

ПРЕПРИНТЫ ПОМИ РАН

ГЛАВНЫЙ РЕДАКТОР

С.В. Кисляков

РЕДКОЛЛЕГИЯ

**В.М.Бабич, Н.А.Вавилов, А.М.Вершик, М.А.Всемирнов, А.И.Генералов, И.А.Ибрагимов,
Л.Ю.Колотилина, Б.Б.Лурье, Ю.В.Матиясевич, Н.Ю.Нецветаев, С.И.Репин, Г.А.Серегин**

**Учредитель: Санкт-Петербургское отделение Математического института
им. В. А. Стеклова Российской академии наук**

**Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16
октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых
коммуникаций**

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны: (812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

<http://www.pdmi.ras.ru/preprint/>

Заведующая информационно-издательским сектором Симонова В.Н

Dynamics of metrics in measure spaces and their asymptotic invariants

A. M. Vershik

St.Petersburg Department Steklov Mathematical Institute RAN, Fontanka
27, 191011, St.Petersburg, Russia

vershik@pdmi.ras.ru

On the 50th anniversary of the Kolmogorov entropy

Abstract

We discuss the Kolmogorov's entropy and Sinai's definition of it; and then define a deformation of the entropy, called *scaling entropy*; this is also a metric invariant of the measure preserving actions of the group, which is more powerful than the ordinary entropy. To define it, we involve the notion of the ε -entropy of a metric in a measure space, also suggested by A. N. Kolmogorov slightly earlier. We suggest to replace the techniques of measurable partitions, conventional in entropy theory, by that of iterations of metrics or semi-metrics. This leads us to the key idea of this paper which as we hope is the answer on the old question: what is the natural context in which one should consider the entropy of measure-preserving actions of groups? the same question about its generalizations—scaling entropy, and more general problems of ergodic theory. Namely, we propose a certain research program, called *asymptotic dynamics of metrics in a measure space*, in which, for instance, the generalized entropy is understood as *the asymptotic Hausdorff dimension of a sequence of metric spaces associated with dynamical system*. As may be supposed, the metric isomorphism problem for dynamical systems as a whole also gets a new geometric interpretation. ^{1 2}

¹This article is completed mathematical addendum to the paper of the author “Information, Entropy, Dynamics” in the volume “Mathematical events of the XX century: look from St.Petersburg”. MCCME. 2009.(In Russian),[23] in which a historical survey of the discovery of the mathematical entropy by Shannon and Kolmogorov and its influence on mathematics was given.

²The author acknowledges support from RFBR (under grant 08-01-00379) and NSh-2460.2008.1 and thanks Alexander von Humboldt Foundation (Germany) and Max-Planck-Institute (Bonn) for providing excellent research facilities.

• •

• • , • • , • • , • • , • • , • • , • • ,
• • , • • , • • , • • , • • , • • , • • , • • ,
• • , • • , • • , • • , • • , • • , • • ,

1 Introduction

More than fifty years ago a A.N.Kolmogorov after serious investigation of the notion of Shannon information theory introduced a new metric invariant in ergodic theory *entropy of automorphism of the Lebesgue space*. That event had drastically changed the theory of dynamical systems in all its aspects - smooth dynamics, as well as topological and measure-theoretical. After the paper by V.Rokhlin, Ya.Sinai, M.Pinsker, L.Abramov, D.Ornstein and their followers the ergodic theory obtained new perspectives and new links. The idea of entropy as an invariant of the various objects in the very different part of mathematics became very popular. Nevertheless, after fifty years of the developing entropy theory there is a room for generalizations and further application of those ideas. Here we present, the natural generalization of this notion which is useful even for automorphisms. Moreover, it seems that the entropy as a very special invariant of automorphism which is very far from others invariants (like spectra) has had up to now no a right framework in the ergodic theory. We suggested a context - *asymptotic invariants of the systems of the compacts with measures* in which entropy and its generalization - *scaling entropy* looks in a very natural way. This idea firstly appeared in the theory of filtrations -decreasing sequences of sigma-fields (or measurable partitions). We believe that in this terms will be possible to clarify some old problems of the ergodic theory.

2 Kolmogorov's and Sinai's definitions of entropy. Why entropy cannot be deformed?

We start with discussing an important question related to the notion of Kolmogorov entropy and its generalizations. In a descriptive form, the question can be stated as follows: what is an appropriate framework for considering entropy? and how, within this framework, can one extend the notion of entropy to the case where the Kolmogorov entropy of an automorphism vanishes? This problem excited the lively interest of specialists from the moment A. N. Kolmogorov had discovered entropy as an invariant of dynamical systems. But at that time, this did not lead to any serious consequences. The attempt to squeeze entropy into the framework of functional analysis, i.e., into the framework of spectral approach, failed, since the nature of entropy is obviously non-spectral and even non-operator. Of course, one can

(and did) artificially recast the notion of entropy in operator terms, but such reformulations do not give any essentially new information: entropy agrees poorly with traditional operator considerations, because it is of a completely different nature. One should search for other appropriate terms and notions.

However, this isolation of entropy is not of course insurmountable. Below we suggest another framework, important in itself, which naturally embraces both the Kolmogorov entropy and its generalizations. Our approach is as follows: given a dynamical system, we consider the *associated dynamics of metrics in a measure space and its asymptotic invariants*. Within this approach, it is natural to consider entropy and its generalizations as *one of the basic and simplest asymptotic invariants* of the associated dynamics of metrics and, consequently, of the dynamical system itself. But for this we must pass from the Kolmogorov entropy to the ε -entropy of metric measure spaces and asymptotic characteristics of their dynamics. This will be done below.

The classical spectral theory of dynamical systems, in particular, the spectral theory of groups of measure-preserving transformations, studies the associated dynamics in function spaces, or, from a more modern point of view, in operator algebras. Our main thesis is as follows: deep properties of a system are reflected in how the group acts on the collection of metrics on the phase space, i.e., in the associated dynamics of metrics, in particular, in the asymptotic properties of this dynamics of metrics. It is this framework that is a natural place for entropy and its generalizations.

This approach first appeared in connection with the theory of decreasing sequences of partitions (filtrations) [18, 19]. There it turns out to provide a fine classification of filtrations. The approach is equally fruitful in the theory of dynamical systems itself.

First recall the basic definitions of entropy theory. The entropy of a discrete measure $\mu = (p_1, \dots, p_n)$, $p_i > 0$, $\sum p_i = 1$, is defined as

$$H(\mu) = - \sum_i p_i \log p_i.$$

Let T be an automorphism, i.e., a measurable invertible transformation with an invariant measure, defined on a standard measure (i.e., in Rokhlin's terms, Lebesgue) space (X, \mathcal{A}, μ) , where \mathcal{A} is the σ -algebra of mod 0 classes of all measurable sets and μ is a probability measure. Now assume that T is realized as the right shift in the space of two-sided sequences of at most countably many symbols. This means that X is the space of infinite

two-sided sequences of symbols (e.g., $X = \mathbb{N}^{\mathbb{Z}}$, T shifts a sequence to the right, and μ is a shift-invariant (stationary) probability measure). So we consider a stationary random process. Thus we can obtain (in many ways) a realization of any automorphism: this is Rokhlin's theorem on the existence of a countable generator [15]. Denote by ζ the partition of the space $\mathbb{N}^{\mathbb{Z}}$ according to the “past” of the process: an element of ζ is a class of all sequences with fixed values of coordinates with negative indices and arbitrary values of coordinates with nonnegative indices. Let us shift ζ one place to the right and consider its average conditional entropy, i.e., the expectation, over all elements of the past, of the entropy of the conditional distribution of the zero coordinate given all coordinates with negative indices: $EH(\zeta | T^{-1}\zeta)$.

Kolmogorov's theorem. *The (finite or infinite) nonnegative number*

$$EH(\zeta | T^{-1}\zeta) \equiv h(T),$$

called the average conditional entropy (or Shannon information) per step of a finite-state stationary random process, is an invariant of the automorphism T . In other words, it does not depend on the particular isomorphic realization of T as the shift in the space of trajectories of such a process.

Let us return for a while to the history of this discovery. There are some events related to the statement of this theorem. In his first paper [10], Kolmogorov interpreted the above statement more widely, but the problem is that for a continual set of symbols (and even for a countable one, but with an infinite entropy), the theorem is not true unless we impose some special conditions on the realization of the automorphism. This was immediately observed by V. A. Rokhlin, who provided a counterexample: an automorphism T of algebraic origin and T -invariant σ -algebras for which the left-hand sides of the above equation are different for different generated partition ζ .³ In his second note [11], A. N. Kolmogorov corrected the statement by imposing

³In [10], the error was in an illegitimate passage to the limit along a decreasing sequence of invariant σ -algebras (in short, along a decreasing filtration). Curiously enough, the same error was made by N. Wiener in an important passage of his well-known book on the nonlinear theory of random processes, as well as by many other authors. The point is that along increasing sequences of σ -algebras (in short, along increasing filtrations), the passage to the limit is obviously legitimate, and this provokes one to assume that the same is true for decreasing filtrations. However, the theory of decreasing filtrations, and especially their classification, is much finer and more interesting than the theory of increasing filtrations. As a rule, the passage to the limit along a decreasing filtration is not possible (for the theory of decreasing filtrations, see [19]).

a priori conditions on automorphisms in the definition of entropy. However, for Bernoulli schemes, i.e., sequences of independent random variables with a finite or countable state space, as well as for many other cases, the invariant was well-defined already in the first paper [10]. For the above statement to be true in full generality, one needed theorems on generators and special invariant σ -algebras which were not yet known. The above-mentioned theorem on the existence of a countable generator for any aperiodic automorphism was proved somewhat later by V. A. Rokhlin: any automorphism can be realized as the shift in the space of trajectories of a process with at most countably many states. This recovered the generality of Kolmogorov's theorem. Somewhat later, W. Krieger proved the existence of a finite generator for automorphisms with finite entropy. The same fact, but with a worse estimate, was proved by A. N. Livshits (1950–2008).

A much simpler definition, generally accepted nowadays, was suggested by Ya. G. Sinai soon after the appearance of Kolmogorov's work. It is not related to information, but is rather of geometric, or combinatorial, nature.

Sinai's theorem [16]. *Let T be an automorphism of a standard measure space (X, \mathcal{A}, μ) and ξ be a finite measurable partition of this space. Let $T^0\xi = \xi, T\xi, T^2\xi, \dots, T^{n-1}\xi$ be the successive T -images of ξ and $\xi_T^n = \bigvee_{i=0}^{n-1} T^i\xi$ be the product of the first n images. Then the following (finite or infinite) limit does exist:*

$$\lim_{n \rightarrow \infty} \frac{H(\xi_T^n)}{n} \equiv h(T, \xi);$$

the expression

$$\sup_{\xi} h(T, \xi) = h(T),$$

where the supremum is taken over all finite partitions of X , coincides with the entropy of T defined above (or may be taken as its definition).

Kolmogorov's and Sinai's definitions are of different nature and have different interpretations (see below); their equivalence is not quite obvious. However, without doubt, one should speak about "Kolmogorov's entropy" and "Sinai's definition" rather than about "Kolmogorov and Sinai's entropy," as is often the case.⁴

An important fact: the expression $h(T, \xi)$ is continuous in ξ on the space of all finite partitions endowed with the so-called entropy metric. It is this

⁴Ya. G. Sinai acknowledged his agreement with this formulation (see [17]).

continuity that allows one to effectively compute the entropy via approximations. As mentioned above, the positivity of entropy distinguishes an important class of automorphisms whose properties differ from those of zero-entropy automorphisms.

Assume that the entropy of an automorphism is zero. Then the sequence $H(\xi_T^n)$ grows sublinearly. Can we replace the linear scaling by another one so that to obtain a new invariant of the automorphism? In other words, can we deform Sinai's definition in such a way that the new invariant would distinguish at least some zero-entropy automorphisms? It turns out that Kolmogorov's entropy in Sinai's definition cannot be deformed in the following literal sense.

Theorem 1. *For every ergodic automorphism T and any sequence of positive numbers $\{c_n\}$ satisfying the condition $\lim_n \frac{c_n}{n} = 0$, there exists a finite partition ξ such that*

$$\lim_n \frac{H(\xi_T^n)}{c_n} = +\infty.$$

In other words, no strictly sublinear growth of the entropy $H(\xi_T^n)$ provides a new invariant. This effect is due to the fact that the partition ξ_T^n in Theorem 1 can be very fine on a set of small measure, thus leading to an artificially high value of the entropy. Not quite accurately, one can formulate this as follows: on a set of small measure, the growth of the entropy can be almost linear (note that it cannot be superlinear). An analogous observation explains why a similar analog of the original Kolmogorov's definition does not lead to reasonable new invariants. This is a manifestation of the general principle of ergodic theory expressed by Rokhlin's lemma: ergodic automorphisms are indistinguishable up to a set of small measure.

Nevertheless, one can refine the idea of scaling the growth. For this, one should consider entropy only up to small changes in the partition ξ_T^n , in other words, involve the notion of ε -entropy. First we will describe new invariants in terms as close as possible to the traditional ones, i.e., using partitions as in Sinai's definition, and then turn to the richer language of metrics.

3 The ε -entropy of a measure space and the definition of scaling entropy

Consider the following function of a partition ξ and a positive number ε :

$$H_\varepsilon(\xi) = \inf_{A: \mu A > 1-\varepsilon} H(\xi|_A).$$

By $\xi|_A$ we mean the partition of a set A of positive measure, with the restriction of the measure μ to A renormalized to unity, whose elements are the intersections of the elements of ξ with A . Observe that this function monotonically decreases as ε grows, taking the value $H(\xi)$ for $\varepsilon = 0$ and vanishing for $\varepsilon = 1$. We will use it to define the scaling entropy of an automorphism. Consider the function

$$H_\varepsilon(\xi_T^n),$$

which depends on n , $\varepsilon > 0$, and a partition ξ , and let us study its growth.

Definition 1 ([20]). *We say that a sequence of positive numbers $\{c_n\}$ is a scaling sequence for an ergodic transformation T if for every finite partition ξ*

$$\lim_{\varepsilon \rightarrow 0} \limsup_n \frac{H_\varepsilon(\xi_T^n)}{c_n} < \infty,$$

and there exists a partition ξ such that

$$\lim_{\varepsilon \rightarrow 0} \liminf_n \frac{h_T(\xi_n, \varepsilon)}{c_n} > 0.$$

(Note that all such sequences $\{c_n\}$ are equivalent as $n \rightarrow \infty$.)

Theorem 2. *The class of scaling sequences for a given transformation is a metric invariant of the transformation. This invariant distinguishes transformations with zero Kolmogorov entropy. The class of sequences $\{c_n\} \sim \{n\}$ corresponds to the Kolmogorov entropy; with this scaling, the function $H_{\varepsilon,n}(\xi)$ for small ε does not depend on ε .*

Sometimes, in a class of equivalent sequences one can choose a sequence suitable for all transformations having this class as the scaling one. Then the invariant we obtain is not merely a class, but a number, called the *scaling entropy*. It is not clear whether one can always make such a choice. In [8], the

so-called “slow” entropy was introduced to distinguish actions of the group \mathbb{Z}^2 ; it resembles our scaling entropy.

Let us give several examples.

Example 1. If the scaling sequence is linear, i.e., $\{c_n\} \sim \{n\}$, then we obtain the Kolmogorov entropy. Of course, this growth is the maximum possible one for the group \mathbb{Z} . In the case of a zero Kolmogorov entropy, the scaling is sublinear.

Example 2. The scaling class corresponding to transformations with discrete spectrum is that of bounded sequences: $\sup c_n < \infty$. Thus for measure-preserving isometries, i.e., shifts on compact groups, the scaling entropy vanishes for any scaling. In a slightly different formulation, this fact was first observed by S. Ferenczi [2]. In appearance, it resembles A. Kushnirenko’s [12] result on the Kirillov–Kushnirenko entropy, also called *sequential entropy*: the class of automorphisms for which the sequential entropy vanishes for any sequence coincides with the class of automorphisms with discrete spectrum.

However, this resemblance is superficial, because the notion of scaling entropy differs crucially from that of Kirillov–Kushnirenko entropy.

Example 3. In exactly the same way as above, a scaling sequence can be also defined for flows. In two independent articles, A. Kushnirenko’s paper [12] mentioned above and M. Ratner’s paper [14] on horocycle flows on surfaces of negative curvature, it was proved that different Cartesian powers of such flows are nonisomorphic. The distinguishing invariant in [12] was the sequential entropy, and in [14] M. Ratner used an idea of J. Feldman [1] and constructed an invariant that resembles a special example of scaling entropy. Comparing the invariant from [14] with our definition, one can conjecture that the scaling sequence for the k th power of a horocycle flow is logarithmic: $\{c_n\} \sim \{(\log n)^k\}$.

Example 4. A challenging problem is to find scaling sequences for adic automorphisms, e.g., the Pascal or Young automorphisms. Presumably, these automorphisms have singular continuous spectra. If the scaling sequence is not bounded, it would follow from above (see Example 2) that the spectrum is not purely discrete; this problem is still open.

4 Admissible metrics instead of partitions

The approach to scaling entropy described above needs further development. Instead of using the theory of partitions, which is a traditional approach in ergodic theory, one should involve the more flexible techniques of metrics and metric spaces, which turn to be useful in many problems of measure theory. Below we illustrate this with the example of the theory of scaling entropy, which includes the ordinary entropy theory. In the author's opinion, this approach must also be fruitful in applications to the general isomorphism problem in ergodic theory.

Any finite or countable measurable partition ξ determines a semimetric:

$$\rho(x, y) = \delta_{C(x), C(y)},$$

where $C(x)$ stands for the element of ξ that contains x .

Thus we can replace manipulations with measurable partitions by the analysis of the corresponding semimetrics; in other words, the transition to (semi)metrics tautologically includes the theory of partitions as a special case. But considering general metrics and semimetrics also opens up new possibilities.

Our approach is as follows: *instead of studying the collection of Borel measures on a fixed metric (or topological) space, which is a usual practice, we consider a set of metrics on a fixed measure space.* From the viewpoint of ergodic theory and probabilistic considerations, such a shift is quite natural. Let us introduce the notion of an *admissible (semi)metric* on a Lebesgue space (X, μ) .

Definition 2. *A (semi)metric ρ on a Lebesgue space (X, μ) with continuous measure is called admissible if the following conditions are satisfied:*

- 1) *The function $\rho(\cdot, \cdot)$ is a measurable function of two variables, defined on a set of full measure (depending on the metric) in the Cartesian square of the space (X, μ) , that satisfies the axioms of a (semi)metric on this set.*
- 2) *The space (X, ρ) , regarded as a (semi)metric space, is quasi-compact, i.e., it turns into a compact space after taking the quotient by the equivalence relation $x \sim y \Leftrightarrow \rho(x, y) = 0$.*

Thus a well-defined notion is not an individual (semi)metric on a Lebesgue space, but a class of mod 0 coinciding (semi)metrics, so that one should speak about classes of mod 0 coinciding (semi)metric spaces. As a rule,

checking that assertions under consideration are well-defined with respect to mod 0 equivalence presents no problem. Nevertheless, there are some subtleties, e.g., in the understanding of the triangle inequality (it should hold for all triples of points from the set of full measure on which the metric is defined, and not for almost all triples of points). The admissible (classes of) (semi)metrics form a convex cone \mathcal{R} in the space of measurable nonnegative functions of two variables on the space (X, μ) . We call \mathcal{R} the cone of (classes of) admissible (semi)metrics; it is closed under supremums of finitely many metrics. This is a canonical object, provided that we restrict ourselves to Lebesgue spaces with continuous measure. The geometry of this cone is of great interest; it is poorly studied.

5 The ε -entropy of measures in metric spaces

The following definition of the ε -entropy of measures in metric spaces is also essentially due to A. N. Kolmogorov (see [9]). We change only one detail, which is not very important; namely, we estimate the closeness of measures in the Kantorovich metric rather than by counting the number of points in an ε -net.

Definition 3. *Let μ be a Borel probability measure on a separable metric space (X, ρ) . Define a function $H(\rho, \mu, \varepsilon)$ as follows:*

$$H(\rho, \mu, \varepsilon) = \inf\{H(\nu) : k_\rho(\mu, \nu) < \varepsilon\},$$

where ν ranges over the set of discrete measures and k_ρ is the Kantorovich distance between measures in the metric space (X, ρ) .

Recall the definition of the Kantorovich (transportation) metric [7] on the space of measures defined on a compact metric space.⁵

Let (X, ρ) be a compact metric space, and let μ_1, μ_2 be two Borel probability measures on X . Then

$$k_\rho(\mu_1, \mu_2) = \inf_{\Psi} \int \int_{X \times X} \rho(x, y) d\Psi(x, y),$$

⁵The necessity to use Kantorovich metric on the set of the probability measures on the metric space, has two explanations: the first — this metric is an analog L^1 -metric for measures which is important in what follows, and secondly, Kantorovich metric is maximal among all metrics on the set of probability measures on the metric space with property $d(\delta_x, \delta_y) = \rho(x, y)$ (see [13]).

where the infimum is taken over all probability measures Ψ on the space $X \times X$ whose projection to the first coordinate coincides with μ_1 and projection to the second coordinate coincides with μ_2 . In other words, Ψ ranges over the set of measures with given marginal projections.

Note that the above definition makes sense also in the case where the space is not compact, because a probability measure in a complete separable space is supported, up to any positive ε , by compact subsets. In the case of semi-metrics, the definitions also remain meaningful. If we are given a finite partition ξ of a measure space (X, μ) , then its entropy coincides with the ε -entropy (for sufficiently small ε) of the space $(X/\xi, \mu/\xi)$ of elements of ξ with the discrete (semi)metric ρ_ξ :

$$H(\rho_\xi, \mu, \varepsilon) = H_\varepsilon(\xi).$$

6 Dynamics of metrics in a measure space as an appropriate framework for entropy

The classical functional analysis suggests to consider, instead of various objects, the spaces of functions on these objects. The spectral theory of dynamical systems is the result of following this recommendation: instead of a transformation of the phase space one considers a unitary operator in the corresponding L^2 space. But somehow these considerations have been hitherto limited to functions of one variable running over the phase space of the system. However, one may consider actions of Cartesian powers of the dynamical system in spaces of functions of several variables, while preserving the separation of variables; for instance, in the space of functions of two variables, namely, on the cone of admissible metrics. Clearly, in this way we obtain much more information on the system than when considering an action in the space of functions of one variable, and thus increase the possibilities of analyzing the properties of dynamical systems.⁶ This leads us to new interesting and important problems.

Let ρ be a (semi)metric and T be an automorphism. Denote by ρ_T the (semi)metric $\rho_T(x, y) = \rho(Tx, Ty)$. The image of an admissible metric is an

⁶Of course, Cartesian powers are widely used in ergodic theory, but usually one considers the Cartesian square merely as an automorphism of a measure space, not fixing the structure of a direct product.

admissible metric. Thus there is a natural action of the group of measure-preserving transformations on the cone \mathcal{R} .

Our main thesis is as follows: it is the *asymptotic theory of iterations of metrics in a space with a fixed measure* under automorphisms that is an appropriate framework for considering both Kolmogorov and scaling entropies (and their generalizations), as well as other invariants of automorphisms.

Given an admissible metric ρ on a space (X, μ) and an automorphism T , we can construct a sequence of new metrics. For this, we must take the orbit of ρ in the cone of admissible metrics under the action of T and then form symmetric combinations of the first n elements of the orbit. The following two sequences of metrics associated with a given metric ρ and a given automorphism T are especially important: the *uniform metric*

$$\rho_T^n = \sup_{i=0, \dots, n-1} \rho_{T^i} \quad \text{where} \quad \rho_{T^i}(x, y) = \rho(T^i x, T^i y)$$

(in the ordinary setting, it corresponds to the product of partitions: $\rho_{\xi_T^n} = \sup_{i=0, \dots, n-1} \rho_{T^i \xi}$) and the *average metric*

$$\widehat{\rho}_T^n = \frac{1}{n} \sum_{i=0}^{n-1} \rho_{T^i}.$$

Below we restrict ourselves to the first of them. *The problem we set out is to study the asymptotic behavior of this sequence of metrics as n goes to infinity.*

A scaling sequence. Consider the growth of the ε -entropy of a measure μ in the sequence of compact metric spaces (X, ρ_T^n) and introduce the class of monotone sequences of positive numbers $\{c_n\}$ such that

$$0 < \lim_{\varepsilon \rightarrow 0} \liminf_n \frac{H(\rho_T^n, \mu, \varepsilon)}{c_n} \leq \lim_{\varepsilon \rightarrow 0} \limsup_n \frac{H(\rho_T^n, \mu, \varepsilon)}{c_n} < \infty.$$

We say that a sequence c_n from this class is a *scaling sequence for the automorphism T and the metric ρ* . This normalization corresponds to the one we considered above when defining the scaling entropy of an automorphism via partitions.

Theorem 3 (On the scaling entropy of an ergodic automorphism). *For an ergodic automorphism T , the class of scaling sequences $\{c_n\}$ does not*

depend on the choice of a metric ρ in the class of metrics satisfying, along with conditions 1) and 2) above, the following condition:

3) The automorphism T of the metric space (X, ρ) is topologically transitive.

If for some canonical choice of a sequence from the equivalence class, the limit

$$\lim_{\varepsilon \rightarrow 0} \lim_n \frac{H(\rho_T^n, \mu, \varepsilon)}{c_n}$$

does exist, we call it the *scaling entropy* of T , indicating the scaling sequence.

Thus the scaling entropy (and, in particular, the Kolmogorov entropy) is a natural asymptotic invariant of a sequence of compact measure spaces. The methodological advantage of passing from partitions to continuous (semi)metrics is that passing to the limit and taking the supremum over all finite partitions is now replaced by considering an appropriate semi-metric. But it is much more important that now the original problem reduces to a circle of asymptotic questions concerning the behavior of a sequence of metrics. And the scaling entropy is merely one of the asymptotic characteristics of these metrics (the coarsest one). It characterizes the growth of the “dimension” of the compact space, or, in other words, the *asymptotics of its Hausdorff dimension*. It is the asymptotics of the dynamics of the metric measure spaces (X, μ, ρ_T^n) that is the appropriate framework for considering entropy and its generalizations we mentioned above.

Let us briefly describe the program we propose. Consider any admissible metric on a Lebesgue space (X, μ) with a fixed continuous measure and a measure-preserving automorphism T (or a group of automorphisms G). We suggest to study asymptotic invariants of the sequence of metrics ρ_T^n introduced above. The asymptotic characteristics of this sequence do not depend on the original metric and thus characterize only invariant properties of the automorphism.

As n grows, the metrics change in a quite complicated way, but presumably there is a number of coarse asymptotic invariants such as entropy. The scaling entropy is a simplest asymptotic invariant, which describes the growth of the cardinality of ε -nets, or the growth of the Hausdorff dimension of the compact space. More involved asymptotic invariants characterize not only the asymptotics of individual compact spaces, but also the asymptotics of their relative position. It is not yet known whether for the sequence of compact spaces there exists a limit object (in the example with filtrations

considered below, such a limit object does exist). It may happen that in this setting limit objects also do exist and can be characterized in a more or less explicit way. Presumably, the study of such objects would allow one to solve the isomorphism problem for automorphisms with completely positive entropy.

7 Relation to invariants of metric triples and their dynamics

Let us relate our considerations to the theory of metric triples, or Gromov triples, or *mm*-spaces in Gromov's terminology. Recall that M. Gromov [3]⁷ suggested a complete invariant of triples (X, ρ, μ) with respect to μ -preserving isometries of the space (X, ρ) . Here X is a space, ρ is a metric on X that turns it into a Polish space, and μ is a fully supported Borel probability measure. In the formulation due to the author of the present paper (see [21]), this invariant looks as a probability measure on the cone of nonnegative infinite symmetric matrices with countably many rows and columns satisfying the triangle inequality, i.e., on the cone of so-called distance matrices. This invariant can be interpreted as a random (semi)metric on the positive integers, or as a random distance matrix. A large variety of known invariants of metric triples can easily be computed via this invariant, i.e., via a random matrix. For example, in these terms one can easily describe the ε -entropy of the triple (or, say it another way, of the measure μ in the metric space (X, ρ)).

In the above scheme, we considered a sequence of metrics on the same measure space, i.e., a sequence of metric triples that differed only by metrics. And the problem was to find asymptotic invariants of these triples. The complete invariant of triples we have just described allows one to easily compute the ε -entropy, and thus to find the asymptotics of the ε -entropies of the sequences of triples, which, in our definition, is exactly the scaled entropy. Note that, as mentioned above, the result does not in fact depend on the original metric. One can also suggest other coarse asymptotic characteristics of sequences of triples; however, the choice of an appropriate characteristic should be determined by the problem under consideration.

For example, how can one formulate Ornstein's results on the classifica-

⁷We quote from the previous edition of the book, published in 2001.

tion and characterization of Bernoulli automorphisms (via the \bar{d} -metric or the VWB -property) as an assertion about some special asymptotic type of a sequence of metric triples? According to Ornstein, each such asymptotic type is determined by just one positive number, the entropy of the Bernoulli automorphism. In other words, the problem is to find a geometric description of the Bernoulli type of sequences of metrics. However, this problem is apparently still far from being solved. One may hope that examples of non-Bernoulli automorphisms with completely positive entropy would be more fully explained in terms of the asymptotic dynamics of metric spaces.

8 A parallel with the theory of filtrations and the dynamics of iterations of the Kantorovich metric

Another dynamics of metrics, which had appeared much earlier, is related to the theory of filtrations and, in particular, to the entropy of filtrations. Though this dynamics is more complicated, an attractive feature of this approach is the existence of limit objects; this allows one to develop the study much further than is currently done for the project described above. Let us briefly mention some definitions and examples.

A filtration is a decreasing sequence of σ -algebras. An example of a filtration is the sequence of pasts of a random process $\{T^{-n}\mathcal{A}\}$, $n = 0, 1, \dots$, where \mathcal{A} is a T -invariant σ -algebra (i.e., $T^{-1}\mathcal{A} \subset \mathcal{A}$). A filtration is called ergodic if the intersection of the σ -algebras $T^{-n}\mathcal{A}$ is the trivial algebra.

If T is a one-sided Bernoulli shift, then the corresponding filtration of pasts (which is ergodic by Kolmogorov's zero-one law) is called *standard*. There exist ergodic filtrations whose finite parts are isomorphic to Bernoulli filtrations but that are not isomorphic to Bernoulli filtrations as a whole. The following *dynamics of (semi)metrics generated by a filtration* is of fundamental importance. It is more convenient to pass from a filtration of σ -algebras to a (decreasing) filtration of partitions $\xi_1 \succ \xi_2 \succ \dots$. Suppose we have a filtration ξ_n with trivial intersection - $\bigcap_n \xi_n = \nu$ (ν is trivial partition). Consider an admissible metric ρ on a measure space (X, μ) and construct the sequence of metrics ρ_n in the different way than we had for the case of automorphisms in the previous paragraph. Namely we use the Kantorovich iterations constructed with the help of the filtration.

Let

$$\begin{aligned}\rho_0 &= \rho, & \rho_1(x, y) &= k_{\rho_1}(\mu^{C_1(x)}, \mu^{C_1(y)}), \dots, \\ \rho_m(x, y) &= k_{\rho_{m-1}}(\mu^{C_m(x)}, \mu^{C_m(y)}), \dots,\end{aligned}$$

where $C_m(x)$ is the element of ξ_m that contains x , μ^C is the conditional measure on an element C , and k_ρ is the Kantorovich distance between measures on the metric measure space (X, ρ) . In other words, the distance between points x and y in the n th semi-metric is the Kantorovich distance between the conditional measures on the elements $C_n(x)$ and $C_n(y)$ of the n th partition with respect to the $(n - 1)$ th semi-metric.

We obtain a sequence $\{\rho_m\}_0^\infty$ of semi-metrics on the space (X, μ) . In terms of the asymptotic behavior of this sequence, one can express many (possibly all) invariant asymptotic properties of the filtration. Remarkably, these asymptotic properties do not depend on the choice of the original metric from a very wide class of metrics — exactly as in the program considered above.

The main example is related to the standardness criterion, see the author's papers [18, 19].

Theorem 4. *The sequence of iterated metrics tends to a degenerate metric (i.e., the metric space contracts to a point) if and only if the filtration is standard (Bernoulli).*

This means that the scaling entropy of the sequence of metrics vanishes for any increasing sequence $\{c_n\}$. One may compare this fact with the dynamics of metrics under automorphisms with discrete spectrum (see above).

EXAMPLE: scaling entropy for the random walk in a random environment.

Let us give a newer example. Consider a random walk in a random environment: namely, the simple random walk (and the corresponding Markov process) on the set of all $\{0, 1\}$ -configurations of the lattice \mathbb{Z}^d equipped with Bernoulli measure $(1/2, 1/2)$.

For $d = 1$, it is so called (T, T^{-1}) transformation, where T is Bernoulli automorphism; the Markov shift in this case is a non-Bernoulli K -automorphism even not loosely Bernoulli - (S. Kalikow [6]). As F. Hollander and J. Steif [5] shown the same holds for $d = 2$; for $d > 2$ the Markov shift is already Bernoulli, but in the case $d = 3, 4$ the natural generator is very weak Bernoulli

but not weak, and is a weak Bernoulli for $d > 4$. All this results simulated the question: what can be said about the filtration of pasts of these processes? The conjecture that in the case $d = 1$ the filtration of the past is not standard was formulated by A. Vershik in 70-th. Now we can answer on properties of the filtration of the pasts and calculate scaling entropy.

Theorem 5. *The scaling entropy for the filtration of the past in the case of random walk in the space of configuration on the lattice \mathbb{Z}^d of dimension d is normalized by sequences which are equivalent to $c_n = n^{\frac{d}{2}}$.⁸*

D. Hecklen and C. Hoffman [4] proved the nonstandardness for the dimension $d = 1$, and then essentially computed the entropy for $d = 1$. A. Vershik and A. Gorbulsky in [22] proved the nonstandardness for $d > 1$, and, found the scaling entropy in general case. It showed, by the way, that the filtrations are nonisomorphic for different d . The latter result implies that if the dimensions d of lattices are different, than the Markov process of the random walk on one lattice cannot be encoded in an invertible way into the shift on the other lattice, though (for $d > 3$) all these Markov shifts are Bernoulli.

The dynamics of metrics in the case of filtrations is closely related to the construction of the so-called tower of measures, which allows one to construct limit metric spaces for a sequence of iterated compact spaces (see [19, 20, 22]). The corresponding combinatorics is quite interesting and has relevance to actions of groups of automorphisms of trees and close groups.

Translated by N. V. Tsilevich.

References

- [1] J. Feldman, r -entropy, equipartition, and Ornstein's isomorphism theorem in \mathbf{R}^n . *Israel Math. J.* **36** (1980), 321–345.

⁸The growth $c_n = n^k$ in this situation corresponds to a logarithmic growth for the group $\mathbb{Z}(\ln)^k$, since here we consider the group $\sum \mathbb{Z}_{2d}$, which has infinitely many generators, and has exponential growth of the number of words, so usual (Kolmogorov) entropy for this group must be normalized with the growth $c_n = (2d)^n$; the result above can be compared with growth $c_n = (\ln n)^{d/2}$ for the group \mathbb{Z} , which is the same as in the example of horocycle flow in the examples 3 above.

- [2] S. Ferenczi, Measure-theoretic complexity of ergodic systems. *Israel Math. J.* **100** (1997), 180–207.
- [3] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces*. Birkhäuser Boston, Boston, MA, 2007.
- [4] D. Heicklen and C. Hoffman, T, T^{-1} is not standard. *Ergodic Theory Dynam. Systems* **18** (1998), No. 4, 875–878.
- [5] F. Hollander and J. Steif, Random walk in random scenery. *IMS Lect. Notes* **48** (2006), 53–65.
- [6] S. A. Kalikow, T, T^{-1} transformation is not loosely Bernoulli. *Ann. of Math. (2)* **115** (1982), No. 2, 393–409.
- [7] L. V. Kantorovich, On the translocation of masses. *Dokl. Akad. Nauk SSSR* **37** (1942), No. 7–8, 227–229.
- [8] A. Katok and J.-P. Thouvenot, Slow entropy type invariants and smooth realization of commuting measure preserving transformation. *Ann. Inst. H. Poincaré* **33** (1997), 323–338.
- [9] A. N. Kolmogorov, Information transmission theory. In: *Session of the Academy of Sciences of the USSR on Scientific Problems of Industrial Automation*, October 15–20, 1956. Izdat. Akad. Nauk, 1957, pp. 66–99.
- [10] A. N. Kolmogorov, A new metric invariant of transient dynamical systems and automorphisms in Lebesgue spaces. *Dokl. Akad. Nauk SSSR* **119** (1958), 861–864.
- [11] A. N. Kolmogorov, Entropy per unit time as a metric invariant of automorphisms. *Dokl. Akad. Nauk SSSR* **124** (1959), 754–755.
- [12] A. G. Kushnirenko, Metric invariants of entropy type. *Uspekhi Mat. Nauk* **22** (1967), No. 5 (137), 57–65.
- [13] J. Melleray, F. Petrov, A. Vershik, Linearly rigid metric space and the embedding problem. *Fund. Math.* **199** (2008), No. 2, 177–194.
- [14] M. Ratner, Some invariants of Kakutani equivalence. *Israel Math. J.* **38** (1981), 232–240.

- [15] V. A. Rokhlin, Lectures on the entropy theory of transformations with invariant measure. *Uspekhi Mat. Nauk* **22** (1967), No. 5 (137), 3–56.
- [16] Ya. G. Sinai, On the concept of entropy for a dynamic system. *Dokl. Akad. Nauk SSSR* **124** (1959), 768–771.
- [17] Ya. G. Sinai, About A. N. Kolmogorov’s work on the entropy of dynamical systems. *Ergodic Theory Dynam. Systems* **8** (1988), 501–502.
- [18] A. M. Vershik, Decreasing sequences of measurable partitions and their applications, *Dokl. Akad. Nauk SSSR*, **193** (1970), No. 4, 748–751.
- [19] A. M. Vershik, Theory of decreasing sequences of measurable partitions. *St. Petersburg Math. J.* **6** (1995), No. 4, 705–761.
- [20] A. M. Vershik, Dynamic theory of growth in groups: entropy, boundaries, examples. *Russian Math. Surveys* **55** (2000), No. 4, 667–733.
- [21] A. M. Vershik, Random metric spaces and universality. *Russian Math. Surveys* **59** (2004), No. 2, 259–295.
- [22] A. M. Vershik and A. D. Gorbulsky, The scaling entropy of filtrations of σ -algebras, *Theory of prob and appl.* **52** (2007) No.3, 446–467.
- [23] *Mathematical events of the XX century: look from St.Petersburg*. Ed. A.Vershik. MCCME, Moscow 2009.(In Russian),