}(S))$, $\boldsymbol{v}_0\in W_2^{l+1}(\Omega)$;

the following compatibility conditions are satisfied:

$$\nabla\cdot\boldsymbol{v}_0=f(x,0),\quad x\in\Omega,$$

and in the case $l>1/2$ also

$$S(\boldsymbol{v}_0)\boldsymbol{n}-\boldsymbol{n}(\boldsymbol{n}\cdot S(\boldsymbol{v}_0)\boldsymbol{n})=\boldsymbol{d}(x,0)-\boldsymbol{n}(\boldsymbol{n}\cdot\boldsymbol{d}(x,0)),\quad x\in S.$$

Then the problem (2.27) has a unique solution $\boldsymbol{v}\in W_2^{l+2,l/2+1}(D_T)$, $\nabla p\in W_2^{l,l/2}(D_T)$, and

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(D_T)}+\|\nabla p\|_{W_2^{l,l/2}(D_T)}+\|p\|_{W_2^{l+1/2,0}(\Sigma_T)}+|p|_{l/2,1/2,\Sigma_T}\\ &\leq c(T)\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)}+\|f\|_{W_2^{l+1/2,0}(D_T)}+\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(D_T)}\\ &+\|\boldsymbol{d}-\boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n})\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}+\|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{l+1/2,0}(\Sigma_T)}+|\boldsymbol{d}\cdot\boldsymbol{n}|_{l/2,1/2,\Sigma_T}+\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega)}\big),\end{aligned}\tag{2.35}$$

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(D_T)}+\|\nabla p\|_{W_2^{l,l/2}(D_T)}+\|p\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}+[p]_{l/2,1/2,\Sigma_T}\\ &\leq c_1\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)}+\|f\|_{W_2^{l+1/2,0}(D_T)}+\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(D_T)}\\ &+\|\boldsymbol{d}-\boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n})\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}+\|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{l+1/2,0}(\Sigma_T)}+[\boldsymbol{d}\cdot\boldsymbol{n}]_{l/2,1/2,\Sigma_T}\\ &+\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega)}\big)+c_2\|\boldsymbol{v}\|_{L_2(D_T)}\end{aligned}\tag{2.36}$$

with the constants c_1,c_2 independent of T.

If, in addition, $\boldsymbol{d}\cdot\boldsymbol{n}\in W_2^{0,l/2+1/4}(\Sigma_T)$, then

$$\begin{aligned}&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(D_T)}+\|\nabla p\|_{W_2^{l,l/2}(D_T)}+\|p\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}\\ &\leq c(T)\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)}+\|f\|_{W_2^{l+1/2,0}(D_T)}+\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega)}+\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(D_T)}\\ &+\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}\big),\end{aligned}\tag{2.37}$$

$$
\begin{aligned}
&\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(D_T)} + \|\nabla p\|_{W_2^{l,l/2}(D_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)} \\
&\le c_1\big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(D_T)} + \|f\|_{W_2^{l+1/2,0}(D_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega)} + \|\boldsymbol{F}_t\|_{W_2^{0,l/2}(D_T)} \\
&+ \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}\big) + c_2\|\boldsymbol{v}\|_{L_2(D_T)}.
\end{aligned} \tag{2.38}
$$

Proof. We reduce (2.27) to a similar problem with zero divergence by construction of an auxiliary vector field $\boldsymbol{w}_1(x,t) = \nabla\Phi(x,t)$, where Φ is a solution of the Dirichlet problem

$$\nabla^2\Phi(x,t) = f(x,t), \quad x \in \Omega, \quad \Phi(x,t)|_{x\in S} = 0.$$

This function satisfies the inequality

$$\|\Phi\|_{W_2^{l+3,0}(Q_T)} \le c\|f\|_{W_2^{l+1}(Q_T)} \tag{2.39}$$

(see Proposition 8.19); in addition, since

$$\nabla^2\Phi_t(x,t) = f_t(x,t) = \nabla\cdot\boldsymbol{F}_t(x,t), \quad x \in \Omega, \quad \Phi_t(x,t) = 0, \quad x \in S$$

and

$$\nabla^2\Delta_t(-h)\Phi_t(x,t) = \nabla\cdot\Delta_t(-h)\boldsymbol{F}_t(x,t), \quad x \in \Omega, \quad \Delta_t(-h)\Phi_t(x,t) = 0, \quad x \in S,$$

where $\Delta_t(-h)\Phi(x,t) = \Phi(x,t-h) - \Phi(x,t)$, $t > h$, we have, by the energy estimate,

$$\|\nabla\Phi_t\|_{W_2^{0,l/2}(D_T)} \le c\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(D_T)}.$$

Hence

$$\|\boldsymbol{w}_1\|_{W_2^{l+2,l/2+1}(D_T)} \le c\Big(\|f\|_{W_2^{l+1}(D_T)} + \|\boldsymbol{F}_t\|_{W_2^{0,l/2}(D_T)}\Big). \tag{2.40}$$

The difference $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{w}_1$ is a solution of the problem

$$
\begin{cases}
\boldsymbol{v}_{1t} - \nu\nabla^2\boldsymbol{v}_1 + \nabla p = \boldsymbol{f}_1(x,t), \quad \nabla\cdot\boldsymbol{v}_1 = 0, \quad x \in \Omega, \\
T(\boldsymbol{v}_1, p)\boldsymbol{n} = \boldsymbol{d}_1(x,t), \quad x \in S \\
\boldsymbol{v}_1|_{t=0} = \boldsymbol{v}_0(x) - \boldsymbol{w}_1(x) = \boldsymbol{v}_{01}(x), \quad x \in \Omega,
\end{cases} \tag{2.41}
$$

where

$$
\begin{cases}
\boldsymbol{f}_1 = \boldsymbol{f} - \boldsymbol{w}_{1t} + \nu\nabla^2\boldsymbol{w}_1, \\
\boldsymbol{d}_1 = \boldsymbol{d} - \nu S(\boldsymbol{w}_1)\boldsymbol{n}.
\end{cases}
$$

The problem (2.41) can be reduced to the problem of the same type but with zero initial data. Let $l < 1/2$. In this case we construct the solenoidal vector field $\boldsymbol{w}_2$ satisfying the condition $\boldsymbol{w}_2(x,0) = \boldsymbol{v}_{01}(x)$ and the inequality

$$\|\boldsymbol{w}_2\|_{W_2^{l+2,l/2+1}(D_T)} \le c\|\boldsymbol{v}_{01}\|_{W_2^{l+1}(\Omega)}. \tag{2.42}$$

For the difference $\boldsymbol{v}_2 = \boldsymbol{v}_1 - \boldsymbol{w}_2$ we have

$$
\begin{cases}
\boldsymbol{v}_{2t} - \nu\nabla^2\boldsymbol{v}_2 + \nabla p = \boldsymbol{f}_2, \quad \nabla\cdot\boldsymbol{v}_2 = 0, \quad x \in \Omega, \\
T(\boldsymbol{v}_2, p)\boldsymbol{n} = \boldsymbol{d}_2, \quad x \in S, \\
\boldsymbol{v}_2(x,0) = 0,
\end{cases}
$$

where

$$\begin{cases} \boldsymbol{f}_2 = \boldsymbol{f}_1 - \boldsymbol{w}_{2t} + \nu\nabla^2\boldsymbol{w}_2, \\ \boldsymbol{d}_2 = \boldsymbol{d}_1 - \nu S(\boldsymbol{w}_2)\boldsymbol{n}. \end{cases}$$

These functions satisfy the assumptions of Proposition 2.6, and the solution $(\boldsymbol{v}_2, p)$ can be estimated by (2.28), (2.30). It is easily verified (in view of (2.40), (2.42)), that these estimates imply (2.35), (2.36) for the solution of (2.27). The remaining estimates (2.37), (2.38) are deduced from (2.35), (2.36) and from the boundary condition $p = \nu\boldsymbol{n} \cdot S(\boldsymbol{v})\boldsymbol{n} - \boldsymbol{d} \cdot \boldsymbol{n}$.

The construction of $\boldsymbol{w}_2$ is carried out in the following way. We find $\boldsymbol{w}_2(x,0)$ in the form $\boldsymbol{w}_2(x,0) = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1$ is the extension of $\boldsymbol{v}_{01}$ with the preservation of class, as in Proposition 8.10; we assume that $\boldsymbol{u}_1$ has a compact support. Then, using the result of Bogovskii [9], we can find $\boldsymbol{u}_2$, also with a compact support, satisfying the equation $\nabla \cdot \boldsymbol{u}_2 = -\nabla \cdot \boldsymbol{u}_1$ and the inequality

$$\|\boldsymbol{u}_2\|_{W_2^{l+1}(\mathbb{R}^3)} \le c\|\boldsymbol{u}_1\|_{W_2^{l+1}(\mathbb{R}^3)} \le c\|\boldsymbol{v}_{01}\|_{W_2^{l+1}(\Omega)}.$$

Finally, we define the extension of $\boldsymbol{w}_2(x,0)$, $\boldsymbol{w}_2(x,t)$, in the half-space $\mathbb{R}^3 \times \mathbb{R}_+$ by the formula (8.25); it is easily verified that the extended vector field remains solenoidal and satisfies (2.42).

If $l \in (1/2, 1)$, then the functions $\boldsymbol{f} \in W_2^{l,l/2}(D_T)$, $\boldsymbol{d} \cdot \boldsymbol{n} \in W_2^{l/2}(0,T; W_2^{1/2}(S))$ can be extended by zero without loss of regularity. The same is true for $\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d} \cdot \boldsymbol{n}) \in W_2^{l+1/2,l/2+1/4}(\Sigma_T)$, since $\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d} \cdot \boldsymbol{n})|_{t=0} = 0$, in view of the compatibility conditions. Hence the above arguments remain in force for $l \in (1/2, 1)$.

In the case $l \in (1, 5/2)$ we calculate $\boldsymbol{v}_{1t}|_{t=0} \equiv \boldsymbol{v}_{11}(x)$. We have

$$\boldsymbol{v}_{11}(x) = \nu\nabla^2\boldsymbol{v}_{10} - \nabla p_0 + \boldsymbol{f}_1(x,0),$$

where p_0 is defined as a solution of the problem

$$\nabla^2 p_0(x) = \nabla \cdot \boldsymbol{f}(x,0), \quad x \in \Omega, \quad p_0(x) = \nu\boldsymbol{n} \cdot S(\boldsymbol{v}_0)\boldsymbol{n} - \boldsymbol{d}_0(x) \cdot \boldsymbol{n}(x), \quad x \in S.$$

According to Proposition 8.20,

$$\|p_0\|_{W_2^l(\Omega)} \le c\Big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega)} + \|\boldsymbol{f}(\cdot,0)\|_{W_2^{l-1}(\Omega)} + \|\boldsymbol{d}(\cdot,0) \cdot \boldsymbol{n}\|_{W_2^{l-1/2}(S)}\Big),$$

hence

$$\|\boldsymbol{v}_{11}\|_{W_2^l(\Omega)} \le c\Big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\Omega)} + \|\boldsymbol{f}(\cdot,0)\|_{W_2^{l-1}(\Omega)} + \|\boldsymbol{d}(\cdot,0) \cdot \boldsymbol{n}\|_{W_2^{l-1/2}(S)}\Big). \tag{2.43}$$

We construct solenoidal extensions of $\boldsymbol{v}_{01}$ and $\boldsymbol{v}_{11}$ in $\mathbb{R}^3$ with preservation of class in the way described above, and then we find a solenoidal $\boldsymbol{w}_2(x,t)$, $t > 0$, such that

$$\boldsymbol{w}_2(x,0) = \boldsymbol{v}_{01}(x), \quad \boldsymbol{w}_{2t}(x,0) = \boldsymbol{v}_{11}(x)$$

and

$$\|\boldsymbol{w}_2\|_{W_2^{l+2,l/2+1}(D_T)} \le c\Big(\|\boldsymbol{v}_{01}\|_{W_2^{l+1}(\Omega)} + \|\boldsymbol{v}_{11}\|_{W_2^{l-1}(\Omega)}\Big). \tag{2.44}$$

We can define $\boldsymbol{w}_2$ by the formula (8.25).

Finally, we find $p_1 \in W_2^{l+1,l/2+1/2}(D_T)$ such that $p_1(x,0) = p_0(x)$ and

$$\|p_1\|_{W_2^{l+1,l/2+1/2}(D_T)} \leq c\|p_0\|_{W_2^{l-1}(\Omega)}. \tag{2.45}$$

For the differences $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{w} - \boldsymbol{w}_2$, $q = p - p_1$ we obtain the problem with zero initial data treated in Proposition 2.6. By using (2.29), (2.30) we complete the proof of (2.35)-(2.38) and of the theorem.

The cases $l = 1/2$, $l = 1$ are treated in [10].

3 On the linear problem related to the stability of uniformly rotating liquid ($\sigma > 0$).

In this section we consider the non-homogeneous problem (1.24), i.e.,

$$\begin{cases} \boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\ \nabla \cdot \boldsymbol{v}(x,t) = f(x,t), \qquad x \in \mathcal{F}, \quad t > 0, \\ T(\boldsymbol{v},p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = \boldsymbol{d}(x,t), \\ \rho_t = \boldsymbol{v}(x,t) \cdot \boldsymbol{N}(x) + g(x,t), \quad x \in \mathcal{G}, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0(x) \quad x \in \mathcal{F}, \quad \rho(x,0) = \rho_0(x), \quad x \in \mathcal{G}, \end{cases} \tag{3.1}$$

with $B_0\rho$ defined in (1.25).

For the moment, we omit the orthogonality conditions of the type (1.26), (1.27) and we do not use (1.22).

Theorem 3.1. *Assume that $l \in [0, 5/2)$, $l \neq 1/2$, $l \neq 1$ and that the data of the problem (3.1) possess the following regularity properties: $\boldsymbol{f} \in W_2^{l,l/2}(Q_T)$, $f \in W_2^{l+1,0}(Q_T)$, $f(x,t) = \nabla \cdot \boldsymbol{F}(x,t)$, $\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T)$, $\boldsymbol{d} \cdot \boldsymbol{N} \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{1/2}\mathcal{G}))$, $\boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d} \cdot \boldsymbol{N}) \in W_2^{l+1/2,l/2+1/4}(G_T)$, $g \in W_2^{l+3/2,l/2+3/4}(G_T)$, $\boldsymbol{v}_0 \in W_2^{l+1}(\mathcal{F})$, $\rho_0 \in W_2^{l+2}(\mathcal{G})$ where $T < \infty$, $Q_T = \mathcal{F} \times (0,T)$, $G_T = \mathcal{G} \times (0,T)$. Moreover, let the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}_0(x) = f(x,0), \quad x \in \mathcal{F}, \tag{3.2}$$

and, if $l > 1/2$,

$$\nabla \cdot \boldsymbol{v}_0(x) = f(x,0), \quad x \in \mathcal{F}, \quad \nu\Pi_{\mathcal{G}}S(\boldsymbol{v}_0)\boldsymbol{N} = \Pi_{\mathcal{G}}\boldsymbol{d}(x,0), \quad x \in \mathcal{G} \tag{3.3}$$

be satisfied, where $\Pi_{\mathcal{G}}\boldsymbol{d} = \boldsymbol{d} - \boldsymbol{N}(\boldsymbol{d} \cdot \boldsymbol{N})$ is the projection of $\boldsymbol{d}$ on the tangent plane to $\mathcal{G}$. Then the problem (3.1) has a unique solution $\boldsymbol{v}, p, \rho$ such that $\boldsymbol{v} \in W_2^{l+2,l/2+1}(Q_T)$, $\nabla p \in W_2^{l,l/2}(Q_T)$, $p \in W_2^{l+1/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{1/2}(\mathcal{G}))$, $\rho \in W_2^{l+5/2,0}(G_T) \cap W_2^{l/2}(0,T; W_2^{5/2}(\mathcal{G}))$, $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, and the solution satisfies the inequality

$$\begin{aligned} Y_T(\boldsymbol{v},p,\rho) \equiv &\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,0}(G_T)} + |p|_{l/2,1/2,G_T} \\ &+ \|\rho\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(\Gamma_T)} + \sup_{t<T} \|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})} + |\rho|_{l/2,5/2,G_T} \\ &\leq c(T)\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)} \\ &+ \|\Pi_{\mathcal{G}}\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d} \cdot \boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} + |\boldsymbol{d} \cdot \boldsymbol{N}|_{l/2,1/2,G_T} \\ &+ \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big). \end{aligned} \tag{3.4}$$

As above in Sec.2, we invoke the corresponding parameter-dependent problem:

$$\begin{cases} s\boldsymbol{v} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x), \\ \nabla \cdot \boldsymbol{v}(x) = 0, \qquad x \in \mathcal{F}, \\ T(\boldsymbol{v},p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = \boldsymbol{d}(x), \\ s\rho = \boldsymbol{v}(x) \cdot \boldsymbol{N}(x) + g(x), \quad x \in \mathcal{G}, \end{cases} \tag{3.5}$$

Proposition 3.1. *Let* $\mathrm{Re}\,s \geq a \gg 1$, $\boldsymbol{f} \in W_2^l(\mathcal{F})$, $\boldsymbol{d} \in W_2^{l+1/2}(\mathcal{G})$, $g \in W_2^{l+3/2}(\mathcal{G})$ *with* $l \in [0, 5/2)$. *Then the problem* (3.5) *has a unique solution* $\boldsymbol{v} \in W_2^{2+l}(\Omega)$, $p \in W_2^{1+l}(\Omega)$, $\rho \in W_2^{l+5/2}(\mathcal{G})$, *and*

$$\begin{aligned}
&|s|^{1+l/2}\|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|\boldsymbol{v}\|_{W_2^{2+l}(\mathcal{F})} + \|p\|_{W_2^{l+1}(\mathcal{F})} + |s|^{l/2}\|p\|_{W_2^1(\mathcal{F})} + |s|^{1+l/2}\|\rho\|_{W_2^{3/2}(\mathcal{G})} \\
&+ |s|\|\rho\|_{W_2^{l+3/2}(\mathcal{G})} + |s|^{l/2}\|\rho\|_{W_2^{5/2}(\mathcal{G})} + \|\rho\|_{W_2^{l+5/2}(\mathcal{G})} \leq C\big(\|\boldsymbol{f}\|_{W_2^l(\mathcal{F})} + |s|^{l/2}\|\boldsymbol{f}\|_{L_2(\mathcal{F})} \\
&+ |s|^{1/4+l/2}\|\boldsymbol{d} - \boldsymbol{n}(\boldsymbol{d}\cdot\boldsymbol{n})\|_{L_2(\mathcal{G})} + \|\boldsymbol{d}\|_{W_2^{l+1/2}(\mathcal{G})} + |s|^{l/2}\|\boldsymbol{d}\cdot\boldsymbol{n}\|_{W_2^{1/2}(\mathcal{G})} \\
&+ |s|^{l/2}\|g\|_{W_2^{3/2}(\mathcal{G})} + \|g\|_{W_2^{l+3/2}(\mathcal{G})}\big).
\end{aligned} \tag{3.6}$$

with the constant independent of $|s|$.

Proof. We restrict ourselves with the sketch of the proof of the estimate (3.6) in the simplest case $l = 0$. We refer to [11,12] for the ideas of the proof of the solvability of (3.5). To shorten the notation, we set

$$|s|^{l/2}\|u\|_{L_2(\Omega)} + \|u\|_{W_2^l(\Omega)} = |||u|||_{l,\Omega}.$$

STEP 1. We consider the model problem in the half-space $\mathbb{R}_+^3 = \{x_3 > 0\}$:

$$\begin{cases}
s\boldsymbol{v}(x) - \nu\nabla^2\boldsymbol{v}(x) + \nabla p(x) = 0, \\
\nabla\cdot\boldsymbol{v}(x) = 0, \quad x \in \mathbb{R}_+^3, \\
\nu\Big(\dfrac{\partial v_3}{\partial x_j} + \dfrac{\partial v_j}{\partial x_3}\Big) = b_j, \quad j = 1, 2, \\
-p + 2\nu\dfrac{\partial v_3}{\partial x_3} - \sigma\Delta'\rho = b_3(x), \\
s\rho + v_3 = g(x), \quad x_3 = 0,
\end{cases} \tag{3.7}$$

where $\Delta' = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

Using the Fourier transform in x_1, x_2, we reduce (3.7) to the boundary value problem on the half-axis $\mathbb{R}_+ = \{x_3 > 0\}$:

$$\begin{cases}
\nu\Big(r^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + i\xi_j\widetilde{p} = 0 \quad j = 1, 2, \\
\nu\Big(r^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_3 + \dfrac{d\widetilde{p}}{dx_3} = 0, \quad i\xi_1\widetilde{v}_1 + i\xi_2\widetilde{v}_2 + \dfrac{d\widetilde{v}_3}{dx_3} = 0, \quad x_3 > 0, \\
\nu\Big(\dfrac{dv_j}{\partial x_3} + i\xi_j\widetilde{v}_3\Big) = \widetilde{b}_j, \quad j = 1, 2, \\
-\widetilde{p} + 2\nu\dfrac{d\widetilde{v}_3}{dx_3} + \sigma|\xi|^2\widetilde{\rho} = \widetilde{b}_3, \\
s\widetilde{\rho} + \widetilde{v}_3 = \widetilde{g}, \quad x_3 = 0, \\
\widetilde{\boldsymbol{v}} \to 0, \quad \widetilde{p} \to 0, \quad (x_3 \to \infty),
\end{cases} \tag{3.8}$$

where $\xi = (\xi_1, \xi_2)$, $r = r(s, \xi) = \sqrt{s\nu^{-1} + |\xi|^2}$, $-\pi \leq arg\ r < \pi$.

It is convenient to exclude the function $\widetilde{\rho}$ from (3.8) and write this problem in the form

$$\begin{cases} \nu\Big(r^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + i\xi_j\widetilde{p} = 0 \quad j=1,2, \\ \nu\Big(r^2 - \dfrac{d^2}{dx_3^2}\Big)\widetilde{v}_j + \dfrac{d\widetilde{p}}{dx_3} = 0, \quad i\xi_1\widetilde{v}_1 + i\xi_2\widetilde{v}_2 + \dfrac{d\widetilde{v}_3}{dx_3} = 0, \quad x_3>0, \\ \nu\Big(\dfrac{d\widetilde{v}_j}{\partial x_3} + i\xi_j\widetilde{v}_3\Big) = \widetilde{b}_j, \quad j=1,2, \\ -\widetilde{p} + 2\nu\dfrac{d\widetilde{v}_3}{dx_3} - \dfrac{\sigma}{s}|\xi|^2\widetilde{v}_3 = \widetilde{b}_3 - \dfrac{\sigma}{s}|\xi|^2\widetilde{g}, \quad x_3=0, \\ \widetilde{\boldsymbol{v}} \to 0, \quad \widetilde{p}\to 0, \quad (x_3\to\infty). \end{cases} \tag{3.9}$$

In the paper [11] the explicit formula for the solution of (3.9) is obtained, in particular, it is shown that

$$\begin{aligned} \widetilde{v}_i = &-\frac{1-\delta_{i3}}{\nu r}e_0(x_3)\widetilde{b}_i + \frac{e_0(x_3)}{\nu^2 r(r+|\xi|)P}\sum_{j=1}^3 U_{ij}\widetilde{b}_j + \frac{e_1(x_3)}{\nu^2(r+|\xi|)P}\sum_{j=1}^3 V_{ij}\widetilde{b}_j \\ &- \frac{\sigma|\xi|^2 e_0(x_3)}{\nu^2 sr(r+|\xi|)P}U_{i3}\widetilde{g} - \frac{\sigma|\xi|^2 e_1(x_3)}{\nu^2 s(r+|\xi|)P}V_{i3}\widetilde{g}, \end{aligned} \tag{3.10}$$

where

$$e_0(x_3) = e^{-rx_3}, \quad e_1(x_3) = \frac{e^{-rx_3} - e^{-|\xi|x_3}}{r-|\xi|}, \tag{3.11}$$

$$P = (r^2+|\xi|^2)^2 - 4r|\xi|^2 + \frac{\sigma}{\nu^2}|\xi|^3 = \frac{s}{\nu}\Big(\frac{s}{\nu} + 4|\xi|^2\Big(1 - \frac{|\xi|}{r+|\xi|}\Big) + \frac{\sigma|\xi|^3}{\nu s}\Big), \tag{3.12}$$

and U_{ij}, V_{ij} are the elements of the matrices

$$\mathcal{U} = \begin{pmatrix} \xi_1^2((3r-|\xi|)s + \frac{\sigma}{\nu}|\xi|^2) & \xi_1\xi_2((3r-|\xi|)s + \frac{\sigma}{\nu}|\xi|^2) & i\xi_1 rs(r-|\xi|) \\ \xi_1\xi_2((3r-|\xi|)s + \frac{\sigma}{\nu}|\xi|^2) & \xi_2^2((3r-|\xi|)s + \frac{\sigma}{\nu}|\xi|^2) & i\xi_1 rs(r-|\xi|) \\ -i\xi_1 rs(r-|\xi|) & -i\xi_2 rs(r-|\xi|) & -|\xi|rs(r+|\xi|) \end{pmatrix},$$

$$\mathcal{V} = \begin{pmatrix} -\xi_1^2(2rs + \frac{\sigma}{\nu}|\xi|^2) & -\xi_1\xi_2(2rs + \frac{\sigma}{\nu}|\xi|^2) & -i\xi_1 s(r^2+|\xi|^2) \\ -\xi_1\xi_2(2rs + \frac{\sigma}{\nu}|\xi|^2) & -\xi_2^2(2rs + \frac{\sigma}{\nu}|\xi|^2) & -i\xi_2 s(r^2+|\xi|^2) \\ -i\xi_1|\xi|(2rs + \frac{\sigma}{\nu}|\xi|^2) & -i\xi_2\xi|(2rs + \frac{\sigma}{\nu}|\xi|^2) & |\xi|s(r^2+|\xi|^2) \end{pmatrix}.$$

If $\mathrm{Re}s \geq \gamma > 0$, then $c|r(s,\xi)| \leq \sqrt{|s|+|\xi|^2} \leq c'|r(s,\xi)|$ and

$$\frac{\gamma^2}{\nu^2} + |s||\xi|^2 + |s|^2 + \sigma|\xi|^3 \leq c(\gamma)|P|, \tag{3.13}$$

moreover,

$$\int_0^\infty \Big|\frac{d^j e_0(x_3)}{dx_3^j}\Big|^2 dx_3 \leq \frac{1}{\sqrt{2}}|r|^{2j-1},$$

$$\int_0^\infty \Big|\frac{d^j e_1(x_3)}{dx_3^j}\Big|^2 dx_3 \leq c\frac{|r|^{2j-1} + |\xi|^{2j-1}}{|r^2|}, \quad j=0,1,2.$$

Making use of these inequalities and of the results of Sec. 8 concerning the equivalent norms in the Sobolev spaces, we obtain

$$\begin{aligned}&|||\boldsymbol{v}|||_{2,\mathbb{R}^3_+}+\|\nabla p\|_{L_2(\mathbb{R}^3_+)}\leq c\big(|||b_1|||_{1/2,\mathbb{R}^2}\\&+\big(|||b_2|||_{1/2,\mathbb{R}^2}+\|b_3\|_{W_2^{1/2}(\mathbb{R}^2)}+\|g\|_{W_2^{3/2}(\mathbb{R}^2)}\big).\end{aligned}\tag{3.14}$$

(see details in [11,12]).

From (3.10) it follows that

$$\widetilde{v}_3(0)=\sum_{i=1}^{3}\frac{U_{3i}\widetilde{b}_i}{\nu^2 r(r+|\xi|)P}+\frac{\sigma|\xi|^3\widetilde{g}}{\nu^2 P}$$

and, as a consequence,

$$\widetilde{v}_3(0)-\widetilde{g}=-\frac{s}{\nu}\Big(\frac{s}{\nu}+4|\xi|^2(1-\frac{|\xi|}{r+|\xi|}\Big)\frac{\widetilde{g}}{P}.$$

Using the fourth equation in (3.9) and taking $\frac{d\widetilde{v}_3}{dx_3}=-(\xi_1\widetilde{v}_1+\xi_2\widetilde{v}_2)$ into account, we obtain after simple calculations

$$\Big|\frac{d\widetilde{v}_3}{dx_3}\Big|+|\widetilde{p}|\Big|_{x_3=0}\leq c\Big(|\widetilde{\boldsymbol{b}}|+|\widetilde{g}|(|\xi|+1)\Big).$$

To estimate the norms of ρ, we use the equations

$$s\widetilde{\rho}=\widetilde{g}-\widetilde{v}_3(0),\quad \sigma|\xi|^2\widetilde{\rho}=\widetilde{b}_3+(\widetilde{p}-2\nu\frac{d\widetilde{v}_3}{dx_3})\Big|_{x_3=0}.$$

and the theorem on the equivalent norms mentioned above. This leads to

$$\begin{aligned}&\|p\|_{W_2^{1/2}(\mathbb{R}^3)}+|s|\|\rho\|_{W_2^{3/2}(\mathbb{R}^2)}+\|\rho\|_{W_2^{5/2}(\mathbb{R}^2)}\\&\leq c\big(|||b_1|||_{1/2,\mathbb{R}^2}+|||b_2|||_{1/2,\mathbb{R}^2}+\|b_3\|_{W_2^{1/2}(\mathbb{R}^2)}+\|g\|_{W_2^{3/2}(\mathbb{R}^2)}\big).\end{aligned}\tag{3.15}$$

Inequalities (3.14), (3.15) imply estimate (3.6) for the problem (3.7) in the case $l=0$.

STEP 2. We consider the non-homogeneous problem

$$\begin{cases}s\boldsymbol{v}(x)-\nu\nabla^2\boldsymbol{v}(x)+\nabla p(x)=\boldsymbol{f}(x),\\ \nabla\cdot\boldsymbol{v}(x)=h(x),\quad x\in\mathbb{R}^3_+,\\ \nu\Big(\dfrac{\partial v_3}{\partial x_j}+\dfrac{\partial v_j}{\partial x_3}\Big)=b_j,\quad j=1,2,\\ -p+2\nu\dfrac{\partial v_3}{\partial x_3}-\sigma\Delta'\rho=b_3(x),\\ s\rho+v_3=g(x),\quad x_3=0.\end{cases}\tag{3.16}$$

Assuming that $\boldsymbol{f}$ and h decay at infinity sufficiently rapidly, and

$$h=\nabla\cdot\boldsymbol{H}(x)+h'(x)\tag{3.17}$$

with compactly supported h', we reduce (3.16) to (3.7). We extend $\boldsymbol{f}$ into the entire space $\mathbb{R}^3$ in such a way that

$$\|\boldsymbol{f}\|_{L_2(\mathbb{R}^3)} \leq c\|\boldsymbol{f}\|_{L_2(\mathbb{R}^3_+)},$$

and we introduce $\boldsymbol{w}_1$ as a solution of the equation

$$s\boldsymbol{w}_1(x) - \nu\nabla^2\boldsymbol{w}_1(x) = \boldsymbol{f}(x), \quad x \in \mathbb{R}^3,$$

i.e.,

$$\widetilde{\boldsymbol{v}}_1 = \frac{\widetilde{\boldsymbol{f}}}{s + \nu(\xi_1^2 + \xi_2^2 + \xi_3^2)},$$

where $\widetilde{u}$ means the Fourier transform in all the space variables x_1, x_2, x_3. It is clear that

$$|||\boldsymbol{w}_1|||_{2,\mathbb{R}^3} \leq c|||\boldsymbol{f}|||_{l,\mathbb{R}^3_+} \tag{3.18}$$

Further, let $\boldsymbol{w}_2 = \nabla\Phi$, where Φ is a solution of the Dirichlet problem

$$\nabla^2\Phi(x) = h(x) - \nabla\cdot\boldsymbol{w}_1, \quad x \in \mathbb{R}^3_+, \quad \Phi(x)|_{x_3=0} = 0. \tag{3.19}$$

By the Green identity,

$$\begin{aligned}
&\int_{\mathbb{R}^3_+} |\nabla\Phi(x)|^2 dx = -\int_{\mathbb{R}^3_+} \Phi(x)\nabla^2\Phi(x)dx = \int_{\mathbb{R}^3_+} \nabla\Phi(x)(\boldsymbol{H} - \boldsymbol{w}_1)dx - \int_{\mathbb{R}^3_+} h'(x)\Phi(x)dx \\
&\leq c\Big(\|\boldsymbol{H} - \boldsymbol{w}_1\|_{L_2(\mathbb{R}^3_+)}\|\nabla\Phi\|_{L_2(\mathbb{R}^3_+)} + \|h'\|_{L_{6/5}(supp\ h')}\|\Phi\|_{L_6(\mathbb{R}^3_+)}\Big) \\
&\leq c\|\nabla\Phi\|_{L_2(\mathbb{R}^3_+)}\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)} + \|h'\|_{L_2(\mathbb{R}^3_+)} + \|\boldsymbol{w}_1\|_{L_2(\mathbb{R}^3_+)}\Big).
\end{aligned} \tag{3.20}$$

Moreover, coercive estimate for the problem (3.19) (see Proposition 8.19) yields

$$\|\nabla\Phi\|_{\dot{W}^2_2(\mathbb{R}^3_+)} \leq c\Big(\|\boldsymbol{w}_1\|_{\dot{W}^2_2(\mathbb{R}^3_+)} + \|h\|_{\dot{W}^1_2(\mathbb{R}^3_+)}\Big),$$

hence

$$\begin{aligned}
&|||\boldsymbol{w}_2|||_{2,\mathbb{R}^3_+} \\
&\leq c|s|\Big(\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)} + \|h'\|_{L_2(\mathbb{R}^3_+)} + \|\boldsymbol{w}_1\|_{L_2(\mathbb{R}^3_+)}\Big) + c\Big(\|\boldsymbol{w}_1\|_{W^2_2(\mathbb{R}^3_+)} + \|h\|_{W^1_2(\mathbb{R}^3_+)}\Big).
\end{aligned} \tag{3.21}$$

The differences $\boldsymbol{v}_1 = \boldsymbol{v} - \boldsymbol{w}_1 - \boldsymbol{w}_2$, $p_1 = p - \nu\nabla^2\Phi - s\Phi$ represent the solution of the problem (3.7) with the data

$$b'_j = b_j - \nu\Big(\frac{\partial w_{1j}}{\partial x_3} + \frac{\partial w_{3j}}{\partial x_j}\Big) - \nu\Big(\frac{\partial w_{2j}}{\partial x_3} + \frac{\partial w_{23}}{\partial x_j}\Big), \quad j = 1, 2,$$

$$b'_3 = b_3 - 2\nu\frac{\partial w_{13}}{\partial x_3} - 2\nu\frac{\partial w_{23}}{\partial x_3},$$

$$g' = g - w_{13} - w_{23},$$

and they can be estimated by (3.14), (3.15). These estimates, together with (3.18), (3.21), yield

$$
\begin{aligned}
&|||\boldsymbol{v}|||_{2,\mathbb{R}^3_+} + \|\nabla p\|_{L_2(\mathbb{R}^3_+)} + \|p\|_{W_2^{1/2}(\mathbb{R}^2)} + \|\rho\|_{W_2^{5/2}(\mathbb{R}^2)} + |s|\|\rho\|_{W_2^{3/2}(\mathbb{R}^2)} \\
&\le c\big(\|\boldsymbol{f}\|_{L_2(\mathbb{R}^3_+)} + |||b_1|||_{1/2,\mathbb{R}^2} + |||b_2|||_{1/2,\mathbb{R}^2} + \|b_3\|_{W_2^{1/2}(\mathbb{R}^2)} + \|g\|_{W_2^{3/2}(\mathbb{R}^2)} \\
&+ |s|\|\boldsymbol{H}\|_{L_2(\mathbb{R}^3_+)} + |s|\|h'\|_{L_2(\mathbb{R}^3_+)} + \|h\|_{W_2^1(\mathbb{R}^3_+)}\big).
\end{aligned} \tag{3.22}
$$

STEP 3. We estimate the solution of (3.5) in the neighborhood of an arbitrary fixed point $x_0 \in \mathcal{G}$ by Schauder's localization method. Without loss of generality we may assume that $x_0 = 0$ and that the interior normal $-\boldsymbol{N}(0)$ is parallel to $\boldsymbol{e}_3$. Let $\zeta(x)$ be a smooth cut-off function equal to 1 for $|x| \le \delta/2$ and to zero in the domain $|x| \ge \delta$. The functions $\boldsymbol{w} = \zeta(x)\boldsymbol{v}(x)$, $\boldsymbol{q} = \zeta \boldsymbol{p}$, $r = \zeta\rho$ satisfy the equations

$$
\begin{cases}
s\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = \boldsymbol{f}(x)\zeta(x) + \boldsymbol{m}_1(\boldsymbol{v}, p), \\
\nabla\cdot\boldsymbol{w}(x) = \nabla\zeta\cdot\boldsymbol{v}(x), \quad x \in \mathcal{F}, \\
T(\boldsymbol{w}, q)\boldsymbol{N} - \sigma\boldsymbol{N}\Delta_{\mathcal{G}} r = \zeta(x)\boldsymbol{d}(x) + \boldsymbol{m}_2(\boldsymbol{v}, \rho) - \zeta b_0\rho\boldsymbol{N}, \\
sr(x) = \boldsymbol{w}(x)\cdot\boldsymbol{N}(x) + g(x)\zeta(x), \quad x \in \mathcal{G},
\end{cases} \tag{3.23}
$$

where

$$
\begin{aligned}
\boldsymbol{m}_1(\boldsymbol{v}, p) &= -2\nu\nabla\zeta(x)\nabla\boldsymbol{v} - \nu\boldsymbol{v}\nabla^2\zeta + p\nabla\zeta - 2\omega(\boldsymbol{e}_3\times\zeta\boldsymbol{v}), \\
\boldsymbol{m}_2(\boldsymbol{v}, \rho) &= \nu\Big(\boldsymbol{v}(x)\frac{\partial\zeta}{\partial N} + \nabla\zeta(x)(\boldsymbol{v}\cdot\boldsymbol{N})\Big) + \boldsymbol{N}\sigma(\zeta(x)\Delta_{\mathcal{G}}\rho - \Delta_{\mathcal{G}}(\zeta\rho)), \\
b_0\rho &= -b(x)\rho - \kappa\int_{\mathcal{G}}\frac{\rho(y)dS_y}{|x-y|}.
\end{aligned} \tag{3.24}
$$

We assume that in the d-neighborhood of the origin ($d \ge 2\delta$) the surface $\mathcal{G}$ is given by the equation

$$x_3 = \phi(x'), \quad x' = (x_1, x_2).$$

The function ϕ is smooth and $\phi(0) = 0$, $\nabla\phi(0) = 0$, which implies

$$|\nabla\phi(x')| \le c|x'|, \quad |\phi(x')| \le c|x'|^2 \tag{3.25}$$

for $|x'| \le d$. The components of $\boldsymbol{N}$ and the Laplace-Beltrami operator $\Delta_{\mathcal{G}}$ are expressed in terms of ϕ as follows:

$$N_\alpha = \frac{\phi_{y_\alpha}}{\sqrt{1+|\nabla\phi|^2}}, \quad \alpha = 1, 2, \quad N_3 = -\frac{1}{\sqrt{1+|\nabla\phi|^2}}, \tag{3.26}$$

$$\Delta_{\mathcal{G}} = \frac{1}{\sqrt{1+|\nabla\phi|^2}}\sum_{\alpha,\beta=1}^{2}\frac{\partial}{\partial y_\alpha}\Big(\delta_{\alpha\beta}\sqrt{1+|\nabla\phi|^2} - \frac{\phi_{y_\alpha}\phi_{y_\beta}}{\sqrt{1+|\nabla\phi|^2}}\Big)\frac{\partial}{\partial y_\beta}. \tag{3.27}$$

We make the change of variables in (3.23):

$$y = F(x): \quad y' = x', \quad y_3 = x_3 - \phi(x').$$

If d is small enough, then the transformation F is invertible, and it establishes one-to one correspondence between the domain $K_d = \{|x| \le d,\ x \in \mathcal{F}\}$ and a certain subdomain

D of $\mathbb{R}^3_+$. The operators ∇_x and $S(\boldsymbol{v})$ are transformed into $\widehat{\nabla} = \nabla_y - \frac{\partial}{\partial y_3}\nabla\phi(y')$ and $\widehat{S}(\boldsymbol{v}) = \widehat{\nabla}v + (\widehat{\nabla}v)^T$, respectively, and it holds

$$\nabla_x \cdot \boldsymbol{f}(x) = \widehat{\nabla} \cdot \boldsymbol{f}(x(y)) = \nabla_y \cdot \widehat{\boldsymbol{f}}(y),$$

where $\widehat{f_i} = f_i - \delta_{i3}\sum_{\alpha=1}^2 \phi_{y_\alpha} f_\alpha$.

We write the equations (3.23) in the variables $\{y\}$, keeping the old notation for all the transformed functions. We have

$$\begin{cases} s\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = \boldsymbol{M}_1(\boldsymbol{w}, q) + \boldsymbol{m}_1(\boldsymbol{v}, p) - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) + \zeta\boldsymbol{f}, \\ \nabla \cdot \boldsymbol{w} = (\nabla - \widehat{\nabla}) \cdot \boldsymbol{w} + \widehat{\nabla}\zeta \cdot \boldsymbol{v}, \end{cases} \tag{3.28}$$

where $\nabla = \nabla_y$,

$$\boldsymbol{M}_1(\boldsymbol{w}, q) = \nu(\widehat{\nabla}^2 - \nabla^2)\boldsymbol{w} + (\nabla - \widehat{\nabla})q. \tag{3.29}$$

We note that the function $\nabla\zeta \cdot \boldsymbol{v}$ can be written in the form

$$\begin{aligned} \nabla\zeta \cdot \boldsymbol{v} &= \frac{1}{s}\nabla\zeta \cdot (\nu\nabla^2\boldsymbol{v} - \nabla p - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) + \boldsymbol{f}) \\ &= \nabla \cdot \boldsymbol{A}_s(\boldsymbol{v}, p) + a_s(\boldsymbol{v}, p) + \frac{1}{s}\nabla\zeta \cdot \boldsymbol{f}, \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} \boldsymbol{A}_s(\boldsymbol{v}, p) &= \frac{1}{s}(\nu\nabla\zeta\nabla\boldsymbol{v} - p\nabla\zeta), \\ a_s(\boldsymbol{v}, p) &= \frac{1}{s}\Big(- \nu D^2\zeta : \nabla\boldsymbol{v} + p\nabla^2\zeta - 2\omega\nabla\zeta \cdot (\boldsymbol{e}_3 \times \boldsymbol{v})\Big), \end{aligned} \tag{3.31}$$

$D^2\zeta = \left(\frac{\partial^2\zeta}{\partial x_i \partial x_j}\right)_{i,j=1,2,3}$, $\nabla\boldsymbol{v} = \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,2,3}$. Hence $h \equiv (\nabla - \widehat{\nabla}) \cdot \boldsymbol{w} + \widehat{\nabla}\zeta \cdot \boldsymbol{v}$ satisfies (3.17) with

$$\boldsymbol{H} = \boldsymbol{e}_3 \sum_{\alpha=1}^2 \phi_{y_\alpha} w_\alpha + \widehat{A}_s(\boldsymbol{v}, p), \quad h' = a_s(\boldsymbol{v}, p) + \frac{1}{s}\widehat{\nabla}\zeta \cdot \boldsymbol{f}. \tag{3.32}$$

We write the boundary condition $T\boldsymbol{N} - \sigma\boldsymbol{N}\Delta_{\mathcal{G}} r = \zeta\boldsymbol{d} + \boldsymbol{m}_2 - \boldsymbol{N}\zeta b_0\rho$ for the tangential and normal components separately, moreover, we can take only the first two components of the tangential part. This gives the system of three equations

$$\begin{aligned} &\nu\Big(\sum_{i=1}^3 \widehat{S}_{\alpha i}(\boldsymbol{w})N_i - N_\alpha(\boldsymbol{N} \cdot \widehat{S}(\boldsymbol{w})\boldsymbol{N})\Big) = \zeta(d_\alpha - N_\alpha(\boldsymbol{d} \cdot \boldsymbol{N})) + m_{2\alpha} - N_\alpha(\boldsymbol{N} \cdot \boldsymbol{m}_2), \quad \alpha = 1, 2, \\ &- q + \nu\boldsymbol{N} \cdot \widehat{S}(\boldsymbol{w})\boldsymbol{N} + B_0 r = \zeta\boldsymbol{d} \cdot \boldsymbol{N} + \boldsymbol{m}_2 \cdot \boldsymbol{N} + \zeta b_0\rho, \end{aligned}$$

i.e.,

$$\begin{cases} \nu S_{\alpha 3}(\boldsymbol{w}) = L_\alpha(\boldsymbol{w}) + l_\alpha(\boldsymbol{v}) + \zeta d'_\alpha(y), \quad \alpha = 1, 2, \\ - q + \nu S_{33}(\boldsymbol{w}) - \sigma\Delta' r = L_3(\boldsymbol{w}) + B'r + l_3(\boldsymbol{v}) + \zeta\boldsymbol{d} \cdot \boldsymbol{N} - \zeta b_0(\rho), \end{cases} \tag{3.33}$$

where $d'_\alpha = d_\alpha - N_\alpha(\boldsymbol{d}\cdot\boldsymbol{N})$,

$$
\begin{aligned}
&L_\alpha(\boldsymbol{w}) = \nu\Big(S_{\alpha 3} - \sum_{j=1}^{3}\widehat{S}_{\alpha j}N_j + N_\alpha(\boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N})\Big),\\
&L_3(\boldsymbol{w}) = \nu\Big(S_{33}(\boldsymbol{w}) - \boldsymbol{N}\cdot\widehat{S}(\boldsymbol{w})\boldsymbol{N}\Big),\\
&B'r = -\sigma(\Delta' + \Delta_{\mathcal{G}})r,\\
&l_\alpha(\boldsymbol{v}) = m_{2\alpha}(\boldsymbol{v}) - N_\alpha(\boldsymbol{m}_2(\boldsymbol{v})\cdot\boldsymbol{N}),\\
&l_3(\boldsymbol{v}) = \boldsymbol{m}_2\cdot\boldsymbol{N}.
\end{aligned}
$$

Finally, we have

$$
sr + w_3 = (w_3 + \boldsymbol{w}\cdot\boldsymbol{N}) + \zeta g. \tag{3.34}
$$

Now, we extend $\boldsymbol{w}$, q, r by zero into $\mathbb{R}^3_+$ and $\mathbb{R}^2$ and consider (3.28), (3.33), (3.34) as the problem of the type (3.7) in the half-space. We estimate $\boldsymbol{w}, q, r$ by (3.22). We note that in view of (3.25) the leading coefficients of the operators $\boldsymbol{M}_1$, $\nabla - \widehat{\nabla}$, L_i, B' are small in the case of a small δ. We have

$$
\|\boldsymbol{M}_1\|_{L_2(\mathbb{R}^3_+)} \le c\delta\Big(\|\boldsymbol{w}\|_{W_2^2(\mathbb{R}^3_+)} + \|\nabla q\|_{L_2(\mathbb{R}^3_+)}\Big) + c\|\boldsymbol{w}\|_{W_2^1(\mathbb{R}^3_+)},
$$

$$
\|(\nabla - \widehat{\nabla})\cdot\boldsymbol{w}\|_{W_2^1(\mathbb{R}^3_+)} \le c(\delta\|\boldsymbol{w}\|_{W_2^2(\mathbb{R}^3_+)} + \|\boldsymbol{w}\|_{W_2^1(\mathbb{R}^3_+)});
$$

similar estimates hold for the norms of $L_i(\boldsymbol{w})$ and $\boldsymbol{w}_3 + \boldsymbol{w}\cdot\boldsymbol{N}$. We also have

$$
\|\boldsymbol{m}_1\|_{L_2(\mathbb{R}^3_+)} \le c(\delta)\Big(\|\boldsymbol{v}\|_{W_2^1(K_{2\delta})} + \|p\|_{L_2(K_{2\delta})}\Big),
$$

$$
\|m_2\|_{W_2^{1/2}(\mathbb{R}^2)} \le c(\delta)\Big(\|\boldsymbol{v}\|_{W_2^1(K_{2\delta})} + \|\rho\|_{W_2^{3/2}(S_{2\delta})}\Big),
$$

where $S_{2\delta} = \bar{K}_{2\delta}\cup S$, and similar inequalities for $l_i(\boldsymbol{v})$. Finally,

$$
\|B'r\|_{W_2^{1/2}(\mathbb{R}^2)} \le c\Big(\delta\|r\|_{W_2^{5/2}(\mathbb{R}^2)} + \|r\|_{W_2^2(\mathbb{R}^2)}\Big).
$$

In view of all these estimates it is not hard to verify that application of (3.22) to our problem leads to

$$
\begin{aligned}
&|||\boldsymbol{w}|||_{2,\mathbb{R}^3_+} + \|\nabla q\|_{L_2(\mathbb{R}^3_+)} + \|q\|_{W_2^{1/2}(\mathbb{R}^2)} + |s|\|r\|_{W_2^{3/2}(\mathbb{R}^2)} + \|r\|_{W_2^{5/2}(\mathbb{R}^2)}\\
&\le c\Big(\|\zeta\boldsymbol{f}\|_{L_2(\mathbb{R}^3_+)} + |||\Pi_{\mathcal{G}}\zeta\boldsymbol{d}|||_{1/2,\mathbb{R}^2} + \|\zeta\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathbb{R}^2)} + \|\zeta g\|_{W_2^{3/2}(\mathbb{R}^2)}\\
&+\|\zeta b_0(\rho)\|_{W_2^{1/2}(\mathbb{R}^2)}\Big) + c(\delta)\Big(\|p\|_{L_2(K_{2\delta})} + \|\boldsymbol{v}\|_{W_2^1(K_{2\delta})} + \|\rho\|_{W_2^{3/2}(S_{2\delta})}\Big),
\end{aligned} \tag{3.35}
$$

provided δ is sufficiently small. Inequalities of this type can be obtained in a neighborhood of any point of $\mathcal{G}$ and of any interior point of $\mathcal{F}$ as well, if the distance of this point to $\mathcal{G}$ is larger that $\delta_1 > 0$ (in this case the norms of g do not occur in the estimate). If we cover $\mathcal{F}$ by a finite number of such neighborhoods and add the squares of the corresponding estimates (3.35) together, we obtain

$$
\begin{aligned}
&|||\boldsymbol{v}|||_{2,\mathcal{F}} + \|\nabla p\|_{L_2(\mathcal{F})} + \|p\|_{W_2^{1/2}(\mathcal{G})} + |s|\|\rho\|_{W_2^{3/2}(\mathcal{G})} + \|\rho\|_{W_2^{5/2}(\mathcal{G})}\\
&\le c\Big(\|\boldsymbol{f}\|_{L_2(\mathcal{F})} + |||\Pi_{\mathcal{G}}\boldsymbol{d}|||_{1/2,\mathcal{G}} + \|\boldsymbol{d}\cdot\boldsymbol{N}\|_{W_2^{1/2}(\mathcal{G})} + \|g\|_{W_2^{3/2}(\mathcal{G})}\Big)\\
&+ c(\delta)\Big(\|\boldsymbol{v}\|_{W_2^1(\mathcal{F})} + \|p\|_{L_2(\mathcal{F})} + \|\rho\|_{W_2^2(\mathcal{G})} + \|b_0(\rho)\|_{W_2^{1/2}(\mathcal{G})}\Big).
\end{aligned} \tag{3.36}
$$

The next step is the estimate of $\|p\|_{L_2(\mathcal{F})}$.

STEP 4. We consider p as a solution of the problem

$$\nabla^2 p = \nabla \cdot (\boldsymbol{f} - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v})), \quad x \in \mathcal{F}, \quad p = \nu \boldsymbol{N} \cdot \boldsymbol{S}(\boldsymbol{v})\boldsymbol{N} + B_0\rho - \boldsymbol{d} \cdot \boldsymbol{N}, \quad x \in \mathcal{G},$$

where the equation is understood in a weak sense. Let φ be a solution of the Dirichlet problem

$$\nabla^2 \varphi(x) = p(x), \quad x \in \mathcal{F}, \quad \varphi|_{\mathcal{G}} = 0.$$

By the Green identity,

$$\begin{aligned}\int_{\mathcal{F}} |p|^2 dx &= \int_{\mathcal{F}} p\nabla^2\bar{\varphi}dx = \int_{\mathcal{F}} \nabla^2 p\bar{\varphi}dx + \int_{\mathcal{G}} p\frac{\partial\bar{\varphi}}{\partial N}dS \\ &= -\int_{\mathcal{F}} (\boldsymbol{f} - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v})) \cdot \nabla\bar{\varphi}dx + \int_{\mathcal{G}} (\nu\boldsymbol{N} \cdot S(\boldsymbol{v})\boldsymbol{N} + B_0\rho - \boldsymbol{d} \cdot \boldsymbol{N})\frac{\partial\bar{\varphi}}{\partial N}dS.\end{aligned} \tag{3.37}$$

This formula and the coercive estimate

$$\|\varphi\|_{W_2^2(\mathcal{F})} \leq c\|p\|_{L_2\mathcal{F})}$$

for φ imply

$$\|p\|_{L_2(\mathcal{F})} \leq c\Big(\|\boldsymbol{f}\|_{L_2(\mathcal{F})} + \|\boldsymbol{v}\|_{L_2(\mathcal{F})} + \|\nabla\boldsymbol{v}\|_{L_2(\mathcal{G})} + \|\rho\|_{W_2^2(\mathcal{G})}\Big). \tag{3.38}$$

When we estimate the norm of p in (3.36) by (3.38), use the boundedness of the integral operator b_0 (see [13]), then estimate the norms of $\boldsymbol{v}$ and ρ by interpolation inequalities

$$\|\boldsymbol{v}\|_{W_2^1(\mathcal{F})} \leq \epsilon_1\|\boldsymbol{v}\|_{W_2^2(\mathcal{F})} + (c(\epsilon_1)|s|^{-1})|s|\|\boldsymbol{v}\|_{L_2(\mathcal{F})},$$

$$\|\rho\|_{W_2^2(\mathcal{G})} \leq \epsilon_2\|\rho\|_{W_2^{5/2}(\mathcal{G})} + (c(\epsilon_2)|s|^{-1})|s|\|\rho\|_{W_2^{3/2}(\mathcal{G})},$$

fix δ sufficiently small and after this choose $a = \inf \mathrm{Re}s$ sufficiently large, we obtain (3.6) in the case $l = 0$.

Proof of Theorem 3.1. We reduce (3.1) to a similar problem with zero divergence by construction of an auxiliary vector field $\boldsymbol{u}_1(x,t) = \nabla\Phi(x,t)$, where Φ is a solution of the Dirichlet problem

$$\nabla^2\Phi(x,t) = f(x,t), \quad x \in \mathcal{G}, \quad \Phi(x,t)|_{x\in\mathcal{G}} = 0.$$

According to Proposition 8.19, this function satisfies the inequality

$$\|\Phi\|_{W_2^{l+3,0}(Q_T)} \leq c\|f\|_{W_2^{l+1}(Q_T)}; \tag{3.39}$$

in addition, since

$$\nabla^2\Phi_t(x,t) = f_t(x,t) = \nabla \cdot \boldsymbol{F}_t(x,t), \quad x \in \mathcal{F}, \quad \Phi_t(x,t) = 0, \quad x \in \mathcal{G},$$

we have

$$\|\nabla\Phi_t\|_{W_2^{0,l/2}(Q_T)} \leq c\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(Q_T)}, \tag{3.40}$$

hence

$$\|\boldsymbol{u}_1\|_{W_2^{l+2,l/2+1}(Q_T)} \leq c\Big(\|f\|_{W_2^{l+1}(Q_T)} + \|\boldsymbol{F}_t\|_{W_2^{0,l/2}(Q_T)}\Big). \tag{3.41}$$

The functions $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}_1$, p, ρ satisfy the relations

$$\begin{cases} \boldsymbol{w}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu\nabla^2\boldsymbol{w} + \nabla p = \boldsymbol{f}_1(x,t), \\ \nabla \cdot \boldsymbol{w}(x,t) = 0, \qquad x \in \mathcal{F}, \quad t > 0, \\ T(\boldsymbol{w},p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = \boldsymbol{d}_1(x,t), \\ \rho_t = \boldsymbol{w}(x,t) \cdot \boldsymbol{N}(x) + g_1(x,t), \quad x \in \mathcal{G}, \\ \boldsymbol{w}(x,0) = \boldsymbol{v}_0 - \boldsymbol{u}_1(x,0) \equiv \boldsymbol{w}_0(x) \quad x \in \mathcal{F}, \quad \rho(x,0) = \rho_0(x), \quad x \in \mathcal{G}, \end{cases} \tag{3.42}$$

where

$$\begin{cases} \boldsymbol{f}_1 = \boldsymbol{f} - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}_1) - \boldsymbol{u}_{1t} + \nu\nabla^2\boldsymbol{u}_1, \\ \boldsymbol{d}_1 = \boldsymbol{d} - \nu S(\boldsymbol{u}_1)\boldsymbol{N}, \quad g_1 = g + \boldsymbol{u}_1 \cdot \boldsymbol{N}. \end{cases} \tag{3.43}$$

In particular,

$$\boldsymbol{d}_1 \cdot \boldsymbol{N} = \boldsymbol{d} \cdot \boldsymbol{N} - \nu\boldsymbol{N} \cdot S(\boldsymbol{u}_1)\boldsymbol{N}|_{x\in\mathcal{G}}.$$

Now we reduce (3.43) to a similar problem with zero initial data in the same way as it has been done in the proof of Theorem 2.1. To be definite, we consider the case $l \in (1, 3/2)$ (this assumption is made in the analysis of the nonlinear problem). Let

$$\boldsymbol{w}_1(x) = \nu\nabla^2\boldsymbol{w}_0 - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}_0) - \nabla p_0(x) + \boldsymbol{f}_1(x,0),$$

where p_0 is a solution of the problem

$$\begin{cases} \nabla^2 p_0(x) = \nabla \cdot \Big(\boldsymbol{f}_1(x,0) - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}_0)\Big), \quad x \in \mathcal{F}, \\ p_0(x) = \nu\boldsymbol{N} \cdot S(\boldsymbol{w}_0)\boldsymbol{N} + B_0(\rho_0) - \boldsymbol{d}_1 \cdot \boldsymbol{N}, \quad x \in \mathcal{G}. \end{cases}$$

By Proposition 8.20,

$$\|p_0\|_{W_2^l(\mathcal{F})} \le c\Big(\|\boldsymbol{f}_1(\cdot,0)\|_{W_2^{l-1}(\mathcal{F})} + \|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \|\boldsymbol{d}_1(\cdot,0) \cdot \boldsymbol{N}\|_{W_2^{l-1/2}(\mathcal{G})}\Big).$$

It follows that

$$\|\boldsymbol{w}_1\|_{W_2^{l-1}(\mathcal{F})} \le c\Big(\|\boldsymbol{f}_1\|_{W_2^{l-1}(\mathcal{F})} + \|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} + \|\boldsymbol{d}_1 \cdot \boldsymbol{N}\|_{W_2^{l-1/2}(\mathcal{G})}\Big). \tag{3.44}$$

We introduce the solenoidal vector field $\boldsymbol{u}_2(x,t)$ such that

$$\boldsymbol{u}_2(x,0) = \boldsymbol{w}_0(x), \quad \boldsymbol{u}_{2t}(x,0) = \boldsymbol{w}_1(x)$$

and

$$\|\boldsymbol{u}_2\|_{W_2^{l+2,l/2+1}(Q_T)} \le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{w}_1\|_{W_2^{l+1}(\mathcal{F})}\Big). \tag{3.45}$$

Moreover, we construct $p_1(x,t)$ and $\rho_1(x,t)$ such that $p_1(x,0) = p_0(x)$,

$$\rho_1(x,0) = \rho_0(x), \quad \rho_{1t}(x,0) = \boldsymbol{w}_0(x) \cdot \boldsymbol{N} + g_1(x,0) \equiv \rho_1'(x) \tag{3.46}$$

and

$$\|p_1\|_{W_2^{l+1,l/2+1/2}(Q_T)} \le c\|p_0\|_{W_2^l(\mathcal{F})},$$

$$\|\rho 0_1\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_{1t}\|_{W_2^{l+3/2,l/2+1/4}(G_T)} + |\rho|_{l/2,5/2,G_T} \leq c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_1'\|_{W_2^{l+1/2}(\mathcal{G})}\Big). \tag{3.47}$$

The construction of these functions is described in the proof of Theorem 2.1 and in Proposition 8.18.

For the differences $\boldsymbol{u} = \boldsymbol{w} - \boldsymbol{u}_2$, $\pi = p - q_1$, $r = \rho - \rho_1$ we obtain the problem with zero initial data

$$\begin{cases} \boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla\pi = \boldsymbol{f}_2(x,t), \\ \nabla \cdot \boldsymbol{u}(x,t) = 0, \qquad x \in \mathcal{F}, \quad t > 0, \\ T(\boldsymbol{u},\pi)\boldsymbol{N} + \boldsymbol{N}B_0 r = \boldsymbol{d}_2(x,t), \\ r_t = \boldsymbol{u}(x,t)\cdot\boldsymbol{N}(x) + g_2(x,t), \quad x \in \mathcal{G}, \\ \boldsymbol{w}(x,0) = 0, \quad x \in \mathcal{F}, \quad r(x,0) = 0, \quad x \in \mathcal{G}, \end{cases} \tag{3.48}$$

where

$$\begin{aligned} &\boldsymbol{f}_2 = \boldsymbol{f}_1 - \Big(\boldsymbol{w}_{2t} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}_2) - \nu\nabla^2\boldsymbol{w}_2 + \nabla p_1\Big), \\ &\boldsymbol{d}_2 = \boldsymbol{d}_1 - (T(\boldsymbol{w}_2, p_1)\boldsymbol{N} + B_0(\rho_1)\boldsymbol{N}), \\ &g_2 = g_1 + \boldsymbol{w}_1\cdot\boldsymbol{N} - \rho_{1t}. \end{aligned}$$

Since $\boldsymbol{f}_2$, $\boldsymbol{d}_2$, g_2 vanish for $t = 0$ and $l \leq 3/2$, we can extend these functions by zero to the domain $t < 0$ and apply the Laplace transform, as it has been done in Sec 2. The problem (3.47) is then converted in (3.5). From (3.6) and the Parceval equality we obtain the estimate (3.4) for the problem (3.48),

$$\begin{aligned} Y_T(\boldsymbol{u},\pi,r) \leq c_1\Big(&\|\boldsymbol{f}_2\|_{W_2^{l,l/2}(Q_T)} + \|\boldsymbol{d}_2 - \boldsymbol{N}(\boldsymbol{d}_2\cdot\boldsymbol{N})\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\boldsymbol{d}_2\cdot\boldsymbol{N}\|_{W_2^{l+1/2,0}(G_T)} \\ &+|\boldsymbol{d}_2\cdot\boldsymbol{N}|_{l/2,1/2,G_T} + \|g_2\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\Big) + c_2\Big(\|\boldsymbol{u}\|_{L_2(Q_T)} + \|r\|_{L_2(G_T)}\Big), \end{aligned}$$

with the constants independent of T. From this estimate of and from (3.41)-(3.47) inequality (3.4) follows. This completes the proof of Theorem 3.1.

The case $l \in (3/2, 5/2)$ is considered in a similar way. The modification concerns the function $\rho_1(x,t)$: in addition to (3.46), it should satisfy the condition

$$\rho_{1tt}(x,0) = \boldsymbol{w}_t(x,0)\cdot\boldsymbol{N} + g_t(x,0) \equiv \rho_1'' \in W_2^{l-3/2}(\mathcal{G})$$

and the inequality

$$\begin{aligned} &\|\rho_1\|_{W_2^{l+5/2,0}(G_T)} + \|\rho_{1t}\|_{W_2^{l+3/2,l/2+1/4}(G_T)} + |\rho|_{l/2,5/2,G_T} \\ &\leq c\Big(\|\rho_0\|_{W_2^{l+2}(\mathcal{G})} + \|\rho_1'\|_{W_2^{l+1/2}(\mathcal{G})} + \|\rho_1''\|_{W_2^{l-3/2}(\mathcal{G})}\Big). \end{aligned}$$

The construction of this function is carried out in the same way as in Proposition 8.18.

Now we turn our attention to the homogeneous problem (1.24) supplemented with the orthogonality conditions (1.26), (1.27), and to the corresponding spectral problem

$$\begin{cases} s\boldsymbol{v} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v}(x) = 0, \qquad x \in \mathcal{F}, \\ T(\boldsymbol{v},p)\boldsymbol{N} + \boldsymbol{N}B_0\rho = 0, \\ s\rho = \boldsymbol{v}(x)\cdot\boldsymbol{N}(x), \quad x \in \mathcal{G}, \end{cases} \tag{3.49}$$

$$
\begin{aligned}
&\int_{\mathcal{G}} \rho(y)dS = 0, \quad \int_{\mathcal{G}} y_i\rho(y)dS = 0, \\
&\int_{\mathcal{F}} \boldsymbol{v}(x)dx = 0, \\
&\int_{\mathcal{F}} \boldsymbol{v}(x)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\mathcal{G}} \rho(x)\boldsymbol{\eta}_i(x)\cdot\boldsymbol{\eta}_3(x)dS = 0, \quad i = 1,2,3.
\end{aligned}
\tag{3.50}
$$

We write (3.49) in an equivalent way. By the classical result of Weyl, arbitrary $\boldsymbol{u} \in L_2(\mathcal{F})$ is representable in the form

$$
\boldsymbol{u}(x) = \boldsymbol{w}(x) + \nabla\varphi(x)
$$

where $\boldsymbol{w}$ is divergence free and $\varphi(x)$ vanishes on $\mathcal{G}$. It follows that

$$
\nabla q(x) = \nabla\psi(x) + \nabla\varphi(x)
$$

where ψ is the harmonic function and $\psi - q|_{\mathcal{G}} = 0$. We denote by $J(\mathcal{G})$ the space of divergence free vector fields in $\mathcal{F}$ and by P_J the orthogonal projection to this space. It is easily verified that (3.49) is equivalent to

$$
\begin{cases}
s\boldsymbol{v} + 2\omega P_J(\boldsymbol{e}_3\times\boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla\psi = 0, \\
\nabla\cdot\boldsymbol{v}(x) = 0, \quad x\in\mathcal{F}, \\
s\rho = \boldsymbol{v}(x)\cdot\boldsymbol{N}(x), \quad x\in\mathcal{G}, \\
\nabla^2\psi(x) = 0, \quad x\in\mathcal{F}, \quad \psi(x) = \nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} + B_0\rho, \quad x\in\mathcal{G}.
\end{cases}
\tag{3.51}
$$

The pressure is excluded, and our problem can be written as

$$
\mathcal{A}U = sU, \tag{3.52}
$$

where $U = (\boldsymbol{v},\rho)^T$, and $\mathcal{A}$ is a 2×2 matrix integro-differential operator:

$$
\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix},
$$

(cf. [12]) with A_{ij} defined by

$$
\begin{gathered}
A_{11}\boldsymbol{v} = \nu\nabla^2\boldsymbol{v} - \nabla\psi_1 - 2\omega\widetilde{P}(\boldsymbol{e}_3\times\boldsymbol{v}), \quad A_{12}\rho = -\nabla\psi_2, \\
A_{21}\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{N}, \\
\nabla^2\psi_1 = 0, \quad \nabla^2\psi_2 = 0, \quad x\in\mathcal{F}, \\
\psi_1 = 2\nu\boldsymbol{N}\cdot S(\boldsymbol{u})\boldsymbol{N}, \quad \psi_2 = \widehat{B}\rho, \quad x\in\mathcal{G}.
\end{gathered}
$$

As the domain of $\mathcal{A}$, $D(\mathcal{A})$, we take the lineal $U = (\boldsymbol{u},\rho)^T$ with $\boldsymbol{u}\in W_2^2(\mathcal{F})$ and $\rho\in W_2^{3/2}(\mathcal{G})$, satisfying (3.50) and the boundary condition

$$
\Pi_{\mathcal{G}}S(\boldsymbol{v})\boldsymbol{N} = S(\boldsymbol{v})\boldsymbol{N} - \boldsymbol{N}(\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N})|_{\mathcal{G}} = 0.
$$

In this linear set we introduce the norm

$$
\|U\|_D = \left(\|\boldsymbol{v}\|^2_{W_2^2(\mathcal{F})} + \|\rho\|_{W_2^{3/2}(\mathcal{G})}\right)^{1/2}.
$$

The operator $\mathcal{A}$ possesses the following properties:

1. It acts in the space defined by the conditions (3.50): if these conditions hold for the components $(\boldsymbol{v},\rho)$ of $U\in D(\mathcal{A})$, then they are satisfied for the components $(\boldsymbol{f},g)$ of $F=\mathcal{A}U$. This is verified by a direct computation (cf. [14], Proposition 3.1).

2. If $\mathrm{Re}s\geq a\gg 1$, then the equation $\mathcal{A}U-sU=F$ is uniquely solvable, and

$$\|U\|_D\leq c\|F\|_R\equiv\Big(\|f\|^2_{L_2(\mathcal{F})}+\|g\|_{W_2^{3/2}(\mathcal{G})}\Big)^{1/2} \tag{3.53}$$

with c independent of s, i.e., there exists a bounded $(\mathcal{A}-sI)^{-1}$. Since $D(\mathcal{A})$ is compactly imbedded in $L_2(\mathcal{F})\times W_2^{3/2}(\mathcal{G})$, $(\mathcal{A}-sI)^{-1}$ is completely continuous. The equation (3.50) is equivalent to $(s-a)U=(\mathcal{A}-aI)^{-1}U$, hence the spectrum of $\mathcal{A}$ consists of a countable number of eigenvalues with the only accumulation point at infinity. It follows that only a finite number of eigenvalues with positive real part may exist.

3. **Proposition 3.2.** *If the quadratic form* (1.4) *is positive definite for arbitrary ρ satisfying* (1.26)*, then all the eigenvalues of $\mathcal{A}$ have negative real part.*

Proof. We note that the orthogonality conditions (3.50) for $\boldsymbol{v}$ imply

$$\boldsymbol{v}=\boldsymbol{v}^{\perp}+\sum_{i=1}^{3}d_i(\rho)\boldsymbol{\eta}_i(x),$$

where $\boldsymbol{v}^{\perp}(x)$ is the vector field orthogonal to all $\boldsymbol{\eta}(x)=\boldsymbol{a}+(\boldsymbol{b}\times x)$, $\boldsymbol{a},\boldsymbol{b}=const$ and

$$d_i(\rho)=-\frac{\omega}{\|\boldsymbol{\eta}_i\|^2_{L_2(\mathcal{F})}}\int_{\mathcal{G}}\rho(x)\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx.$$

We introduce the vector field $\boldsymbol{u}=\boldsymbol{v}-d_3(\rho\boldsymbol{\eta}_3(x)$. Since $2(\boldsymbol{e}_3\times\boldsymbol{\eta}_3)=-\nabla|x'|^2$ and $\boldsymbol{\eta}_3\cdot\boldsymbol{N}|_{\mathcal{G}}=0$, $\boldsymbol{u}$ and ρ satisfy the equations

$$\begin{cases} s\boldsymbol{u}+2\omega P_J(\boldsymbol{e}_3\times\boldsymbol{u})-\nu\nabla^2\boldsymbol{u}+\nabla\phi=-sd_3(\rho)\boldsymbol{\eta}_3(x),\\ \nabla\cdot\boldsymbol{u}(x)=0,\quad x\in\mathcal{F},\\ s\rho=\boldsymbol{u}(x)\cdot\boldsymbol{N}(x),\quad x\in\mathcal{G},\\ \nabla^2\phi(x)=0,\quad x\in\mathcal{F},\quad \phi(x)=\nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N}+B\rho,\quad x\in\mathcal{G}\end{cases} \tag{3.54}$$

with

$$B\rho=B_0\rho+\frac{\omega^2|x'|^2}{\|\boldsymbol{\eta}_3\|_{L_2(\mathcal{F})}}\int_{\mathcal{G}}\rho(y)|y'|^2dS$$

and

$$\begin{gathered}\int_{\mathcal{F}}\boldsymbol{u}(x)dx=0,\quad \int_{\mathcal{F}}\boldsymbol{u}\cdot\boldsymbol{\eta}_3dx=0,\\ \int_{\mathcal{F}}\boldsymbol{v}(x)\cdot\boldsymbol{\eta}_\alpha(x)dx+\omega\int_{\mathcal{G}}\rho(x)\boldsymbol{\eta}_\alpha(x)\cdot\boldsymbol{\eta}_3(x)dS=0,\quad \alpha=1,2.\end{gathered} \tag{3.55}$$

It is important to notice that the integral $\int_{\mathcal{G}}\rho B\rho dS$ coincides with the quadratic form (1.4): $\int_{\mathcal{G}}\rho B\rho dS=\delta^2\mathcal{R}(\rho)$.

We multiply the first equation in (3.54) by $\boldsymbol{u}$ and integrate over $\mathcal{F}$. Upon integrating by parts we obtain

$$s\|\boldsymbol{u}\|^2_{L_2(\mathcal{F})}+2\omega\int_{\mathcal{F}}(u_1\bar{u}_2-u_2\bar{u}_1)dx+\bar{s}^{-1}\int_{\mathcal{G}}B\rho\bar{\rho}dS+\frac{\nu}{2}\|S(\boldsymbol{u})\|^2_{L_2(\mathcal{F})}=0. \tag{3.56}$$

In view of the positivity of the form (1.4) we conclude from (3.56) that $\boldsymbol{u}=0$ and $\rho=0$, if $\mathrm{Re}s>0$.

If $\mathrm{Re}s=0$ and $s\neq 0$, then, by the Korn inequality, the same calculation yields $\boldsymbol{v}^{\perp}=0$, $\boldsymbol{u}=\sum_{\alpha=1}^{2} d_\alpha(\rho)\boldsymbol{\eta}_\alpha(x)$. It follows that

$$s\sum_{i=1}^{3} d_i(\rho)\boldsymbol{\eta}_i(x)+2\omega(\boldsymbol{e}_3\times\sum_{\alpha=1}^{2} d_\alpha\boldsymbol{\eta}_\alpha)+\nabla(\phi-\omega d_3|x'|^2)=0,\quad x\in\mathcal{F}, \tag{3.57}$$

$$s\rho(x)=\sum_{\alpha=1}^{2} d_\alpha\boldsymbol{\eta}_\alpha(x)\cdot\boldsymbol{N}(x),\quad x\in\mathcal{G}.$$

When we compute $d_i(\rho)$, using the last equation, we obtain $d_3(\rho)=0$ and also

$$sd_1=-\omega\frac{\widetilde{S}}{S}d_2,\quad sd_2=\omega\frac{\widetilde{S}}{S}d_2, \tag{3.58}$$

where

$$S=\|\boldsymbol{\eta}_\alpha\|^2_{L_2(\mathcal{F})},\quad \widetilde{S}=\int_{\mathcal{F}}(x_\alpha^2-x_3^2)dx,\quad \alpha=1,2.$$

Moreover, applying the operator *rot* to (3.57), we obtain

$$sd_1(\rho)=\omega d_2(\rho),\quad sd_2(\rho)=-\omega d_1(\rho). \tag{3.59}$$

From (3.58), (3.59) it follows that $d_1=d_2=0$, hence $\boldsymbol{u}=0$, $\rho=0$.

It remains to consider the case $s=0$. As above, we deduce from (3.54) that $\boldsymbol{u}=\sum_{\alpha=1}^{2} d_\alpha(\rho)\boldsymbol{\eta}_\alpha(x)$ and, as a consequence, $\sum_{\alpha=1}^{2} d_\alpha(\rho)\boldsymbol{\eta}_\alpha\cdot\boldsymbol{N}|_G=0$. This implies $d_1=d_2=0$, $\boldsymbol{u}=0$, $q=q_0=const$ and $-q_0+B\rho(x)=0$ on the boundary. Hence $\int_{\mathcal{G}}B\rho\bar{\rho}dS=0$ and $\rho=0$. This completes the proof of Proposition 3.2.

Finally we consider the evolution problem (1.24) with the initial data satisfying (1.25), (1.26).

Proposition 3.3. *If the condition* (1.22) *is satisfied, then the problem* (1.24)-(1.26) *with the initial data* $\boldsymbol{v}_0\in W_2^{l+1}(\mathcal{F})$, $\rho_0\in W_2^{l+2}(\mathcal{G})$, $l\in[0,5/2)$, *satisfying the compatibility conditions* (3.2), (3.3) *(with* $f=0$, $\boldsymbol{d}=0$*) has a unique solution* $\boldsymbol{v}$, p, ρ, *as in Theorem 3.1, and this solution satisfies the inequalities*

$$Y_T(\boldsymbol{v},p,\rho)\le c\Big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F})}+\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big), \tag{3.60}$$

$$\|\boldsymbol{v}(\cdot,t)\|_{W_2^{l+1}(\mathcal{F})}+\|\rho(\cdot,t)\|_{W_2^{l+2}(\mathcal{G})}\le ce^{-\beta t}\Big(\|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F})}+\|\rho_0\|_{W_2^{l+2}(\mathcal{G})}\Big) \tag{3.61}$$

with the constant $c,\beta>0$ *independent of* T.

Proof. The problem (1.24)-(1.26) is equivalent to

$$U_t=\mathcal{A}U,\quad U|_{t=0}=U_0=(\boldsymbol{v}_0,\rho_0)^T \tag{3.62}$$

and $U(t)=e^{-\mathcal{A}t}U_0$. Since the spectrum of $\mathcal{A}$ is located in the left complex half-plane, we have

$$\|U(t)\|_R\le ce^{-\beta_1 t}\|U_0\|_R \tag{3.63}$$

and

$$\int_0^T e^{2\beta t}\|U(t)\|_R^2 dt \le c\|U_0\|_R^2 \tag{3.64}$$

with some $\beta_1 > 0$, $\beta \in (0, \beta_1)$. The solution of the non-homogeneous problem

$$U_t = \mathcal{A}U + F, \quad U|_{t=0} = U_0 = (\boldsymbol{v}_0, \rho_0)^T$$

that is expressed by the formula

$$U(t) = e^{-\mathcal{A}t}U_0 + \int_0^t e^{-\mathcal{A}(t-\tau)}F(\tau)d\tau$$

satisfies the inequality

$$\int_0^T e^{2\beta t}\|U(t)\|_R^2 dt \le c\Big(\|U_0\|_R^2 + \int_0^T e^{2\beta t}\|F(t)\|_R^2 dt\Big).$$

To shorten the arguments, we restrict ourselves with the case $l = 0$. If U is a solution of (3.62), then $U_\beta(t) = e^{\beta t}U(t)$ satisfies

$$U_{\beta t} = \mathcal{A}U_\beta + \beta U_\beta, \quad U_\beta|_{t=0} = U_0.$$

It is not difficult to see that for this problem the estimate of the type (2.30) holds (although the initial conditions are not homogeneous). In the case $l = 0$ it is equivalent to

$$\int_0^T \|U_{\beta t}(t)\|_R^2 dt + \int_0^T \|U_\beta(t)\|_D^2 dt + \sup_{t<T}\|\boldsymbol{v}_\beta(\cdot,t)\|_{W_2^1(\mathcal{F})}^2 + \sup_{t<T}\|\rho_\beta(\cdot,t)\|_{W_2^2(\mathcal{G})}^2$$
$$\le c_1\int_0^T \|U_\beta(t)\|_R^2 dt + c_2\Big(\|\boldsymbol{v}_0\|_{W_2^1(\mathcal{F})}^2 + \|\rho_0\|_{W_2^2(\mathcal{G})}^2\Big)$$

with the constants c_1, c_2 independent of T. Taking (3.64) into account we obtain

$$\int_0^T \|U_{\beta t}(t)\|_R^2 dt + \int_0^T \|U_\beta(t)\|_D^2 dt + \sup_{t<T}\|\boldsymbol{v}_\beta(\cdot,t)\|_{W_2^1(\mathcal{F})}^2 + \sup_{t<T}\|\rho_\beta(\cdot,t)\|_{W_2^2(\mathcal{G})}^2$$
$$\le c_3\Big(\|\boldsymbol{v}_0\|_{W_2^1(\mathcal{F})}^2 + \|\rho_0\|_{W_2^2(\mathcal{G})}^2\Big).$$

This implies (3.60) and (3.61). The norms of p can be estimated as follows. The problem (3.62) is equivalent to (1.24)-(1.26) with p satisfying

$$\nabla^2 p(x,t) = -2\omega\nabla(\boldsymbol{e}_3 \times \boldsymbol{v}), \quad x \in \mathcal{F}, \quad p(x,t)|_{x\in\mathcal{G}} = \nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} + B_0\rho.$$

We have

$$\|p(\cdot,t)\|_{W_2^1(\mathcal{F})} \le c\Big(\|\boldsymbol{v}\|_{W_2^1(\mathcal{F})} + \|\nabla\boldsymbol{v}\|_{W_2^{1/2}(\mathcal{G})} + \|\rho(\cdot,t)\|_{W_2^{5/2}(\mathcal{G})}\Big),$$

which yields all the missing estimates for p.

The case $l > 0$ is analyzed by the same kind of arguments; they are only slightly more complicated.

4 On the linear problem related to the stability of uniformly rotating liquid ($\sigma = 0$).

In this section we consider the problem (3.1) with

$$B_0\rho = b(x)\rho - \kappa\int_{\mathcal{G}}\frac{\rho(y,t)dS_y}{|x-y|}, \quad x \in \mathcal{G}, \tag{4.1}$$

$$b(x) \geq b_0 = 0.$$

The main result of the section is as follows:

Theorem 4.1 *Let $l \in [0,5/2)$, $l \neq 1/2$, $l \neq 1$. For arbitrary $\boldsymbol{f} \in W_2^{l,l/2}(Q_T)$, $f \in W_2^{1+l,0}(Q_T)$, such that $f = \nabla\cdot\boldsymbol{F}$, $\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T)$, $\boldsymbol{v}_0 \in W_2^{1+l}(\Omega)$, $\boldsymbol{d} \in W_2^{l+1/2,l/2+1/4}(G_T)$, $g \in W_2^{l+3/2,l/2+3/4}(G_T)$ satisfying the compatibility conditions (3.2), (3.3), the problem (3.1) has a unique solution $\boldsymbol{v},p,\rho$, such that $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_T)$, $\nabla p \in W_2^{l,l/2}(Q_T)$, $p|_{G_T} \in W_2^{l+1/2,l/2+1/4}(G_T)$, $\rho \in W_2^{l+1/2,0}(G_T)$, $\rho_t \in W_2^{l+3/2,l/2+3/4}(G_T)$, $\rho(\cdot,t) \in W_2^{l+1}(\mathcal{G})$, $\forall t \in (0,T)$, and the solution satisfies the inequality*

$$\begin{aligned}&\|\boldsymbol{v}\|^2_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|^2_{W_2^{l,l/2}(Q_T)} + \|p\|^2_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\rho\|^2_{W_2^{l+1/2,0}(G_T)}\\ &+ \|\rho_t\|^2_{W_2^{l+3/2,l/2+3/4}(G_T)} + \sup_{t<T}\|\rho(\cdot,t)\|^2_{W_2^{l+1}(S)} \leq c(N_1^2(T) + \|\boldsymbol{v}\|^2_{L_2(Q_T)} + \|\rho\|^2_{L_2(G_T)}),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}N_1^2(T) = &\|\boldsymbol{f}\|^2_{W_2^{l,l/2}(Q_T)} + \|f\|^2_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|^2_{W_2^{0,1+l/2}(Q_T)} + \|\boldsymbol{v}_0\|^2_{W_2^{1+l}(\mathcal{G})}\\ &+ \|\boldsymbol{d}\|^2_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|g\|^2_{W_2^{l+3/2,l/2+3/4}(G_T)}\end{aligned} \tag{4.3}$$

c is the constant independent of T.

Estimate (4.2) holds also without additional L_2-norms of the solution in the right hand side, but then $c = c(T)$.

We outline the proof of Theorem 4.1, following the paper [10]. The solvability of the problem (3.1) in any finite time interval can be easily deduced from Theorem 2.1. Indeed, since

$$\rho(x,t) = \rho_0(x) + \int_0^t \Big(\boldsymbol{v}(x,\tau)\cdot\boldsymbol{N}(x) + g(x,\tau)\Big)d\tau,$$

this problem can be written in the form (2.27) with $\boldsymbol{f}$ and $\boldsymbol{d}$ replaced by $\boldsymbol{f} - 2\omega(\boldsymbol{e}_3\times\boldsymbol{v})$ and $\boldsymbol{d} - \boldsymbol{n}\Big(B_0(\rho_0 + \int_0^t g d\tau) + \ell(\boldsymbol{v})\Big)$, respectively, where

$$\ell(\boldsymbol{v}) = B_0\int_0^t \boldsymbol{v}(x,\tau)\cdot\boldsymbol{N}(x)d\tau.$$

The lower order terms $2\omega(\boldsymbol{e}_3\times\boldsymbol{v})$ and $\ell(\boldsymbol{v})$ can be estimated by interpolation inequalities, which permits to solve the problem by successive approximations and prove the estimate

$$\|\boldsymbol{v}\|^2_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|^2_{W_2^{l,l/2}(Q_T)} + \|p\|^2_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|\rho\|^2_{W_2^{l+1/2,0}(G_T)}$$

$$+\|\rho_t\|^2_{W_2^{l+3/2,l/2+3/4}(G_T)}+\sup_{t<T}\|\rho(\cdot,t)\|^2_{W_2^{l+1}(S)}\le c(T)N_1^2,$$

using the Gronwall inequality (details are omitted).

We concentrate on obtaining the estimate (4.3). We consider the model problem

$$\begin{cases} \boldsymbol{v}_t-\nu\nabla^2\boldsymbol{v}+\nabla p=\boldsymbol{f}(x,t), \quad \nabla\cdot\boldsymbol{v}=f(x,t) \qquad x\in\mathbb{R}^3_+,\\ T_{i3}(\boldsymbol{v},p)+\delta_{i3}\beta\rho=-d_i(x,t), \quad \beta=const>0, \quad i=1,2,3,\\ \rho_t+v_3=g(x,t), \quad \rho(x,0)=\rho_0(x), \quad x\in\mathbb{R}^2=\{x_3=0\},\\ \boldsymbol{v}(x,0)=\boldsymbol{v}_0(x), \qquad x\in\mathbb{R}^3_+ \end{cases} \tag{4.4}$$

Proposition 4.1 *Let $\mathbb{R}_T=\mathbb{R}^3_+\times(0,T)$, $\mathbb{R}'_T=\mathbb{R}^2\times(0,T)$ and let $\boldsymbol{v}\in W_2^{2+l,1+l/2}(\mathbb{R}_T)$, $\nabla p\in W_2^{l,l/2}(\mathbb{R}_T)$, $\rho\in W_2^{l+1/2,l/2+1/4}(\mathbb{R}'_T)$ be a solution of the model problem (4.4) having for all $t\le T$ a compact support contained in $C_\lambda=B_\lambda\times(0,\lambda)$, where B_λ is a disc $|x'|\le\lambda$ in $\mathbb{R}^2$ and $\lambda\in(0,1)$. The solution satisfies the inequality*

$$\begin{aligned} &\sup_{t<T}\Big(\langle\langle\boldsymbol{v}(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}+\|\rho(\cdot,t)\|^2_{W_2^{l+1}(\mathbb{R}^2)}\Big)\\ &\quad+\int_0^T\Big(\langle\langle\nabla\boldsymbol{v}(\cdot,\tau)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}+\|\rho(\cdot,\tau)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}\Big)d\tau\\ &\le c\Big(\langle\langle\boldsymbol{v}_0\rangle\rangle^2_{l+1,\mathbb{R}^3_+}+\|\rho_0\|^2_{W_2^{l+1}(\mathbb{R}^2)}\Big)+c\int_0^T\Big(\langle\langle\boldsymbol{f}(\cdot,t)\rangle\rangle^2_{l,\mathbb{R}^3_+}+\langle\langle f(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}\\ &\quad+\|\boldsymbol{d}(\cdot,t)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}+\|g(\cdot,t)\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}\Big)dt\\ &\quad+c\Big(\int_0^T\langle\langle f(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}dt\Big)^{1/2}\Big(\int_0^T\langle\langle p(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}dt\Big)^{1/2} \end{aligned} \tag{4.5}$$

where

$$\langle\langle u\rangle\rangle_{l,R^3_+}=\Big(\int_0^\infty\|u(\cdot,x_3\|^2_{W_2^l(\mathbb{R}^2)}dx_3\Big)^{1/2}$$

and c is a constant independent of T.

Proof. At first we present some auxiliary propositions.

1. We recall that the norm

$$\|u\|_{H^l(\mathbb{R}^n)}=\Big(\int_{\mathbb{R}^2}(1+|\xi|^2)^l|\widetilde{u}(\xi)|^2d\xi\Big)^{1/2} \tag{4.6}$$

where

$$\widetilde{u}(\xi)=\int_{\mathbb{R}^2}u(x')e^{-i\xi\cdot x'}dx', \quad x'=(x_1,x_2)\in\mathbb{R}^2,$$

is the Fourier transform of u, is equivalent to $\|u\|_{W_2^l(\mathbb{R}^2)}$ (see Proposition 8.3). Another equivalent norm is

$$\Big(\int_{|z'|<a}\|\Delta^m(z')u\|^2_{L_2(\mathbb{R}^2)}\frac{dz'}{|z'|^{2+2l}}+\|u\|^2_{L_2(\mathbb{R}^2)}\Big)^{1/2}, \tag{4.7}$$

where $\Delta^m(\xi)u=\sum_{i=0}^m(-1)^{m-i}C^i_m u(x'+iz')$ is a finite difference of order m of the function $u(x')$, $m>l$, $a>0$. This is proved by expressing the norm (4.7) in terms of $\widetilde{u}$.

Both statements extend to 2π- periodic functions. In the periodic case the norm equivalent to $\|u\|_{W_2^l(\Sigma)}$, $\Sigma = (-\pi, \pi)$, is

$$\|u\|_{H^l(\Sigma)} = \Big(\sum_{\xi \in \mathbb{Z}^2} |\xi|^{2l} |\widehat{u}(\xi)|^2 + |\widehat{u}(0)|^2 \Big)^{1/2}$$

where

$$\widehat{u}(\xi) = \int_\Sigma u(x') e^{-i\xi \cdot x'} dx',$$

are Fourier coefficients of u with respect to the system of functions $e^{i\xi\cdot x'}$, $\xi \in \mathbb{Z}^2$. They are eigenfunctions of the Laplace operator $-\Delta' = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ in Σ with periodic boundary conditions. The corresponding eigenvalues are equal to $|\xi|^2$; hence,

$$\|u\|_{H^l(\Sigma)} = \Big((2\pi)^{-2} \|(-\Delta')^{l/2} u\|^2_{L_2(\Sigma)} + \Big| \int_\Sigma u dx' \Big|^2 \Big)^{1/2}.$$

2. We introduce the norms $W_2^{-l}(\Sigma)$, $l > 0$, of periodic functions by a standard formula

$$\|u\|_{W_2^{-l}(\Sigma)} = \sup_{\|\varphi\|_{W_2^l(\Sigma)}=1} \Big| \int_\Sigma u(x')\varphi(x') dx' \Big|$$

and

$$\begin{aligned} \|u\|_{H^{-l}(\Sigma)} &= \Big(\sum_{\xi \in \mathbb{Z}^2 \setminus 0} |\xi|^{-2l} |\widehat{u}(\xi)|^2 + \Big| \int_\Sigma u(x) dx \Big|^2 \Big)^{1/2} \\ &= \Big((2\pi)^{-2} \|(-\Delta')^{-l/2} u'\|^2_{L_2(\Sigma)} + \Big| \int_\Sigma u dx' \Big|^2 \Big)^{1/2}, \end{aligned}$$

where $u' = u - (2\pi)^{-2} \int_\Sigma u(x')dx'$. They are equivalent to each other. In what follows we deal only with the space $W_2^{-1/2}(\Sigma)$.

If $u \in W_2^{-1/2}(\Sigma)$, then $(2\pi)^{-2}(-\Delta')^{-1/2}u' + \int_\Sigma u(x')dx' \in W_2^{1/2}(\Sigma)$, and

$$\|(-\Delta')^{-1/2} u'\|_{W_2^{l/2}(\Sigma)} + \Big| \int_\Sigma u(x') dx' \Big| \le c \|u\|_{W_2^{-1/2}(\Sigma)}. \tag{4.8}$$

3. We recall the trace theorem (Proposition 8.6): every function $v \in W_2^1(\Omega)$ given in $\Omega = \Sigma \times (0, 2\pi) \subset \mathbb{R}^3_+$ has a trace $v(\cdot, 0) \in W_2^{1/2}(\Sigma)$ and

$$\|v(\cdot, 0)\|_{W_2^{1/2}(\Sigma)} \le c \|v\|_{W_2^1(\Omega)}.$$

For arbitrary $u \in W_2^{1/2}(\Sigma)$ there exists its extension into Ω, $v \in W_2^1(\Omega)$, such that

$$\|v\|_{W_2^1(\Omega)} \le c \|u\|_{W_2^{1/2}(\Sigma)}.$$

For periodic functions, the extension can be defined by

$$v(x', x_3) = \sum_{\xi \in \mathbb{Z}^2} \widehat{u}(\xi) e^{i\xi\cdot x' - |\xi| x_3} \zeta(x_3)$$

where ζ is a smooth cut-off function equal to 1 for small $x_3 > 0$ and for zero for $x_3 > 1$.

4. Let $\varphi(x')$ be a periodic function from $W_2^{1/2}(\Sigma)$. There exists a vector field $\boldsymbol{w}$ given in $\mathbb{R}^2 \times (0, 2\pi)$, periodic with respect to x_1, x_2 such that

$$\nabla \cdot \boldsymbol{w}(x) = 0, \quad \boldsymbol{w}(x', 0) = \boldsymbol{e}_3 \varphi(x')$$

and

$$\|\boldsymbol{w}\|_{W_2^1(\Omega)} \le c\|\varphi\|_{W_2^{1/2}(\Sigma)}, \quad \|\boldsymbol{w}\|_{L_2(\Omega)} \le c\|\varphi\|_{L_2(\Sigma)}. \tag{4.9}$$

Indeed, we can take $\boldsymbol{w}$ in the form $\boldsymbol{w}(x) = \sum_{\xi\in\mathbb{Z}}^2 \widehat{\boldsymbol{w}}(\xi, x_3) e^{\xi\cdot x'}$ where

$$\widehat{w}_j(\xi, x_3) = -i\xi_j \widetilde{\varphi}(\xi) x_3 e^{-|\xi| x_3}, \quad j = 1, 2,$$

$$\widehat{w}_3(\xi, x_3) = |\xi| \widetilde{\varphi}(\xi) x_3 e^{-|\xi| x_3} + \widetilde{\varphi}(\xi) e^{-|\xi| x_3}.$$

We pass to the proof of (4.5). We write the first equation in (4.4) in the form

$$\boldsymbol{v}_t - \nabla \cdot T(\boldsymbol{v}, p) = \boldsymbol{f}(x, t) - \nu \nabla f \equiv \boldsymbol{f}_1, \tag{4.10}$$

multiply it by $\boldsymbol{v}$, integrate over $\mathbb{R}^3_+$ and make use of the boundary conditions. This leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \beta\|\rho\|^2_{L_2(\mathbb{R}^2)}\Big) + \frac{\nu}{2}\|S(\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} \\ &\quad = \int_{\mathbb{R}^3_+} \Big(\boldsymbol{f}_1(x,t)\cdot \boldsymbol{v}(x,t) + f(x,t)p(x,t)\Big)dx \\ &\quad + \int_{\mathbb{R}^2} \Big(\boldsymbol{d}(x',t)\cdot \boldsymbol{v}(x',0,t) + \beta g(x',t)\rho(x',t)\Big)dx'. \end{aligned} \tag{4.11}$$

Next, we extend ρ from Σ to $\mathbb{R}^2$ as a periodic function and introduce $\boldsymbol{w}$ as in n.4. with

$$\varphi = -(2\pi)^{-2}(-\Delta')^{-1/2}\rho' - \int_\Sigma \rho(x')dx', \quad \rho' = \rho - (2\pi)^{-2}\int_\Sigma \rho(x')dx'.$$

By (4.9),

$$\|\boldsymbol{w}\|_{W_2^1(\Omega)} \le c\|\rho\|_{W_2^{-1/2}(\Sigma)}, \quad \|\boldsymbol{w}_t\|_{L_2(\Omega)} \le c\Big(\|v_3\|_{L_2(\Sigma)} + \|g\|_{L_2(\Sigma)}\Big), \tag{4.12}$$

moreover, applying (4.12) to $\Delta^m(z)\boldsymbol{w}$, we obtain

$$\begin{aligned} &\langle\langle \nabla \boldsymbol{w}\rangle\rangle_{l+1,\Omega} \le c\|\varphi\|_{W_2^{l+3/2}(\Sigma)} \le c\|\rho\|_{W_2^{l+1/2}(\Sigma)}, \\ &\|\boldsymbol{w}_t(\cdot,t)\|_{W_2^l(\Omega)} \le c\Big(\|v_3(\cdot,t)\|_{W_2^l(\Sigma)} + \|g\|_{W_2^l(\Sigma)}\Big). \end{aligned} \tag{4.13}$$

We multiply (4.10) by $\boldsymbol{w}$ and integrate over Ω. Since *supp* $\boldsymbol{v}$, *supp* $p \subset C_\lambda \subset \Omega$, we obtain

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}dx - \int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}_t dx + \frac{\nu}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{w})dx \\ &+ \beta\int_\Sigma \rho\Big((2\pi)^{-2}(-\Delta')^{-1/2}\rho' + \int_\Sigma \rho dy'\Big)dx' = \int_\Omega \boldsymbol{f}_1\cdot\boldsymbol{w}dx + \int_\Sigma \boldsymbol{d}\cdot\boldsymbol{w}dx'. \end{aligned} \tag{4.14}$$

Finally, we multiply (4.14) by a small $\gamma > 0$ and add to (4.11). This gives $\frac{dE(t)}{dt} + E_1(t) = F(t)$, and, as a consequence,

$$E(t) + \int_0^t E_1(\tau)d\tau = E(0) + \int_0^t F(\tau)d\tau, \tag{4.15}$$

where

$$E(t) = \frac{1}{2}\Big(\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho\|^2_{L_2(\mathbb{R}^2)}\Big) + \gamma\int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}dx,$$

$$E_1(t) = \frac{\nu}{2}\|S(\boldsymbol{v})\|^2_{L_2(\mathbb{R}^3_+)} - \gamma\int_\Omega \boldsymbol{v}\cdot\boldsymbol{w}_t dx + \frac{\nu\gamma}{2}\int_\Omega S(\boldsymbol{v}) : S(\boldsymbol{w})dx$$
$$+\beta\gamma\|\rho(\cdot,t)\|^2_{H^{-1/2}(\Sigma)},$$

$$F(t) = \int_\Omega (\boldsymbol{f}_1(x,t)\cdot\boldsymbol{v}(x,t) + f(x,t)p(x,t))dx + \int_\Sigma \Big(\boldsymbol{d}(x',t)\cdot\boldsymbol{v}(x',0,t)$$
$$+\beta g(x',t)\rho(x',t)\Big)dx' + \gamma\int_\Omega \boldsymbol{f}_1\cdot\boldsymbol{w}dx + \gamma\int_\Sigma \boldsymbol{d}\cdot\boldsymbol{w}dx'.$$

By virtue of (4.12) and of the Korn inequality

$$\|\boldsymbol{v}\|_{W^1_2(\mathbb{R}^3_+)} \le c\|S(\boldsymbol{v})\|_{L_2(\mathbb{R}^3_+)}$$

that holds for vector fields supported in C_λ, $E(t)$ and $E_1(t)$ satisfy the estimates

$$c_1\Big(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho(\cdot,t)\|^2_{L_2(\mathbb{R}^2)}\Big) \le E(t) \le c_2\Big(\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho(\cdot,t)\|^2_{L_2(\mathbb{R}^2)}\Big),$$

$$E_1(t) \ge c_3\Big(\|\boldsymbol{v}\|^2_{W^1_2(\mathbb{R}^3_+)} + \|\rho\|^2_{W_2^{-1/2}(\Sigma)}\Big) - c_4\|g\|_{L_2(\Sigma)}\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)},$$

provided γ is small enough. Moreover,

$$|F(t)| \le \|\boldsymbol{f}_1\|_{L_2(\mathbb{R}^3_+)}(\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)} + \gamma\|\boldsymbol{w}\|_{L_2(\mathbb{R}^3_+)}) + \|f\|_{L_2(\mathbb{R}^3_+)}\|p\|_{L_2(\mathbb{R}^3_+)}$$
$$+\|\boldsymbol{d}\|_{L_2(\mathbb{R}^2)}(\|\boldsymbol{v}|_{x_3=0}\|_{L_2(\mathbb{R}^2)} + \gamma\|\varphi\|_{L_2(\mathbb{R}^2)}) + \beta\|g\|_{W_2^{1/2}(\Sigma)}\|\rho\|_{W_2^{-1/2}(\Sigma)},$$

so (4.15) implies

$$\begin{aligned}
&\|\boldsymbol{v}(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho(\cdot,t)\|^2_{L_2(\mathbb{R}^2)} + \int_0^t \Big(\|\nabla\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho\|^2_{W_2^{-1/2}(\Sigma)}\Big)d\tau\\
&\le c\Big(\|\boldsymbol{v}_0\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho_0\|^2_{L_2(\mathbb{R}^2)}\Big) + c\int_0^t \Big(\|\boldsymbol{f}_1\|_{L_2(\mathbb{R}^3_+)}(\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)} + \gamma\|\boldsymbol{w}\|_{L_2(\Omega)})\\
&+ \|f\|_{L_2(\mathbb{R}^3_+)}\|p\|_{L_2(\mathbb{R}^3_+)} + \|\boldsymbol{d}\|_{L_2(\mathbb{R}^2)}(\|\boldsymbol{v}|_{x_3=0}\|_{L_2(\mathbb{R}^2)} + \gamma\|\varphi\|_{L_2(\Sigma)})\\
&+ \beta\|g\|_{W_2^{1/2}(\Sigma)}\|\rho\|_{W_2^{-1/2}(\Sigma)} + \|g\|_{L_2(\Sigma)}\|\boldsymbol{v}\|_{L_2(\mathbb{R}^3_+)}\Big)d\tau, \quad \forall t \le T.
\end{aligned} \tag{4.16}$$

Now, we take a finite difference $\Delta^m(z)$ of (4.10) assuming that $m > 2 + l$, $z = z' \in \mathbb{R}^2$ and $|z| \le a < 1/m$, so that the supports of finite differences of all functions in (4.10) are contained in Ω. We obtain

$$\begin{cases} \boldsymbol{v}_t^{(m)} - \nu\nabla^2\boldsymbol{v}^{(m)} + \nabla p^{(m)} = \boldsymbol{f}^{(m)}(x,t), \quad \nabla\cdot\boldsymbol{v}^{(m)} = f^{(m)}(x,t) \qquad x\in\mathbb{R}^3_+, \\ T_{i3}(\boldsymbol{v}^{(m)},p^{(m)}) + \delta_{i3}\beta\rho^{(m)} = -d_i^{(m)}(x,t), \quad \beta = const, \quad i=1,2,3, \\ \rho_t^{(m)} + v_3^{(m)} = g^{(m)}(x,t), \quad \rho^{(m)}(x,0) = \rho_0^{(m)}(x), \quad x\in\mathbb{R}^2, \\ \boldsymbol{v}^{(m)}(x,0) = \boldsymbol{v}_0^{(m)}(x), \qquad x\in\mathbb{R}^3_+, \end{cases}$$

where $\boldsymbol{v}^{(m)} = \Delta^m(z)\boldsymbol{v}$ etc. Repeating step by step the above arguments we arrive at the inequality of the type (4.16), i.e.

$$\begin{aligned} &\|\boldsymbol{v}^{(m)}(\cdot,t)\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho^{(m)}(\cdot,t)\|^2_{L_2(\mathbb{R}^2)} + \int_0^t \Big(\|\nabla\boldsymbol{v}^{(m)}\|^2_{L_2(\mathbb{R}^3_+)} + |||\rho^{(m)}|||^2_{-1/2}\Big)d\tau \\ &\le c\Big(\|\boldsymbol{v}_0^{(m)}\|^2_{L_2(\mathbb{R}^3_+)} + \|\rho_0^{(m)}\|^2_{L_2(\mathbb{R}^2)}\Big) + c\int_0^t \Big(\|\boldsymbol{f}_1^{(m)}\|_{L_2(\mathbb{R}^3_+)}(\|\boldsymbol{v}^{(m)}\|_{L_2(\mathbb{R}^3_+)} \\ &\quad + \gamma\|\boldsymbol{w}^{(m)}\|_{L_2(\Omega)}) + \|f^{(m)}\|_{L_2(\mathbb{R}^3_+)}\|p^{(m)}\|_{L_2(\mathbb{R}^3_+)} + \|\boldsymbol{d}^{(m)}\|_{L_2(\mathbb{R}^2)}(\|\boldsymbol{v}^{(m)}|_{x_3=0}\|_{L_2(\mathbb{R}^2)} \\ &\quad + \gamma\|\varphi^{(m)}\|_{L_2(\Sigma)}) + \beta\|g^{(m)}\|_{W_2^{1/2}(\Sigma)}\|\rho^{(m)}\|_{W_2^{-1/2}(\Sigma)} + \|g^{(m)}\|_{L_2(\Sigma)}\|\boldsymbol{v}^{(m)}\|_{L_2(\mathbb{R}^3_+)}\Big)d\tau, \end{aligned} \tag{4.17}$$

for any $t\le T$. We divide (4.17) by $|z|^{2+2(1+l)}$, integrate over the disc $|z|\le a$ and add the resulting inequality to (4.16). We estimate from below the left hand side of the inequality obtained using the properties of the Sobolev norms listed above:

$$\|\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \int_{|z|<a}\|\boldsymbol{v}^{(m)}\|^2_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2(l+1)}} \ge c\langle\langle\boldsymbol{v}(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+},$$

$$\|\rho\|^2_{L_2(\mathbb{R}^2)} + \int_{|z|<a}\|\rho^{(m)}\|^2_{L_2(\mathbb{R}^2)}\frac{dz}{|z|^{2+2(l+1)}} \ge c\|\rho\|^2_{W_2^{l+1}(\Sigma)},$$

$$\|\nabla\boldsymbol{v}\|^2_{L_2(\mathbb{R}^3_+)} + \int_{|z|<a}\|\nabla\boldsymbol{v}^{(m)}\|^2_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2(l+1)}} \ge c\langle\langle\nabla\boldsymbol{v}(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}.$$

Finally, since $(-\Delta')^{-1/4}\rho^{(m)} = ((-\Delta')^{-1/4}\rho)^{(m)}$ and

$$\begin{aligned} \|\rho\|^2_{H^{l+1/2}(\Sigma)} &= \Big|\int_\Sigma \rho dx'\Big|^2 + (2\pi)^{-2}\|(-\Delta')^{(l+1)/2-1/4}\rho\|^2_{L_2(\Sigma)} \\ &\le \Big|\int_\Sigma \rho dx'\Big|^2 + c\Big(\|(-\Delta')^{-1/4}\rho'\|^2_{L_2(\Sigma)} + \int_{|z|<a}\|((-\Delta'^{-1/4})\rho)^{(m)}\|^2_{L_2(\Sigma)}\frac{dz}{|z|^{2+2(l+1)}}, \end{aligned}$$

we have

$$\|\rho\|^2_{H^{-1/2}(\Sigma)} + \int_{|z|<a}\|(-\Delta^{-1/4})\rho^{(m)}\|^2_{L_2(\Sigma)}\frac{dz}{|z|^{2+2(l+1)}} \ge c\|\rho\|^2_{W_2^{l+1/2}(\Sigma)}.$$

The integrals of scalar products in the right hand side of (4.17) are estimated from above in the following manner:

$$\int_{|z|<a}\|\boldsymbol{f}_1^{(m)}\|_{L_2(\mathbb{R}^3_+)}\|\boldsymbol{v}^{(m)}\|_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2(l+1)}}$$
$$\leq\Big(\int_{|z|<a}\|\boldsymbol{f}_1^{(m)}\|^2_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2l}}\Big)^{1/2}\Big(\int_{|z|<a}\|\boldsymbol{v}_1^{(m)}\|^2_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2(l+2)}}\Big)^{1/2}$$
$$\leq c\langle\langle\boldsymbol{f}_1\rangle\rangle_{l,\mathbb{R}^3_+}\langle\langle\boldsymbol{v}\rangle\rangle_{l+2,\mathbb{R}^3_+},$$

$$\int_{|z|<a}\|\boldsymbol{f}_1^{(m)}\|_{L_2(\mathbb{R}^3_+)}\|\boldsymbol{w}^{(m)}\|_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2(l+1)}}$$
$$\leq c\langle\langle\boldsymbol{f}_1\rangle\rangle_{l,\mathbb{R}^3_+}\langle\langle\boldsymbol{w}\rangle\rangle_{l+2,\mathbb{R}^3_+}\leq c\langle\langle\boldsymbol{f}_1\rangle\rangle_{l,\mathbb{R}^3_+}\|\rho\|_{W_2^{l+1/2}(\Sigma)},$$

$$\int_{|z|<a}\|\boldsymbol{d}_1^{(m)}\|_{L_2(\mathbb{R}^2)}\|\varphi^{(m)}\|_{L_2(\Sigma)}\frac{dz}{|z|^{2+2(l+1)}}$$
$$\leq c\|\boldsymbol{d}\|_{W_2^{l+1/2}(\Sigma)}\|\varphi\|_{W_2^{l+3/2}(\Sigma)}\leq c\|\boldsymbol{d}\|_{W_2^{l+1/2}(\Sigma)}\|\rho\|_{W_2^{l+1/2}(\Sigma)},$$

$$\int_{|z|<a}\|f^{(m)}\|_{L_2(\mathbb{R}^3_+)}\|p^{(m)}\|_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2(l+1)}}\leq\langle\langle f\rangle\rangle_{l+1,\mathbb{R}^3_+}\langle\langle p\rangle\rangle_{l+1,\mathbb{R}^3_+},$$

$$\int_{|z|<a}\|\boldsymbol{d}_1^{(m)}\|_{L_2(\mathbb{R}^2)}\|\boldsymbol{v}^{(m)}\|_{L_2(\mathbb{R}^2)}\frac{dz}{|z|^{2+2(l+1)}}\leq\|\boldsymbol{d}\|_{W_2^{l+1/2}(\mathbb{R}^2)}\|\boldsymbol{v}\|_{W_2^{l+3/2}(\mathbb{R}^2)},$$

$$\int_{|z|<a}\|g^{(m)}\|_{W_2^{1/2}(\mathbb{R}^2)}\|\rho^{(m)}\|_{W_2^{-1/2}(\Sigma)}\frac{dz}{|z|^{2+2(l+1)}}\leq\|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\|\rho\|_{W_2^{l+1/2}(\Sigma)},$$

$$\int_{|z|<a}\|g^{(m)}\|_{L_2(\mathbb{R}^2)}\|\boldsymbol{v}^{(m)}\|_{L_2(\mathbb{R}^3_+)}\frac{dz}{|z|^{2+2(l+1)}}\leq\|g\|_{W_2^{l+3/2}(\mathbb{R}^2)}\|\boldsymbol{v}\|_{W_2^{l+1/2}(\mathbb{R}^3_+)}.$$

After simple calculations, using the Cauchy inequality, we arrive at (4.5). The proposition is proved.

Now we estimate the function ρ in our basic problem (3.1) (with $\sigma=0$), using Proposition 4.1 and Schauder's localization method. Let $x_0\in\mathcal{G}$. Without restriction of generality we can assume that $x_0=0$ and the x_3-axis is directed along the interior normal $-\boldsymbol{N}(0)$. In the d_0-neighborhood of the point $x_0=0$ the surface $\mathcal{G}$ is given by the equation

$$x_3=\phi(x'),\quad |x'|\leq d_0$$

where $\phi(x')$ is a smooth function of $x'=(x_1,x_2)$. It is clear that $\phi(0)=0$, $\nabla\phi(0)=0$, and

$$\boldsymbol{N}(x)=\Big(\frac{\phi_{x_1}}{\sqrt{1+|\nabla\phi|^2}},\frac{\phi_{x_2}}{\sqrt{1+|\nabla\phi|^2}},-\frac{1}{\sqrt{1+|\nabla\phi|^2}}\Big).$$

Let $\chi\in C_0^\infty(\mathbb{R}^3)$, $\chi(x)=1$ for $|x|\leq 1/2$, $\chi(x)=1$ for $|x|\geq 1$. We set $\chi_\lambda(x)=\chi(x/\lambda)$, $\lambda\in(0,1)$ and introduce the functions

$$\boldsymbol{u}=\boldsymbol{v}\chi_\lambda,\quad q=p\chi_\lambda,\quad r=\rho\chi_\lambda.$$

They satisfy the relations

$$\begin{cases}\boldsymbol{u}_t-\nu\nabla^2\boldsymbol{u}+\nabla q=\boldsymbol{f}\chi_\lambda+\boldsymbol{f}_1,\quad \nabla\cdot\boldsymbol{u}=f\chi_\lambda+f_1,\\ T(\boldsymbol{u},q)\boldsymbol{N}+\boldsymbol{N}\chi_\lambda B_0\rho=\boldsymbol{d}\chi_\lambda+\boldsymbol{d}_1,\\ r_t=\boldsymbol{u}\cdot\boldsymbol{N}+g\chi_\lambda,\quad r(x,0)=\rho_0(x)\chi_\lambda\equiv r_0(x),\\ \boldsymbol{u}(x,0)=\boldsymbol{v}_0(x)\chi_\lambda\equiv\boldsymbol{u}_0,\end{cases}\tag{4.18}$$

where

$$\boldsymbol{f}_1 = -2\omega(\boldsymbol{e}_3 \times \chi_\lambda \boldsymbol{v}) - \nu(\nabla^2(\boldsymbol{v}\chi_\lambda) - \chi_\lambda \nabla^2 \boldsymbol{v}) + p\nabla\chi_\lambda, \quad f_1 = \boldsymbol{v}\cdot\nabla\chi_\lambda,$$
$$\boldsymbol{d}_1 = (S(\chi_\lambda \boldsymbol{v}) - \chi_\lambda S(\boldsymbol{v}))\boldsymbol{n}.$$

We make the change of variables $x = \Phi(y)$:

$$y' = x', \quad y_3 = x_3 - \phi(x')$$

"rectifying the boundary" near the origin and transforming the gradient with respect to x into the operator $\widehat{\nabla} = \nabla - \nabla\phi\frac{\partial}{\partial y_3}$ where $\nabla = \Big(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}\Big)$, $\nabla\phi = \Big(\frac{\partial\phi}{\partial y_1}, \frac{\partial\phi}{\partial y_2}, 0\Big)$. Since $N_3 < 0$ in the neighborhood of the origin, we can rewrite relations (4.18), without changing notations of the transformed functions, in the equivalent form

$$\begin{cases} \boldsymbol{u}_t - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{f}_2, \quad \nabla\cdot\boldsymbol{u} = f_2, \\ T_{i3}(\boldsymbol{u}, q) + \delta_{i3} b(0) r = d^{(i)}, \quad i = 1,2,3, \\ r_t + u_3 = g_2, \quad r(y', 0) = r_0(y'), \\ \boldsymbol{u}(y,0) = \boldsymbol{u}_0(y), \end{cases} \tag{4.19}$$

where

$$\begin{aligned}
&\boldsymbol{f}_2 = \boldsymbol{f}\chi_\lambda + \boldsymbol{f}_1 - \nu(\nabla^2 - \widehat{\nabla}^2)\boldsymbol{u} + (\nabla - \widehat{\nabla})q, \\
&f_2 = f\chi_\lambda + f_1 + (\nabla - \widehat{\nabla})\cdot\boldsymbol{u}, \\
&d^{(3)} = (\boldsymbol{d}\chi_\lambda + \boldsymbol{d}_1)\cdot\boldsymbol{N} + \nu\Big(S_{33}(\boldsymbol{u}) - \boldsymbol{N}\cdot\widehat{S}(\boldsymbol{u})\boldsymbol{N}\Big) + (b(0) - b(y))r + \chi_\lambda K\rho, \\
&d^{(j)} = -(d_j\chi_\lambda + d_{1j}) + N_j(\boldsymbol{N}\cdot(\boldsymbol{d}\chi_\lambda + \boldsymbol{d}_1)) \\
&\qquad + \nu(S_{j3} + \sum_{k=1}^{3}\widehat{S}_{jk}N_k) - \nu N_j(\boldsymbol{N}\cdot\widehat{S}\boldsymbol{N}), \quad j = 1,2, \\
&g_2 = g\chi_\lambda + (\boldsymbol{u}\cdot\boldsymbol{N} + u_3), \qquad K\rho = \kappa\int_{\mathcal{G}}\frac{\rho(z,t)dS_z}{|y-z|},
\end{aligned}$$

$\widehat{S}(\boldsymbol{u}) = \widehat{\nabla}\boldsymbol{u} + (\widehat{\nabla})^T\boldsymbol{u}$ is the transformed rate-of-strain tensor. It can be assumed that $\boldsymbol{u}$, q are given in the half-space $\mathbb{R}^3_+$ and r in $\mathbb{R}^2$.

Now, we use Proposition 4.1. By (4.5),

$$\begin{aligned}
&\sup_{t<T}\Big(\langle\langle\boldsymbol{u}(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+} + \|r(\cdot,t)\|^2_{W_2^{l+1}(\mathbb{R}^2)}\Big) \\
&\quad + \int_0^T\Big(\langle\langle\nabla\boldsymbol{u}(\cdot,\tau)\rangle\rangle^2_{l+1,\mathbb{R}^3_+} + \|r(\cdot,\tau)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}\Big)d\tau \\
&\le c\Big(\langle\langle\boldsymbol{u}_0\rangle\rangle^2_{l+1,\mathbb{R}^3_+} + \|r_0\|^2_{W_2^{l+1}(\mathbb{R}^2)}\Big) + c\int_0^T\Big(\langle\langle\boldsymbol{f}_2(\cdot,t)\rangle\rangle^2_{l,\mathbb{R}^3_+} + \langle\langle f_2(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+} \\
&\quad + \|\boldsymbol{D}(\cdot,t)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)} + \|g_2(\cdot,t)\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}\Big)dt \\
&\quad + c\Big(\int_0^T\langle\langle f_2(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}dt\Big)^{1/2}\Big(\int_0^T\langle\langle q(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}dt\Big)^{1/2},
\end{aligned} \tag{4.20}$$

where $\boldsymbol{D} = (d^{(1)}, d^{(2)}, d^{(3)})$. A similar estimate can be obtained in the neighborhood of arbitrary point $x \in \mathcal{G}$. We introduce a finite "$\lambda/4$-net" of points $x^{(k)} \in \mathcal{G}$ and a set of cut-off functions $\chi_k(x)$ such that $\chi_k(x) = 1$, if $|x - x^{(k)}| \leq \lambda/2$, $\chi_k(x) = 0$, if $|x - x^{(k)}| \geq \lambda$, $|D^j \chi_k| \leq c\lambda^{-|j|}$,

$$c_1 \leq \sum_k \chi_k^2(x) \leq c_2, \quad x \in \mathcal{G},$$

with constants independent of λ. We set

$$\boldsymbol{u}_k = \boldsymbol{v}\chi_k, \quad q_k = p\chi_k, \quad r_k = \rho\chi_k,$$

write estimates (4.20) for all these functions and add the estimates together. The sum in the left hand side can be estimated from below by

$$\begin{aligned} &c\Big(\sup_{t<T} \|\rho(\cdot,t)\|^2_{W_2^{l+1}(\mathcal{G})} + \int_0^T \|\rho(\cdot,\tau)\|^2_{W_2^{l+1/2}(\mathcal{G})} d\tau\Big) \\ &\qquad - c(\lambda)\Big(\sup_{t<T} \|\rho(\cdot,t)\|^2_{W_2^{l}(\mathcal{G})} + \int_0^T \|\rho(\cdot,\tau)\|^2_{W_2^{l-1/2}(\mathcal{G})} d\tau\Big) \end{aligned} \tag{4.21}$$

This is evident for the case of integral l and follows from the inequality

$$|\chi_k(x)v(x) - \chi_k(y)v(y)|^2 \geq 2^{-1}\chi_k^2(x)|v(x) - v(y)|^2 - |v(y)|^2|\chi_k(x) - \chi_k(y)|^2,$$

if l is not integral. The functions $\boldsymbol{f}_2$, f_2, $\boldsymbol{D}$, g_2 in (4.19) depend on the data $\boldsymbol{f}$, f, $\boldsymbol{d}$, g of problem (4.1), and also on $\boldsymbol{v}$, p, ρ and their derivatives. The higher order derivatives appear with small coefficients proportional to $\nabla\phi$ or $b(0) - b(x)$, not exceeding $c\lambda$ in the λ-neighborhoods of $x^{(k)}$. Therefore the above calculation leads to the inequality

$$\begin{aligned} R^2(T) \equiv \; & \sup_{t<T} \|\rho(\cdot,t)\|^2_{W_2^{l+1}(\mathcal{G})} + \int_0^T \|\rho(\cdot,\tau)\|^2_{W_2^{l+1/2}(\mathcal{G})} d\tau \\ \leq \; & c\Big(N_1^2(T) + \lambda V^2(T) + \lambda R^2(T)\Big) + c(\lambda)\Big(\|\boldsymbol{v}\|^2_{W_2^{1+l,1/2+l/2}(Q_T)} \\ & + \|p\|^2_{W_2^{l,l/2}(Q_T)} + \|\rho\|^2_{W_2^{l,0}(G_T)} + \|K\rho\|^2_{W_2^{0,l/2+1/4}(G_T)}\Big), \end{aligned} \tag{4.22}$$

where $N_1(T)$ is defined in (4.3) and

$$V^2(T) = \|\boldsymbol{v}\|^2_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|^2_{W_2^{l,l/2}(Q_T)} + \|p\|^2_{W_2^{l+1/2,l/2+1/4}(G_T)}.$$

This is the desired estimate for ρ.

Next, we consider (3.1) as the problem (2.27) with $\boldsymbol{d} - \boldsymbol{N}B_0\rho$ instead of $\boldsymbol{d}$, $\boldsymbol{f} - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v})$ instead of $\boldsymbol{f}$, and we apply the inequality (2.31). This leads to

$$\begin{aligned} V^2(T) \leq c\Big(&\|\boldsymbol{f} - 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v})\|^2_{W_2^{l,l/2}(Q_T)} + \|\boldsymbol{d} - \boldsymbol{N}B_0\rho\|^2_{W_2^{l+1/2,l/2+1/4}(G_T)} \\ &+ \|\boldsymbol{v}_0\|^2_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{v}\|_{L_2(Q_T)}\Big) \end{aligned}$$

hence, by (4.22),

$$\begin{aligned} V^2(T) + R^2(T) \leq c\Big(&N_1^2(T) + \|\boldsymbol{v}\|^2_{W_2^{1+l,1/2+l/2}(Q_T)} + \|p\|^2_{W_2^{l,l/2}(Q_T)} \\ &+ \|\rho\|^2_{W_2^{0,l/2+1/4}(G_T)} + \|\rho\|^2_{W_2^{l,0}(G_T)} + \|K\rho\|^2_{W_2^{l+1/2,l/2+1/4}(G_T)}\Big), \end{aligned} \tag{4.23}$$

if λ is sufficiently small.

Now it is necessary to estimate the norms of the solution on the right-hand side. We use the interpolation inequality

$$\|p\|^2_{W_2^{l,0}(Q_T)} \leq \epsilon_1 \|\nabla p\|^2_{W_2^{l,0}(Q_T)} + c(\epsilon_1)\|p\|^2_{L_2(Q_T)} \tag{4.24}$$

with arbitrarily small $\epsilon_1 > 0$. To estimate the L_2-norm of p, we regard p as a solution of the problem

$$\nabla^2 p(x,t) = \nabla \cdot (\boldsymbol{f}(x,t) + \nu\nabla f - \boldsymbol{F}), \quad x \in \mathcal{F},$$
$$p(x,t) = \nu \boldsymbol{N} \cdot S(\boldsymbol{w}_1)\boldsymbol{N} + B_0\rho - \boldsymbol{d}\cdot\boldsymbol{N}, \quad x \in \mathcal{G}.$$

Repeating the arguments in the proof of Proposition 3.1, Step 4, we obtain

$$\|p\|_{L_2(\mathcal{F})} \leq c\Big(\|\boldsymbol{f}\|_{L_2(\mathcal{F})} + \|\nabla f\|_{L_2(\mathcal{F})} + \|\boldsymbol{F}\|_{L_2(\mathcal{F})} + \|\boldsymbol{v}\|_{W_2^1(\mathcal{F})} + \|\rho\|_{L_2(\mathcal{G})}\Big).$$

Similar inequalities hold for the finite difference of p with respect to t, hence

$$\begin{aligned}\|p\|_{W_2^{0,l/2}(Q_T)} \leq c\Big(\|\|\boldsymbol{f}\|_{W_2^{0,l/2}(Q_T)} &+ \|\nabla f\|_{W_2^{0,l/2}(Q_T)} \\ &+ \|\boldsymbol{F}\|_{W_2^{0,l/2}(Q_T)} + \|\nabla\boldsymbol{v}\|_{W_2^{0,l/2}(Q_T)} + \|\rho\|_{W_2^l(G_T)}\Big).\end{aligned} \tag{4.25}$$

In view of the equation $\rho_t = \boldsymbol{v}\cdot\boldsymbol{N} + g$, the $W_2^{0,l/2+1/4}$-norm of ρ and $K\rho$ can be estimated as follows:

$$\begin{aligned}\|\rho\|_{W_2^{0,l/2+1/4}(G_T)} + \|K\rho\|_{W_2^{0,l/2+1/4}(G_T)} &\leq c\Big(\|\rho\|_{L_2(G_T)} + \|\rho_t\|_{L_2(G_T)}\Big) \\ \leq c\Big(\|\rho\|_{L_2(G_T)} + \|\boldsymbol{v}\|_{L_2(G_T)} &+ \|g\|_{L_2(G_T)}\Big);\end{aligned} \tag{4.26}$$

moreover, as shown in [13],

$$\|K\rho\|_{W_2^{l+1/2,0}(G_T)} \leq c\|\rho\|_{W_2^{l-1/2,0}(G_T)}$$

hence, by interpolation inequalities, it holds

$$\|K\rho\|_{W_2^{l+1/2,0}(G_T)} + \|\rho\|_{W_2^{l,0}(G_T)} \leq \epsilon_2\|\rho\|_{W_2^{l+1/2,0}(G_T)} + c(\epsilon_2)\|\rho\|_{L_2(G_T)}. \tag{4.27}$$

Finally,

$$\|\nabla\boldsymbol{v}\|_{W_2^{0,l/2}(G_T)} + \|\boldsymbol{v}\|_{W_2^{l+1,l/2+1/2}(Q_T)} \leq \epsilon_3\|\boldsymbol{v}\|_{W_2^{l+2,l/2+1}(Q_T)} + c(\epsilon_3)\|\boldsymbol{v}\|_{L_2(Q_T)}. \tag{4.28}$$

Inequalities (4.22), (4.23), (4.25)-(4.28) with sufficiently small ϵ_i yield

$$V^2(T) + R^2(T) \leq c\big(N_1^2(T) + \|\boldsymbol{v}\|_{L_2(Q_T)} + \|\rho\|_{L_2(G_T)}\big). \tag{4.29}$$

To estimate ρ_t, we use the equation $\rho_t = \boldsymbol{v}\cdot\boldsymbol{N} + g$ and inequality (4.29). As a result, we obtain (4.2).

We also need to estimate the solution of (4.1) in weighted spaces $\widetilde{W}_2^{l,l/2}(Q_T)$.

Theorem 4.2 *If $\boldsymbol{f} \in \widetilde{W}_2^{l,l/2}(Q_T)$, $f \in \widetilde{W}_2^{1+l,0}(Q_T)$, $f = \nabla \cdot \boldsymbol{F}$, $\boldsymbol{F} \in \widetilde{W}_2^{0,1+l/2}(Q_T)$, $\boldsymbol{v}_0 \in W_2^{1+l}(\Omega)$, $\boldsymbol{d} \in \widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)$, $g \in \widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)$ and the conditions* (3.2), (3.3) *are satisfied, then the problem* (3.1) *has a unique solution $\boldsymbol{v} \in \widetilde{W}_2^{2+l,1+l/2}(Q_T)$, $\nabla p \in \widetilde{W}_2^{l,l/2}(Q_T)$ with $p|_{G_T} \in \widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)$, and the inequality*

$$\begin{aligned}
&\|\boldsymbol{v}\|^2_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|^2_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|p\|^2_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} \\
&\quad + \|\rho\|^2_{\widetilde{W}_2^{l+1/2,0}(G_T)} + \sup_{t<T} \|\rho(\cdot,t)\|^2_{W_2^{l+1}(S)} + \sup_{t<T} t\|\rho(\cdot,t)\|^2_{W_2^{l}(S)} \\
&\le c(\widetilde{N}_1^2 + \|(1+t)\boldsymbol{v}\|^2_{L_2(Q_T)} + \|(1+t)\rho\|^2_{L_2(G_T)})
\end{aligned} \tag{4.30}$$

holds with

$$\begin{aligned}
\widetilde{N}_1^2 \;=\; &\|\boldsymbol{f}\|^2_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|f\|^2_{\widetilde{W}_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|^2_{\widetilde{W}_2^{0,1+l/2}(Q_T)} \\
&+ \|\boldsymbol{v}_0\|^2_{W_2^{1+l}(\mathcal{G})} + \|\boldsymbol{d}\|^2_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|g\|^2_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)}
\end{aligned}$$

and with the constant c independent of T.

Multiplying the relations (3.1) by t, we obtain

$$\begin{cases}
(t\boldsymbol{v})_t - \nu\nabla^2 t\boldsymbol{v} + \nabla tp = t\boldsymbol{f}(x,t) + \boldsymbol{v}, \quad \nabla \cdot \boldsymbol{v} = tf(x,t), \\
T(t\boldsymbol{v}, tp)\boldsymbol{n} + \boldsymbol{n}B_0 t\rho = t\boldsymbol{d}(x,t), \\
(t\rho)_t = t\boldsymbol{v} \cdot \boldsymbol{n} + tg(x,t) + \rho, \\
t\rho|_{t=0} = 0, \quad t\boldsymbol{v}|_{t=0} = 0.
\end{cases} \tag{4.31}$$

It remains to apply inequality (4.2) with $l-1$ instead of l to $(t\boldsymbol{v}, tp, t\rho)$ (this is possible, because (4.2) is valid also for $l \in (0,1/2)$). The functions $\boldsymbol{v}$ and ρ in the right hand side of (4.31) can be estimated by the same inequality (4.2).

Remark. In view of the interpolation inequality

$$\|\rho(\cdot,t)\|_{L_2(\mathcal{G})} \le \delta\|\rho(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})} + c(\delta)\|\rho\|_{W_2^{-1/2}(\mathcal{G})}, \quad \forall \delta \in (0,1),$$

the norm $\|\rho\|_{L_2(G_T)}$ in the right hand side of (4.2) can be substituted by

$$\Big(\int_0^T \|\rho(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G})} dt\Big)^{1/2}.$$

The norm $\|\rho\|_{W_2^{-1/2}(\mathcal{G})}$ is defined in a standard way:

$$\|\rho\|_{W_2^{-1/2}(\mathcal{G})} = \sup_{\varphi \in W_2^{1/2}(\mathcal{G})} \frac{\Big|\int_{\mathcal{G}} \rho(x)\varphi(x)dS\Big|}{\|\varphi\|_{W_2^{1/2}(\mathcal{G})}}.$$

5 Proof of the estimates (1.23) and (1.38)

In this section main ideas of the proof of stability of rotating liquid are presented. As above, we consider the cases $\sigma > 0$ and $\sigma = 0$ separately.

1. The case $\sigma > 0$.

We modify the equations (1.18). We write the condition $\widetilde{T}(\boldsymbol{u}, q)\boldsymbol{n} = M\boldsymbol{n}$ separately for the tangential and normal components and we notice that in view of (1.3)

$$M = \sigma(H(x) - \mathcal{H}(y)) + \frac{\omega^2}{2}(|x'|^2 - |y'|^2) + \kappa(U(x) - \mathcal{U}(y)), \quad x = e_\rho(y).$$

Using (8.54), (8.55), we compute the first variation of M with respect to ρ. It coincides with the expression $-B_0\rho$ defined in (1.25). Hence

$$M = -B_0\rho + B_1(\rho),$$

where $B_1(\rho)$ is a nonlinear remainder. It follows that (1.18) can be written in the form

$$\begin{cases} \boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q, \rho), \\ \nabla \cdot \boldsymbol{u}(y,t) = l_2(\boldsymbol{u}, \rho), \quad y \in \mathcal{F}, \quad t > 0, \\ \Pi_{\mathcal{G}} S(\boldsymbol{u})\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}, \rho), \\ -q + \nu \boldsymbol{N} \cdot S(\boldsymbol{u})\boldsymbol{N} + B_0\rho = l_4(\boldsymbol{u}, \rho) + B_1(\rho), \\ \rho_t = \boldsymbol{u} \cdot \boldsymbol{N} + l_5(\boldsymbol{u}, \rho), \quad y \in \mathcal{G}, \\ \boldsymbol{w}(y, 0) = \boldsymbol{w}_0(y), \quad y \in \mathcal{F}, \quad \rho(y, 0) = \rho_0(y), \quad y \in \mathcal{G}, \end{cases} \tag{5.1}$$

where

$$\begin{aligned} &\boldsymbol{l}_1(\boldsymbol{u}, q, \rho) = \frac{\partial\rho^*}{\partial t}(\mathcal{L}^{-1}\boldsymbol{N}^* \cdot \nabla)\boldsymbol{u} - (\mathcal{L}^{-1}\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nu(\widetilde{\nabla} \cdot \widetilde{\nabla} - \nabla^2)\boldsymbol{u} + (\nabla - \widetilde{\nabla})q, \\ &l_2(\boldsymbol{u}, \rho) = (I - \widehat{\mathcal{L}}^T)\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{L}_2(\boldsymbol{u}, \rho), \quad \boldsymbol{L}_2 = (I - \widehat{\mathcal{L}})\boldsymbol{u}, \\ &\boldsymbol{l}_3(\boldsymbol{u}, \rho) = \Pi_{\mathcal{G}}(\Pi_{\mathcal{G}} S(\boldsymbol{u})\boldsymbol{N} - \Pi\widetilde{S}(\boldsymbol{u})\boldsymbol{n}), \\ &l_4(\boldsymbol{u}, \rho) = \nu(\boldsymbol{N} \cdot S(\boldsymbol{u})\boldsymbol{N} - \boldsymbol{n} \cdot \widetilde{S}(\boldsymbol{u})\boldsymbol{n}), \\ &l_5(\boldsymbol{u}, \rho) = \boldsymbol{u} \cdot (\boldsymbol{N} - \boldsymbol{a}), \quad \boldsymbol{a} = \frac{\widehat{\mathcal{L}}^T \boldsymbol{N}}{\Lambda(y, \rho)}, \end{aligned} \tag{5.2}$$

$$B_1(\rho) = \frac{\omega^2}{2}\rho^2|\boldsymbol{N}'|^2 + \sigma\int_0^1 (1 - s)\frac{d^2H_s}{ds^2}ds + \kappa\int_0^1 (1 - s)\frac{d^2U_s}{ds^2}ds, \tag{5.3}$$

H_s is the doubled mean curvature of the surface $\Gamma_t^{(s)} = \{x = e_{s\rho}(y), \quad y \in \mathcal{G}\}$,

$$U_s(y, t) = \int_{\mathcal{F}} \frac{L_s(\zeta)d\zeta}{|e_{s\rho}(y) - e_{s\rho}(\zeta)|},$$

and $L_s(y)$ is the Jacobian of the transformation $x = e_{s\rho}(y)$.

The orthogonality conditions (1.17), (1.19) can be written in a similar way:

$$
\begin{aligned}
&\int_{\mathcal{G}} \rho dS = \int_{\mathcal{G}} (\rho - \varphi(y,\rho)) dS \equiv l(t), \\
&\int_{\mathcal{G}} \rho y_i dS = \int_{\mathcal{G}} (\rho y_i - \psi_i(y,\rho)) dS \equiv l_i(t), \\
&\int_{\mathcal{F}} \boldsymbol{u} dx = \int_{\mathcal{F}} \boldsymbol{u}(1-L) dx \equiv \boldsymbol{m}(t), \\
&\int_{\mathcal{F}} \boldsymbol{u} \cdot \boldsymbol{\eta}_i dx + \omega \int_{\mathcal{G}} \rho \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i dS = \int_{\mathcal{F}} \boldsymbol{u} \cdot (\boldsymbol{\eta}_i(y) - \boldsymbol{\eta}_i(e_\rho(y))) L dy \\
&+ \omega \Big(\int_{\mathcal{G}} \rho \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i dS - \int_{\mathcal{F}} \boldsymbol{\eta}_3(e_\rho) \cdot \boldsymbol{\eta}_i(e_\rho) L dy + \int_{\mathcal{F}} \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i dy \Big) \equiv M_i(t).
\end{aligned}
\tag{5.4}
$$

Following [15], where the problem (5.1) was studied in the Hölder spaces of functions, we look for $(\boldsymbol{u}, q, \rho)$ in the form

$$
\boldsymbol{u} = \boldsymbol{u}' + \boldsymbol{u}'', \quad q = q' + q'', \quad \rho = \rho' + \rho'', \tag{5.5}
$$

where $\boldsymbol{u}'$, $\boldsymbol{u}''$, q', q'', ρ', ρ'' are defined as follows. First we find $\boldsymbol{u}_0''$, ρ_0'' such that

$$
\begin{aligned}
&\int_{\mathcal{G}} \rho_0'' dS = l(0), \quad \int_{\mathcal{G}} \rho_0'' y_i dS = l_i(0), \\
&\int_{\mathcal{F}} \boldsymbol{u}_0'' dy = \boldsymbol{m}(0), \quad \int_{\mathcal{F}} \boldsymbol{u}_0'' \cdot \boldsymbol{\eta}_i dy + \omega \int_{\mathcal{G}} \rho_0'' \boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_3 dS = M_i(0),
\end{aligned}
\tag{5.6}
$$

and, in addition,

$$
\nabla \cdot \boldsymbol{u}_0''(y) = l_2(\boldsymbol{u}_0, \rho_0), \quad y \in \mathcal{F}, \quad \Pi_{\mathcal{G}} S(\boldsymbol{u}_0'') \boldsymbol{N}(y) = \boldsymbol{l}_3(\boldsymbol{u}_0, \rho_0), \quad y \in \mathcal{G}. \tag{5.7}
$$

It is clear that $\boldsymbol{u}_0' = \boldsymbol{u}_0 - \boldsymbol{u}_0''$, $\rho_0' = \rho_0 - \rho_0''$ satisfy (3.50) and the compatibility conditions

$$
\nabla \cdot \boldsymbol{u}_0'(y) = 0, \quad y \in \mathcal{F}, \quad \Pi_{\mathcal{G}} S(\boldsymbol{u}_0') \boldsymbol{N}(y) = 0, \quad y \in \mathcal{G},
$$

hence we can define $\boldsymbol{u}'$, q', ρ' as the solution of the problem (1.24) with the initial data

$$
\boldsymbol{u}'(y,0) = \boldsymbol{u}_0'(y), \quad y \in \mathcal{F}, \quad \rho'(y,0) = \rho_0'(y), \quad y \in \mathcal{G}.
$$

For $\boldsymbol{u}''$, q'', ρ'' we obtain the equations

$$
\begin{cases}
\boldsymbol{u}_t'' + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}'') - \nu \nabla^2 \boldsymbol{u}'' + \nabla q'' = \boldsymbol{l}_1(\boldsymbol{u}' + \boldsymbol{u}'', q' + q'', \rho' + \rho''), \\
\nabla \cdot \boldsymbol{u}'' = l_2(\boldsymbol{u}' + \boldsymbol{u}'', \rho' + \rho''), \qquad x \in \mathcal{F}, \ t > 0, \\
\Pi_{\mathcal{G}} S(\boldsymbol{u}'') \boldsymbol{N}_0 = \boldsymbol{l}_3(\boldsymbol{u}' + \boldsymbol{u}'', \rho' + \rho''), \\
-q'' + \nu \boldsymbol{N} \cdot S(\boldsymbol{u}'') \boldsymbol{N} + B_0 \rho'' = l_4(\boldsymbol{u}' + \boldsymbol{u}'', \rho' + \rho'') + B_1(\rho' + \rho''), \quad x \in \mathcal{G}, \\
\rho_t'' - \boldsymbol{u}'' \cdot \boldsymbol{N} = l_5(\boldsymbol{u}' + \boldsymbol{u}'', \rho' + \rho''), \\
\boldsymbol{u}''(y,0) = \boldsymbol{u}_0''(y), \quad y \in \mathcal{F}, \quad \rho''(y,0) = \rho_0''(y), \quad y \in \mathcal{G}.
\end{cases}
\tag{5.8}
$$

Now we pass to the estimates of $(\boldsymbol{u}', q', \rho')$ and $(\boldsymbol{u}'', q'', \rho'')$.

Proposition 5.1. *There exist the functions $\boldsymbol{u}_0''$ and ρ_0'', satisfying* (5.6), (5.7) *and the inequality*

$$\begin{aligned} &\|\boldsymbol{u}_0''\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0''\|_{W_2^{l+2}(\mathcal{G})} \\ &\le c\Big(|l(0)| + |\boldsymbol{l}(0)| + |m(0)| + |\boldsymbol{M}(0)| + \|l_2(\boldsymbol{u}_0, \rho_0)\|_{W_2^l(\mathcal{F})} + \|\boldsymbol{l}_3(\boldsymbol{u}_0, \rho_0)\|_{W_2^{l-1/2}(\mathcal{G})}\Big). \end{aligned} \tag{5.9}$$

Proof. We put

$$\rho_0''(y) = \frac{l(0)\boldsymbol{N}(y) \cdot \boldsymbol{y}}{3|\mathcal{F}|} + \frac{1}{|\mathcal{F}|}\boldsymbol{l}(0) \cdot \boldsymbol{N}(y), \quad y \in \mathcal{G}.$$

It is easily seen that

$$\int_{\mathcal{G}} \rho_0'' dS = l(0), \quad \int_{\mathcal{G}} \rho_0'' y_i dS = l_i(0).$$

Next, we introduce the vector field $\boldsymbol{u}_1(y) = \nabla\phi(y)$ where ϕ is a solution of the Neumann problem

$$\nabla^2\Phi = f_0(y), \quad y \in \mathcal{F}, \quad \frac{\partial\Phi(y)}{\partial N}\Big|_{\mathcal{G}} = f_1(y)$$

with $f_0(y) = l_2(\boldsymbol{u}_0, \rho_0)$ and

$$f_1(y) = \frac{\boldsymbol{N}(y) \cdot \boldsymbol{y}}{3|\mathcal{F}|} \int_{\mathcal{F}} f_0(z)dz + \frac{1}{|\mathcal{F}|}\Big(\int_{\mathcal{F}} f_0(z)\boldsymbol{z}dz + \boldsymbol{m}\Big) \cdot \boldsymbol{N}(y).$$

The necessary compatibility condition $\int_{\mathcal{F}} f_0(y)dy = \int_{\mathcal{G}} f_1(y)dS$ is easily verified, in addition, we have

$$\int_{\mathcal{G}} f_1(y)y_i dS = \int_{\mathcal{F}} f_0(y)y_i dy + m_i.$$

On the other hand,

$$\int_{\mathcal{G}} f_1(y)y_i dS = \int_{\mathcal{G}} \boldsymbol{u}_1 \cdot \boldsymbol{N} y_i dS = \int_{\mathcal{F}} f_0(y)y_i dy + \int_{\mathcal{F}} u_{1i}(y)dy, \quad i = 1, 2, 3,$$

hence,

$$\int_{\mathcal{F}} \boldsymbol{u}_1(y)dy = \boldsymbol{m}.$$

By the well known coercive estimate for the Neumann problem,

$$\|\boldsymbol{u}_1\|_{W_2^{l+1}(\mathcal{F})} \le c\Big(\|f_0\|_{W_2^l(\mathcal{F})} + \|f_1\|_{W_2^{l+1/2}(\mathcal{G})}\Big) \le c\Big(\|f_0\|_{W_2^l(\mathcal{F})} + |\boldsymbol{m}|\Big).$$

Next, we construct $\boldsymbol{u}_2 \in W_2^{l+1}(\mathcal{F})$ such that

$$\Pi_{\mathcal{G}} S(\boldsymbol{u}_2)\boldsymbol{N}(y) = \boldsymbol{l}_3(\boldsymbol{u}_0, \rho_0) - \Pi_{\mathcal{G}} S(\boldsymbol{u}_1)\boldsymbol{N}(y) \equiv \boldsymbol{b}(y).$$

We set $\boldsymbol{u}_2(y) = \mathrm{rot}\boldsymbol{\Phi}(y, t)$ with $\boldsymbol{\Phi} \in W_2^{l+2}(\mathcal{F})$ satisfying the conditions

$$\boldsymbol{\Phi}(y) = \frac{\partial\boldsymbol{\Phi}}{\partial N} = 0, \quad \frac{\partial^2\boldsymbol{\Phi}}{\partial N^2} = \boldsymbol{b}(y) \times \boldsymbol{N}(y), \quad y \in \mathcal{G},$$

and the estimate

$$\|\boldsymbol{\Phi}\|_{W_2^{l+2}(\mathcal{F})} \le c\|\boldsymbol{b}\|_{W_2^{l-1/2}(\mathcal{G})}.$$

It is clear that $\boldsymbol{u}_2(y) = 0$ on $\mathcal{G}$ and $\int_{\mathcal{F}} \boldsymbol{u}_2(y)dy = 0$; moreover,

$$\frac{\partial \boldsymbol{u}_2(y)}{\partial N} = \boldsymbol{N}(y) \times \frac{\partial^2 \boldsymbol{\Phi}(y,t)}{\partial N^2}, \qquad y \in \mathcal{G},$$

which implies $\boldsymbol{N} \cdot \frac{\partial \boldsymbol{u}_2}{\partial N}\Big|_{\mathcal{G}} = 0$ and

$$\Pi_{\mathcal{G}} S(\boldsymbol{u}_2)\boldsymbol{N} = \frac{\partial \boldsymbol{u}_2}{\partial N} = \boldsymbol{N} \times [\boldsymbol{b} \times \boldsymbol{N}] = \boldsymbol{b}, \quad y \in \mathcal{G}.$$

Finally, we introduce the vector field

$$\boldsymbol{u}_3(y) = \sum_{i=1}^{3} \widehat{M}_i \mathrm{rot} \boldsymbol{e}_i A(y)$$

where $A \in C_0^\infty(\mathcal{F})$, $\int_{\mathcal{F}} A(y)dy = 1/2$, and

$$\widehat{M}_i = M_i - \int_{\mathcal{F}} (\boldsymbol{u}_1(y) + \boldsymbol{u}_2(y)) \cdot \boldsymbol{\eta}_i(y)dy - \omega \int_{\mathcal{G}} \rho_0'' \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_i dS$$

Since $\mathrm{rot}\boldsymbol{\eta}_i = 2\boldsymbol{e}_i$, we have

$$\int_{\mathcal{F}} \boldsymbol{u}_3(y) \cdot \boldsymbol{\eta}_j(y)dy = \sum_{i=1}^{3} \widehat{M}_i \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \widehat{M}_i,$$

in addition,

$$\|\boldsymbol{u}_3\|_{W_2^{l+1}(\mathcal{F})} \le c \sum_{j=1}^{3} |\widehat{M}_j|.$$

The function $\rho_0''(y)$ defined above and $\boldsymbol{u}_0''(y) = \boldsymbol{u}_1 + \boldsymbol{u}_2 + \boldsymbol{u}_3$ satisfy all the necessary requirements. The proposition is proved.

Let us estimate $(\boldsymbol{u}', q', \rho')$. It follows from (5.4) and from the formula (8.48) that the right hand side in (5.9) does not exceed cN_0^2, where

$$N_0 = \|\boldsymbol{u}_0\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{\rho}_0\|_{W_2^{l+2}(\mathcal{G})},$$

hence

$$N_0'' \equiv \|\boldsymbol{u}_0''\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{\rho}_0''\|_{W_2^{l+2}(\mathcal{G})} \le cN_0^2$$

and

$$N_0' = \|\boldsymbol{u}_0'\|_{W_2^{l+1}(\mathcal{F})} + \|\rho_0'\|_{W_2^{l+2}(\mathcal{G})} \le cN_0.$$

By Proposition 3.3,

$$Y_T(\boldsymbol{u}', q', \rho') \le cN_0, \tag{5.10}$$

$$\|\boldsymbol{u}'(\cdot,T)\|_{W_2^{l+1}(\mathcal{F})} + \|\rho'(\cdot,T)\|_{W_2^{l+2}(\mathcal{G})} \le ce^{-\beta T} N_0 \tag{5.11}$$

with the constants independent of T.

Now we turn to the estimate of $(\boldsymbol{u}'', q'', \rho'')$. We are in a position to apply Theorem 3.1, but before this we should estimate the nonlinear terms (5.2). We assume that $\boldsymbol{N}^*(y)$ is a sufficiently regular function of $y \in \mathcal{F}$ (this can be achieved,for instance, by setting $\boldsymbol{N}^*(z) = \boldsymbol{N}(y)\zeta(z)$ where $y \in \mathcal{G}$, $z = y + \boldsymbol{N}(y)\lambda$, $0 < -\lambda < \delta$, $\delta > 0$, and ζ is a smooth cut-off function equal to one near $\mathcal{G}$ and vanishing for $|\lambda| > \delta/2$). Concerning ρ^* we assume that $\rho^* = E\rho$ where E is a linear extension operator with the following properties:

$$\frac{\partial \rho^*(x,t)}{\partial N}\Big|_{\mathcal{G}} = 0,$$

$$\|\rho^*(\cdot,t)\|_{W_2^{r+1/2}(\mathcal{F})} \le c\|\rho\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+5/2]. \tag{5.12}$$

It follows that the time derivatives of ρ^* satisfy similar inequalities:

$$\|\rho_t^*(\cdot,t)\|_{W_2^{r+1/2}(\mathcal{F})} \le c\|\rho_t\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+3/2],$$

$$\|\rho_{tt}^*(\cdot,t)\|_{W_2^{r+1/2}(\mathcal{F})} \le c\|\rho_{tt}\|_{W_2^r(\mathcal{G})}, \quad r \in (0, l+1/2].$$

Proposition 5.2. *If*

$$Y_T(\boldsymbol{u}, q, \rho) \le \delta \ll 1,$$

then

$$\begin{aligned}&\|\boldsymbol{l}_1(\boldsymbol{u},q,\rho)\|_{W_2^{l,l/2}(Q_T)} + \|l_2(\boldsymbol{u},\rho)\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{L}_2(\boldsymbol{u},\rho)\|_{W_2^{0,1+l/2}(Q_T)}\\ &+ \|\boldsymbol{l}_3(\boldsymbol{u},\rho)\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4(\boldsymbol{u},\rho)\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|l_5(\boldsymbol{u},\rho)\|_{W_2^{l+3/2,l/2+3/4}(G_T)}\\ &+ \|B_1(\rho)\|_{W_2^{l+1/2,0}(G_T)} + |B_1|_{l/2,5/2,G_T} \le cY_T^2(\boldsymbol{u},q,\rho).\end{aligned} \tag{5.13}$$

The proof of this proposition is rather technical and is omitted. It particular, it contains the estimates of the surface and volume potentials that occur in $B_1(\rho)$. We present here one of the typical estimates of the single layer potential

$$V(x) = \int_S \frac{h(y)dS}{|x-y|}.$$

Proposition 5.3 [13]. *Let S be a bounded closed surface of class $W_2^{l+1-\epsilon}$ with $\epsilon \in (0,1)$, $l \in (1, 3/2)$. If $h \in W_2^{l-1/2}(S)$, then $V \in W_2^{l+1}(\Omega)$, where Ω is a domain bounded by S, and*

$$\|V\|_{W_2^{l+1}(\Omega)} \le c\|h\|_{W_2^{l-1/2}(S)}.$$

We notice that the compatibility conditions

$$\nabla \cdot \boldsymbol{u}_0''(y) = l_2(\boldsymbol{u}_0, \rho_0), \quad \Pi_{\mathcal{G}} S(\boldsymbol{u}_0'')\boldsymbol{N} = \boldsymbol{l}_3(\boldsymbol{u}_0, \rho_0)$$

are satisfied, so we can make use of Theorem 3.1 and prove the following existence theorem for the problem (5.8).

Proposition 5.4. *Given $T > 0$, there exists such $\epsilon > 0$ that in the case $N_0 \le \epsilon$ the problem* (5.8) *has a unique solution defined for $t \le T$ and satisfying the estimate*

$$Y_T(\boldsymbol{u}'', q'', \rho'') \le cN_0^2. \tag{5.14}$$

We give the sketch of the proof of (5.14). We set

$$Y' = Y_T(\boldsymbol{u}', q', \rho'), \quad Y'' = Y_T(\boldsymbol{u}'', q'', \rho''), \quad Y = Y_T(\boldsymbol{u}, q, \rho).$$

By Theorem 3.1 and Proposition 5.2,

$$Y'' \leq c_1 N_0^2 + c_2 Y^2 \leq c_1 N_0^2 + 2c_2(Y'^2 + Y''^2) \leq c_3 N_0^2 + 2c_2 Y''^2. \tag{5.15}$$

If $2c_2c_3N_0^2 < 1$, then it can be shown that $Y'' \leq \xi$, where ξ is the root of the equation $2c_2\xi^2 - \xi + c_3N_0^2 = 0$ given by the formula

$$\xi = \frac{1}{2c_2} - \sqrt{\frac{1}{4c_2^2} - \frac{c_3N_0^2}{2c_2}} = \frac{2c_3N_0^2}{1 + \sqrt{1 - 2c_2c_3N_0^2}},$$

which implies

$$Y'' \leq 2c_3N_0^2 \tag{5.16}$$

The existence of the solution of the problem (5.8) can be proved by successive approximations (we omit the details).

Thus, the solution of the problem (5.1) is constructed in the time interval $(0, T)$. Let

$$N(T) = \|\boldsymbol{u}(\cdot, T)\|_{W_2^{l+1}(\mathcal{F})} + \|\rho(\cdot, T)\|_{W_2^{l+2}(\mathcal{G})},$$

$$N'(T) = \|\boldsymbol{u}'(\cdot, T)\|_{W_2^{l+1}(\mathcal{F})} + \|\rho'(\cdot, T)\|_{W_2^{l+2}(\mathcal{G})}$$

By (5.11), (5.16),

$$N(T) \leq N'(T) + cY'' \leq c_4e^{-\beta T}N_0 + c_5N_0^2 \leq (c_4e^{-\beta T} + c_5(T)\epsilon)N_0.$$

Choosing T sufficiently large and ϵ sufficiently small, we obtain

$$N(T) \leq \theta N_0, \quad \theta < 1.$$

By the same arguments the solution of (5.1) can be extended to the time interval $t \in (T, 2T)$ and the inequality

$$N(2T) \leq \theta N(T)$$

can be proved. By repeating these arguments we arrive at the estimate

$$N(kT) \leq \theta^k N_0, \quad k = 1, 2...$$

which yields the exponential decay of the solution. Other estimates announced in Theorem 1.1 are obtained in a similar way, also on the base of Propositions 3.3 and 5.4.

2. The case $\sigma = 0$.

We consider the problem (1.10) written in the Lagrangian coordinates in the form (1.33). As we have seen above, the function M in the boundary condition $T_u \boldsymbol{n} = M\boldsymbol{n}$ can be written in the form $M = -B_0\rho + B_1(\rho)$; the expressions $B_0\rho$ and $B_1(\rho)$ are given by (1.25) and (5.3) (with $\sigma = 0$). We pass to the Lagrangian coordinates and obtain the relation

$$M = -B_0' r + B_1'(r, \boldsymbol{u}),$$

where

$$\begin{aligned} B_0'(\xi) r &= b(\bar{\xi}) r(\xi, t) - \kappa \int_{\Gamma_0} \frac{r(\eta, t) dS_\eta}{|\bar{\xi} - \bar{\eta}|}, \quad \xi \in \Gamma_0, \\ B_1'(\boldsymbol{u}, r) &= \frac{\omega}{2} |\boldsymbol{N}'(\bar{X})|^2 r^2 + \kappa \int_0^1 (1-s) \frac{\partial^2 U_s}{\partial s^2} ds - (b(\bar{X}) - b(\bar{\xi})) r \\ &+ \kappa \int_{\Gamma_0} \frac{|A(\eta, t) \boldsymbol{n}_0(\eta)|}{|\widehat{\mathcal{L}}(\bar{X}, r) \boldsymbol{N}(\bar{X})|} \frac{r dS}{|\bar{X}(\xi, t) - \bar{X}(\eta, t)|} - \kappa \int_{\Gamma_0} \frac{r dS}{|\bar{\xi} - \bar{\eta}|} \Big). \end{aligned} \tag{5.17}$$

Hence (1.33) is equivalent to

$$\begin{cases} \boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu \nabla^2 \boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q), \\ \nabla \cdot \boldsymbol{u} = l_2(\boldsymbol{u}), \qquad \xi \in \Omega_0, \ \ t > 0, \\ \Pi_0 S(\boldsymbol{u}) \boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}), \\ -q + \nu \boldsymbol{n}_0 \cdot S(\boldsymbol{u}) \boldsymbol{n}_0 + B_0' r = l_4(\boldsymbol{u}) + B_1'(\boldsymbol{u}, r), \\ r_t(\xi, t) = \boldsymbol{N}(\bar{\xi}) \cdot \boldsymbol{u} + l_5(\boldsymbol{u}), \quad \xi \in \Gamma_0, \\ \boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \quad \xi \in \Omega_0, \quad r(\xi, 0) = \rho_0(\bar{\xi}), \quad \xi \in \Gamma_0, \end{cases} \tag{5.18}$$

where $\Pi_0 \boldsymbol{d} = \boldsymbol{d} - \boldsymbol{n}_0(\boldsymbol{n}_0 \cdot \boldsymbol{d})$,

$$\begin{aligned} &\boldsymbol{l}_1(\boldsymbol{u}, q) = \nu(\nabla_u^2 \boldsymbol{u} - \nabla^2 \boldsymbol{u}) + \nabla q - \nabla_u q, \\ &l_2(\boldsymbol{u}) = (\nabla - \nabla_u) \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{L}(\boldsymbol{u}), \quad \boldsymbol{L} = (I - A^T) \boldsymbol{u}, \\ &\boldsymbol{l}_3(\boldsymbol{u}) = \Pi_0(\Pi_0 S(\boldsymbol{u}) \boldsymbol{n}_0 - \Pi S_u(\boldsymbol{u}) \boldsymbol{n}), \\ &l_4(\boldsymbol{u}) = \nu(\boldsymbol{n}_0 \cdot S(\boldsymbol{u}) \boldsymbol{n}_0 - \boldsymbol{n} \cdot S_u(\boldsymbol{u}) \boldsymbol{n}), \\ &l_5(\boldsymbol{u}) = (\boldsymbol{N}(\bar{X}) - \boldsymbol{N}(\bar{\xi})) \cdot \boldsymbol{u}(\xi, t). \end{aligned} \tag{5.19}$$

The proof of solvability of (5.18) is based on the analysis of the linear problem

$$\begin{cases} \boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu \nabla^2 \boldsymbol{v} + \nabla p = \boldsymbol{f}(\xi, t), \\ \nabla \cdot \boldsymbol{v}(\xi, t) = f(\xi, t), \quad \xi \in \Omega_0, \ \ t > 0, \\ \Pi_0 S(\boldsymbol{v}) \boldsymbol{N}_0 = \Pi_0 \boldsymbol{d}, \quad \xi \in \Gamma_0, \\ -p + \nu \boldsymbol{n}_0 \cdot S(\boldsymbol{u}) \boldsymbol{n}_0 + B_0' r = \boldsymbol{d} \cdot \boldsymbol{n}_0 \\ r_t - \boldsymbol{n}_0(\xi) \cdot \boldsymbol{v}(\xi, t) = g(\xi, t), \quad \xi \in \Gamma_0, \\ \boldsymbol{v}(\xi, 0) = \boldsymbol{v}_0(\xi), \quad \xi \in \Omega_0, \qquad r(\xi, 0) = r_0(\xi), \quad \xi \in \Gamma_0 \end{cases} \tag{5.20}$$

In contrast to (3.1), this problem is written in the domain Ω_0 that can be transformed in $\mathcal{F}$ by the mapping inverse to $\xi = e_{\rho_0}(y), \quad y \in \mathcal{F}$; the function ρ_0 should belong to $W_2^{l+3/2}(\mathcal{G})$. In this way we reduce (5.20) to (3.1) and prove the following theorem (see details in [5]):

Theorem 5.1. *Let $l \in (1,3/2)$, $Q_T^0 = \Omega_0 \times (0,T)$, $G_T^0 = \Gamma_0 \times (0,T)$ and let the data of the problem* (5.20) *possess the following regularity properties: $\boldsymbol{f} \in W_2^{l,l/2}(Q_T^0)$, $f \in W_2^{1+l,0}(Q_T^0)$, $f = \nabla \cdot \boldsymbol{F}$, $\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T^0)$, $\boldsymbol{v}_0 \in W_2^{1+l}(\Omega_0)$, $r_0 \in W_2^{l+1}(\Gamma_0)$, $\boldsymbol{d} \in W_2^{l+1/2,l/2+1/4}(G_T^0)$, $g \in W_2^{l+3/2,l/2+3/4}(G_T^0)$. Assume also that the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}_0 = f(\xi,0), \quad \xi \in \Omega_0, \quad \nu \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{n}_0 = \Pi_0 \boldsymbol{d}(\xi,0), \quad \xi \in \Gamma_0$$

are satisfied. Then the problem (5.20) *has a unique solution $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_T^0)$, $\nabla p \in W_2^{l,l/2}(Q_T^0)$, $r \in W_2^{l+1/2,0}(G_T^0)$, $r_t \in W_2^{l+3/2,l/2+3/4}(G_T^0)$ such that $p|_{G_T^0} \in W_2^{l+1/2,l/2+1/4}(G_T^0)$, $r(\cdot,t) \in W_2^{l+1}(\Gamma_0)$ for arbitrary $t \in (0,T)$, and*

$$\begin{aligned} Y(T) &\equiv \|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T^0)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T^0)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_T^0)} \\ &\quad + \|r\|_{W_2^{l+1/2,0}(G_T^0)} + \|r_t\|_{W_2^{l+3/2,l/2+3/4}(G_T^0)} + \sup_{t<T} \|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} \\ &\le c\Big(N(T) + \Big(\int_0^T (\|\boldsymbol{v}\|^2_{L_2(\Omega_0)} + \|r\|^2_{W_2^{-1/2}(\Gamma_0)})dt\Big)^{1/2}\Big), \end{aligned} \tag{5.21}$$

where

$$\begin{aligned} N(T) \;\; = \;\; & \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T^0)} + \|f\|_{W_2^{l+1,0}(Q_T^0)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T^0)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)} \\ & + \|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_0)} + \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T^0)} + \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T^0)}. \end{aligned}$$

Moreover, if $\boldsymbol{f} \in \widetilde{W}_2^{l,l/2}(Q_T^0)$, $\boldsymbol{d} \in \widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)$, $g \in \widetilde{W}_2^{l+3/2,l/2+3/4}(G_T^0)$, $f \in \widetilde{W}_2^{1+l,0}(Q_T^0)$, $\boldsymbol{F} \in \widetilde{W}_2^{0,1+l/2}(Q_T^0)$ (this means that $f \in W_2^{1+l,0}(Q_T^0)$, $tf \in W_2^{l,0}(Q_T^0)$), $\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T^0)$, $t\boldsymbol{F} \in W_2^{0,(l+1)/2}(Q_T^0)$), then

$$\begin{aligned} \widetilde{Y}(T) &\equiv \|\boldsymbol{v}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T^0)} + \|\nabla p\|_{\widetilde{W}_2^{l,l/2}(Q_T^0)} + \|p\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)} + \|r\|_{\widetilde{W}_2^{l+1/2,0}(G_T^0)} \\ &\quad + \|r_t\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T^0)} + \sup_{t<T} \|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} + \sup_{t<T} t\|r(\cdot,t)\|_{W_2^{l}(\Gamma_0)} \\ &\le c\Big(\widetilde{\mathcal{N}}(T) + \Big(\int_0^T (1+t^2)(\|\boldsymbol{v}\|^2_{L_2(\Omega_0)} + \|r\|^2_{W_2^{-1/2}(\Gamma_0)})dt\Big)^{1/2}\Big), \end{aligned} \tag{5.22}$$

where

$$\begin{aligned} \widetilde{\mathcal{N}}(T) \;\; = \;\; & \|\boldsymbol{f}\|_{\widetilde{W}_2^{l,l/2}(Q_T^0)} + \|f\|_{\widetilde{W}_2^{l+1,0}(Q_T^0)} + \|\boldsymbol{F}\|_{\widetilde{W}_2^{0,1+l/2}(Q_T^0)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)} \\ & + \|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{G})} + \|\boldsymbol{d}\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)} + \|g\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T^0)}. \end{aligned}$$

The constants in (5.21), (5.22) *are independent of T.*

As in the case $\sigma > 0$, we estimate the nonlinear terms (5.19) and apply Theorem 5.1.

Proposition 5.5. *If*

$$\widetilde{Y}(T) \le \delta_1 \tag{5.23}$$

with a certain small $\delta_1 > 0$ and $\rho_0 \in W_2^{l+3/2}(\mathcal{G})$ is extended in $\mathcal{F}$ so that

$$\|\rho_0^*\|_{W_2^{l+2}(\mathcal{F})} \le c\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})},$$

Then

$$\begin{aligned}
&\|\boldsymbol{l}_1\|_{\widetilde{W}_2^{l,l/2}(Q_T^0)} + \|l_2\|_{\widetilde{W}_2^{1+l,0}(Q_T^0)} + \|\boldsymbol{L}\|_{\widetilde{W}_2^{0,1+l/2}(Q_T^0)} + \|\boldsymbol{l}_3\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)} \\
&\quad + \|l_4\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)} + \|l_5\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)} + \|l_6\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T^0)} \\
&\le c\Big(\|\boldsymbol{u}\|^2_{\widetilde{W}_2^{2+l,1+l/2}(Q_T^0)} + \|\nabla q\|^2_{\widetilde{W}_2^{l,l/2}(Q_T^0)} + \|r\|^2_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T^0)}\Big) \\
&\le c\delta_1 \widetilde{Y}(T)
\end{aligned} \tag{5.24}$$

with the constant c independent of $T \ge 1$.

We also need to estimate weak norms of $\boldsymbol{u}$ and ρ that occur in the inequalities (5.21), (5.22). This is achieved by construction of a special Lyapunov function that is often referred to as a generalized energy.

Proposition 5.6 [16] *Assume that the problem* (1.10) *has a strong solution defined for $t \in (0, T)$, and Γ_t is representable in the form* (5.3) *with $\rho(y, t)$ satisfying*

$$\sup_{t<T} |\rho(\cdot, t)|_{C^1(\mathcal{G})} \le \delta_2 \ll 1. \tag{5.25}$$

Then there exist two positive functions, $E(t)$ and $E_1(t)$, such that

$$\frac{dE(t)}{dt} + E_1(t) = 0, \tag{5.26}$$

$$c_1\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big) \le E(t) \le c_2\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big), \tag{5.27}$$

and

$$\begin{aligned}
E_1(t) &\ge c_3\Big(\|S(\boldsymbol{w}(\cdot,t))\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G})}\Big) \\
&\ge c_4\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{W_2^1(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G})}\Big)
\end{aligned} \tag{5.28}$$

with the constants independent of T.

The proof of this proposition is given in Sec. 6.

It follows that

$$\begin{aligned}
&\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega_0)} + \|r(\cdot,t)\|^2_{L_2(\Gamma_0)} + \int_0^t \Big(\|\boldsymbol{u}(\cdot,\tau)\|^2_{L_2(\Omega_0)} + \|r(\cdot,\tau)\|^2_{W_2^{-1/2}(\Gamma_0)}\Big)d\tau \\
&\le c\Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_0)} + \|\rho_0(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big).
\end{aligned} \tag{5.29}$$

Theorem 5.1 and Proposition 5.5, 5.6 allow us to obtain the following uniform estimate for the solution of the problem (5.18).

Proposition 5.7. *Assume that the solution of* (5.18) *is defined for $t \in (0, T)$ and satisfies* (5.23). *Then*

$$\widetilde{Y}(T) \le cN_0, \tag{5.30}$$

where

$$N_0 = \|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}.$$

Proof. Making use of (5.21), (5.23), (5.29) we obtain

$$Y(T) \le c(\widetilde{Y}^2(T) + N_0) \le c\Big(\delta_1 \widetilde{Y}(T) + N_0\Big).$$

Now , we multiply (5.26) by $1 + t^2$ which leads to

$$\frac{d(1+t^2)E(t)}{dt} + (1+t^2)E_1(t) = 2tE(t),$$

and, as a consequence, to

$$(1+t^2)E(t) + \int_0^t (1+\tau^2)E_1(\tau)d\tau = E(0) + 2\int_0^t \tau E(\tau)d\tau$$

$$\le E(0) + 2\sqrt{\int_0^t \tau^2 E(\tau)d\tau}\sqrt{\int_0^t E(\tau)d\tau} \le E(0) + c\sqrt{\widetilde{Y}(T)}\sqrt{Y(T)}.$$

By (5.22), we have

$$\widetilde{Y}(T) \le c(\delta_1 \widetilde{Y}(T) + N_0) + c\sqrt{\widetilde{Y}(T)}\sqrt{Y(T)},$$

which implies

$$\widetilde{Y}(T) \le c_1\sqrt{\delta_1}\widetilde{Y}(T) + c_2 N_0,$$

and if

$$\sqrt{\delta_1} \le \frac{1}{2c_1},$$

then

$$\widetilde{Y}(T) \le 2c_2 N_0, \tag{5.31}$$

q.e.d. ■

The solvability of the problem (5.18) is proved step by step, first in the time interval $t \in (0,1)$, then $t \in (1,2)$ and so forth. We outline the construction of the solution in the time interval $t \in (T, T+1)$ under the assumption that it is already constructed for $t \in (0,T)$ and the estimate (5.30) is obtained (the details can be found in [5]).

We reduce the construction to the solution of the problem (1.28) for $t \in (T, T+1)$ with the initial condition $\boldsymbol{u}(\xi, T) = \lim_{\tau \to +0} \boldsymbol{u}(\xi, T-\tau)$. We introduce the functions $\boldsymbol{u}_0$ and q_0 that coincide with $\boldsymbol{u}$ and q for $t < T$ and are defined by

$$\begin{aligned} \boldsymbol{u}_0(\xi,t) &= -3\boldsymbol{u}(\xi, 2T-t) + 4\boldsymbol{u}(\xi, 3T/2 - t/2), \\ q_0(\xi,t) &= -3q(\xi, 2T-t) + 4q(\xi, 3T/2 - t/2), \end{aligned} \tag{5.32}$$

for $t > T$ (this extension guarantees preservation of class). For $t \in (T, T+1)$ these functions satisfy the relations

$$\begin{cases} \boldsymbol{u}_{0t} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}_0) - \nu\nabla^2\boldsymbol{u}_0 + \nabla q_0 = \boldsymbol{l}_1^{(0)}(\boldsymbol{u}_0, q_0) + \boldsymbol{u}_{0t} - (\boldsymbol{u}_t)^{(0)}, \\ \nabla \cdot \boldsymbol{u}_0 = l_2^{(0)}(\boldsymbol{u}_0), \quad \xi \in \Omega_0, \\ \Pi_0 S(\boldsymbol{u}_0)\boldsymbol{n}_0 = \boldsymbol{l}_3^{(0)}(\boldsymbol{u}_0) \\ -q_0 + \nu\boldsymbol{n}_0 \cdot S(\boldsymbol{u}_0)\boldsymbol{n}_0 = l_4^{(0)}(\boldsymbol{u}_0) + M^{(0)}(\xi,t), \quad \xi \in \Gamma_0, \end{cases} \tag{5.33}$$

where $f^{(0)}(\xi,t)$ is a function given for $t\in(0,T)$ and extended into the time interval $(T,T+1)$ according to the rule (5.32). For $t\in(T,T+1)$, we introduce new unknown functions $\boldsymbol{v}=\boldsymbol{u}-\boldsymbol{u}_0$, $p=q-q_0$ and we set $X_w(\xi,t)=\xi+\int_0^t \boldsymbol{w}(\xi,\tau)d\tau$. Since $M=m(X_u)+p_0$, where

$$m(X_u)=\kappa U(X_u)+\frac{\omega^2}{2}|X_u'|^2,$$

we have

$$M-M^{(0)}=m-m^{(0)}=m(X_{u_0+v})-m(X_{u_0})+m(X_{u_0})-m^{(0)}(X_{u_0}),$$

moreover,

$$m(X_{u_0+v})-m(X_{u_0})=\frac{\partial m(X_{u_0+sv})}{\partial s}\Big|_{s=0}+\int_0^1(1-s)\frac{\partial^2 m(X_{u_0+sv})}{\partial s^2}ds,$$

where

$$\begin{aligned}
&\frac{\partial m(X_{u_0+sv})}{\partial s}\Big|_{s=0}\equiv\lambda(\boldsymbol{v})=\omega^2X_{u_0}'(\xi,t)\cdot\int_T^t \boldsymbol{v}'(\xi,\tau)d\tau\\
&-\kappa\int_{\Omega_0}D_0(\eta,t)\frac{X_{u_0}(\xi,t)-X_{u_0}(\eta,t)}{|X_{u_0}(\xi,t)-X_{u_0}(\eta,t)|^3}d\eta\cdot\int_T^t\boldsymbol{v}'(\xi,\tau)d\tau\\
&+\kappa\int_{\Gamma_0}\frac{A_0(\eta,t)\boldsymbol{n}_0(\eta)}{|X_{u_0}(\xi,t)-X_{u_0}(\eta,t)|}\cdot\Big(\int_T^t\boldsymbol{v}'(\eta,\tau)d\tau\Big)dS
\end{aligned}$$

is the first variation of $m(X_{u_0+v})-m(X_{u_0})$ with respect to $\boldsymbol{v}$ (it is computed in the same way as δU in Proposition 8.26). By D_0 we mean the Jacobian of the transformation $x=X_{u_0}(\xi,t)$ (it is equal to one for $t<T$), and A_0 is the cofactors matrix corresponding to this transformation.

We subtract (5.33) from the equations (5.18) for $(\boldsymbol{u},q)$ and arrive at the following problem for $\boldsymbol{v}$, p:

$$\begin{cases}
\boldsymbol{v}_t+2\omega(\boldsymbol{e}_3\times\boldsymbol{v})-\nu\nabla^2\boldsymbol{v}+\nabla p=\boldsymbol{l}_1(\boldsymbol{u}_0+\boldsymbol{v},q_0+p)-\boldsymbol{l}_1(\boldsymbol{u}_0,q_0)+\boldsymbol{f}(\xi,t),\\
\nabla\cdot\boldsymbol{v}=l_2(\boldsymbol{u}_0+\boldsymbol{v})-l_2(\boldsymbol{u}_0)+f(\xi,t),\qquad \xi\in\Omega_0,\ \ t>T,\\
\Pi_0S(\boldsymbol{v})\boldsymbol{n}_0=\boldsymbol{l}_3(\boldsymbol{u}_0+\boldsymbol{v})-\boldsymbol{l}_3(\boldsymbol{u}_0)+\boldsymbol{d}(\xi,t),\\
-p+f\nu\boldsymbol{n}_0\cdot S(\boldsymbol{v})\boldsymbol{n}_0-\lambda(\boldsymbol{v})=l_4(\boldsymbol{u}_0+\boldsymbol{v})-l_4(\boldsymbol{u}_0)+l_5(\boldsymbol{v})+d(\xi,t),\quad x\in\Gamma_0,\\
\boldsymbol{v}(\xi,T)=0,\quad \xi\in\Omega_0,
\end{cases}\tag{5.34}$$

where

$$\begin{aligned}
&\boldsymbol{f}=\boldsymbol{l}_1(\boldsymbol{u}_0,q_0)-\boldsymbol{l}_1^{(0)}(\boldsymbol{u}_0,q_0)+(\boldsymbol{u}_t)^{(0)}-\boldsymbol{u}_{0t},\\
&f(\xi,t)=l_2(\boldsymbol{u}_0)-l_2^{(0)}(\boldsymbol{u}_0)=\nabla\cdot\boldsymbol{F}(\xi,t),\\
&\boldsymbol{F}(\xi,t)=\boldsymbol{L}(\boldsymbol{u}_0)-\boldsymbol{L}^{(0)}(\boldsymbol{u}_0),\\
&\boldsymbol{d}(\xi,t)=\boldsymbol{l}_3(\boldsymbol{u}_0)-\boldsymbol{l}_3^{(0)}(\boldsymbol{u}_0),\\
&d(\xi,t)=l_4(\boldsymbol{u}_0)-l_4^{(0)}(\boldsymbol{u}_0)+m(X[\boldsymbol{u}_0])-m^{(0)}(X[\boldsymbol{u}_0]),\\
&l_5(\boldsymbol{v})=\int_0^1(1-s)\frac{\partial^2m(X_{u_0+sv})}{\partial s^2}ds.
\end{aligned}$$

These functions vanish for $t=T$, which implies $p|_{t=T}=0$, $\boldsymbol{v}_t|_{t=T}=0$.

Given the solution of (5.18) in the time interval $(T,T+1)$, we are looking for the solution of (5.34) with finite norm

$$
\begin{aligned}
&V(\boldsymbol{v},p)=\|\boldsymbol{v}\|_{W_2^{l+12,l/2+1}(Q^0_{T,T+1})}+T\|\boldsymbol{v}\|_{W_2^{l+1,l/2+1|2}(Q^0_{T,T+1})}\\
&+\|\nabla p\|_{W_2^{l,l/2}(Q^0_{T,T+1})}+T\|\nabla p\|_{W_2^{l-1,l/2-1/2}(Q^0_{T,T+1})},\quad Q^0_{T,T+1}=\Omega_0\times(T,T+1).
\end{aligned}
$$

The differences $\boldsymbol{l}_1(\boldsymbol{u}_0+\boldsymbol{v},q+p)-\boldsymbol{l}_1(\boldsymbol{u}_0,q_0)$, $l_2(\boldsymbol{u}_0+\boldsymbol{v})-l_2(\boldsymbol{u}_0)$ etc. satisfy the inequalities

$$
\begin{aligned}
&\|\boldsymbol{l}_1(\boldsymbol{u}_0+\boldsymbol{v},q_0+p)-\boldsymbol{l}_1(\boldsymbol{u}_0,q_0)\|_{W_2^{l,l/2}(Q^0_{T,T+1})}\\
&+T\|\boldsymbol{l}_1(\boldsymbol{u}_0+\boldsymbol{v},q_0+p)-\boldsymbol{l}_1(\boldsymbol{u}_0,q_0)\|_{W_2^{l-1,(l-1)/2}(Q^0_{T,T+1})}\\
&\leq c\delta_1\Big(\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q^0_{T,T+1})}+T\|\boldsymbol{v}\|_{W_2^{1+l,1/2+l/2}(Q^0_{T,T+1})}\\
&+\|\nabla p\|_{W_2^{l,l/2}(Q^0_{T,T+1})}+T\|\nabla p\|_{W_2^{l-1,l/2-1/2}(Q^0_{T,T+1})}\Big),\\
&\|l_2(\boldsymbol{u}_0+\boldsymbol{v})-l_2(\boldsymbol{u}_0)\|_{W_2^{l+1,0}(Q^0_{T,T+1})}+\|\boldsymbol{L}(\boldsymbol{u}_0+\boldsymbol{v})-\boldsymbol{L}(\boldsymbol{u}_0)\|_{W_2^{0,1+l/2}(Q^0_{T,T+1})}\\
&+\|\boldsymbol{l}_3(\boldsymbol{u}_0+\boldsymbol{v})-\boldsymbol{l}_3(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G^0_{T,T+1})}+\|l_4(\boldsymbol{u}_0+\boldsymbol{v})-l_4(\boldsymbol{u}_0)\|_{W_2^{l+1/2,l/2+1/4}(G^0_{T,T+1})}\\
&+T\Big(\|l_2(\boldsymbol{u}_0+\boldsymbol{v})-l_2(\boldsymbol{u}_0)\|_{W_2^{l,0}(Q^0_{T,T+1})}+\|\boldsymbol{L}(\boldsymbol{u}_0+\boldsymbol{v})-\boldsymbol{L}(\boldsymbol{u}_0)\|_{W_2^{0,1/2+l/2}(Q^0_{T,T+1})}\\
&+\|\boldsymbol{l}_3(\boldsymbol{u}_0+\boldsymbol{v})-\boldsymbol{l}_3(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G^0_{T,T+1})}+\|l_4(\boldsymbol{u}_0+\boldsymbol{v})-l_4(\boldsymbol{u}_0)\|_{W_2^{l-1/2,l/2-1/4}(G^0_{T,T+1})}\\
&\leq c\delta_1\Big(\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q^0_{T,T+1})}+T\|\boldsymbol{v}\|_{W_2^{1+l,(l+1)/2}(Q^0_{T,T+1})}\Big),
\end{aligned}
\tag{5.35}
$$

provided $\widetilde{Y}_T\leq\delta_1$. Moreover, we have

$$
\begin{aligned}
&\|l_5(\boldsymbol{v})\|_{W_2^{l+1/2,l/2+1/4}(G^0_{T,T+1})}+T\|l_5(\boldsymbol{v})\|_{W_2^{l-1/2,l/2-1/4}(G^0_{T,T+1})}\\
&\leq c\Big(\|\boldsymbol{v}\|_{W_2^{l+1/2,l/2+1/4}(G^0_{T,T+1})}+T\|\boldsymbol{v}\|_{W_2^{l-1/2,l/2-1/4}(G^0_{T,T+1})}\Big)^2.
\end{aligned}
\tag{5.36}
$$

We are in a position to apply the estimate (4.2) to the problem (5.32), although this problem somewhat differs from (5.19): instead of B_0' we have the operator $\lambda(\boldsymbol{v})$ in the boundary condition. But this is also a lower order operator of the Volterra type, so the inequality (4.2) is still applicable. Using (4.2), (5.35), (5.36), we prove the following proposition.

Proposition 5.8. *There exist $\epsilon_1>0$ such that if*

$$\widehat{Y}(T)\leq c\epsilon_1,$$

then the problem (5.32) *has a unique solution, and*

$$V(\boldsymbol{v},p)\leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q^0_T)}+\|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q^0_T)}\Big)\leq c\widetilde{Y}(T). \tag{5.37}$$

To estimate $r(\xi,t)$, we use (5.37) and the formula

$$r(\xi,t)=r(\xi,T)+\int_T^t\boldsymbol{N}(\bar{X})\cdot\boldsymbol{u}(\xi,\tau)d\tau,\quad t\in(T,T+1).$$

In this way we arrive at the inequality $\widetilde{Y}(T+1) \leq c_1\widetilde{Y}(T)$, and, as a consequence, at $\widetilde{Y}(T+1) \leq c_1 c_0 N_0 \leq c\epsilon$, where c_0 is the constant in (5.31) ($c_0 = 2c_2$). When we impose on the initial data one more restriction $c_0 c_1 \epsilon \leq \delta_1$, we obtain for $\widetilde{Y}(T+1)$ the estimate (5.31) with the constant c_0.

The solvability of the problem (1.28) in the time interval $t \in (0,1)$ is established by similar (even easier) arguments.

Now we can extend the solution step by step to the infinite time interval $t > 0$ and prove (5.31) for $T = \infty$. This completes the proof of Theorem 1.2.

6 Estimate of generalized energy

This section contains the proof of Proposition 5.6. We consider the problem (1.10)-(1.13), i.e.,

$$\begin{cases} \boldsymbol{w}_t + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + 2\omega(\boldsymbol{e}_3\times\boldsymbol{w}) - \nu\nabla^2\boldsymbol{w} + \nabla s = 0, \\ \nabla\cdot\boldsymbol{w}(y,t) = 0, \qquad y\in\Omega_t, \quad t>0, \\ T(\boldsymbol{w},s)\boldsymbol{n} = (\frac{\omega^2}{2}|y'|^2 + \kappa U(y,t) + p_0)\boldsymbol{n}, \\ V_n = \boldsymbol{w}\cdot\boldsymbol{n}, \quad y\in\Gamma_t, \\ \boldsymbol{w}(y,0) = \boldsymbol{v}_0(y) - \boldsymbol{V}(y) \equiv \boldsymbol{w}_0(y), \quad y\in\Omega_0, \end{cases} \tag{6.1}$$

$$\begin{gathered} \int_{\Omega_t}\boldsymbol{w}(x,t)dx = 0, \\ \int_{\Omega_t}\boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_i(x)dx + \omega\int_{\Omega_t}\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx = \omega\int_{\mathcal{F}}\boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_i(x)dx, \quad i=1,2,3. \end{gathered} \tag{6.2}$$

$$\int_{\mathcal{G}}\varphi(y,\rho)dS = 0, \quad \int_{\mathcal{G}}\psi(y,\rho)dS = 0, \quad i=1,2,3. \tag{6.3}$$

We assume that the solution of this problem is defined for all $t>0$ and possesses all the derivatives that occur in (1.10). The free boundary Γ_t is given by the equation (1.14), the extension ρ^* of the function ρ in $\mathcal{F}$ satisfies the conditions formulated in Sec. 5. We start with the following auxiliary propositions.

Proposition 6.1 *Given the function $f_0(y,t)$, $y\in\mathcal{G}$, belonging to $W_2^{1/2}(\mathcal{G})$ and satisfying the orthogonality condition*

$$\int_{\mathcal{G}} f_0(y,t)dS = 0,$$

there exists a vector field $\boldsymbol{W}(x,t)$, $x\in\Omega_t$, such that $\boldsymbol{W}\in W_2^1(\Omega_t)$,

$$\nabla\cdot\boldsymbol{W}(x,t) = 0, \quad x\in\Omega_t, \quad \boldsymbol{W}\cdot\boldsymbol{n}\Big|_{\Gamma_t} = \frac{f_0(y,t)}{|\widehat{\mathcal{L}}^T(y,\rho)\boldsymbol{N}|}\Big|_{y=e_\rho^{-1}(x)},$$

$$\int_{\Omega_t}\boldsymbol{W}(x,t)\cdot\boldsymbol{\eta}_i(x)dx = 0, \quad i=1,2,3,$$

$$\|\boldsymbol{W}(\cdot,t)\|_{W_2^1(\Omega_t)} \le c\|f_0\|_{W_2^{1/2}(\mathcal{G})}, \tag{6.4}$$

$$\|\boldsymbol{W}(\cdot,t)\|_{L_2(\Omega_t)} \le c\|f_0\|_{L_2(\mathcal{G})}, \tag{6.5}$$

$$\|\boldsymbol{W}_t(\cdot,t)\|_{L_2(\Omega_t)} \le c(\|f_{0t}\|_{L_2(\mathcal{G})} + \|f_0\|_{W_2^{1/2}(\mathcal{G})}). \tag{6.6}$$

Proof. The vector field $\boldsymbol{W}_0 = \nabla\phi(y,t)$, $y\in\mathcal{F}$, where ϕ is a solution of the Neumann problem

$$\nabla^2\phi(y,t) = 0, \quad y\in\mathcal{F}, \quad \frac{\partial\phi}{\partial\boldsymbol{N}}\Big|_{\mathcal{G}} = f_0,$$

is divergence free, satisfies the boundary condition $\boldsymbol{W}_0 \cdot \boldsymbol{N} = f_0$ and the inequalities

$$\|\boldsymbol{W}_0\|_{W_2^1(\mathcal{F})} \le c\|f_0\|_{W_2^{1/2}(\mathcal{G})}, \quad \|\boldsymbol{W}_0\|_{L_2(\mathcal{F})} \le c\|f_0\|_{L_2(\mathcal{G})},$$

$$\|\boldsymbol{W}_{0t}\|_{L_2(\mathcal{F})} \le c\|f_{0t}\|_{L_2(\mathcal{G})}.$$

We introduce the vector field

$$\boldsymbol{W}_1(x,t) = \frac{1}{L(y,\rho)}\mathcal{L}(y,\rho)\boldsymbol{W}_0(y,t)|_{y=e_\rho^{-1}(x)}. \tag{6.7}$$

Since

$$0 = \nabla_y \cdot \boldsymbol{W}_0 = \nabla_y \cdot (\widehat{\mathcal{L}}\boldsymbol{W}_1) = \widehat{\mathcal{L}}^T\nabla_y \cdot \boldsymbol{W}_1(e_\rho(y),t) = L\nabla_x \cdot \boldsymbol{W}_1|_{x=e_\rho(y)},$$

$$f_0 = \boldsymbol{W}_0 \cdot \boldsymbol{N} = \boldsymbol{N} \cdot \widehat{\mathcal{L}}\boldsymbol{W}_1(e_\rho(y),t) = \widehat{\mathcal{L}}^T\boldsymbol{N} \cdot \boldsymbol{W}_1 = |\widehat{\mathcal{L}}^T\boldsymbol{N}|\boldsymbol{n} \cdot \boldsymbol{W}_1\big|_{x=e_\rho(y)},$$

we see that $\nabla \cdot \boldsymbol{W}_1 = 0$ and

$$\boldsymbol{W}_1 \cdot \boldsymbol{n}|_{\Gamma_t} = \frac{f_0}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|}\Big|_{y=e_\rho^{-1}(x)}. \tag{6.8}$$

Now we pass to the estimates of $\boldsymbol{W}_1$. By (6.7),

$$\|\boldsymbol{W}_1\|_{L_2(\Omega_t)} \le c\|\boldsymbol{W}_1(e_\rho,t)\|_{L_2(\mathcal{F})} \le c\|\boldsymbol{W}_0\|_{L_2(\mathcal{F})} \le c\|f_0\|_{L_2(\mathcal{G})};$$

moreover, using the Hölder inequality and the imbedding theorem (Proposition 8.12), we obtain

$$\|\nabla\boldsymbol{W}_1\|_{L_2(\Omega_t)} \le c\big(\max|\frac{\mathcal{L}}{L}|\|\nabla\boldsymbol{W}_0\|_{L_2(\mathcal{F})} + \|\nabla\frac{\mathcal{L}}{L}\|_{L_3(\mathcal{F})}\|\boldsymbol{W}_0\|_{L_6(\mathcal{F})}\big)$$
$$\le c\|\boldsymbol{W}_0\|_{W_2^1(\mathcal{F})} \le c\|f_0\|_{W_2^{1/2}(\mathcal{G})},$$

where the constants depend on $\|\rho^*\|_{W_2^{5/2+\alpha}(\mathcal{F})}$, i.e, on $\|\rho\|_{W_2^{2+\alpha}(\mathcal{G})}$, $\alpha > 0$. In addition, we have

$$\frac{\partial\boldsymbol{W}_1(e_\rho(y),t)}{\partial t} = \frac{\partial\boldsymbol{W}_1(x,t)}{\partial t}\Big|_{x=e_\rho(y)} + \nabla\boldsymbol{W}_1(x,t)|_{x=e_\rho(y)}\boldsymbol{N}^*\rho_t^*,$$

$$\frac{\partial\boldsymbol{W}_1(e_\rho(y),t)}{\partial t} = \frac{\mathcal{L}}{L}\frac{\partial\boldsymbol{W}_0(y,t)}{\partial t} + \frac{\partial}{\partial t}(\frac{\mathcal{L}}{L})\boldsymbol{W}_0,$$

which permits to estimate the time derivative $\boldsymbol{W}_{1t}(x,t)$ as follows:

$$\|\boldsymbol{W}_{1t}(\cdot,t)\|_{L_2(\Omega_t)} \le c\Big(\sup_{\mathcal{F}}|L^{-1}\mathcal{L}|\|\boldsymbol{W}_{0t}(\cdot,t)\|_{L_2(\mathcal{F})} + \|(L^{-1}\mathcal{L})_t\|_{L_3(\mathcal{F})}\|\boldsymbol{W}_0(\cdot,t)\|_{L_6(\mathcal{F})}$$
$$+ \sup_{\mathcal{F}}|\boldsymbol{N}^*\rho_t^*|\|\nabla\boldsymbol{W}_1\|_{L_2(\Omega_t)}\Big) \le c\Big(\|f_{0t}\|_{L_2(\mathcal{G})} + \|f_0\|_{W_2^{1/2}(\mathcal{G})}\Big).$$

Now we define $\boldsymbol{W}$ by

$$\boldsymbol{W} = \boldsymbol{W}_1 - \mathrm{rot}\boldsymbol{A}, \quad \boldsymbol{A} = a(x)\sum_{k=1}^3 \boldsymbol{e}_k \int_{\Omega_t} \boldsymbol{W}_1 \cdot \boldsymbol{\eta}_k dx,$$

where $a \in C_0^\infty(\Omega_t)$, $\int_{\Omega_t} a(x)dx = \frac{1}{2}$. Since $\mathrm{rot}\boldsymbol{\eta}_j = 2\boldsymbol{e}_j$, $\boldsymbol{W}$ satisfies the condition

$$\int_{\Omega_t} \boldsymbol{W} \cdot \boldsymbol{\eta}_j dx = \int_{\Omega_t} \boldsymbol{w}_1 \cdot \boldsymbol{\eta}_j dx - \int_{\Omega_t} \boldsymbol{A} \cdot \mathrm{rot}\boldsymbol{\eta}_j dx = 0.$$

Inequalities (6.4)-(6.6) follow from the estimates of $\boldsymbol{W}_1$. The proposition is proved.

The next proposition concerns estimates of functions satisfying the conditions (1.17).

Proposition 6.2. *Assume that ρ is subject to* (1.17) *and $|\rho(\cdot,t)|_{C^1(\mathcal{G})} \leq \delta$. Then*

$$\|\rho - \rho^\perp\|_{L_2(\mathcal{G})} \leq c\delta\|\rho\|_{L_2(\mathcal{G})}, \tag{6.9}$$

$$\|\rho - \rho^\perp\|_{W_2^{-1/2}(\mathcal{G})} \leq c\delta\|\rho\|_{W_2^{-1/2}(\mathcal{G})}, \tag{6.10}$$

where $\rho^\perp$ is the projection of ρ on the space of function satisfying (1.6).

If δ is sufficiently small and (1.37) *holds, then*

$$c_1\|\rho\|_{L_2(\mathcal{G})} \leq \|\rho^\perp\|_{L_2(\mathcal{G})} \leq c_2\|\rho\|_{L_2(\mathcal{G})}, \tag{6.11}$$

$$c_1'\|\rho\|_{W_2^{-1/2}(\mathcal{G})} \leq \|\rho^\perp\|_{W_2^{-1/2}(\mathcal{G})} \leq c_2'\|\rho\|_{W_2^{-1/2}(\mathcal{G})}, \tag{6.12}$$

$$\delta^2\mathcal{R}(\rho) \geq c\|\rho\|^2_{L_2(\mathcal{G})} \tag{6.13}$$

Proof. Let $\chi_i(y) = y_i$, $i = 1,2,3$, and $\chi_4(y) = 1$. It is clear that

$$\rho = \rho^\perp + \sum_{j=1}^4 b_j\chi_j(y),$$

where $b = (b_1, b_2, b_3, b_4)$ is a solution of the system

$$\int_{\mathcal{G}} \rho\chi_j dS = \sum_{i=1}^4 b_i \int_{\mathcal{G}} \chi_i\chi_j dS \stackrel{\mathrm{def}}{=} \sum_{i=1}^4 C_{ij}b_i.$$

Since the matrix $C = \big(\int_{\mathcal{G}} \chi_i\chi_j dS\big)_{i,j=1,2,3,4}$ is invertible, we have

$$b_k = \sum_{j=1}^4 C^{kj} \int_{\mathcal{G}} \rho\chi_j dS,$$

where C^{kj} are the elements of C^{-1}. By (1.17),

$$\int_{\mathcal{G}} \rho\chi_4 dS = \int_{\mathcal{G}} \rho dS = \int_{\mathcal{G}} \big(\rho - \varphi(y,\rho)\big)dS = \int_{\mathcal{G}} \rho\mathcal{E}_4 dS,$$

$$\int_{\mathcal{G}} \rho\chi_i dS = \int_{\mathcal{G}} \rho y_i dS = \int_{\mathcal{G}} \big(\rho y_i - \psi_i(y,\rho)\big)dS = \int_{\mathcal{G}} \rho\mathcal{E}_i dS, \quad i = 1,2,3,$$

where $\mathcal{E}_4 = \frac{\rho}{2}\mathcal{H} - \frac{\rho^2}{3}\mathcal{K}$, $\mathcal{E}_i = \rho\Big(y_i\mathcal{E}_4 + N_i\big(\frac{\rho}{2} - \frac{\rho^2}{3}\mathcal{H} + \frac{\rho^3}{4}\mathcal{K}\big)\Big)$. It follows that

$$|\int_{\mathcal{G}} \rho\chi_k dS| \leq \|\rho\|_{L_2(\mathcal{G})}\|\mathcal{E}_k\|_{L_2(\mathcal{G})} \leq c\delta\|\rho\|_{L_2(\mathcal{G})},$$

$$|\int_{\mathcal{G}} \rho\chi_k dS| \leq \|\rho\|_{W_2^{-1/2}(\mathcal{G})}\|\mathcal{E}_k\|_{W_2^{1/2}(\mathcal{G})} \leq c\delta\|\rho\|_{W_2^{-1/2}(\mathcal{G})}.$$

These estimates imply (6.9)-(6.12). Finally, (6.13) follows from

$$\begin{aligned}\delta^2\mathcal{R}(\rho) = \delta^2\mathcal{R}(\rho^\perp + (\rho - \rho^\perp)) \geq \delta^2\mathcal{R}(\rho^\perp) - \delta\|\rho\|^2_{L_2(\mathcal{G})} \\ \geq c\|\rho^\perp\|^2_{L_2(\mathcal{G})} - \delta\|\rho\|^2_{L_2(\mathcal{G})}\end{aligned}$$

and from (6.11). The proposition is proved.

Proof of Proposition 5.6.

We multiply the first equation in (6.1) by $\boldsymbol{w}$ and integrate over Ω. We make use of the relation

$$\frac{d}{dt}\int_{\Omega_t} f(x,t)dx = \int_{\Omega_t} (f_t(x,t) + \boldsymbol{w}\cdot\nabla)f(x,t)dx$$

that can be easily verified by the passage to the Lagrangian coordinates. We observe that $-\nu\nabla^2\boldsymbol{w} + \nabla s = -\nabla\cdot T(\boldsymbol{w}, s)$ and we integrate by parts, using the boundary conditions. As a result, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L_2(\Omega_t)} - \int_{\Gamma_t}\Big(\kappa U + \frac{\omega^2}{2}|x'|^2\Big)\boldsymbol{w}\cdot\boldsymbol{n}dS + \frac{\nu}{2}\|S(\boldsymbol{w})\|^2_{L_2(\Omega)} = 0.$$

We have

$$\int_{\Gamma_t}|x'|^2\boldsymbol{w}\cdot\boldsymbol{n}dS = \int_{\Omega_t}(\boldsymbol{w}\cdot\nabla)|x'|^2dx = \frac{d}{dt}\int_{\Omega_t}|x'|^2dx,$$

$$\int_{\Gamma_t}U\boldsymbol{w}\cdot\boldsymbol{n}dS = \int_{\Omega_t}(\boldsymbol{w}\cdot\nabla)U(x,t)dx = \frac{d}{dt}\int_{\Omega_t}Udx - \int_{\Omega_t}U_tdx.$$

Since

$$\int_{\Omega_t}U_tdx = \int_{\Omega_t}dx\int_{\Omega_t}(\boldsymbol{w}(y,t)\cdot\nabla_y)\frac{1}{|x-y|}dy = \int_{\Omega_t}(\boldsymbol{w}(y,t)\cdot\nabla_y)U(y,t)dy,$$

we can conclude that

$$\int_{\Gamma_t}U\boldsymbol{w}\cdot\boldsymbol{n}dS = \frac{1}{2}\frac{d}{dt}\int_{\Omega_t}U(x,t)dx.$$

Putting all the terms together, we obtain the energy relation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\|\boldsymbol{w}\|^2_{L_2(\Omega_t)} - \frac{\omega^2}{2}\int_{\Omega_t}|x'|^2dx - \frac{\kappa}{2}\int_{\Omega_t}U(x,t)dx\Big) \\ + \frac{\nu}{2}\|S(\boldsymbol{w})\|^2_{L_2(\Omega_t)} = 0.\end{aligned} \tag{6.14}$$

We can write (6.14) in another way. Since $\int_{\Omega_t}x_idx = 0$, (6.2) implies

$$\boldsymbol{w}(x,t) = \boldsymbol{w}^\perp(x,t) + \sum_{k=1}^{3}g_k(t)\boldsymbol{\eta}_k(x), \tag{6.15}$$

where $\boldsymbol{w}^\perp$ is the vector field orthogonal to all the vectors of rigid motion

$$\boldsymbol{\eta}(x) = \boldsymbol{a} + \boldsymbol{b}\times x,$$

i.e.,

$$\int_{\Omega_t} \boldsymbol{w}^{\perp}(x,t)dx = 0, \quad \int_{\Omega_t} \boldsymbol{w}^{\perp}(x,t)\cdot\boldsymbol{\eta}_i(x)dx = 0, \quad i=1,2,3.$$

These conditions yield the following relations for $g_k(t)$:

$$\begin{aligned}\sum_{k=1}^{3} S_{jk}(t)g_k(t) &= \int_{\Omega_t} \boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_j(x)dx \\ &= -\omega S_{3j}(t) + \omega\int_{\mathcal{F}} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_j(x)dx = \beta\delta_{3j} - \omega S_{3j}(t), \quad j=1,2,3,\end{aligned}$$

where $\beta = \omega\int_{\mathcal{F}} |x'|^2dx$ and $S_{jk}(t) = \int_{\Omega_t} \boldsymbol{\eta}_j(x)\cdot\boldsymbol{\eta}_k(x)dx$. The matrix $\mathcal{S}(t)$ with the elements S_{jk} is symmetric and positive definite, hence

$$g_k(t) = \sum_{m=1}^{3} S^{km}(t)(\beta\delta_{3m} - \omega S_{3m}(t)) = \omega\sum_{m=1}^{3} S^{km}(t)\Big(\int_{\mathcal{F}} \boldsymbol{\eta}_3(y)\cdot\boldsymbol{\eta}_m(y)dy - \int_{\Omega_t} \boldsymbol{\eta}_3(x)\cdot\boldsymbol{\eta}_m(x)dx\Big), \tag{6.16}$$

where S^{km} are the elements of $\mathcal{S}^{-1}$.

By (6.15) and (6.16),

$$\begin{aligned}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L_2(\Omega_t)} &= \frac{d}{dt}\Big(\|\boldsymbol{w}^{\perp}\|^2_{L_2(\Omega_t)} + \sum_{j,k=1}^{3} g_k g_j S_{jk}\Big) \\ &= \frac{d}{dt}\Big(\|\boldsymbol{w}^{\perp}\|^2_{L_2(\Omega_t)} + \sum_{j,k=1}^{3} S^{kj}(\beta\delta_{3k} - \omega S_{3k}(t))(\beta\delta_{3j} - \omega S_{3j}(t))\Big) \\ &= \frac{d}{dt}\Big(\|\boldsymbol{w}^{\perp}\|^2_{L_2(\Omega_t)} + S^{33}(t)\beta^2 + S_{33}(t)\omega^2 - 2\beta\omega\Big) \\ &= \frac{d}{dt}\Big(\|\boldsymbol{w}^{\perp}\|^2_{L_2(\Omega_t)} + \frac{\beta^2}{S_{33}} + \beta^2(S^{33}(t) - \frac{1}{S_{33}}) + S_{33}(t)\omega^2 - 2\beta\omega\Big).\end{aligned}$$

The expression

$$\beta^2(S^{33}(t) - \frac{1}{S_{33}}) = -\frac{\beta^2}{S_{33}}\sum_{j=1}^{2} S^{j3}S_{j3} = \frac{\beta^2}{S_{33}\mathrm{det}S}(S_{11}S_{23}^2 + S_{22}S_{13}^2 - 2S_{12}S_{13}S_{23}) \overset{\mathrm{def}}{=} 2Q(t)$$

is a positive definite quadratic form with respect to S_{13}, S_{23}, since $2S_{12} < \sqrt{S_{11}}\sqrt{S_{22}}$. Hence (6.14) may be written in the form

$$\frac{d}{dt}\big(\frac{1}{2}\|\boldsymbol{w}^{\perp}\|^2_{L_2(\Omega_t)} + Q(t) + \mathcal{R}(t) - \mathcal{R}_0\big) + \frac{\nu}{2}\|\boldsymbol{S}(w^{\perp})\|^2_{L_2(\Omega_t)} = 0 \tag{6.17}$$

where

$$\mathcal{R}(t) = \frac{\beta^2}{2\int_{\Omega_t}|x'|^2dx} - \frac{\kappa}{2}\int_{\Omega_t}\int_{\Omega_t}\frac{dxdy}{|x-y|} - p_0|\Omega_t|, \tag{6.18}$$

is the energy functional and

$$\mathcal{R}_0 = \frac{\beta^2}{2\int_{\mathcal{F}}|x'|^2dx} - \frac{\kappa}{2}\int_{\mathcal{F}}\int_{\mathcal{F}}\frac{dxdy}{|x-y|} - p_0|\mathcal{F}|.$$

Since Γ_t is given by the equation (1.14), we can consider $\mathcal{R}(t)$ as the functional depending on ρ and write

$$\mathcal{R}(t) = \mathcal{R}(\rho), \quad \mathcal{R}_0 = \mathcal{R}(0).$$

The difference $\mathcal{R}(\rho) - \mathcal{R}(0)$ can be expressed as

$$\mathcal{R}(\rho) - \mathcal{R}(0) = \delta\mathcal{R}(\rho) + \frac{1}{2}\delta^2\mathcal{R}(\rho) + \int_0^1 (1-s)\Big(\frac{d^2\mathcal{R}(s\rho)}{ds^2} - \frac{d^2\mathcal{R}(s\rho)}{ds^2}\Big|_{s=0}\Big)ds. \tag{6.19}$$

As shown is Sec. 8, $\delta\mathcal{R}(\rho) = 0$, hence

$$\frac{1}{2}\frac{d}{dt}\big(\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)} + 2Q(t) + \delta^2\mathcal{R}(\rho) + 2\mathcal{R}_1\big) + \frac{\nu}{2}\|\boldsymbol{S}(w^\perp)\|^2_{L_2(\Omega_t)} = 0. \tag{6.20}$$

It can be proved that the remainder

$$\mathcal{R}_1(\rho) = \int_0^1 (1-s)\Big(\frac{d^2\mathcal{R}(s\rho)}{ds^2} - \frac{d^2\mathcal{R}(s\rho)}{ds^2}\Big|_{s=0}\Big)ds$$

satisfies the inequality

$$|\mathcal{R}_1| \le c\|\rho\|_{C^1(\mathcal{G})}\|\rho\|^2_{L_2(\mathcal{G})} \le c\delta\|\rho\|^2_{L_2(\mathcal{G})}. \tag{6.21}$$

We supplement (6.17) with one more relation. We write the first equation in (6.1) in the form

$$\begin{aligned} &\boldsymbol{w}_t^\perp + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w}^\perp + (\boldsymbol{w}^\perp\cdot\nabla)\boldsymbol{w}' + (\boldsymbol{w}'\cdot\nabla)\boldsymbol{w}' + 2\omega(\boldsymbol{e}_3\times\boldsymbol{w}^\perp) \\ &\quad + 2\omega(\boldsymbol{e}_3\times\boldsymbol{w}') - \nu\nabla^2\boldsymbol{w}^\perp + \nabla s = -\boldsymbol{w}'_t, \end{aligned} \tag{6.22}$$

where $\boldsymbol{w}' = \boldsymbol{w} - \boldsymbol{w}^\perp = \sum_{k=1}^3 g_k(t)\boldsymbol{\eta}_k(x)$. We notice that

$$(\boldsymbol{w}'\cdot\nabla)w'_i = \sum_{k=1}^3 w'_k\frac{\partial w'_i}{\partial x_k} = -\sum_{k=1}^3 w'_k\frac{\partial w'_k}{\partial x_i},$$

hence

$$(\boldsymbol{w}'\cdot\nabla)\boldsymbol{w}' = -\frac{1}{2}\nabla|\boldsymbol{w}'|^2,$$

moreover,

$$2(\boldsymbol{e}_3\times\boldsymbol{\eta}_i) = -\nabla(\boldsymbol{\eta}_i\cdot\boldsymbol{\eta}_3) + \boldsymbol{\eta}^i, \quad i = 1,2,3$$

with $\boldsymbol{\eta}^1 = \boldsymbol{\eta}_2$, $\boldsymbol{\eta}^2 = -\boldsymbol{\eta}_1$, $\boldsymbol{\eta}^3 = 0$. It follows that (6.22) is equivalent to

$$\begin{aligned} &\boldsymbol{w}_t^\perp + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w}^\perp + (\boldsymbol{w}^\perp\cdot\nabla)\boldsymbol{w}' + 2\omega(\boldsymbol{e}_3\times\boldsymbol{w}^\perp) - \nu\nabla^2\boldsymbol{w}^\perp \\ &\quad + \nabla\Big(s - \omega\sum_{j=1}^3 g_j(t)\boldsymbol{\eta}_j\cdot\boldsymbol{\eta}_3 - \frac{1}{2}|\boldsymbol{w}'|^2\Big) = -\boldsymbol{w}'_t - \omega\sum_{j=1}^3 g_k(t)\boldsymbol{\eta}^k. \end{aligned} \tag{6.23}$$

We multiply this equation by the vector field $\boldsymbol{W}$ constructed in Proposition 6.1 (the function f_0 is going be chosen later) and integrate over Ω_t. Since $\int_{\Omega_t}\big(\boldsymbol{w}'_t + \omega\sum_{j=1}^3 g_k(t)\boldsymbol{\eta}^k\big)\cdot\boldsymbol{W}\,dx = 0$, we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{W}dx-\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\big(\boldsymbol{W}_t+(\boldsymbol{w}\cdot\nabla)\boldsymbol{W})\big)dx+2\omega\int_{\Omega_t}(\boldsymbol{e}_3\times\boldsymbol{w}^\perp)\cdot\boldsymbol{W}dx\\
&\quad+\frac{\nu}{2}\int_{\Omega_t}S(\boldsymbol{w}^\perp):S(\boldsymbol{W})dx+\int_{\Omega_t}(\boldsymbol{w}^\perp\cdot\nabla)\boldsymbol{w}'\cdot\boldsymbol{W}dx\\
&\quad-\int_{\Gamma_t}(M+\omega\sum_{j=1}^3 g_j\boldsymbol{\eta}_j\cdot\boldsymbol{\eta}_3+\frac{1}{2}|\boldsymbol{w}'|^2)\boldsymbol{W}\cdot\boldsymbol{n}dS=0;
\end{aligned}
\tag{6.24}
$$

we recall that $M=-B_0\rho+B_1\rho$ where B_0 and B_1 are given by (4.1) and (5.3) (with $\sigma=0$).

Now we multiply (6.24) by a small $\gamma>0$ and add to (6.21). This leads to

$$
\frac{dE(t)}{dt}+E_1(t)=0,
\tag{6.25}
$$

where

$$
E(t)=\frac{1}{2}\big(\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+2Q(t)+\delta^2\mathcal{R}(\rho)+2\mathcal{R}_1(\rho)\big)+\gamma\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{W}dx,
\tag{6.26}
$$

$$
\begin{aligned}
E_1(t)=&\frac{\nu}{2}\|S(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)}-\gamma\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\big(\boldsymbol{W}_t+(\boldsymbol{w}\cdot\nabla)\boldsymbol{W}\big)dx\\
&+2\omega\gamma\int_{\Omega_t}(\boldsymbol{e}_3\times\boldsymbol{w}^\perp)\cdot\boldsymbol{W}dx+\frac{\nu\gamma}{2}\int_{\Omega_t}S(\boldsymbol{w}^\perp):S(\boldsymbol{W})dx\\
&+\gamma\int_{\Omega_t}(\boldsymbol{w}^\perp\cdot\nabla)\boldsymbol{w}'\cdot\boldsymbol{W}dx-\gamma J(t),
\end{aligned}
\tag{6.27}
$$

and $J(t)$ is the surface integral in (6.24).

We pass to the estimates of $E(t)$ and $E_1(t)$. We have

$$
|\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{W}dx|\le\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\|\boldsymbol{W}\|_{L_2(\Omega_t)}\le c\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\|\rho^\perp\|_{L_2(\mathcal{G})},
$$

hence, by Proposition 6.2 and inequalities (1.37), (6.21),

$$
E(t)\ge c\Big(\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+\|\rho\|^2_{L_2(\mathcal{G})}\Big)\ge c\Big(\|\boldsymbol{w}\|^2_{L_2(\Omega_t)}+\|\rho\|^2_{L_2(\mathcal{G})}\Big),
$$

if $\gamma>0$ and $\delta>0$ are small.

Now we consider the surface integral J that can be written as the integral over $\mathcal{G}$ (see Proposition 8.25):

$$
J=\int_{\mathcal{G}}(M+\omega\sum_{j=1}^3 g_j(t)\boldsymbol{\eta}_j\cdot\boldsymbol{\eta}_3+\frac{1}{2}|\boldsymbol{w}'|^2)f_0(y)dS.
$$

In view of (6.16), $g_k(t)=-\omega\sum_{k=1}^3 S_0^{km}\int_{\mathcal{G}}\rho\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_m dS+g_k'(t)$, where

$$
g_k'(t)=-\omega\sum_{k=1}^3(S^{km}(t)\Big(\int_{\Omega_t}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_m dx-\int_{\mathcal{F}}\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_m\Big)+S_0^{km}(t)\int_{\mathcal{G}}\rho\boldsymbol{\eta}_3\cdot\boldsymbol{\eta}_m dS
$$

and S_0^{km} are the elements of the matrix inverse to $\mathcal{S}_0 = (\int_{\mathcal{F}} \boldsymbol{\eta}_k \cdot \boldsymbol{\eta}_m dx)_{k,m=1,2,3}$, i.e.,

$$S_0^{km} = \delta_{km} \|\boldsymbol{\eta}_m\|_{L_2(\mathcal{F})}^{-2}.$$

It follows that

$$-J = \int_{\mathcal{G}} B'\rho f_0 dS - J' = \int_{\mathcal{G}} \rho B' f_0 dS - J'$$

where

$$B'f = B_0 f + \omega^2 \sum_{k=1}^{3} \frac{\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_k(x)}{\|\eta_k\|_{L_2(\mathcal{F})}^2} \int_{\mathcal{G}} f(y,t) \boldsymbol{\eta}_3(y) \boldsymbol{\eta}_k(y) dS,$$

$$J' = \int_{\mathcal{G}} \Big(B_1 \rho + \frac{1}{2} |\boldsymbol{w}'|^2 + \sum_{k=1}^{3} g_k'(t) \boldsymbol{\eta}_k \cdot \boldsymbol{\eta}_3(x) \Big) f_0 dS.$$

By (1.37), the integral operator B' possesses the property $\int_{\mathcal{G}} \rho B' \rho dS \geq c\|\rho\|_{L_2(\mathcal{G})}^2$ for arbitrary ρ satisfying (1.6). Hence the equation

$$P_0 B' P_0 f = g \tag{6.28}$$

is uniquely solvable for arbitrary $g \in L_2(\mathcal{G})$ satisfying the same condition; moreover, if $g \in W_2^{1/2}(\mathcal{G})$, then

$$\|f\|_{W_2^{1/2}(\mathcal{G})} \leq c\|g\|_{W_2^{1/2}(\mathcal{G})}.$$

We define f_0 as a solution of (6.28) with $g = P_0(-\Delta_{\mathcal{G}})^{-1/2} \rho^{\perp}$. Then

$$\|f_0\|_{W_2^{1/2}(\mathcal{G})} \leq c\|\rho^{\perp}\|_{W_2^{-1/2}(\mathcal{G})} \leq c\|\rho\|_{W_2^{-1/2}(\mathcal{G})};$$

moreover, using the equation (8.48) for the estimate of g_j' and $\boldsymbol{w}'$ and the inequality

$$\Big| \int_{\mathcal{G}} B_1 \rho f_0 dS \Big| \leq c\delta \|\rho\|_{W_2^{-1/2}(\mathcal{G})}^2$$

(whose proof is omitted), we can show that

$$|J'| \leq c\delta \|\rho\|_{W_2^{-1/2}(\mathcal{G})}^2.$$

This implies

$$-J \geq \int_{\mathcal{G}} \rho P_0 (-\Delta)^{-1/2} \rho^{\perp} dS - c\delta \|\rho\|_{W_2^{-1/2}(\mathcal{G})}^2.$$

Since the expression $\int_{\mathcal{G}} \rho^{\perp} P_0 (-\Delta)^{-1/2} \rho^{\perp} dS$ is equivalent to $\|\rho^{\perp}\|_{W_2^{-1/2}(\mathcal{G})}^2$, we have

$$-J \geq c\|\rho\|_{W_2^{-1/2}(\mathcal{G})}^2,$$

if δ is small.

Now we estimate $E_1(t)$ from below. It is easily seen that

$$\Big| \int_{\Omega_t} (\boldsymbol{e}_3 \times \boldsymbol{w}^{\perp}) \cdot \boldsymbol{W} dx \Big| + \Big| \int_{\Omega_t} S(\boldsymbol{w}^{\perp}) : S(\boldsymbol{W}) dx \Big| + \Big| \int_{\Omega_t} (\boldsymbol{w}^{\perp} \cdot \nabla) \boldsymbol{w}' \cdot \boldsymbol{W} dx \Big|$$

$$\leq c\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)}\|\boldsymbol{W}\|_{W_2^1(\Omega_t)} \leq c\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)}\|\rho\|_{W_2^{-1/2}(\mathcal{G})},$$

$$\Big|\int_{\mathcal{F}} \boldsymbol{w}^\perp \cdot (\boldsymbol{w}\cdot\nabla)\boldsymbol{W}dx\Big| \leq c\|\boldsymbol{w}^\perp\|_{L_6(\Omega_t)}\|\boldsymbol{W}\|_{W_2^1(\Omega_t)}\|\boldsymbol{w}\|_{L_3(\Omega_t)} \leq c\|\boldsymbol{w}^\perp\|_{L_6(\Omega_t)}\|\boldsymbol{W}\|_{W_2^1(\Omega_t)}$$

with the constant proportional to $\|\boldsymbol{w}\|_{L_3(\Omega_t)}$. Moreover, in view of (6.6) and of the equation

$$\rho_t = \frac{\boldsymbol{w}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{\Lambda(\rho)}$$

($\Lambda(\rho) = \boldsymbol{N}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}$ is computed in Sec. 8), we have

$$\|\boldsymbol{W}_t\|_{L_2(\Omega_t)} \leq c\|f_{0t}\|_{L_2(\mathcal{G})} \leq c\|(-\Delta)^{-1/2}P_0\rho_t\|_{L_2(\mathcal{G})} \leq c\|\boldsymbol{w}\|_{L_2(\Gamma_t)}$$

$$\leq c\Big(\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)} + \|\rho\|_{W_2^{-1/2}(\mathcal{G})}\Big)$$

and, as a consequence,

$$\Big|\int_{\Omega_t} \boldsymbol{w}^\perp\cdot\boldsymbol{W}_t dx\Big| \leq c\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\Big(\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)} + \|\rho\|_{W_2^{-1/2}(\mathcal{G})}\Big).$$

These estimates imply

$$E_1(t) \geq \frac{\nu}{2}\|S(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)} + \gamma\|\rho\|^2_{W_2^{-1/2}(\mathcal{G})} - c\gamma\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\Big(\|\boldsymbol{w}^\perp\|_{W_2^1(\Omega_t)} + \|\rho\|_{W_2^{-1/2}(\mathcal{G})}\Big).$$

Due to the Korn inequality, $\|w^\perp\|_{W_2^1(\Omega_t)} \leq \|S(\boldsymbol{w}^\perp)\|_{L_2(\Omega_t)}$ we obtain

$$E_1(t) \geq c\Big(\|\boldsymbol{w}^\perp\|^2_{W_2^1(\Omega_t)} + \gamma\|\rho\|^2_{W_2^{-1/2}(\mathcal{G})}\Big),$$

if γ is sufficiently small. This completes the proof of Proposition 5.6.

The generalized energy can be also estimated in the case $\sigma > 0$, which gives the inequality

$$\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^1(\mathcal{G})} \leq ce^{-\beta t}\Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_0)} + \|\rho_0\|^2_{W_2^1(\mathcal{G})}\Big) \tag{6.29}$$

with $\beta > 0$. But the inequality of this type has been already obtained in Sec. 3, and we omit the proof of (6.29).

It should be noted that in the case $\sigma > 0$

$$\mathcal{R}(t) = \sigma|\Gamma_t| + \frac{\beta^2}{2\int_{\Omega_t}|x'|^2dx} - \frac{\kappa}{2}\int_{\Omega_t}\int_{\Omega_t}\frac{dxdy}{|x-y|} - p_0|\Omega_t|, \tag{6.30}$$

7 Inversion of the Lagrange theorem in the problem of stability of rotating viscous incompressible liquid

In this section we assume that the quadratic form (1.4) can take negative values for some ρ satisfying (1.6), and we show that in this case the linear homogoneous problem (1.24)-(1.27) has solutions growing exponentially as $t \to \infty$. By analogy with classical mechanics, statements of such type are referred to as "inversion of the Lagrange theorem".

The proof is based on the construction of the Lyapunov function, as it has been done in Sec. 6. We follow the paper [17].

We require that

$$\mathcal{S} = \int_{\mathcal{F}} (x_1^2 - x_3^2)dx = \int_{\mathcal{F}} (x_2^2 - x_3^2)dx > 0 \tag{7.1}$$

which means that $\mathcal{F}$ is an oblate spheroid.

First we need to prove some auxiliary propositions.

Proposition 7.1 *For arbitrary vector field of rigid motion $\boldsymbol{\eta}(x) = \boldsymbol{a} + \boldsymbol{b} \times x$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constant vectors, the equation*

$$B_0(\boldsymbol{\eta}(x) \cdot \boldsymbol{N}(x)) = -\omega^2 \boldsymbol{\eta}(x) \cdot x', \tag{7.2}$$

holds, where $x' = (x_1, x_2, 0)$.

Proof. We take an arbitrary small smooth function $r(x)$ and consider the integral

$$I[r] = \int_\Gamma \Big(\sigma H(x) + \frac{\omega^2}{2}|x'|^2 + \kappa U(x) + p_0\Big)\boldsymbol{\eta}(x) \cdot \boldsymbol{n} dS,$$

where $U(x) = \int_\Omega |x-y|^{-1}dy$ and Ω is a domain whose boundary Γ is given by the equation

$$x = y + \boldsymbol{N}(y)r(y), \quad y \in \mathcal{G}.$$

By $\boldsymbol{n}$ and H we mean the exterior normal to Γ and the doubled mean curvature of Γ, respectively. It can be shown that only the term containing ω^2 is different from zero; indeed, we have $\int_\Gamma \boldsymbol{\eta} \cdot \boldsymbol{n} dS = 0$,

$$\begin{aligned}
&\int_\Gamma H\boldsymbol{\eta} \cdot \boldsymbol{n} dS = \int_\Gamma \Delta_\Gamma x \cdot \boldsymbol{\eta}(x) dS = -\int_\Gamma \nabla_\Gamma x : \nabla_\Gamma \boldsymbol{\eta}(x) dS = 0,\\
&\int_\Gamma U(x)\boldsymbol{n}(x) dS = \int_\Omega \nabla U(x) dx = \int_\Omega\int_\Omega \frac{y-x}{|y-x|^3} dx dy = 0,\\
&\int_\Gamma U(x)\boldsymbol{\eta}_i(x) \cdot \boldsymbol{n}(x) dS = \int_\Omega \nabla U(x) \cdot \boldsymbol{\eta}_i(x) dx\\
&\quad = \int_\Omega\int_\Omega \frac{y-x}{|y-x|^3} \cdot \boldsymbol{\eta}_i(x-y) dx dy + \int_\Omega\int_\Omega \frac{y-x}{|y-x|^3} \cdot \boldsymbol{\eta}_i(y) dx dy\\
&\quad = -\int_\Omega \nabla U(y) \cdot \boldsymbol{\eta}_i(y) dy,
\end{aligned}$$

from which we can conclude that

$$\int_\Gamma U(x)\boldsymbol{\eta}_i(x) \cdot \boldsymbol{n}(x) dS = 0.$$

Hence

$$I[r] = \frac{\omega^2}{2}\int_\Gamma |x'|^2 \boldsymbol{\eta}(x)\cdot \boldsymbol{n}(x)dS = \omega^2 \int_\Omega \boldsymbol{\eta}(x)\cdot x' dx$$

and $I[0] = 0$. Taking the first variation of both parts of this equation with respect to r we find (see Sec. 8)

$$-\int_{\mathcal{G}} B_0 r \boldsymbol{\eta}(x)\cdot \boldsymbol{N}(x)dS = \omega^2 \int_{\mathcal{G}} r(x)\boldsymbol{\eta}(x)\cdot x' dS,$$

i.e.,

$$\int_{\mathcal{G}} r(x) B_0(\boldsymbol{\eta}(x)\cdot \boldsymbol{N}(x))dS = -\omega^2 \int_{\mathcal{G}} r(x)\boldsymbol{\eta}(x)\cdot x' dS.$$

Since $r(x)$ is arbitrary, we can conclude that (7.2) holds. The proposition is proved. ■

From

$$\int_{\mathcal{G}} |x'|^2 \boldsymbol{\eta}(x)\cdot \boldsymbol{N}(x)dS = 2\int_{\mathcal{F}} \boldsymbol{\eta}(x)\cdot x' dx = 0$$

it follows that also

$$B(\boldsymbol{\eta}\cdot \boldsymbol{N}) = -\omega^2 \boldsymbol{\eta}(x)\cdot x'. \tag{7.3}$$

Direct computations show that

$$\begin{gathered}\int_{\mathcal{G}} \boldsymbol{\eta}_1(x)\cdot \boldsymbol{N}(x)x_2x_3 dS = \mathcal{S}, \quad \int_{\mathcal{G}} \boldsymbol{\eta}_2(x)\cdot \boldsymbol{N}(x)x_1x_3 dS = -\mathcal{S}, \\ \int_{\mathcal{G}} \boldsymbol{\eta}_1(x)\cdot \boldsymbol{N}(x)x_1x_3 dS = \int_{\mathcal{G}} \boldsymbol{\eta}_2(x)\cdot \boldsymbol{N}(x)x_2x_3 dS = 0.\end{gathered} \tag{7.4}$$

Indeed,

$$\begin{aligned}\int_{\mathcal{G}} \boldsymbol{\eta}_1(x)\cdot \boldsymbol{N}(x)x_2x_3 dS &= \int_{\mathcal{F}} \boldsymbol{\eta}_1(x)\cdot \nabla(x_2x_3)dx \\ = \int_{\mathcal{G}} (\boldsymbol{e}_3x_2 - \boldsymbol{e}_2x_3)(\boldsymbol{e}_3x_2 + \boldsymbol{e}_2x_3)dx &= \int_{\mathcal{F}} (x_2^2 - x_3^2)dx,\end{aligned}$$

and other equations are verified in the same way.

As a consequence, we obtain

$$\int_{\mathcal{G}} \boldsymbol{\eta}_1 \cdot \boldsymbol{N} B(\boldsymbol{\eta}_1\cdot \boldsymbol{N}\Big)dS = \omega^2 \int_{\mathcal{G}} x_2x_3\boldsymbol{\eta}_1\cdot \boldsymbol{N}dS = \omega^2\mathcal{S},$$

which shows that (7.1) is necessary for the positivity of $\int_{\mathcal{G}} \rho B\rho dS = \delta^2\mathcal{R}(\rho)$.

We denote by H the subspace of functions from $L_2(\mathcal{G})$ satisfying (1.6), and we set $(f,g) = \int_{\mathcal{G}} f(x)g(x)dS$.

Using the relations (7.4), we can easily prove

Proposition 7.2 *An arbitrary $\rho \in L_2(\mathcal{G})$ can be represented in the form*

$$\rho(x) = \rho_1(x) + \rho_2(x) \tag{7.5}$$

where

$$\rho_1(x) = \mathcal{S}^{-1}(\boldsymbol{\eta}_1(x)\cdot \boldsymbol{N}(x)I_2(\rho) - \boldsymbol{\eta}_2(x)\cdot \boldsymbol{N}(x)I_1(\rho)),$$

$$I_\alpha(\rho) = \int_{\mathcal{G}} \rho(x) x_3 x_\alpha dS, \quad \alpha = 1, 2,$$

and ρ_2 satisfies the orthogonality conditions

$$\int_{\mathcal{G}} \rho_2(y) y_3 y_1 dS = \int_{\mathcal{G}} \rho_2(y) y_3 y_2 dS = 0. \tag{7.6}$$

If $\rho \in H$, then $\rho_2 \in H$. If $\rho = \boldsymbol{\eta}_\beta \cdot \boldsymbol{N}$, $\beta = 1, 2$, then $\rho_2 = 0$.

The equation (7.5) defines a non-orthogonal projection Q on the space (7.6):

$$Q\rho = \rho - \mathcal{S}^{-1}\Big(\boldsymbol{\eta}_1(x) \cdot \boldsymbol{N}(x) I_2(\rho) - \boldsymbol{\eta}_2(x) \cdot \boldsymbol{N}(x) I_1(\rho)\Big) = \rho_2,$$

and it is easily seen that $Q\rho \in H$, if $\rho \in H$.

By (7.3) and (7.4), we have

$$(B\rho, \rho) = (B\rho_1, \rho_1) + (B\rho_2, \rho_2) \tag{7.7}$$

and

$$(B\rho_1, \rho_1) = \omega^2 \mathcal{S}^{-1} \sum_{\alpha=1}^{2} I_\alpha^2(\rho).$$

By (7.1), this quadratic form is positive, whence

$$(B\rho, \rho) \geq (B\rho_2, \rho_2) = (BQ\rho, Q\rho).$$

This inequality shows that $(BQ\rho, Q\rho) < 0$, if $(B\rho, \rho) < 0$.

Let H_1 be a subspace of functions from $L_2(\mathcal{G})$ orthogonal to 1, x_1, x_2, x_3, x_3x_1, x_3x_2. We also introduce the space $H_2 \equiv H_1 \ominus Ker|_{H_1}B$, where $Ker|_{H_1}B$ is a finite dimensional space of elements of H_1 satisfying the equation $B\rho = 0$. Since $(BQ\rho, Q\rho) = (BP_2Q\rho, P_2Q\rho)$, where P_2 is the orthogonal projection on H_2, we see that if the quadratic form $(B\rho, \rho)$ can take negative values for some $\rho \in H$, then the same is true for the form (Br, r), $r \in H_2$. This is equivalent to the fact that the operator B restricted to the space H_2, i.e., the operator $B_2 = P_2BP_2$, has negative eigenvalues.

It is easily seen that only a finite number of such eigenvalues may exist. Indeed, in the case $\sigma > 0$ B_2 is a self-adjoint elliptic integro-differential operator with the principal part $-\sigma\Delta_{\mathcal{G}}$, and its spectrum consists of a countable number of real eigenvalues having an accumulation point at $+\infty$.

If $\sigma = 0$, then $B_2 r = b(x)r(x) + \int_{\mathcal{G}} K(x, y)\rho(y) dS$ with $b(x) > 0$, and with a weakly singular symmetric kernel K. The operators of this type also have at most a finite number of negative eigenvalues.

Let H_- be a finite-dimensional subspace of H_2 spanned by the eigenfunctions of B_2 corresponding to the negative eigenvalues, and $H_+ = H_2 \ominus H_-$. Since the elements of H_- are regular functions, $\rho \in W_2^1(\mathcal{G})$, $\rho \in H_2$ implies $P_+\rho = \rho - P_-\rho \in W_2^1(\mathcal{G})$, where $P_\pm$ are projections on $H_\pm$. It is clear that $(B_2\rho, \rho) < 0$ for arbitrary non-zero $\rho \in H_-$ and $(B_2\rho, \rho) > 0$ for $\rho \in H_+$ (in the case $\sigma > 0$ for $\rho \in W_2^1(\mathcal{G}) \cap H_+$).

The quadratic forms $\pm(B\rho_\pm, \rho_\pm)$ are equivalent to $\|\rho_\pm\|^2_{W_2^1(\mathcal{G})}$, if $\sigma > 0$, and to $\|\rho_\pm\|^2_{L_2(\mathcal{G})}$, if $\sigma = 0$.

Arbitrary $r, s \in H_2$ ($r, s \in H_2 \cap W_2^1(\mathcal{G})$, if $\sigma > 0$,) satisfy the relation

$$(Br, s) = (BP_+r, P_+s) + (BP_-r, P_-s).$$

Now, we pass to the construction of the Lyapunov function. We transform the problem (1.24)-(1.27). By (1.26),

$$\boldsymbol{v}(x,t) = \boldsymbol{v}^\perp(x,t) + \sum_{i=1}^{3} d_i(\rho)\boldsymbol{\eta}_i(x),$$

where $\boldsymbol{v}^\perp$ is a vector field orthogonal to all the vectors of rigid motion $\boldsymbol{\eta} = \boldsymbol{a} + \boldsymbol{b} \times x$ and

$$d_i(\rho) = -\frac{\omega}{\|\boldsymbol{\eta}_i\|^2_{L_2(\mathcal{F})}} \int_{\mathcal{G}} \rho(y,t)\boldsymbol{\eta}_i(y) \cdot \boldsymbol{\eta}_3(y)dS.$$

We introduce the functions

$$\boldsymbol{u}(x,t) = \boldsymbol{v}(x,t) - d_3(\rho)\boldsymbol{\eta}_3(x), \quad q(x,t) = p(x,t) - \omega d_3(\rho)|x'|^2$$

and write (1.24) in the form

$$\begin{cases} \boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = -d_3(\rho_t)\boldsymbol{\eta}_3(x), \\ \nabla \cdot \boldsymbol{u}(x,t) = 0, \quad x \in \mathcal{F}, \ \ t > 0, \\ T(\boldsymbol{u}, q)\boldsymbol{N} + \boldsymbol{N}B\rho = 0, \\ \rho_t = \boldsymbol{N}(x) \cdot \boldsymbol{u}(x,t), \quad x \in \mathcal{G}, \\ \boldsymbol{u}(x,0) = \boldsymbol{v}_0(x) - \boldsymbol{u}(x,0) \equiv \boldsymbol{u}_0(x), \quad x \in \mathcal{F}, \quad \rho(x,0) = \rho_0(x), \quad x \in \mathcal{G}. \end{cases} \tag{7.8}$$

Orthogonality conditions (1.26) remain invariant and (1.27) are converted into

$$\begin{aligned} &\int_{\mathcal{F}} \boldsymbol{u}(x,t)dx = 0, \quad \int_{\mathcal{F}} \boldsymbol{u}(x,t) \cdot \boldsymbol{\eta}_3(x)dx = 0, \\ &\int_{\mathcal{F}} \boldsymbol{u}(x,t) \cdot \boldsymbol{\eta}_\alpha(x)dx + \omega \int_{\mathcal{G}} \rho(x,t)\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_\alpha(x)dS = 0, \quad \alpha = 1, 2. \end{aligned} \tag{7.9}$$

We multiply the first equation in (7.8) by $\boldsymbol{u}$ and integrate over $\mathcal{F}$. Then we integrate by parts and make use of the boundary conditions. This leads to

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\mathcal{F})} + (B\rho, \boldsymbol{u} \cdot \boldsymbol{N}) + \frac{\nu}{2}\|S(\boldsymbol{u})\|^2_{L_2(\mathcal{F})} = 0.$$

Since $(B\rho, \boldsymbol{u} \cdot \boldsymbol{N}) = (B\rho, \rho_t)$, we obtain the energy relation

$$\frac{1}{2}\frac{d}{dt}\Big(\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\mathcal{F})} + (B\rho, \rho)\Big) + \frac{\nu}{2}\|S(\boldsymbol{u})\|^2_{L_2(\mathcal{F})} = 0. \tag{7.10}$$

From the equation $\boldsymbol{u} = \boldsymbol{v}^\perp + \sum_{\alpha=1}^{2} d_\alpha(\rho)\boldsymbol{\eta}_\alpha(x)$ it follows that

$$\|\boldsymbol{u}\|^2_{L_2(\mathcal{F})} = \|\boldsymbol{v}^\perp\|^2_{L_2(\mathcal{F})} + \sum_{\alpha=1}^{2} d_\alpha^2(\rho)\|\boldsymbol{\eta}_\alpha\|^2_{L_2(\mathcal{F})}$$

and

$$\|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\mathcal{F})}+(B\rho,\rho)=\|\boldsymbol{v}^\perp(\cdot,t)\|^2_{L_2(\mathcal{F})}+(B\rho_1,\rho_1)+\sum_{\alpha=1}^2 d_\alpha^2\|\boldsymbol{\eta}_\alpha\|^2_{L_2(\mathcal{F})}$$
$$+(BP_+Q\rho,P_+Q\rho)+(BP_-Q\rho,P_-Q\rho),$$

Now, we use the relations

$$2\boldsymbol{e}_3\times\boldsymbol{\eta}_\alpha=2\boldsymbol{e}_\alpha x_3=\boldsymbol{e}_\alpha x_3-\boldsymbol{e}_3 x_\alpha+\nabla x_3 x_\alpha,\quad \alpha=1,2,$$

and write the first equation in (7.8) in the form

$$\boldsymbol{v}_t^\perp+2\omega(\boldsymbol{e}_3\times\boldsymbol{v}^\perp)-\nu\nabla^2\boldsymbol{v}^\perp+\nabla(q+\omega\sum_{\alpha=1}^2 d_\alpha x_\alpha x_3)=\boldsymbol{R}, \tag{7.11}$$

where $\boldsymbol{R}$ is a linear combination of $\boldsymbol{\eta}_i$.

We multiply (7.11) by the divergence free vector field $\boldsymbol{W}\in W_2^1(\mathcal{F})$, possessing the following properties: $\boldsymbol{W}\cdot\boldsymbol{N}|_{\mathcal{G}}=f(\cdot,t)\in W_2^{1/2}(\mathcal{G})$, $\int_{\mathcal{G}} f dS=0$,

$$\|\boldsymbol{W}(\cdot,t)\|_{W_2^1(\mathcal{F})}\le c\|f\|_{W_2^{1/2}(\mathcal{G})},\quad \|\boldsymbol{W}(\cdot,t)\|_{L_2(\mathcal{F})}\le c\|f\|_{L_2(\mathcal{G})},$$
$$\|\boldsymbol{W}_t(\cdot,t)\|_{L_2(\mathcal{F})}\le c\|f_t\|_{L_2(\mathcal{G})},$$
$$\int_{\mathcal{F}}\boldsymbol{W}\cdot\boldsymbol{\eta}_i(x)dx=0,\quad i=1,2,3.$$

(cf. Proposition 6.1). For the moment, we leave the function f indefinite. Upon integration by parts we arrive at

$$\begin{aligned}&\frac{d}{dt}\int_{\mathcal{F}}\boldsymbol{v}^\perp\cdot\boldsymbol{W}dx-\int_{\mathcal{F}}\boldsymbol{v}^\perp\cdot\boldsymbol{W}_t dx+2\omega\int_{\mathcal{F}}(\boldsymbol{e}_3\times\boldsymbol{v}^\perp)\cdot\boldsymbol{W}dx\\&+\frac{\nu}{2}\int_{\mathcal{F}}S(\boldsymbol{v}^\perp):S(\boldsymbol{W})dx+\int_{\mathcal{G}}(B\rho+\omega\sum_{\alpha=1}^2 d_\alpha(\rho)x_\alpha x_3)f dS=0.\end{aligned} \tag{7.12}$$

Now we multiply (7.12) by a small positive γ and add to (7.10). This leads to

$$\frac{d}{dt}E(t)+E_1(t)=0 \tag{7.13}$$

with

$$\begin{aligned}E(t)&=\frac12\Big(\|\boldsymbol{v}^\perp(\cdot,t)\|^2_{L_2(\mathcal{F})}+(B\rho_1,\rho_1)+\sum_{\alpha=1}^2 d_\alpha^2(\rho)\|\boldsymbol{\eta}_\alpha\|^2_{L_2(\mathcal{F})}\\&\quad+(BP_+Q\rho,P_+Q\rho)+(BP_-Q\rho,P_-Q\rho)+2\gamma\int_{\mathcal{F}}\boldsymbol{v}^\perp\cdot\boldsymbol{W}dx\Big),\\E_1(t)&=\frac{\nu}{2}\|S(\boldsymbol{v}^\perp)\|^2_{L_2(\mathcal{F})}-\gamma\int_{\mathcal{F}}\boldsymbol{v}^\perp\cdot\boldsymbol{W}_t dx+2\omega\gamma\int_{\mathcal{F}}(\boldsymbol{e}_3\times\boldsymbol{v}^\perp)\cdot\boldsymbol{W}dx\\&\quad+\frac{\nu\gamma}{2}\int_{\mathcal{F}}S(\boldsymbol{v}^\perp):S(\boldsymbol{W})dx+\gamma J_{\mathcal{G}},\end{aligned}$$

where $J_{\mathcal{G}}$ is the surface integral in (7.12).

Our next objective is the proof of the inequalities

$$E_1(t) \geq -\beta E(t), \quad E(0) < 0$$

with an appropriate choice of f. This is done in a different way in the cases $\sigma > 0$ and $\sigma = 0$.

1. $\sigma > 0$. We choose

$$f = I_2(\rho)\boldsymbol{\eta}_1 \cdot \boldsymbol{N} - I_1(\rho)\boldsymbol{\eta}_2 \cdot \boldsymbol{N} + P_+Q\rho - P_-Q\rho \tag{7.14}$$

and consider the surface integral $J_{\mathcal{G}}$. By (7.2), (7.5),

$$B\rho + \omega \sum_{\alpha=1}^{2} d_\alpha x_3 x_\alpha = BQ\rho + \omega^2 \mathcal{S}^{-1} \sum_{\alpha=1}^{2} m_\alpha I_\alpha(\rho) x_3 x_\alpha$$

with $m_\alpha = \|\boldsymbol{\eta}_3\|^2_{L_2(\mathcal{F})} \|\boldsymbol{\eta}_\alpha\|^{-2}_{L_2(\mathcal{F})}$. It follows that $J_{\mathcal{G}}$ can be written in the form

$$\begin{aligned} J_{\mathcal{G}} &= \omega^2 \sum_{\alpha=1}^{2} m_\alpha I_\alpha^2(\rho) + (BP_+Q\rho, P_+Q\rho) - (BP_-Q\rho, P_-Q\rho). \\ &\geq c\Big(\sum_{\alpha=1}^{2} I_\alpha^2(\rho) + \|P_+Q\rho\|^2_{W_2^1(\mathcal{G})} + \|P_-Q\rho\|^2_{W_2^1(\mathcal{G})}\Big). \end{aligned} \tag{7.15}$$

We also need to estimate $\boldsymbol{W}_t$. Since

$$f_t = I_2(\rho_t)\boldsymbol{\eta}_1 \cdot \boldsymbol{N} - I_1(\rho_t)\boldsymbol{\eta}_2 \cdot \boldsymbol{N} + P_+Q\rho_t - P_-Q\rho_t$$

and

$$\rho_t = \boldsymbol{v}^\perp \cdot \boldsymbol{N} + \sum_{\alpha=1}^{2} d_\alpha(\rho)\boldsymbol{\eta}_\alpha \cdot \boldsymbol{N},$$

we have

$$\|\rho_t\|_{L_2(\mathcal{G})} \leq c\Big(\|\boldsymbol{v}^\perp\|^2_{L_2(\mathcal{G})} + \sum_{\alpha=1}^{2} d_\alpha^2(\rho)\Big)^{1/2} \leq c\Big(\|S(\boldsymbol{v}^\perp)\|^2_{L_2(\mathcal{F})} + \sum_{\alpha=1}^{2} I_\alpha^2(\rho)\Big)^{1/2} \tag{7.16}$$

and

$$\|\boldsymbol{W}_t\|_{L_2(\mathcal{F})} \leq c\|f_t\|_{L_2(\mathcal{G})} \leq c\Big(\sum_{\alpha=1}^{2} I_\alpha^2(\rho) + \|S(\boldsymbol{v}^\perp)\|^2_{L_2(\mathcal{F})}\Big)^{1/2}. \tag{7.17}$$

Finally, it follows from (7.14) that

$$\|\boldsymbol{W}\|_{W_2^1(\mathcal{F})} \leq c\|f\|_{W_2^{1/2}(\mathcal{G})} \leq c\Big(\sum_{\alpha=1}^{2} I_\alpha^2(\rho) + \|P_2Q\rho\|^2_{W_2^{1/2}(\mathcal{G})}\Big)^{1/2}. \tag{7.18}$$

Estimates (7.15)-(7.18) enable us to show, using the Cauchy-Schwartz inequality, that in the case of a small γ

$$\begin{aligned} &\Big| -\gamma \int_{\mathcal{F}} \boldsymbol{v}^\perp \cdot \boldsymbol{W}_t dx + 2\omega\gamma \int_{\mathcal{F}} (\boldsymbol{e}_3 \times \boldsymbol{v}^\perp) \cdot \boldsymbol{W} dx + \frac{\nu\gamma}{2} \int_{\mathcal{F}} S(\boldsymbol{v}^\perp) : S(\boldsymbol{W}) dx \Big| \\ &\leq \theta\Big(\frac{\nu}{2}\|S(\boldsymbol{v}^\perp)\|^2_{L_2(\mathcal{F})} + \gamma J_{\mathcal{G}}\Big) \end{aligned} \tag{7.19}$$

with $\theta \in (0,1)$, from which it follows that

$$E_1(t) \geq (1-\theta)\Big(\frac{\nu}{2}\|S(\boldsymbol{v}^{\perp})\|^2_{L_2(\mathcal{G})} + \gamma J_{\mathcal{G}}(t)\Big) \geq c\|P_-Q\rho\|^2_{W^1_2(\mathcal{G})}. \tag{7.20}$$

Now we estimate $E(t)$ from below. Since

$$\Big|\int_{\mathcal{F}} \boldsymbol{v}^{\perp}\cdot \boldsymbol{W}\,dx\Big| \leq c\|\boldsymbol{v}^{\perp}\|_{L_2(\mathcal{F})}\Big(\sum_{\alpha=1}^{2} I^2_{\alpha}(\rho) + \|P_2 Q\rho\|^2_{W^1_2(\mathcal{G})}\Big)^{1/2},$$

it is not hard to see that for small γ

$$E(t) \geq -\|P_-Q\rho\|^2_{W^1_2(\mathcal{G})}.$$

Together with (7.20), this inequality implies

$$E_1(t) \geq -\beta E(t), \quad \beta > 0.$$

We set $z(t) = -E(t)$ and obtain

$$\frac{dz(t)}{dt} = E_1(t) \geq \beta z(t),$$

hence

$$z(t) \geq e^{\beta t} z(0). \tag{7.21}$$

At the initial moment $t=0$ we have

$$\begin{aligned} E(0) &= \frac{1}{2}\Big(\|\boldsymbol{v}^{\perp}(\cdot,0)\|^2_{L_2(\mathcal{F})} + (B\rho_{01},\rho_{01}) + \sum_{\alpha=1}^{2} d^2_{\alpha}(\rho_0)\|\boldsymbol{\eta}_{\alpha}\|^2_{L_2(\mathcal{F})} \\ &\quad +(BP_+Q\rho_0, P_+Q\rho_0) + (BP_-Q\rho_0, P_-Q\rho_0) + 2\gamma\int_{\mathcal{F}} \boldsymbol{v}^{\perp}\cdot\boldsymbol{W}\,dx\Big|_{t=0}\Big) \\ &= (BP_-Q\rho, P_-Q\rho)\Big|_{t=0} < 0, \end{aligned}$$

if $\boldsymbol{v}^{\perp}(x,0) = 0$ and $\rho(x,0) = P_-Q\rho(x,0)$. Hence $z(0) > 0$, and (7.21) proves the exponential growth of the solution of (1.24)-(1.27) with the initial data chosen above.

2. $\sigma = 0$**.** We set

$$f = I_2(\rho)\boldsymbol{\eta}_1\cdot\boldsymbol{N} - I_1(\rho)\boldsymbol{\eta}_2\cdot\boldsymbol{N} + s_+ + s_-,$$

where $s_{\pm} \in H_{\pm}$ are solution of the equations

$$Bs_{\pm} = P_{\pm}(-\Delta_{\mathcal{G}})^{-1/2} r_{\pm}, \quad r_{\pm} = P_{\pm}Q\rho. \tag{7.22}$$

Instead of (7.15), we have

$$\begin{aligned} J_{\mathcal{G}} &= \omega^2\sum_{\alpha=1}^{2} m_{\alpha} I^2_{\alpha}(\rho) + (r_+, Bs_+) - (r_-, Bs_+) \\ &\geq c\Big(\sum_{\alpha=1}^{2} I^2_{\alpha}(\rho) + \|r_+\|^2_{W_2^{-1/2}(\mathcal{G})} + \|r_-\|^2_{W_2^{-1/2}(\mathcal{G})}\Big). \end{aligned} \tag{7.23}$$

The time derivatives $(s_\pm)_t$ satisfy the equations

$$B(s_\pm)_t = P_\pm(-\Delta_{\mathcal{G}})^{-1/2}P_\pm\rho_t;$$

as a consequence,

$$\|s_{+,t}\|_{W_2^{1/2}(\mathcal{G})} + \|s_{-,t}\|_{W_2^{1/2}(\mathcal{G})} \leq c\|\rho_t\|_{L_2(\mathcal{G})} \leq c\Big(\|\boldsymbol{v}^\perp\|_{L_2(\mathcal{G})} + \textstyle\sum_{\alpha=1}^2 I_\alpha^2(\rho)\Big).$$

It follows that

$$f_t = I_2(\rho_t)\boldsymbol{\eta}_1 \cdot \boldsymbol{N} - I_1(\rho_t)\boldsymbol{\eta}_2 \cdot \boldsymbol{N} + s_{+t} + s_{-t}$$

satisfies the same inequality as in the case $\sigma > 0$:

$$\|f_t\|_{L_2(\mathcal{G})} \leq c\|\rho_t\|_{L_2\mathcal{G})} \leq c\Big(\sum_{\alpha=1}^2 I_\alpha^2(\rho) + \|S(\boldsymbol{v}^\perp)\|^2_{L_2(\mathcal{F})}\Big)^{1/2};$$

moreover, in view of (7.23) inequality (7.19) holds also in the case $\sigma = 0$. Finally,

$$E_1(t) \geq c\|r_-\|^2_{W_2^{-1/2}(\mathcal{G})} \geq c\|r_-\|^2_{L_2(\mathcal{G})} \geq -\beta E(t),$$

because r_- is an element of a finite-dimensional space H_-. Hence (7.21) is satisfied.

We have obtained the following result.

Theorem 7.1. *If* (7.1) *holds and if the form* $(B\rho, \rho)$ *can take negative values for some* ρ *satisfying* (1.6), *then the problem* (1.24) *has solutions growing exponentially as* $t \to \infty$.

This means that the corresponding spectral problem has eigenvalues with a positive real part. This is essential for the proof of instability of solutions of the complete nonlinear problem (see [18]).

8 Auxiliary material

1. Sobolev-Slobodetskii spaces

We start with S.L.Sobolev's definition of generalized derivatives.

Definition 8.1 *Let $j = (j_1, ...j_n)$, $j_i \geq 0$, be a multi-index of the length $|j| = j_1 + ...j_n$ and let Ω be a domain in $\mathbb{R}^n$. The function $v \in L_{1,loc}(\Omega)$ is said to be a generalized derivative of $u \in L_{1,loc}(\Omega)$: $v = D^j u \equiv \frac{\partial^{|j|} u(x)}{\partial x_1^{j_1} ... \partial x_n^{j_n}}$ in Ω, if*

$$\int_\Omega u(x) D^j \varphi(x) dx = (-1)^{|j|} \int_\Omega v(x)\varphi(x) dx, \quad \forall \varphi \in C_0^\infty(\Omega).$$

Definition 8.2 *$W_2^l(\Omega)$ is the space of functions belonging to $L_2(\Omega)$ together with all their generalized derivatives of order $\leq [l]$ equipped with the norm*

$$\|u\|_{W_2^l(\Omega)} \stackrel{def}{=} \big(\sum_{|j| \leq l} \|D^j u\|^2_{L^2(\Omega)}\big)^{1/2}, \tag{8.1}$$

if $l > 0$ is an integer,

$$\|u\|_{W_2^l(\Omega)} \stackrel{def}{=} \big(\|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{n+2\lambda}} dxdy\big)^{1/2}, \tag{8.2}$$

if $l = [l] + \lambda$ with $\lambda \in (0, 1)$.

We introduce special notation for the principal parts of the norms (8.1), (8.2):

$$\|u\|_{\dot{W}_2^l(\Omega)} = \big(\sum_{|j|=l} \|D^j u\|^2_{L_2(\Omega)}\big)^{1/2}, \tag{8.3}$$

if $l > 0$ is an integer,

$$\|u\|_{\dot{W}_2^l(\Omega)} = \big(\sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{n+2\lambda}} dxdy\big)^{1/2}, \tag{8.4}$$

if $l = [l] + \lambda$ with $\lambda \in (0, 1)$.

The spaces $W_2^l(\Omega)$ with integral l are introduced and studied by S.L.Sobolev [19] and in the case of arbitrary $l > 0$ by L.N.Slobodetskii [20]. They are Hilbert spaces with the inner product

$$(u, v)_{W_2^l(\Omega)} = \sum_{|j| \leq l} (D^j u, D^j v)_{L_2(\Omega)}.$$

if l is an integer, and

$$(u, v)_{W_2^l(\Omega)} = \sum_{|j| \leq [l]} (D^j u, D^j v)_{L_2(\Omega)} + \sum_{|j|=[l]} \int_\Omega \int_\Omega \frac{(D^j u(x) - D^j u(y))(D^j v(x) - D^j v(y))}{|x-y|^{n+2\lambda}} dxdy,$$

if $l = [l] + \lambda$, $\lambda \in (0, 1)$.

Now we consider the spaces $W_2^l(\mathbb{R}^n)$ more closely.

Proposition 8.1. *Every $u \in W_2^l(\mathbb{R}^n)$ can be approximated by the functions from $C_0^\infty(\mathbb{R}^n)$.*

For the approximation one can use the sequence of regularizations (mollifications) of $u(x)$ [19] multiplied by appropriate cut-off functions with expanding supports.

If $\Omega = \mathbb{R}^n$, then the norms (8.1), (8.2) can be expressed in terms of the Fourier transform of u.

Definition 8.3 [21]. *The space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions is the set of functions $u \in C^\infty(\mathbb{R}^n)$ such that*

$$|D^j u(x)| \le \frac{C_{j,m}}{(1+|x|^2)^m}, \quad \forall j, \forall m \ge 0.$$

Definition 8.4 *For arbitrary $u \in L_1(\mathbb{R}^n)$, the Fourier transform on u is defined as*

$$\tilde{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)dx \equiv F(u)$$

Proposition 8.2 [21]. *If $u \in \mathcal{S}(\mathbb{R}^n)$, then $\widetilde{u} \in \mathcal{S}(\mathbb{R}^n)$, and u is expressed in terms of $\tilde{u}$ by the inverse Fourier transform:*

$$u(x) \equiv F^{-1}\tilde{u} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\tilde{u}(\xi)d\xi.$$

Moreover,

$$FD^j u(\xi) = (i\xi)^j \tilde{u}(\xi), \quad Fu(\cdot + z) = e^{iz\cdot\xi}\tilde{u}(\xi),$$

where $(i\xi)^j = (i\xi_1)^{j_1}\cdots(i\xi_n)^{j_n}$; finally,

$$\int_{\mathbb{R}^n} |\tilde{u}(\xi)|^2 d\xi = (2\pi)^n \int_{\mathbb{R}^n} |u(x)|^2 dx.$$

(Parserval equality).

Definition 8.5 *For $l \ge 0$, $H^l(\mathbb{R}^n)$ is the closure of $\mathcal{S}(\mathbb{R}^n)$ in the norm*

$$\|u\|_{H^l(\mathbb{R}^n)} = \Big(\int_{\mathbb{R}^n} (1+|\xi|^2)^l |\tilde{u}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}. \tag{8.5}$$

Proposition 8.3. *The norm (8.5) is equivalent to the norms (8.1) or (8.2), i.e.,*

$$c^{-1}\|u\|_{H^l(\mathbb{R}^n)} \le \|u\|_{W_2^l(\mathbb{R}^n)} \le c\|u\|_{H^l(\mathbb{R}^n)}. \tag{8.6}$$

where c is independent of u.

Proof. By the Parseval equality, we have in the case of integral l

$$\|u\|^2_{W_2^l(\mathbb{R}^n)} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} |\tilde{u}(\xi)|^2 \sum_{|j|\le l} |\xi^j|^2 d\xi,$$

which implies (8.6). So, for l a noninteger, we only need to prove

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{n+2\lambda}} dxdy = C\int_{\mathbb{R}^n} |\xi^j|^2 |\xi|^{2\lambda} |\tilde{u}(\xi)|^2 d\xi \tag{8.7}$$

where $l=[l]+\lambda$, $\lambda\in(0,1)$ and $|j|=[l]$.

It is easy to see that

$$
\begin{aligned}
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|D^j u(x)-D^j u(y)|^2}{|x-y|^{n+2\lambda}}dxdy &= \int_{\mathbb{R}^n}\frac{dz}{|z|^{n+2\lambda}}\int_{\mathbb{R}^n}|D^j u(x+z)-D^j u(x)|^2dx\\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\frac{dz}{|z|^{n+2\lambda}}\int_{\mathbb{R}^n}|(i\xi)^j|^2|e^{iz\cdot\xi}-1|^2|\tilde{u}(\xi)|^2d\xi \\
&= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi^j|^2|\tilde{u}(\xi)|^2d\xi\int_{\mathbb{R}^n}\frac{|e^{iz\cdot\xi}-1|^2}{|z|^{n+2\lambda}}dz.
\end{aligned}
\tag{8.8}
$$

We make the change of variables $\tau=Tz$, where T is an orthogonal matrix such that $T\xi=|\xi|\,(1,0,\cdots,0)^T$. Then $z\cdot\xi=Tz\cdot T\xi=\tau_1|\xi|$ and

$$
\begin{aligned}
\int_{\mathbb{R}^n}\frac{|e^{iz\cdot\xi}-1|^2}{|z|^{n+2\lambda}}dz &= \int_{\mathbb{R}^n}\frac{|e^{i\tau_1|\xi|}-1|^2}{|\tau|^{n+2\lambda}}d\tau\\
&= \int_{-\infty}^{+\infty}|e^{i\tau_1|\xi|}-1|^2d\tau_1\int_{\mathbb{R}^{n-1}}\frac{d\tau_2\cdots d\tau_n}{(\tau_1^2+\tau_2^2+\cdots+\tau_n^2)^{\frac{n+2\lambda}{2}}}\\
&= \int_{-\infty}^{+\infty}\frac{|e^{i\tau_1|\xi|}-1|^2}{|\tau_1|^{1+2\lambda}}d\tau_1\int_{\mathbb{R}^{n-1}}\frac{ds_2\cdots ds_n}{(1+s_2^2+\cdots+s_n^2)^{\frac{n+2\lambda}{2}}}\\
&= C_1|\xi|^{2\lambda}\int_{-\infty}^{+\infty}\frac{|e^{is_1}-1|^2}{|s_1|^{1+2\lambda}}ds_1\\
&= C_1C_2|\xi|^{2\lambda},
\end{aligned}
\tag{8.9}
$$

where we took $\tau_i=|\tau_1|s_i$, $i=2,\cdots,n$ in the third equality and $s_1=\tau_1|\xi|$ in the fourth equality. Clearly,

$$
C_1=\int_{\mathbb{R}^{n-1}}\frac{ds_2\cdots ds_n}{(1+s_2^2+\cdots+s_n^2)^{\frac{n+2\lambda}{2}}},\quad C_2=\int_{-\infty}^{+\infty}\frac{|e^{is_1}-1|^2}{|s_1|^{1+2\lambda}}ds_1.
$$

Thanks to (8.8) and (8.9), we obtain

$$
\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|D^j u(x)-D^j u(y)|^2}{|x-y|^{n+2\lambda}}dxdy=\frac{C_1C_2}{(2\pi)^n}\int_{\mathbb{R}^n}|\xi^j|^2|\xi|^{2\lambda}|\tilde{u}(\xi)|^2d\xi,
$$

which completes the proof of the proposition. ■

From Propositions 8.1 and 8.3 it follows that the spaces $W_2^l(\mathbb{R}^n)$ and $H^l(\mathbb{R}^n)$ coincide.

Proposition 8.4 *The norm* $\|u\|_{W_2^l(\mathbb{R}^n)}$ *is equivalent to*

$$
\Big(\|u\|^2_{L_2(\mathbb{R}^n)}+\int_{\mathbb{R}^n}\frac{dz}{|z|^{n+2l}}\int_{\mathbb{R}^n}|\triangle^k(z)u(x)|^2dx\Big)^{1/2},
\tag{8.10}
$$

where $k>l$, $\triangle(z)u=u(x+z)-u(x)$ *and*

$$
\triangle^k(z)u=\triangle^{k-1}(z)\triangle(z)u(x)=\sum_{j=1}^{k}(-1)^{k-j}\binom{k}{j}u(x+jz).
$$

Indeed, it is easily seen that (8.10) is equivalent to

$$\frac{1}{(2\pi)^{n/2}}\big(\|\tilde{u}\|^2_{L_2(\mathbb{R}^n)} + \int_{\mathbb{R}^n}\frac{dz}{|z|^{n+2l}}\int_{\mathbb{R}^n}|e^{iz\cdot\xi}-1|^{2k}|\tilde{u}(\xi)|^2 d\xi\big)^{1/2}$$

$$= \frac{1}{(2\pi)^{n/2}}\big(\|\tilde{u}\|^2_{L_2(\mathbb{R}^n)} + c\int_{\mathbb{R}^n}|\xi|^{2l}|\tilde{u}(\xi)|^2 d\xi\big)^{1/2},$$

which proves the proposition.

Proposition 8.5 *If $u \in H^l(\mathbb{R}^n)$ and $|j| < l$, then $D^j u \in H^{l-|j|}(\mathbb{R}^n)$ and*

$$\|D^j u\|_{H^{l-|j|}(\mathbb{R}^n)} \le c\|u\|_{H^l(\mathbb{R}^n)};$$

moreover, the following interpolation inequality holds:

$$\|u\|_{H^{l_1}(\mathbb{R}^n)} \le \epsilon^{l-l_1}\|u\|_{H^{l_1}(\mathbb{R}^n} + c\epsilon^{-l_1}\|u\|_{L_2(\mathbb{R}^n)}, \tag{8.11}$$

where $l_1 \in (0,l)$ and ϵ is an arbitrary positive (usually small) number.

Both estimates follow from algebraic inequalities $|\xi^j|^2(1+|\xi|^2)^{l-|j|} \le c(1+|\xi|^2)^l$ and $(1+|\xi|^2)^{l_1} \le \epsilon^{2(l-l_1)}(1+|\xi|^2)^l + c\epsilon^{-2l_1}$.

Now we start analysis of the properties of the functions from $H^l(\mathbb{R}^n)$ on hyperplanes of lower dimensions. The following proposition is referred to as the trace theorem.

Proposition 8.6 *If $l > \frac{1}{2}$, then there exists a continuous restriction operator $\mathcal{R}$: $H^l(\mathbb{R}^n) \to H^{l-\frac{1}{2}}(\mathbb{R}^{n-1})$.*

Proof. We consider the restriction to the plane $x_n = 0$. Let $\widehat{u}(\xi', x_n)$ be the Fourier transform of $u \in \mathcal{S}(\mathbb{R}^n)$ with respect to $x' = (x_1, ..., x_{n-1}$. Obviously,

$$\hat{u}(\xi', x_n) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix_n\xi_n}\tilde{u}(\xi',\xi_n)d\xi_n,$$

and

$$\begin{aligned}
|\hat{u}(\xi',0)| &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty}(1+|\xi'|^2+\xi_n^2)^{\frac{l}{2}}|\tilde{u}(\xi,\xi_n)|\frac{d\xi_n}{(1+|\xi'|^2+\xi_n^2)^{\frac{l}{2}}} \\
&\le \frac{1}{2\pi}\Big(\int_{-\infty}^{+\infty}(1+|\xi'|^2+\xi_n^2)^l|\tilde{u}(\xi,\xi_n)|^2 d\xi_n\Big)^{\frac{1}{2}}\Big(\int_{-\infty}^{+\infty}\frac{d\xi_n}{(1+|\xi'|^2+\xi_n^2)^l}\Big)^{\frac{1}{2}} \\
&\le \frac{1}{2\pi}\Big(\int_{-\infty}^{+\infty}(1+|\xi'|^2+\xi_n^2)^l|\tilde{u}(\xi,\xi_n)|^2 d\xi_n\Big)^{\frac{1}{2}}\Big(\frac{\sqrt{1+|\xi'|^2}}{(1+|\xi'|^2)^l}\int_{-\infty}^{+\infty}\frac{dt}{(1+t^2)^l}\Big)^{\frac{1}{2}},
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\|u(x',0)\|^2_{H^{l-\frac{1}{2}}(\mathbb{R}^{n-1})} &= \int_{\mathbb{R}^{n-1}}(1+|\xi'|^2)^{l-\frac{1}{2}}|\hat{u}(\xi',0)|^2 d\xi' \\
&\le C\int_{\mathbb{R}^{n-1}}\int_{-\infty}^{+\infty}(1+|\xi'|^2+\xi_n^2)^l|\tilde{u}(\xi,\xi_n)|^2 d\xi_n d\xi' \\
&\le C\|u\|^2_{H^l(\mathbb{R}^n)}.
\end{aligned}$$

Since $\mathcal{S}(\mathbb{R}^n)$ is dense in $H^l(\mathbb{R}^n)$, the proposition is proved. ■

There holds also the inverse trace theorem:

Proposition 8.7 *There exists a continuous linear extension operator $T : H^{l-\frac{1}{2}}(\mathbb{R}^{n-1}) \to H^l(\mathbb{R}^n)$ such that $T(\varphi)(x',0) = \varphi$.*

Proof. We assume that $\varphi \in \mathcal{S}(\mathbb{R}^{n-1})$ and we define $T(\varphi) = u$ by

$$\widehat{u}(\xi', x_n) = \widetilde{\varphi}(\xi')\Phi(x_n\sqrt{1+|\xi'|^2}),$$

where $\Phi \in C_0^\infty(\mathbb{R})$, $\Phi(t) = 1$ for $|t| \le t_0$ (by $\hat{u}$, as above, we mean the Fourier transform of u with respect to the tangential variables $x' = (x_1, ..., x_{n-1})$). Hence

$$u(x', x_n) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \tilde{\varphi}(\xi')\Phi(x_n\sqrt{1+|\xi'|^2})d\xi',$$
$$u(x', 0) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix'\cdot\xi'} \tilde{\varphi}(\xi')d\xi' = \varphi(x');$$

moreover,

$$\begin{aligned}
\tilde{u}(\xi', \xi_n) = &\int_{-\infty}^{+\infty} e^{-ix_n\xi_n}\hat{u}(\xi', x_n)dx_n \\
=&\tilde{\varphi}(\xi')\int_{-\infty}^{+\infty} e^{-ix_n\xi_n}\phi(x_n\sqrt{1+|\xi'|^2})dx_n \\
=&\frac{\hat{\varphi}(\xi')}{\sqrt{1+|\xi'|^2}}\int_{-\infty}^{+\infty} e^{-it\frac{\xi_n}{\sqrt{1+|\xi'|^2}}}\phi(t)dt \\
=&\frac{\hat{\varphi}(\xi')}{\sqrt{1+|\xi'|^2}}\tilde{\phi}(\frac{\xi_n}{\sqrt{1+|\xi'|^2}}).
\end{aligned} \tag{8.12}$$

It follows that

$$\begin{aligned}
\int_{\mathbb{R}^n}(1+|\xi|^2)^l|\tilde{u}|^2d\xi &= \int_{\mathbb{R}^{n-1}}\frac{|\hat{\varphi}(\xi')|^2}{1+|\xi'|^2}d\xi'\int_{-\infty}^{+\infty}|\tilde{\phi}(\frac{\xi_n}{\sqrt{1+|\xi'|^2}})|^2(1+|\xi'|^2+|\xi_n|^2)^l d\xi_n \\
&= \int_{\mathbb{R}^{n-1}}|\hat{\varphi}(\xi')|^2(1+|\xi'|^2)^{l-1/2}d\xi'\int_{-\infty}^{+\infty}|\tilde{\phi}(t)|^2(1+t^2)^l dt \\
&= \|\varphi\|^2_{H^{l-1/2}(\mathbb{R}^{n-1})}\|\phi\|^2_{H^l(\mathbb{R})},
\end{aligned}$$

i.e,

$$\|u\|^2_{H^l(\mathbb{R}^n)} = \|\varphi\|^2_{H^{l-1/2}(\mathbb{R}^{n-1})}\|\phi\|^2_{H^l(\mathbb{R})},$$

so the proposition is proved. ■

Corollary 8.1 *If $l - \frac{n-m}{2} > 0$, $m < n$, then there exists a continuous restriction operator $\mathcal{R}_m : H^l(\mathbb{R}^n) \to H^{l-\frac{n-m}{2}}(\mathbb{R}^m)$; moreover, there exists a continuous extension operator $T : H^{l-\frac{n-m}{2}}(\mathbb{R}^m) \to H^l(\mathbb{R}^n)$.*

Corollary 8.2 *If $l - |j| - \frac{n-m}{2} > 0$, then $\mathcal{R}_m(D^j u) \in H^{l-|j|-\frac{n-m}{2}}(\mathbb{R}^m)$ and*

$$\|\mathcal{R}_m(D^j u)\|_{H^{l-|j|-\frac{n-m}{2}}(\mathbb{R}^m)} \le C\|u\|_{H^l(\mathbb{R}^n)}.$$

In particular, if $j = (j_1, \cdots, j_n)$ with $j_m = \cdots = j_n = 0$, then $D^j\mathcal{R}_m(u) = \mathcal{R}_m(D^j u)$.

Proposition 8.8. *Given* $\varphi_i \in H^{l-i-\frac{1}{2}}(\mathbb{R}^{n-1})$, $i = 0, 1, \cdots, k < l - 1/2$, *there exists* $u \in H^l(\mathbb{R}^n)$ *such that*

$$\frac{\partial^i}{\partial x_n^i} u(x', 0) = \varphi_i(x'), \quad i = 0, 1, \cdots, k,$$

and

$$\|u\|_{H^l(\mathbb{R}^n)} \le C \sum_{i=0}^{k} \|\varphi_i\|_{H^{l-i-\frac{1}{2}}(\mathbb{R}^{n-1})}.$$

Proof. Without loss of generality we may assume that $\varphi_i \in \mathcal{S}(\mathbb{R}^n)$. We set

$$\hat{u}(\xi', x_n) = \sum_{i=0}^{k} \tilde{\varphi}_i(\xi') \frac{\Phi_i(x_n \sqrt{1 + |\xi'|^2})}{(\sqrt{1 + |\xi'|^2})^i},$$

where $\phi_0(t) = 1$, if $|t| \le t_0$, and

$$\Phi_i(t) \stackrel{\text{def}}{=} \frac{t^i}{i!} \Phi_0(t)$$

This implies

$$\frac{d^j}{dt^j} \Phi_i(0) = \delta_{ij}. \tag{8.13}$$

Repeating the calculation in (8.12), we obtain

$$\tilde{u}(\xi', \xi_n) = \sum_{i=0}^{k} \frac{\tilde{\varphi}_i(\xi')}{(\sqrt{1 + |\xi'|^2})^{i+1}} \tilde{\Phi}_i\left(\frac{\xi_n}{\sqrt{1 + |\xi'|^2}}\right). \tag{8.14}$$

Since $|\sum_{i=0}^k z_i|^2 \le (k+1) \sum_{i=0}^k |z_i|^2$, we have

$$\begin{aligned}
\|u\|^2_{H^l(\mathbb{R}^n)} &\le (k+1) \sum_{i=0}^{k} \int_{\mathbb{R}^{n-1}} \frac{|\tilde{\varphi}_i(\xi')|^2}{(1 + |\xi'|^2)^{i+1}} d\xi' \int_{-\infty}^{+\infty} |\tilde{\Phi}_i(\frac{\xi_n}{\sqrt{1 + |\xi'|^2}})|^2 (1 + |\xi|^2)^l d\xi_n \\
&\le (k+1) \sum_{i=0}^{k} \int_{\mathbb{R}^{n-1}} |\tilde{\varphi}_i(\xi')|^2 (1 + |\xi'|^2)^{l-i-\frac{1}{2}} d\xi' \int_{-\infty}^{+\infty} |\tilde{\Phi}_i(t)|^2 (1 + t^2)^l dt \\
&\le (k+1) \sum_{i=0}^{k} \|\varphi_i\|^2_{H^{l-i-\frac{1}{2}}(\mathbb{R}^{n-1})} \|\Phi_i\|^2_{H^l(\mathbb{R})}.
\end{aligned}$$

Now we calculate the derivative $\frac{\partial^j}{\partial x_n^j} \hat{u}(\xi', x_n)$, $j \le k$, and take the inverse Fourier transform with respect to ξ', which gives

$$\frac{\partial^j}{\partial x_n^j} u(x', x_n) = \frac{1}{(2\pi)^{n-1}} \sum_{m=0}^{k} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \tilde{\varphi}_m(\xi') \frac{d^j \Phi_m}{dt^j} (x_n \sqrt{1 + |\xi'|^2}) d\xi'.$$

By (8.13),

$$\frac{\partial^j}{\partial x_n^j} u(x', 0) = \frac{1}{(2\pi)^{n-1}} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \hat{\varphi}_j(\xi') d\xi' = \varphi_j(x').$$

Thus, the proposition is proved. ∎

Corollary 8.3 *Let $m \le n$, $k < l - \frac{n-m}{2}$. Given $\varphi_\alpha \in H^{l-|\alpha|-\frac{n-m}{2}}(\mathbb{R}^m)$ numbered by multi-indices $\alpha = (0, ..., 0, \alpha_{n-m+1}, ..., \alpha_{n-m})$ with $|\alpha| \le k$, there exists $u \in H^l(\mathbb{R}^n)$ such that $D^\alpha u|_{x_{n-m+1}=0,...,x_{n-m}=0} = \varphi_\alpha$ and*

$$\|u\|_{H^l(\mathbb{R}^n)} \le \sum_{|\alpha| \le k} \|\varphi_\alpha\|_{H^{l-|\alpha|-\frac{n-m}{2}}(\mathbb{R}^m)}.$$

The next proposition concerns the extension of the functions $u \in W_2^l(\mathbb{R}_+^n)$ in the whole space $\mathbb{R}^n$ with preservation of class.

Proposition 8.9. *For arbitrary $u \in W_2^l(\mathbb{R}_+^n)$ there exists an extension of u denoted by u^*, such that $u^* \in W_2^l(\mathbb{R}^n)$, $u^*|_{\mathbb{R}_+^n} = u$ and*

$$\|u^*\|_{W_2^l(\mathbb{R}^n)} \le C\|u\|_{W_2^l(\mathbb{R}_+^n)}.$$

Proof. We define u^* by

$$u^*(x) = \begin{cases} u(x), & x_n > 0; \\ \sum_{k=1}^{m} \lambda_k u(x', -\frac{x_n}{k}), & x_n < 0, \end{cases} \tag{8.15}$$

where $\lambda_1, \cdots, \lambda_m$ are found as a solution of the algebraic system

$$\sum_{k=1}^{m} (-\frac{1}{k})^j \lambda_k = 1, \quad j = 0, \cdots, m-1, \quad m \ge l.$$

It can be easily verified by direct computation that u^* has generalized derivatives in $\mathbb{R}^n$ up to the order $[l]$ and

$$\|D^j u^*\|_{L_2(\mathbb{R}^n)} \le c\|D^j u\|_{L_2(\mathbb{R}_+^n)},$$

if $|j| \le [l]$. Let $v = D^j u$ with $|j| = [l]$. Then

$$v^*(x) = \begin{cases} v(x), & x_n > 0; \\ \sum_{k=1}^{m} \mu_k v(x', -\frac{x_n}{k}), & x_n < 0, \end{cases}$$

with μ_k satisfying the condition $\sum_{k=1}^m \mu_k = 1$. We need to prove that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v^*(x) - v^*(y)|^2}{|x-y|^{n+2\lambda}} dxdy \le C \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2\lambda}} dxdy \tag{8.16}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|v^*(x) - v^*(y)|^2}{|x-y|^{n+2\lambda}} dxdy &= \int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2\lambda}} dxdy \\ + \int_{\mathbb{R}_-^n}\int_{\mathbb{R}_-^n} \frac{|v^*(x) - v^*(y)|^2}{|x-y|^{n+2\lambda}} dxdy &+ 2\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_-^n} \frac{|v^*(x) - v^*(y)|^2}{|x-y|^{n+2\lambda}} dxdy \end{aligned} \tag{8.17}$$

If $x, y \in \mathbb{R}^n_-$, then

$$v^*(x) - v^*(y) = \sum_{k=1}^m \mu_k \big(v(x', -\frac{x_n}{k}) - v(y', -\frac{y_n}{k}) \big).$$

Since

$$|(x', -\frac{x_n}{k}) - (y', -\frac{y_n}{k})|^2 \le |x-y|^2,$$

we obtain

$$\int_{\mathbb{R}^n_-} \int_{\mathbb{R}^n_-} \frac{|v^*(x) - v^*(y)|^2}{|x-y|^{n+2\lambda}} dxdy \le m \sum_{k=1}^m \mu_k^2 \int_{\mathbb{R}^n_-} \int_{\mathbb{R}^n_-} \frac{|v(x', -\frac{x_n}{k}) - v(y', -\frac{y_n}{k})|^2}{(|x'-y'|^2 + k^{-2}|x_n - y_n|^2)^{n/2+\lambda}} dxdy$$
$$\le C \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2\lambda}} dxdy \tag{8.18}$$

In the case $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n_-$ we have

$$v^*(x) - v^*(y) = \sum_{k=1}^m \mu_k \big(v(x', x_n) - v(y', -\frac{y_n}{k}) \big)$$

and

$$|(x', x_n) - (y', -\frac{y_n}{k})|^2 \le |x-y|^2,$$

hence

$$\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_-} \frac{|v^*(x) - v^*(y)|^2}{|x-y|^{n+2\lambda}} dxdy \le m \sum_{k=1}^m \mu_k^2 \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_-} \frac{|v(x', x_n) - v(y', -\frac{y_n}{k})|^2}{(|x'-y'|^2 + (x_n + k^{-1} y_n)^2)^{n/2+\lambda}} dxdy$$
$$\le C \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{|v(x) - v(y)|^2}{|x-y|^{n+2\lambda}} dxdy \tag{8.19}$$

Inequalities (8.17), (8.18), (8.19), imply (8.16). The proposition is proved. ■

Now, we turn our attention to the functions given in a domain $\Omega \subset \mathbb{R}^n$ and on the boundary S of Ω. We assume that S is a compact sufficiently regular manifold. The space $W_2^l(S)$ is usually defined with the help of local charts and partition of unity. Let $\{S_k\}$ be a covering of S and $\{\varphi_k\}$ be the partition of unity subordinated to this covering. We assume that on every S_k a mapping $T_k : S_k \to \Omega_k \subset \mathbb{R}^{n-1}$ is defined and we set $u_k = u\varphi_k$. We say that $u \in W_2^l(S)$ if $u_k(T_k^{-1}(y)) \in W_2^l(\Omega_k)$, for all k, and we set

$$\|u\|^2_{W_2^l(S)} \overset{\text{def}}{=} \sum_k \|u_k(T_k^{-1}(y))\|^2_{W_2^l(\Omega_k)}.$$

In the same way the spaces $W_2^l(\Gamma)$ are defined on more general manifolds.

Proposition 8.10 *Arbitrary $u \in W_2^l(\Omega)$ can be extended from Ω into $\mathbb{R}^n$ in such a way that the extended function u^* belongs to $\in W_2^l(\mathbb{R}^n)$, $u^*|_\Omega = u$ and*

$$\|u^*\|_{W_2^l(\mathbb{R}^n)} \le C \|u\|_{W_2^l(\Omega)}. \tag{8.20}$$

The idea of the proof is the following: we cover Ω by $\{\Omega_k\}_{k=0}^N$ with $\Omega_0 = \Omega$, $\Omega_k \cap \partial\Omega \neq \emptyset$, $\forall k \geq 1$. Let $\{\phi_k\}$ be the partition of unity subordinated to this covering. We assume that Ω_0 is a strictly interior subdomain of Ω and in every Ω_k, $k > 0$, the mapping $y = T_k x$ is defined that "rectifies" the part $S \cap \bar{\Omega}_k$ of S. We have $u = \sum_{k=0}^N u_k$ where $u_k = u\phi_k$. The function u_0 can be extended in the whole space $\mathbb{R}^n$ by zero, and $u_k(T_k^{-1}y)$ $(k \geq 1)$ can be extended in the way explained in Theorem 2.5. The norm of u^* is estimated by the sum of the norms of u_k, which yields (8.20).

Using Propositions 8.6-8.8, we can prove the trace theorem:

Proposition 8.11 1. *If $l > \frac{1}{2}$, $u \in W_2^l(\Omega)$, then $u|_S \in W_2^{l-\frac{1}{2}}(S)$ and*

$$\|u\|_{W_2^{l-\frac{1}{2}}(S)} \leq C\|u\|_{W_2^l(\Omega)}. \tag{8.21}$$

Moreover, $D^j u|_S \in W_2^{l-|j|-\frac{1}{2}}(S)$, if $l - |j| > \frac{1}{2}$.

2. *Let $l > \frac{1}{2}$. For arbitrary $\varphi \in W_2^{l-\frac{1}{2}}(S)$ there exists $u \in W_2^l(\Omega)$ such that $u|_S = \varphi$ and*

$$\|u\|_{W_2^l(\Omega)} \leq C\|\varphi\|_{W_2^{l-\frac{1}{2}}(S)}. \tag{8.22}$$

Moreover, if $l - k > \frac{1}{2}$, and if $\varphi_0 \in W_2^{l-\frac{1}{2}}(S)$, $\varphi_1 \in W_2^{l-\frac{3}{2}}(S), \cdots, \varphi_k \in W_2^{l-k-\frac{1}{2}}(S)$ are given, then there exists $u \in W_2^l(\Omega)$ such that

$$u|_S = \varphi_0, \quad \frac{\partial^j u}{\partial \boldsymbol{n}^j}|_S = \varphi_j,$$

and

$$\|u\|_{W_2^l(\Omega)} \leq C\sum_{j=0}^{k} \|\varphi_j\|_{W_2^{l-j-\frac{1}{2}}(S)}, \tag{8.23}$$

where $\boldsymbol{n}$ is the unit outward normal to S.

Proposition 8.12 (Sobolev imbedding) *If $l - \frac{n}{2} + \frac{n}{p} \geq 0$, $p \geq 2$, $u \in W_2^l(\Omega)$, then $u \in L_p(\Omega)$; if $l - \frac{n}{2} + \frac{m}{p} \geq 0$, $p \geq 2$, $m < n$ and $n - m < 2l$, then $u \in L_p(S_m)$, where $S_m \subset \bar{\Omega}$ is an m-dimensional manifold.*

Now we introduce the anisotropic Sobolev-Slobodetskii spaces. We restrict ourselves with the spaces $W_2^{l,l/2}(D_T)$ of functions given in a cylindrical domain $D_T = \Omega \times (0, T)$. They are widely used in the analysis of parabolic problems. The standard definition is

$$W_2^{l,l/2}(D_T) = L_2(0, T; W_2^l(\Omega)) \cap W_2^{l/2}(0, T; L_2(\Omega)),$$

and the norm in $W_2^{l,l/2}(D_T)$ is defined by

$$\|u\|^2_{W_2^{l,l/2}(D_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_0^T \sum_{0\leq j\leq l/2} \|D_t^j u(\cdot,t)\|^2_{L_2(\Omega)}dt,$$

if $l/2$ is an integral number, and

$$\|u\|^2_{W_2^{l,l/2}(D_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_0^T \sum_{0\leq j\leq [l/2]} \|D_t^j u(\cdot,t)\|^2_{L_2(\Omega)}dt$$

$$+ \int_0^T \frac{dh}{h^{1+2\mu}} \int_h^T \|D_t^{[l/2]}u(\cdot,t-h) - D_t^{[l/2]}u(\cdot,t)\|^2_{L_2(\Omega)}dt,$$

if $l/2=[l/2]+\mu$, $0<\mu<1$.

In the space $W_2^{l,l/2}(\mathbb{R}^{n+1})$ one can introduce an equivalent norm expressed in terms of the Fourier transform of the elements of this space,

$$\widetilde{u}(\xi,\xi_0)=\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}e^{-i\xi\cdot x-i\xi_0 t}u(x,t)dxdt\equiv Fu.$$

The inverse transformation is given by

$$u(x,t)=\frac{1}{(2\pi)^{n+1}}\int_{-\infty}^{\infty}\int_{\mathbb{R}^n}e^{i\xi\cdot x+i\xi_0 t}\widetilde{u}(\xi,\xi_0)d\xi d\xi_0\equiv F^{-1}\widetilde{u}.$$

We define $H^{l,l/2}(\mathbb{R}^{n+1})$ as the closure of $\mathcal{S}(\mathbb{R}^{n+1})$ in the norm

$$\|u\|_{H^{l,l/2}(\mathbb{R}^{n+1})}=\Big(\int_{\mathbb{R}^{n+1}}(1+|\xi|^2+|\xi_0|)^l|\widetilde{u}(\xi,\xi_0)|^2d\xi d\xi_0.\Big)^{1/2}.$$

The following propositions are proved essentially in the same way as in the isotropic case.

Proposition 8.13 *The norms $\|u\|_{W_2^{l,l/2}(\mathbb{R}^{n+1})}$ and $\|u\|_{H^{l,l/2}(\mathbb{R}^{n+1})}$ are equivalent.*

Proposition 8.14 *If $j\in\mathbb{N}^n$, $k\in\mathbb{N}$ and $l-|j|-2k>0$, then for arbitrary $u\in W_2^{l,l/2}(\mathbb{R}^{n+1})$, $D_x^jD_t^ku\in W_2^{l-|j|-2k,\frac{l}{2}-\frac{|j|}{2}-k}(\mathbb{R}^{n+1})$ and*

$$\|D_x^jD_t^ku\|_{H^{l-|j|-2k,\frac{l}{2}-\frac{|j|}{2}-k}(\mathbb{R}^{n+1})}\le C\|u\|_{H^{l,\frac{l}{2}}(\mathbb{R}^{n+1})},$$

moreover,

$$|u|^2_{l/2,r,\mathbb{R}^{n+1}}\le c\int_{-\infty}^{\infty}(1+|\xi_0|)^ld\xi_0\int_{\mathbb{R}^n}(1+|\xi|^2)^rd\xi\le c\|u\|^2_{H^{l+r,(l+r)/2}(\mathbb{R}^{n+1})}.$$

Proposition 8.15 *There exist continuous restriction operators*

$$\mathcal{R}_{x_n}:W_2^{l,l/2}(\mathbb{R}^{n+1})\to W_2^{l-\frac{1}{2},\frac{l}{2}-\frac{1}{4}}(\mathbb{R}^n),\quad\text{if}\quad l>1/2,$$

and

$$\mathcal{R}_t:W_2^{l,l/2}(\mathbb{R}^{n+1})\to W_2^{l-1}(\mathbb{R}^n),\quad\text{if}\quad l>1,$$

where $\mathcal{R}_{x_n}(u)=u|_{x_n=0}$ and $\mathcal{R}_t(u)=u|_{t=0}$.

Proof. Let $\widehat{u}(\xi,t)$ be the Fourier transform of u with respect to the space variables x. We estimate

$$\hat{u}(\xi,0)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\tilde{u}(\xi,\xi_0)d\xi_0$$

as in Proposition 8.6:

$$\begin{aligned}|\hat{u}(\xi,0)|^2&\le\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}(1+|\xi|^2+|\xi_0|)^l|\tilde{u}(\xi,\xi_0)|^2d\xi_0\int_{-\infty}^{+\infty}\frac{d\xi_0}{(1+|\xi|^2+|\xi_0|)^l}\\&\le\frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}(1+|\xi|^2+|\xi_0|)^l|\tilde{u}(\xi,\xi_0)|^2d\xi_0\frac{1}{(1+|\xi|^2)^{l-1}}\int_{-\infty}^{+\infty}\frac{dt}{(1+|t|)^l},\end{aligned}$$

$$\|u(\cdot,0)\|^2_{H^{l-1}(\mathbb{R}^n)}=\int_{\mathbb{R}^n}(1+|\xi|^2)^{l-1}|\hat{u}(\xi,0)|^2d\xi\le C\|u\|_{H^{l,l/2}(\mathbb{R}^{n+1})}.$$

Thus, $\mathcal{R}_t : W_2^{l,l/2}(\mathbb{R}^{n+1}) \to W_2^{l-1}(\mathbb{R}^n)$ is continuous. Similarly, we can prove that $\mathcal{R}_{x_n}$: $W_2^{l,l/2}(\mathbb{R}^{n+1}) \to W_2^{l-\frac{1}{2},\frac{l}{2}-\frac{1}{4}}(\mathbb{R}^n)$ is continuous. ■

Proposition 8.16 *Given* $\varphi_j \in H^{l-j-1/2,l/2-j/2-1/4}(\mathbb{R}^n)$, $j = 0,...,k < l - 1/2$, *there exists* $u \in H^{l,l/2}(\mathbb{R}^{n+1})$ *such that*

$$\frac{\partial^j}{\partial x_n^j} u(x,t)\Big|_{x_n=0} = \varphi_j(x',t), \quad j \le k,$$

and

$$\|u\|_{H^{l,l/2}(\mathbb{R}^{n+1})} \le c \sum_{j=0}^{k_1} \|\varphi_j\|_{H^{l-j-1/2,l/2-j/2-1/4}(\mathbb{R}^n)}.$$

Proposition 8.17 *Given* $\phi_j \in H^{l-j-1}(\mathbb{R}^n)$, $j = 0,...,k_1 < l - 1$, *there exists* $v \in H^{l,l/2}(\mathbb{R}^{n+1})$ *such that*

$$\frac{\partial^j}{\partial t^j} v(x,t)\Big|_{t=0} = \phi_j(x), \quad j \le k_1,$$

and

$$\|v\|_{H_2^{l,l/2}(\mathbb{R}^{n+1})} \le c \sum_{j=0}^{k_1} \|\phi_j\|_{H^{l-j-1}(\mathbb{R}^n)}.$$

The functions $u(x,t)$ and $v(x,t)$ can be defined by

$$\widehat{u}(\xi', x_n, \xi_0) = \sum_{j=0}^{k} \tilde{\varphi}_j(\xi', \xi_0) \frac{\Phi_j(x_n\sqrt{1+\xi'^2+|\xi_0|})}{(\sqrt{1+\xi'^2+|\xi_0|})^j}, \tag{8.24}$$

$$\widehat{v}(\xi, t) = \sum_{j=0}^{k_1} \widehat{\phi}_j(\xi', \xi_0) \frac{\Psi_j(t(1+\xi'^2+|\xi_0|)}{(1+\xi'^2+|\xi_0|)^j}, \tag{8.25}$$

where $\Phi_j(x_n)$ and $\Psi_j(t)$ are functions with compact supports satisfying the conditions

$$\frac{d^j\Phi(x_n)}{dx_n^j}\Big|_{x_n=0} = \delta_{ij}, \quad i,j = 0,...,k,$$

$$\frac{d^j\Psi(t)}{dt^j}\Big|_{t=0} = \delta_{ij}, \quad i,j = 0,...,k.$$

By $\widehat{u}$ we mean a partial Fourier transform of u (with respect to x', t in (8.24) and with respect to x in (8.25)).

The proof of Propositions 8.13, 8.14, 8.16, 8.17 is left to the reader.

Remark. If, instead of the functions $\phi_j(x)$, we have solenoidal vector fields $\boldsymbol{\phi}_j(x)$ in (8.25), then this formula defines a vector field $\boldsymbol{v}(x,t)$, that is also solenoidal.

As in the case of isotropic spaces, the results presented above enable one to prove the trace and extension theorems for the functions given in a cylinder D_T. Moreover, by Proposition 8.14,

$$|u|_{l/2,r,D_T} \le c\|u\|_{W_2^{l+r,(l+r)/2}(D_T)}.$$

The proofs are omitted; we restrict ourselves with the following extension theorem that is fundamental for the analysis of the problem (3.1):

Proposition 8.18. *Let S be a closed regular surface. For arbitrary $\rho_0 \in W_2^{l+2}(S)$ and $\rho_1 \in W_2^{l+1/2}(S)$ there exists $\rho(x,t)$, given on the cylindrical manifold $\Sigma_T = S \times (0,T)$ and such that*

$$\rho(x,0) = \rho_0(x), \quad \rho_t(x,0) = \rho_1(x)$$

and

$$\|\rho\|_{W_2^{l+5/2,0}(\Sigma_T)} + \|\rho_t\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)} + [\rho]_{l/2,5/2,\Sigma_T}$$
$$\leq c\Big(\|\rho_0\|_{W_2^{l+2}(S)} + \|\rho_1\|_{W_2^{l+2}(S)}\Big)$$

Proof. By Proposition 8.11, we can construct $r_1 \in W_2^{l+5/2}(\Sigma_T)$ such that $r_1(x,0) = \rho_0(x)$, $r_{1t}(x,0) = 0$ and

$$\|r_1\|_{W_2^{l+5/2}(\Sigma_T)} \leq c\|\rho_0\|_{W_2^{l+2}(S)}.$$

By Proposition 8.17, there exists $r_2 \in W_2^{7/2+l,7/4+l/2}(\Sigma_T)$ such that $r_2(x,0) = 0$, $r_{2t}(x,0) = \rho_1(x)$ and

$$\|r_2\|_{W_2^{l+7/2,l/2+7/4}(\Sigma_T)} \leq c\|\rho_1\|_{W_2^{l+1/2}(S)}.$$

It is easily verified that $\rho = r_1 + r_2$ possesses all the necessary properties. The proposition is proved.

2. Auxiliary inequalities and estimates for solutions of the Dirichlet and Neumann problems.

We recall classical results on the solvability of the Dirichlet and Neumann problems.

Proposition 8.19. *Let Ω be a bounded domain in $\mathbb{R}^n$ with a regular boundary S. If $f \in W_2^l(\Omega)$, $\varphi \in W_2^{l+3/2}(S)$, $l \geq 0$, then the Dirichlet problem*

$$\nabla^2 u(x) = f(x), \quad x \in \Omega, \quad u(x) = \varphi(x), \quad x \in S \tag{8.26}$$

has a unique solution $u \in W_2^{l+2}(\Omega)$, and

$$\|u\|_{W_2^{l+2}(\Omega)} \leq c\Big(\|f\|_{W_2^l(\Omega)} + \|\varphi\|_{W_2^{l+3/2}(S)}\Big).$$

For arbitrary $f \in W_2^l(\Omega)$ and $\psi \in W_2^{l+1/2}(S)$, satisfying the compatibility conditions

$$\int_\Omega f(x)dx = \int_S \psi(x)dS$$

the Neumann problem

$$\nabla^2 v(x) = f(x), \quad x \in \Omega, \quad \frac{\partial}{\partial n} v(x) = \psi(x), \quad x \in S$$

has a unique, up to the constant, solution $v \in W_2^{l+2}(\Omega)$, and

$$\|\nabla v\|_{W_2^{l+1}(\Omega)} \leq c\Big(\|f\|_{W_2^l(\Omega)} + \|\psi\|_{W_2^{l+1/2}(S)}\Big).$$

The proof (of a more general result) can be found in [22]. As for the regularity of S, the condition $S \in C^{l_1+2}$, $l_1 > l$ is sufficient (for integral l, one can require $l_1 \geq l$).

For the model problems in the half-space $\mathbb{R}^n_+ = \{x_n > 0\}$

$$\nabla^2 u(x) = f(x), \quad x \in \mathbb{R}^n_+, \quad u(x', 0) = \varphi(x'),$$

$$\nabla^2 v(x) = f(x), \quad x \in \mathbb{R}^n_+, \quad \frac{\partial u}{\partial x_n} = \psi(x')$$

(with f, φ, ψ sufficiently rapidly decaying at infinity) the estimates involving only principal parts of the norms hold:

$$\|u\|_{\dot{W}_2^{l+2}(\mathbb{R}^n_+)} \leq c\Big(\|f\|_{\dot{W}_2^l(\mathbb{R}^n_+)} + \|\varphi\|_{\dot{W}_2^{l+3/2}(\mathbb{R}^{n-1})}\Big),$$

$$\|v\|_{\dot{W}_2^{l+2}(\mathbb{R}^n_+)} \leq c\Big(\|f\|_{\dot{W}_2^l(\mathbb{R}^n_+)} + \|\psi\|_{\dot{W}_2^{l+1/2}(\mathbb{R}^{n-1})}\Big).$$

Proposition 8.20. *For the solution of the Dirichlet problem* (8.26) *with* $f = \nabla \cdot \boldsymbol{F}(x)$ *the estimate*

$$\|u\|_{W_2^{l+1}(\Omega)} \leq c\Big(\|\boldsymbol{F}\|_{W_2^l(\Omega)} + \|\varphi\|_{W_2^{l+1/2}(S)}\Big)$$

holds with $l \geq 0$.

In Sec. 3 we have used the Weyl orthogonal decomposition of arbitrary $\boldsymbol{v} \in L_2(\Omega)$:

$$\boldsymbol{v}(x) = \boldsymbol{w}(x) + \nabla u(x), \quad x \in \Omega,$$

where $\boldsymbol{w}$ is a solenoidal vector field and u is a function from $W_2^1(\Omega)$ vanishing on S. It satisfies the relations (8.26) with $f = \nabla \cdot \boldsymbol{w}$ and $\varphi = 0$ (in general, in a weak sense). If $\boldsymbol{v} \in W_2^l(\Omega)$, then, by Proposition 8.20,

$$\|u\|_{W_2^{l+1}(\Omega)} + \|\boldsymbol{w}\|_{W_2^l(\Omega)} \leq c\|\boldsymbol{v}\|_{W_2^l(\Omega)}.$$

Our next objective is proof of the Korn inequality.

Proposition 8.21. *Arbitrary vector field* $\boldsymbol{u} \in W_2^1(\Omega)$, $\Omega \subset \mathbb{R}^3$, *such that*

$$\int_\Omega \boldsymbol{u}(x)dx = 0, \quad \int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{\eta}_i(x)dx = 0, \quad i = 1, 2, 3, \tag{8.27}$$

satisfies the inequality

$$\|\boldsymbol{u}\|_{W_2^1(\Omega)} \leq c\|S(\boldsymbol{u})\|_{L_2(\Omega)}, \tag{8.28}$$

where

$$S(\boldsymbol{u}) = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T = \Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big)_{i,j=1,2,3}.$$

The proof of (8.28) relies on some auxiliary propositions.

Proposition 8.22. [23] *Let* $f \in L_2(\Omega)$ *satisfy the condition* $\int_\Omega f(x)dx = 0$. *There exists a vector field* $\boldsymbol{v} \in W_2^1(\Omega)$ *such that*

$$\nabla \cdot \boldsymbol{v}(x) = f(x), \quad \boldsymbol{v}(x)|_S = 0 \tag{8.29}$$

and

$$\|\boldsymbol{v}\|_{W_2^1(\Omega)} \leq c\|f\|_{L_2(\Omega)}. \tag{8.30}$$

The relation between f and $\boldsymbol{v}$ is linear.

Proof. We shall give the proof for the case $\Omega \subset \mathbb{R}^3$ and $S \in C^2$. Then $\boldsymbol{v}$ may be taken in the form

$$\boldsymbol{v}(x) = \nabla\Phi(x) + rot\boldsymbol{A}(x),$$

where Φ is a solution of the Neumann problem

$$\nabla^2\Phi(x) = f(x), \quad x \in \Omega, \quad \frac{\partial\Phi}{\partial n}\Big|_S = 0.$$

By Proposition 8.19,

$$\|\nabla\Phi\|_{W_2^1(\Omega)} \leq c\|f\|_{L_2(\Omega)}. \tag{8.31}$$

The vector field $rot\boldsymbol{A}$ should satisfy the boundary condition

$$rot\boldsymbol{A}|_S = -\nabla\Phi + \boldsymbol{n}\frac{\partial\Phi}{\partial n}\Big|_S = -\nabla_S\Phi|_S,$$

then $\boldsymbol{v}|_S = \nabla\Phi + rot\boldsymbol{A}|_S = 0$.

We assume that

$$\boldsymbol{A} = 0, \quad \frac{\partial\boldsymbol{A}}{\partial n} = -\nabla\Phi \times \boldsymbol{n} \tag{8.32}$$

on S. In this case

$$rot\boldsymbol{A}|_S = \boldsymbol{n} \times \frac{\partial\boldsymbol{A}}{\partial n} = -\boldsymbol{n} \times (\nabla\Phi \times \boldsymbol{n}) = -\nabla\Phi + \boldsymbol{n}(\boldsymbol{n}\cdot\nabla\Phi) = -\nabla_S\Phi.$$

We make use of the inverse trace theorem and define $\boldsymbol{A}$ as the element of $W_2^2(\Omega)$ satisfying the boundary conditions (8.32) and the inequality

$$\|\boldsymbol{A}\|_{W_2^2(\Omega)} \leq c\|\nabla_S\Phi\|_{W_2^{1/2}(S)} \leq c\|\Phi\|_{W_2^2(\Omega)} \leq c\|f\|_{L_2(\Omega)}.$$

Together with (8.31), this estimate implies (8.30). The proposition is proved. ■

Proposition 8.23. *Let $p(x)$ be the function in $W_2^1(\Omega)$ satisfying the condition $\int_\Omega p(x)dx = 0$ and the equation*

$$\nabla p(x) = \nabla\cdot\mathbf{F}(\mathbf{x}), \tag{8.33}$$

where $\mathbf{F} = (F_{ij})_{i,j=1,2,3}$ and $\nabla\cdot\mathbf{F} = \left(\sum_{i=1}^3 \frac{\partial F_{ij}}{\partial x_i}\right)_{j=1,2,3}$. Then

$$\|p\|_{L_2(\Omega)} \leq c\|\mathbf{F}\|_{L_2(\Omega)}. \tag{8.34}$$

Proof. We multiply (8.33) by the vector field $\boldsymbol{v}$ satisfying (8.29) and (8.30) with $f = p$, and integrate over Ω. This leads to

$$\int_\Omega p^2(x)dx = \int_\Omega \mathbf{F}(x) : \nabla\boldsymbol{v}(x)dx \leq \|\mathbf{F}\|_{L_2(\Omega)}\|\nabla\boldsymbol{v}\|_{L_2(\Omega)} \leq \|\mathbf{F}\|_{L_2(\Omega)}\|p\|_{L_2(\Omega)},$$

which proves (8.34). ■

Proof of the Korn inequality.

We present the proof given in [24]. Without restriction of generality we may assume that

$$\int_\Omega x_i dx = 0, \quad i = 1, 2, 3. \tag{8.35}$$

At first we prove (8.28) for arbitrary $\boldsymbol{v} \in W_2^2(\Omega)$ satisfying the conditions

$$\int_\Omega \boldsymbol{v}(x)dx = 0, \quad \int_\Omega rot\boldsymbol{v}(x)dx = 0. \tag{8.36}$$

We start with the estimate of $\|\nabla \boldsymbol{v}\|_{L_2(\Omega)}$. Since

$$\nabla \boldsymbol{v} = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T) + \frac{1}{2}(\nabla \boldsymbol{v} - (\nabla \boldsymbol{v})^T),$$

it suffices to estimate $\|rot\boldsymbol{v}\|_{L_2(\Omega)}$. We make use of the relations

$$\frac{\partial}{\partial x_1}\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big) = \frac{\partial}{\partial x_1}\Big(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\Big) - 2\frac{\partial}{\partial x_2}\frac{\partial v_1}{\partial x_1} = \frac{\partial S_{12}}{\partial x_1} - \frac{\partial S_{11}}{\partial x_2},$$

$$\frac{\partial}{\partial x_2}\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big) = \frac{\partial S_{22}}{\partial x_1} - \frac{\partial S_{12}}{\partial x_2},$$

$$\frac{\partial}{\partial x_3}\Big(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big) = \frac{\partial S_{23}}{\partial x_1} - \frac{\partial S_{13}}{\partial x_2}.$$

In view of Proposition 8.23, these relations imply

$$\|\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\|_{L_2(\Omega)} \le c\|S(\boldsymbol{v})\|_{L_2(\Omega)}.$$

In the same way other components of $rot\boldsymbol{v}$ are estimated and we obtain

$$\|\nabla \boldsymbol{v}\|_{L_2(\Omega)} \le c\|S(\boldsymbol{v})\|_{L_2(\Omega)}. \tag{8.37}$$

The L_2-norm of $\boldsymbol{v}$ can be estimated by the Poincaré inequality, so as a result we arrive at (8.28) for the vector field $\boldsymbol{v}$.

Now we take arbitrary $\boldsymbol{u}$ satisfying (8.27) and set

$$\boldsymbol{v}(x) = \boldsymbol{u}(x) - \sum_{k=1}^{3} c_k \boldsymbol{\eta}_k(x) \tag{8.38}$$

with

$$c_k = \frac{1}{2|\Omega|}\boldsymbol{e}_k \cdot \int_\Omega rot\boldsymbol{u}(x)dx.$$

Then $rot\boldsymbol{v} = rot\boldsymbol{u} - 2\sum_{k=1}^3 c_k \boldsymbol{e}_k$ and, as a consequence, $\int_\Omega rot\boldsymbol{v}(x)dx = 0$. Moreover, in view of (8.35), $\int_\Omega \boldsymbol{v}(x)dx = 0$, so $\boldsymbol{v}$ satisfies (8.27). On the other hand, we can express c_k in terms of $\boldsymbol{v}$. Since

$$\int_\Omega \boldsymbol{u}(x) \cdot \boldsymbol{\eta}_i(x)dx = 0,$$

we have

$$\int_\Omega \boldsymbol{v}(x)\cdot\boldsymbol{\eta}_i(x)dx=\sum_{k=1}^3 S_{ik}c_k,$$

where $S_{ik}=\int_\Omega \boldsymbol{\eta}_i(x)\cdot\boldsymbol{\eta}_k(x)dx$. The matrix $\mathcal{S}=(S_{ik})_{i,k=1,2,3}$ is non-degenerate, hence

$$c_k=\sum_{m=1}^3 S^{km}\int_\Omega \boldsymbol{v}(x)\cdot\boldsymbol{\eta}_m(x)dx, \tag{8.39}$$

where S^{km} are elements of $\mathcal{S}^{-1}$. By (8.38) and (8.39),

$$\|\boldsymbol{u}\|_{W_2^1(\Omega)}\le c\Big(\|\boldsymbol{v}\|_{W_2^1(\Omega)}+\sum_{k=1}^3|c_k|\Big)\le c\|S(\boldsymbol{v})\|_{L_2(\Omega)}=c\|S(\boldsymbol{u})\|_{L_2(\Omega)},$$

q.e.d. ■

Corollary. *Arbitrary vector field $\boldsymbol{v}\in W_2^1(\Omega)$ satisfies the inequality*

$$\begin{aligned}\|\boldsymbol{v}\|_{W_2^1(\Omega)}&\le c\Big(\|S(\boldsymbol{v})\|_{L_2(\Omega)}+\Big|\int_\Omega \boldsymbol{v}(x)dx\Big|+\sum_{i=1}^3\Big|\int_\Omega \boldsymbol{v}(x)\cdot\boldsymbol{\eta}_i(x)dx\Big|\Big)\\&\le c\Big(\|S(\boldsymbol{v})\|_{L_2(\Omega)}+\|\boldsymbol{v}\|_{L_2(\Omega)}\Big).\end{aligned} \tag{8.40}$$

Indeed, we can represent $\boldsymbol{v}$ in the form

$$\boldsymbol{v}=\boldsymbol{v}^\perp+\sum_{i=1}^3 c_i\boldsymbol{e}_i+\sum_{i=1}^3 c_i'\boldsymbol{\eta}_i.$$

The constants c_i, c_i' are easily found from the conditions

$$\int_\Omega \boldsymbol{v}^\perp\cdot\boldsymbol{e}_j dx=0,\quad \int_\Omega \boldsymbol{v}^\perp\cdot\boldsymbol{\eta}_j dx=0,\quad j=1,2,3;$$

they are estimated by

$$c\Big(\Big|\int_\Omega \boldsymbol{v}(x)dx\Big|+\sum_{i=1}^3\Big|\int_\Omega \boldsymbol{v}(x)\cdot\boldsymbol{\eta}_i(x)dx\Big|\Big).$$

Since

$$\|\boldsymbol{v}^\perp\|_{W_2^1(\Omega)}\le c\|S(\boldsymbol{v}^\perp)\|_{L_2(\Omega)}=c\|S(\boldsymbol{v})\|_{L_2(\Omega)},$$

we obtain (8.40).

3. Calculation of variations of some functionals.

We have often used the transformation $x=e_\rho(y)$ defined in (1.15). We assume that this transformation establishes one-to-one correspondence between $\mathcal{F}$ and the domain $\Omega\equiv\Omega(\rho)$.

Concerning $\boldsymbol{N}^*$ and ρ^* we assume that these functions satisfy the conditions formulated in Sec.5 (see (5.12)), in particular, that $|\boldsymbol{N}^*| = 1$ in the neighborhood of $\mathcal{G}$ and

$$\frac{\partial \rho^*}{\partial N}\Big|_{\mathcal{G}} = 0, \quad \frac{\partial \boldsymbol{N}^*}{\partial N}\Big|_{\mathcal{G}} = 0. \tag{8.41}$$

We are going to compute the first and the second variations of some functions and functionals depending on ρ, when $\rho = 0$. The standard definition is

$$\delta f(\rho) = \frac{df(s\rho)}{ds}\Big|_{s=0}, \quad \delta^2 f(\rho) = \frac{d^2 f(s\rho)}{ds^2}\Big|_{s=0}.$$

It follows that

$$\begin{aligned} f(\rho) - f(0) &= \int_0^1 \frac{df(s\rho)}{ds} ds = \delta f(\rho) + \int_0^1 (1-s)\frac{d^2 f(s\rho)}{ds^2} ds \\ &= \delta f(\rho) + \frac{1}{2}\delta^2 f(\rho) + \int_0^1 (1-s)\Big(\frac{d^2 f(s\rho)}{ds^2} - \frac{d^2 f(s\rho)}{ds^2}\Big|_{s=0}\Big) ds, \end{aligned}$$

which gives the representation of the difference $f(\rho) - f(0)$ in the form of the sum of linear, quadratic terms with respect to ρ and of a remainder. In this formula f may be a function or a functional depending also on the derivatives of ρ.

As usual, we mean by $\mathcal{L}(y,\rho)$ the Jacobi matrix of the transformation (1.15); it has the elements

$$l_{ij} = \delta_{ij} + \frac{\partial}{\partial y_j} N_i^*(y)\rho^*(y).$$

We set $L(y,\rho) = det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$. We denote by l^{ij} the elements of $\mathcal{L}^{-1}$ and by $\widehat{L}_{ij}$ the elements of $\widehat{\mathcal{L}}$. We shall often deal with the function

$$\Lambda(y,\rho) = \boldsymbol{N}(y) \cdot \widehat{\mathcal{L}}^T(y,\rho)\boldsymbol{N}(y),$$

defined for $y \in \mathcal{G}$. We follow the arguments in [25].

Proposition 8.24. *For arbitrary continuously differentiable function $f(x)$ given in $\mathcal{F}$ and in a certain neighborhood of $\mathcal{F}$ the relation*

$$\int_\Omega f(x)dx - \int_{\mathcal{F}} f(y)dy = \int_0^1 ds \int_{\mathcal{G}} \rho(y)\Lambda(y,s\rho)f(e_{s\rho}(y))dS \tag{8.42}$$

holds.

Proof. It is clear that

$$\begin{aligned} \int_\Omega f(x)dx - \int_{\mathcal{F}} f(y)dy &= \int_{\mathcal{F}} f(e_\rho(y))L(y,\rho)dy - \int_{\mathcal{F}} f(y)dy \\ &= \int_0^1 ds \int_{\mathcal{F}} \frac{d}{ds}\Big(f(e_{s\rho}(y))L(y,s\rho)\Big) dy \end{aligned}$$

By the formula for the derivative of the determinant, we have

$$\frac{d}{ds}L(y,s\rho) = \sum_{i,j=1}^3 \frac{\partial N_i^*\rho^*}{\partial y_j}\widehat{L}_{ji}(y,s\rho), \tag{8.43}$$

hence

$$\int_\Omega f(x)dx - \int_{\mathcal{F}} f(y)dy$$
$$= \int_0^1 ds \int_{\mathcal{F}} \Big(\nabla f(e_{s\rho}(y)) \cdot \boldsymbol{N}^* \rho^* L(y, s\rho) + f(e_{s\rho}(y)) \sum_{i,j=1}^3 \frac{\partial N_i^* \rho^*}{\partial y_j} \widehat{L}_{ji}(y, s\rho)\Big) dy.$$

Now we integrate by parts in the second term and use the identity

$$\sum_{j=1}^3 \frac{\partial}{\partial y_j} \widehat{L}_{ji}(y, s\rho) = 0,$$

which leads to (8.42). The proposition is proved. ■

Our next objective is to calculate $\Lambda(y, \rho)$. It is a second degree polynomial with respect to ρ and the first derivatives of ρ and $\Lambda(y, 0) = 1$. Hence

$$\Lambda(y, \rho) = 1 + \delta\Lambda(y, \rho) + \frac{1}{2}\delta^2\Lambda(y, \rho).$$

The calculation of $\delta\Lambda$ and $\delta^2\Lambda$ reduces to the calculation of the variations of $\widehat{\mathcal{L}}$. First we compute $\frac{d}{ds} l^{ij}(y, s\rho)$, using the relation $\mathcal{L}^{-1}\mathcal{L} = I$. Since

$$\frac{d}{ds} l_{km}(y, s\rho) = \frac{\partial}{\partial y_m} N_k^*(y)\rho^*,$$

we have

$$\frac{d}{ds} l^{ij}(y, s\rho) = -\sum_{k,m=1}^3 l^{ik} \frac{\partial N_k^* \rho^*}{\partial y_m} l^{mj}. \tag{8.44}$$

Taking (8.43) into account, we obtain

$$\frac{d}{ds} \widehat{L}_{ij}(y, s\rho) = \sum_{k,m=1}^3 \Big(-l^{ik} \frac{\partial N_k^* \rho^*}{\partial y_m} \widehat{L}_{mj} + l^{ij} \frac{\partial N_k^* \rho^*}{\partial y_m} \widehat{L}_{mk} \Big). \tag{8.45}$$

It follows that

$$\frac{d}{ds} \Lambda(y, s\rho) = \sum_{i,j,k,m=1}^3 \Big(-l^{ik} N_i \frac{\partial N_k^* \rho^*}{\partial y_m} \widehat{L}_{mj} N_j + l^{ij} N_i N_j \frac{\partial N_k^* \rho^*}{\partial y_m} \widehat{L}_{mk} \Big). \tag{8.46}$$

In view of (8.41),

$$\delta\Lambda = \sum_{m=1}^3 \frac{\partial N_m \rho}{\partial y_m} = -\mathcal{H}\rho.$$

When we differentiate (8.46) with respect to s once more and take account of (8.44), (8.45), we obtain

$$\delta^2\Lambda = \Big(\sum_{m=1}^3 \frac{\partial N_m \rho}{\partial y_m}\Big)^2 - \sum_{k,m=1}^3 \frac{\partial N_k \rho}{\partial y_m} \frac{\partial N_m \rho}{\partial y_k} = 2\mathcal{K}(y)\rho^2.$$

Thus,

$$\Lambda(y,\rho) = 1 - \rho\mathcal{H}(y) + \rho^2\mathcal{K}(y), \tag{8.47}$$

and (8.42) takes the form

$$\int_\Omega f(x)dx - \int_{\mathcal{F}} f(y)dy = \int_0^1 ds \int_{\mathcal{G}} \rho(y)(1 - s\rho\mathcal{H}(y) + s^2\rho^2\mathcal{K}(y))f(e_{s\rho}(y))dS. \tag{8.48}$$

Setting $f = 1$ and $f = x_i$ in this formula, we justify (1.17). In addition, (8.48) implies

$$\delta\int_\Omega dx = \int_{\mathcal{G}} \rho(y)dS, \quad \delta\int_\Omega |x'|^2 dx = \int_{\mathcal{G}} \rho(y)|y'|^2 dS, \tag{8.49}$$

$$\delta^2\int_\Omega dx = -\int_{\mathcal{G}} \rho^2\mathcal{H}(y)dS, \quad \delta^2\int_\Omega |x'|^2 dx = 2\int_{\mathcal{G}} \rho^2 \boldsymbol{y}'\cdot\boldsymbol{N}'dS - \int_{\mathcal{G}} |y'|^2\rho^2\mathcal{H}dS. \tag{8.50}$$

Now, we pass to the calculation of variations of $|\Gamma|$, where Γ is the boundary of Ω.

Proposition 8.25. *Let $x = T(y)$ be a continuously differentiable invertible mapping of a bounded domain $\Omega \subset \mathbb{R}^n$ with the boundary S on the domain $\Omega' \subset \mathbb{R}^n$ with $\partial\Omega' = S'$, and let $\mathcal{L}$ be the Jacobi matrix of the transformation T, $L = \det\mathcal{L}$, $\widehat{\mathcal{L}} = L\mathcal{L}^{-1} = (\widehat{L}_{km})_{k,m=1,\dots,n}$. Arbitrary function $f(y)$ given in Ω satisfies the relation*

$$\int_S f(y)|\widehat{\mathcal{L}}^T\boldsymbol{n}(y)|dS_y = \int_{S'} f(T^{-1}(x))dS_x, \tag{8.51}$$

where $\boldsymbol{n}$ is the exterior normal to S.

Proof Let $\boldsymbol{w}'(x)$ be a vector field given in Ω' such that $f(T^{-1}(x)) = \boldsymbol{w}'\cdot\boldsymbol{n}'$ where $\boldsymbol{n}'$ is the exterior normal to S'; it is connected with $\boldsymbol{n}$ by

$$\boldsymbol{n}'(T(y)) = \frac{\widehat{\mathcal{L}}^T\boldsymbol{n}(y)}{|\widehat{\mathcal{L}}^T\boldsymbol{n}(y)|}.$$

We set $\boldsymbol{w}(y) = \boldsymbol{w}'(T(y))$ and obtain

$$\int_{S'} f(T^{-1}x)dS_x = \int_{\Omega'} \nabla_x\cdot\boldsymbol{w}'dx = \sum_{k,m=1}^n \int_\Omega \frac{\partial w_k}{\partial y_m}\frac{\partial y_m}{\partial x_k}Ldy = \sum_{k,m=1}^n \int_S w_k n_m \widehat{L}_{mk}dS_y$$
$$= \int_S \boldsymbol{w}(y)\cdot\boldsymbol{n}'(T(y))|\widehat{\mathcal{L}}^T\boldsymbol{n}|dS_y = \int_S f(y)|\widehat{\mathcal{L}}^T\boldsymbol{n}(y)|dS_y.$$

The proposition is proved. ■

The formula (8.51) justifies the appearance of the factor $|A\boldsymbol{n}_0||\widehat{\mathcal{L}}^T\boldsymbol{N}|^{-1}$ in (5.17). In addition, it implies

$$|\Gamma| = \int_{\mathcal{G}} |\widehat{\mathcal{L}}^T\boldsymbol{N}(y)|dS.$$

Making use of (8.45), it is possible to prove that

$$\delta|\Gamma| = -\int_{\mathcal{G}} \rho\mathcal{H}dS, \tag{8.52}$$

$$\delta^2|\Gamma| = \int_{\mathcal{G}} (|\nabla_{\mathcal{G}}\rho|^2 + 2\mathcal{K}\rho^2)dS. \tag{8.53}$$

In Sec.5 we have used the formula

$$H(e_\rho(y)) - \mathcal{H}(y) = \frac{dH_s}{ds}\Big|_{s=0} + \int_0^1 (1-s)\frac{d^2 H_s}{ds^2}ds,$$

where H_s is the doubled mean curvature of the surface $\Gamma^s = \{w = e_{s\rho}(y), \quad y \in \mathcal{G}\}$. It may defined by $H_s = -\nabla_{\Gamma^s} \cdot \boldsymbol{n}^s$, where ∇_{Γ^s} is the surface gradient on Γ^s and $\boldsymbol{n}^s$ is the exterior normal to Γ^s. It is connected with $\boldsymbol{N}$ by

$$\boldsymbol{n}^s(e_{s\rho}(y)) = \frac{\widehat{\mathcal{L}}^T(y, s\rho)\boldsymbol{N}(y)}{|\widehat{\mathcal{L}}^T(y, s\rho)\boldsymbol{N}(y)|},$$

which shows that $\boldsymbol{n}^s$ is defined in a certain neighborhood of $\mathcal{G}$. Since $\boldsymbol{n}^s \cdot \frac{\partial}{\partial y_j}\boldsymbol{n}^s = 0$, we have

$$H_s = -\nabla \cdot \boldsymbol{n}^s = -\sum_{k,m=1}^{3} l^{mi}(y, s\rho)\frac{\partial}{\partial y_m}\frac{\widehat{L}_{ki}(y, s\rho)N_k(y)}{|\widetilde{\mathcal{L}}^T\boldsymbol{N}(y)|}$$

and, as a consequence,

$$\delta H = -\sum_{m,i=1}^{3} \delta l^{mi}\frac{\partial N_m}{\partial y_m} - \sum_{i=1}^{3}\frac{\partial}{\partial y_i}\Big(\sum_{k=1}^{3}\delta\widehat{L}_{ki}N_k + N_i\delta\frac{1}{|\widetilde{\mathcal{L}}^T\boldsymbol{N}(y)|}\Big).$$

With the help of (8.44), (8.45) and of the formulas

$$\mathcal{H}(y) = -\sum_{m=1}^{3}\frac{\partial N_m(y)}{\partial y_m}, \qquad 2\mathcal{K} = \Big(\sum_{m=1}^{3}\frac{\partial N_m(y)}{\partial y_m}\Big)^2 - \sum_{m,k=1}^{3}\frac{\partial N_m(y)}{\partial y_k}\frac{\partial N_k(y)}{\partial y_m}$$

one can show that $\delta H = \Delta_{\mathcal{G}}\rho + (\mathcal{H}^2 - 2\mathcal{K})\rho$; hence

$$H - \mathcal{H} = \Delta_{\mathcal{G}}\rho + (\mathcal{H}^2 - 2\mathcal{K})\rho - \int_0^1 (1-s)\frac{d^2}{ds^2}\sum_{k,m,i=1}^{3} l^{mi}(y, s\rho)\frac{\partial}{\partial y_m}\frac{\widehat{L}_{ki}(y, s\rho)N_k(y)}{|\widetilde{\mathcal{L}}^T\boldsymbol{N}(y)|}ds. \tag{8.54}$$

Now we pass to the calculation of variations of the Newtonian potential $U(x)$ and of the integral $J(\rho) = \int_\Omega U(x)dx$.

Proposition 8.26 *The following formulas hold:*

$$\delta U = \frac{\partial \mathcal{U}(y)}{\partial N}\rho(y,t) + \int_{\mathcal{G}}\frac{\rho(z,t)dS_z}{|y-z|}, \tag{8.55}$$

$$\delta J = \frac{dJ(s\rho)}{ds}\Big|_{s=0} = 2\int_{\mathcal{G}}\rho\mathcal{U}(y)dS, \tag{8.56}$$

$$\begin{aligned} \delta^2 J(\rho) = \quad & -2\int_{\mathcal{G}}\rho^2(y)\mathcal{H}(y)\mathcal{U}(y)dS + 2\int_{\mathcal{G}}\rho^2(y)\frac{\partial\mathcal{U}}{\partial N}dS \\ & + 2\int_{\mathcal{G}}\int_{\mathcal{G}}\rho(y)\rho(z)\frac{dS_y dS_z}{|y-z|}. \end{aligned} \tag{8.57}$$

Proof. We have

$$U(e_{s\rho}(y)) = \int_{\mathcal{F}} \frac{L(y,s\rho)dy}{|e_{s\rho}(y)-e_{s\rho}(z)|},$$

hence,

$$\begin{aligned}\frac{d}{ds}U(e_{s\rho}(y)) &= \int_{\mathcal{F}} \sum_{i,j=1}^{3} \frac{\partial N_i^*(z)\rho^*(z,t)}{\partial z_j}\widehat{L}_{ji}(z,s\rho)\frac{dz}{|e_{s\rho}(z)-e_{s\rho}(y)|} \\ &\quad - \int_{\mathcal{F}} L(z,s\rho)\frac{e_{s\rho}(y)-e_{s\rho}(z)}{|e_{s\rho}(z)-e_{s\rho}(y)|^3}\cdot(\boldsymbol{N}^*(y)\rho^*(y,t)-\boldsymbol{N}^*(z)\rho^*(z,t))dz.\end{aligned}$$

The integration by parts leads to

$$\begin{aligned}\frac{d}{ds}U(e_{s\rho}(y)) = & -\int_{\mathcal{F}} L(z,s\rho)\frac{e_{s\rho}(y)-e_{s\rho}(z)}{|e_{s\rho}(z)-e_{s\rho}(y)|^3}dz\cdot\boldsymbol{N}^*(y)\rho^*(y,t) \\ & + \int_{\mathcal{G}} \rho(z,t)\Lambda(z,s\rho)\frac{dS}{|e_{s\rho}(z)-e_{s\rho}(y)|}.\end{aligned} \tag{8.58}$$

Setting $s=0$, we obtain (8.55).

Now we calculate

$$\begin{aligned}\frac{dJ(s\rho)}{ds} &= \sum_{i,j=1}^{3}\int_{\mathcal{F}} \frac{\partial N_i^*\rho^*}{\partial y_j}\widehat{L}_{ji}(y,s\rho)U(e_{s\rho}(y))dy + \int_{\mathcal{F}} L(y,s\rho)dy \\ &\quad \times\Big(\int_{\mathcal{G}}\frac{\Lambda(z,s\rho)\rho(z)dS_z}{|e_{s\rho}(y)-e_{s\rho}(z)|} + \boldsymbol{N}^*(y)\rho^*(y)\cdot\int_{\mathcal{F}} L(z,s\rho)\frac{e_{s\rho}(z)-e_{s\rho}(y)}{|e_{s\rho}(z)-e_{s\rho}(y)|^3}dz\Big).\end{aligned}$$

Integrating by parts in the first term we get

$$\frac{dJ(s\rho)}{ds} = 2\int_{\mathcal{G}} \Lambda(z,s\rho)\rho(z)U(e_{s\rho}(z))dS, \tag{8.59}$$

which implies (8.56).

When we differentiate this formula with respect to s and take (8.58) into account, we obtain (8.57). The proposition is proved.

Now we can compute the variations of $\mathcal{R}$ and of M. From (8.49), (8.56), (8.52), (1.3) it follows that $\delta\mathcal{R}(\rho)=0$. The formulas (8.53), (8.50), (8.57), (1.3) imply (1.4). Finally, the equation $\delta M = -B_0\rho$ used in Sec. 5 follows from (8.54), (8.55).

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