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**On the stability of non-symmetric equilibrium figures of
rotating self-gravitating liquid not subjected to capillary
forces**

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To the memory of Professor A.V.Kazhikhov

Abstract The paper contains the justification of the principle of minimum of potential energy in the problem of stability of rotating viscous incompressible self-gravitating liquid bounded only by a free surface. It is assumed that the domain occupied by a rotating liquid that is referred to as equilibrium figure is not symmetric with respect to the axis of rotation. The surface tension is not taken into account. The proof of stability is based on the analysis of evolution free boundary problem for the perturbations of the velocity and pressure.

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1. Introduction.

In the present article we continue the analysis of the stability of an isolated mass of uniformly rotating viscous incompressible self-gravitating liquid initiated in [1]. As in [1], we do not take into account capillary forces on the free boundary. We recall that the velocity and the pressure of a liquid rotating as a rigid body about the x_3 -axis is given by

$$\mathbf{V}(x) = \omega(\mathbf{e}_3 \times \mathbf{x}) = \omega(-x_2, x_1, 0), \quad P(x) = \frac{\omega^2}{2}|x'|^2 + p_0 \quad (1.1)$$

where $x' = (x_1, x_2, 0)$, $p_0 = \text{const}$, \mathbf{e}_3 is a unit vector directed along the x_3 -axis and ω is the angular velocity of rotation. The domain \mathcal{F} occupied by the liquid, so called equilibrium figure, is defined by the equation

$$\frac{\omega^2}{2}|x'|^2 + \kappa\mathcal{U}(x) + p_0 = 0, \quad x \in \mathcal{G} = \partial\mathcal{F}, \quad (1.2)$$

where

$$\mathcal{U}(x) = \int_{\mathcal{F}} \frac{dz}{|x-z|}$$

is a gravitational potential of the domain \mathcal{F} (the density of the liquid equals one).

We consider the functions (1.1) given in \mathcal{F} as a solution of a free boundary problem governing the evolution of an isolated liquid mass bounded only by a free surface. This problem consists of determination of a bounded domain $\Omega_t \subset \mathbb{R}^3$, $t > 0$, as well as of the vector field of velocities $\mathbf{v}(x, t) = (v_1, v_2, v_3)$ and the pressure function $p(x, t)$, $x \in \Omega_t$, $t > 0$, satisfying the equations

$$\begin{aligned} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} - \nu\nabla^2\mathbf{v} + \nabla p &= 0, \\ \nabla \cdot \mathbf{v} &= 0, \quad x \in \Omega_t, \quad t > 0, \\ T(\mathbf{v}, p)\mathbf{n} &= \kappa U(x, t)\mathbf{n}, \quad V_n = \mathbf{v} \cdot \mathbf{n}, \quad x \in \Gamma_t \equiv \partial\Omega_t, \\ \mathbf{v}(x, 0) &= \mathbf{v}_0(x), \quad x \in \Omega_0, \end{aligned} \quad (1.3)$$

where $\nu, \kappa = \text{const} > 0$,

$$U(x, t) = \int_{\Omega_t} \frac{dz}{|x-z|}$$

is the Newtonian potential depending on an unknown domain Ω_t , $T(\mathbf{v}, p) = -pI + \nu S(\mathbf{v})$ is the stress tensor, $S(\mathbf{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j} \right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, \mathbf{n} is the exterior normal to Γ_t , V_n is the velocity of evolution of Γ_t in the normal direction. The domain Ω_0 is given.

We assume that the equilibrium figure \mathcal{F} is a given bounded domain. If it is axially symmetric with respect to the x_3 -axis (as the Maclaurin ellipsoids), then the functions (1.1) given in the domain \mathcal{F} represent the stationary solution of (1.3). If \mathcal{F} does not possess the symmetry property (as the Jacobi ellipsoids,

pear-formed figures of Poincaré etc., see [2-5]), then there exists a one-parameter family of the equilibrium figures, \mathcal{F}_θ , obtained by rotation of the angle θ about the x_3 -axis of one of them, \mathcal{F}_0 . We assume that $\theta \in \mathbb{R}$ and $\mathcal{F}_\theta = \mathcal{F}_{\theta+2\pi}$. In this case the functions (1.1) defined in the variable domain $\mathcal{F}_{\omega t + \varphi}$ represent a periodic solution of (1.3).

We observe that in the case of non-symmetric \mathcal{F} the function $h(y) = \mathbf{N}(y) \cdot (\mathbf{e}_3 \times y)|_{\mathcal{G}}$, where $\mathbf{N}(y)$ is the exterior normal to $\mathcal{G} = \partial\mathcal{F}$ and \mathbf{e}_3 is a unit vector directed along the x_3 -axis, is different from identical zero, whereas for axially symmetric \mathcal{F} this function vanishes.

We are interested in the problem of stability of these solutions, that is closely related to the well known problem of stability of equilibrium figures. According to the classical theory, the figure is stable, if the quadratic form

$$\begin{aligned} \delta^2 \mathcal{R}[\rho] = & \int_{\mathcal{G}} b(x) \rho^2(x) dS + \frac{\omega^2}{\int_{\mathcal{F}} |z'|^2 dz} \left(\int_{\mathcal{G}} |y'|^2 \rho(y) dS \right)^2 \\ & - \kappa \int_{\mathcal{G}} \int_{\mathcal{G}} \frac{\rho(y) \rho(z)}{|y-z|} dS_y dS_z \end{aligned} \quad (1.4)$$

where

$$b(x) = -\omega^2 \mathbf{x}' \cdot \mathbf{N}(x) - \kappa \frac{\partial \mathcal{U}(x)}{\partial N} \geq b_0 > 0, \quad (1.5)$$

is positive definite, i.e.,

$$c_1 \|\rho\|_{L_2(\mathcal{F})}^2 \leq \delta^2 \mathcal{R}[\rho] \leq c_2 \|\rho\|_{L_2(\mathcal{F})}^2 \quad (1.6)$$

for arbitrary function $\rho(x)$ given on \mathcal{G} and satisfying the conditions

$$\int_{\mathcal{G}} \rho(x) dS = 0, \quad \int_{\mathcal{G}} \rho(x) x_i dS = 0, \quad i = 1, 2, 3, \quad (1.7)$$

$$\int_{\mathcal{G}} \rho(x) h(x) dS_x = 0, \quad (1.8)$$

and unstable, if this form can take negative values. We give the justification of the first statement by the analysis of the evolution free boundary problem for the perturbations $\mathbf{w}(x, t) = \mathbf{v} - \mathbf{V}$, $s = p - P$ of the velocity and pressure. This problem consists of determination of a bounded domain in \mathbb{R}^3 (denoted also by Ω_t) with the boundary Γ_t , $t > 0$, as well as of the functions $\mathbf{w}(x, t)$ and $s(x, t)$, satisfying the relations

$$\begin{aligned} \mathbf{w}_t + (\mathbf{w} \cdot \nabla) \mathbf{w} + 2\omega(\mathbf{e}_3 \times \mathbf{w}) - \nu \nabla^2 \mathbf{w} + \nabla s &= 0, \\ \nabla \cdot \mathbf{w} &= 0, \quad x \in \Omega_t, \quad t > 0, \\ T(\mathbf{w}, s) \mathbf{n} &= \left(\frac{\omega^2}{2} |x'|^2 + \kappa U(x, t) + p_0 \right) \mathbf{n}, \\ V_n &= \mathbf{w} \cdot \mathbf{n}, \quad x \in \Gamma_t, \end{aligned} \quad (1.9)$$

$$\mathbf{w}(x, 0) = \mathbf{w}_0(x), \quad x \in \Omega_0.$$

The vector field $\mathbf{w}_0 = \mathbf{v}_0 - \mathbf{V}$ should satisfy the orthogonality conditions

$$\begin{aligned} \int_{\Omega_0} \mathbf{w}_0(x) dx &= 0, \\ \int_{\Omega_0} \mathbf{w}_0(x) \cdot \boldsymbol{\eta}_i(x) dx + \omega \int_{\Omega_0} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx &= \omega \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx, \end{aligned} \quad (1.10)$$

and it is easily verified that they hold at any moment of time $t \geq 0$:

$$\begin{aligned} \int_{\Omega_t} \mathbf{w}(x, t) dx &= 0, \\ \int_{\Omega_t} \mathbf{w}(x, t) \cdot \boldsymbol{\eta}_i(x) dx + \omega \int_{\Omega_t} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx &= \omega \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx, \end{aligned} \quad (1.11)$$

$i = 1, 2, 3$. In addition, we have

$$|\Omega_t| = |\mathcal{F}|, \quad (1.12)$$

$$\int_{\Omega_t} x_i dx = 0, \quad i = 1, 2, 3.$$

We find it convenient to pass to the Lagrangian coordinates $\xi \in \Omega_0$ connected with the Eulerian coordinates $x \in \Omega_t$ by

$$x = \xi + \int_0^t \mathbf{u}(\xi, \tau) d\tau \equiv X(\xi, t), \quad (1.13)$$

where $\mathbf{u}(\xi, t) = \mathbf{w}(X(\xi, t), t)$. Under this transformation (1.9) is converted to

$$\begin{aligned} \mathbf{u}_t + 2\omega(\mathbf{e}_3 \times \mathbf{u}) - \nu \nabla_u^2 \mathbf{u} + \nabla_u q &= 0, \\ \nabla_u \cdot \mathbf{u}(\xi, t) &= 0, \quad \xi \in \Omega_0, \quad t > 0, \\ T_u(\mathbf{u}, q) \mathbf{n} &= \left(\kappa U(X, t) + \frac{\omega^2}{2} |X'{}^2(\xi, t)|^2 + p_0 \right) \mathbf{n}, \quad \xi \in \Gamma_0, \\ \mathbf{u}(\xi, 0) &= \mathbf{w}_0(\xi), \quad \xi \in \Omega_0, \end{aligned} \quad (1.14)$$

where $q(\xi, t) = s(X(\xi, t), t)$, and ∇_u , T_u are the transformed gradient and the stress tensor, respectively. Since the Jacobian of the transformation (1.13) equals one, we have $\nabla_u = A \nabla_\xi$, $T_u(\mathbf{u}, q) = -qI + \nu S_u(\mathbf{u})$, where $S_u(\mathbf{u}) = A \nabla_u \mathbf{u} + (A \nabla_u \mathbf{u})^T$ is the transformed doubled rate-of-strain tensor, and $A(\xi, t) = (A_{ij})_{i,j=1,2,3}$ is the co-factors matrix corresponding to the transformation (1.13). Finally, $U(X, t) = \int_{\Omega_0} |X(\xi, t) - X(\eta, t)|^{-1} d\eta$ and $\mathbf{n}(x)$ is the exterior normal to the surface $\Gamma_t = X\Gamma_0$ connected with the normal $\mathbf{n}_0(\xi)$ to Γ_0 by

$$\mathbf{n}(X(\xi, t)) = \frac{A(\xi, t) \mathbf{n}_0(\xi)}{|A(\xi, t) \mathbf{n}_0(\xi)|}. \quad (1.15)$$

The problem (1.14) is studied in the weighted anisotropic Sobolev-Slobodetskii spaces introduced by Y.Hataya [6]. Let $Q_T = \Omega_0 \times (0, T)$ and let $W_2^{l,l/2}(Q_T)$, $l \geq 1$, be a standard anisotropic Sobolev-Slobodetskii space. The weighted space $\widetilde{W}_2^{l,l/2}(Q_T)$ is defined as the set of functions (or vector fields) $u(\xi, t)$, such that $u \in W_2^{l,l/2}(Q_T)$, $tu \in W_2^{l-1,l/2-1/2}(Q_T)$ (the weight improves the behavior of u for large t), and supplied with the norm

$$\|u\|_{\widetilde{W}_2^{l,l/2}(Q_T)} = \|u\|_{W_2^{l,l/2}(Q_T)} + \|tu\|_{W_2^{l-1,l/2-1/2}(Q_T)}.$$

We also set

$$\begin{aligned} \|u\|_{\widetilde{W}_2^{l,0}(Q_T)} &= \|u\|_{W_2^{l,0}(Q_T)} + \|tu\|_{W_2^{l-1,0}(Q_T)}, \\ \|u\|_{\widetilde{W}_2^{0,l/2}(Q_T)} &= \|u\|_{W_2^{0,l/2}(Q_T)} + \|tu\|_{W_2^{0,l/2-1/2}(Q_T)}. \end{aligned}$$

The weighted spaces of functions given on smooth manifolds, in particular, on $G_T = \Gamma_0 \times (0, T)$, are defined in a similar way.

The main result of the paper is as follows.

Theorem 1.1. *Assume the following:*

1. $\mathbf{w}_0 \in W_2^{l+1}(\Omega_0)$, $l \in (1, 3/2)$, satisfies the orthogonality conditions (1.10) and the compatibility conditions

$$\nabla \cdot \mathbf{w}_0 = 0, \quad \Pi_0 S(\mathbf{w}_0) \mathbf{n}_0|_{\Gamma_0} = 0, \quad (1.16)$$

where $\Pi_0 \mathbf{f} = \mathbf{f} - \mathbf{n}_0(\mathbf{f} \cdot \mathbf{n}_0)$ is the projection on the tangent plane to Γ_0 .

2. The domain Ω_0 satisfies (1.12), the surface $\Gamma_0 = \partial\Omega_0$ is given by the equation

$$x = y + \mathbf{N}_0(y) \rho_0(y), \quad y \in \mathcal{G}, \quad (1.17)$$

where \mathbf{N}_0 is the unit normal to \mathcal{G}_0 , and $\rho_0(y) \in W_2^{l+3/2}(\mathcal{G})$ satisfies the condition

$$\int_{\Gamma_0} \rho_0(\bar{\xi}) \mathbf{N}(\bar{\xi}) \cdot (\mathbf{e}_3 \times \bar{\xi}) dS_\xi = 0, \quad (1.18)$$

$\bar{\xi}$ being the closest point of \mathcal{G}_0 to ξ .

3. The following smallness condition holds:

$$\|\mathbf{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \epsilon \ll 1. \quad (1.19)$$

4. The quadratic form (1.4) satisfies the condition (1.6), where \mathcal{G} is an arbitrary \mathcal{G}_θ .

Then the problem (1.14) has a unique solution $\mathbf{u} \in \widetilde{W}_2^{2+l,1+l/2}(Q_\infty)$, $\nabla s \in \widetilde{W}_2^{l,l/2}(Q_\infty)$ such that $s|_{\xi \in \Gamma_0} \in \widetilde{W}_2^{1/2+l,1/4+l/2}(G_\infty)$, and

$$\begin{aligned} &\|\mathbf{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_\infty)} + \|\nabla s\|_{\widetilde{W}_2^{l,l/2}(Q_\infty)} + \|s\|_{\widetilde{W}_2^{1/2+l,1/4+l/2}(G_\infty)} \\ &\leq c \left(\|\mathbf{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})} \right). \end{aligned} \quad (1.20)$$

The surface Γ_t is given by the equation

$$x = z + \mathbf{N}_{\theta(t)}(z)\widehat{\rho}(z, t), \quad z \in \mathcal{G}_{\theta(t)}, \quad (1.21)$$

where \mathbf{N}_{θ} is a unit exterior normal to \mathcal{G}_{θ} . The derivative of $\theta(t)$ satisfies the inequality

$$|\theta'(t)| \leq c \int_{\Gamma_0} |\mathbf{u}(\xi, t)| dS_{\xi}, \quad (1.22)$$

whereas

$$\theta(t) = \int_0^t \theta'(\tau) d\tau \rightarrow \theta_{\infty} \quad (1.23)$$

as $t \rightarrow \infty$. The function

$$r(\xi, t) \equiv \widehat{\rho}(z, t), \quad (1.24)$$

where z is the closest point of $\mathcal{G}_{\theta(t)}$ to $X(\xi, t) \in \Gamma_t$, satisfies the condition

$$\int_{\Gamma_0} r(\xi, t) h_{\theta(t)}(z) dS_{\xi} = \int_{\Gamma_0} \widehat{\rho}(z, t) h_{\theta(t)}(z) dS_{\xi} = 0 \quad (1.25)$$

and the inequality

$$\begin{aligned} & \|r\|_{\widetilde{W}_2^{l+1/2, 0}(G_{\infty})} + \sup_{t>0} \|r(\cdot, t)\|_{W_2^{l+1}(\Gamma_0)} + \sup_{t>0} t \|r(\cdot, t)\|_{W_2^l(\Gamma_0)} \\ & \leq c \left(\|w_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})} \right). \end{aligned} \quad (1.26)$$

Thus, $w, s \rightarrow 0$ and $\Omega_t \rightarrow \mathcal{F}_{\theta_{\infty}}$ as $t \rightarrow \infty$, which means the stability of the regime (1.1) of rigid rotation.

The condition (1.8) is trivial in the case of axially symmetric \mathcal{F} (since $h(y) = 0$). In the general case we have (1.25); as we shall see, it may be regarded as the approximate condition (1.18) for $\widehat{\rho}(z, t)$, $z \in \mathcal{G}_{\theta(t)}$.

The quadratic form (1.4) is the second variation of the energy functional

$$\mathcal{R} = \frac{\beta^2}{2 \int_{\Omega} |x'|^2 dx} - \frac{\kappa}{2} \int_{\Omega} \int_{\Omega} \frac{dx dy}{|x - y|} - p_0 |\Omega| \quad (1.27)$$

where $\beta = \omega \int_{\mathcal{F}} |x'|^2 dx$ is the magnitude of the total angular momentum of the rotating liquid and Ω is the domain in \mathbb{R}^3 close to \mathcal{F} and having the same volume and the position of the barycenter as \mathcal{F} . If the boundary of Ω is given by the equation $x = y + \mathbf{N}(y)\rho(y, t)$, $y \in \mathcal{F}$, then the above-mentioned properties of Ω can be expressed in terms of ρ as follows:

$$\int_{\mathcal{G}} \varphi(y, \rho) dS = 0, \quad \int_{\mathcal{G}} \psi_i(y, \rho) dS = 0, \quad i = 1, 2, 3, \quad (1.28)$$

where

$$\varphi(y, \rho) = \rho - \frac{\rho^2}{2} \mathcal{H}(y) + \frac{\rho^3}{3} \mathcal{K}(y),$$

$$\psi_i(y, \rho) = \varphi(y, \rho)y_i + N_i(y) \left(\frac{\rho^2}{2} - \frac{\rho^3}{3} \mathcal{H}(y) + \frac{\rho^4}{4} \mathcal{K}(y) \right), \quad (1.29)$$

$\mathcal{H}(y)$ and $\mathcal{K}(y)$ are the doubled mean curvature and the Gaussian curvature of \mathcal{G} , respectively. In particular, these conditions are satisfied by $\hat{\rho}$. Direct calculation shows that the first variation of \mathcal{R} (considered as a functional defined on the set of small ρ satisfying (1.28)) vanishes in view of (1.2) and the second variation coincides with the form (1.4); moreover, if the form (1.4) is positive definite for arbitrary ρ satisfying (1.7), (1.8), then the difference $\mathcal{R} - \mathcal{R}_0$ where $\mathcal{R}_0 = \mathcal{R}|_{\rho=0}$ is equivalent to $\|\rho\|_{L_2(\mathcal{G})}^2$ for small ρ satisfying (1.28), (1.25).

It should be observed that $\delta^2 \mathcal{R}[h] = 0$.

When the surface tension is taken into account, then the extra term σH appears in the boundary conditions, where σ is a positive constant coefficient of the surface tension and H is the doubled mean curvature of Γ_t . This term is a strong regularizer of the problem, moreover, it guarantees the exponential decay of the solution of (1.9), as $t \rightarrow \infty$. The problem of stability of the rotating capillary viscous incompressible self-gravitating liquid is treated in a series of papers of the author, partly in collaboration with Professor M. Padula. In particular, the analogue of Theorem 1.1 for non-symmetric equilibrium figures is proved in [7].

As it has been pointed out, our main attention is given to the case of non-symmetric \mathcal{F} . Sec. 2 is devoted to the construction of $\theta(t)$ and to the proof of (1.22). In Sec. 3 the general scheme of the proof of Theorem 1.1 is presented and the necessary transformations of the problem (1.14) are carried out. In Sec. 4 the main estimate of $\theta'(t)$ is obtained, as well as some important auxiliary inequalities, whose proof requires additional calculations in the case of non-symmetric \mathcal{F} . Finally, in Sec. 5 the "generalized energy" is estimated, which furnishes uniform bounds for some weak norms of the solution of the problem (1.14). In the case of symmetric \mathcal{F} these bounds are obtained in [8].

2. On the construction of $\theta(t)$.

This section is devoted to the construction of the function $\theta(t)$. At first we introduce some notations (some of them are introduced above).

By \mathcal{F}_θ we mean the family of equilibrium figures obtained by rotation of the angle θ of one of them, \mathcal{F}_0 , about the x_3 -axis, \mathcal{G}_θ is the boundary of \mathcal{F}_θ , \mathbf{N}_θ is the exterior normal to \mathcal{G}_θ .

We set

$$R_\theta(x) = \pm \text{dist}(x, \mathcal{G}_\theta), \quad (2.1)$$

with the signs "+" and "-" corresponding to the cases $x \in \mathbb{R}^3 \setminus \mathcal{F}_\theta$ and $x \in \mathcal{F}_\theta$, respectively. The function R_θ is smooth in a certain neighborhood (δ_1 -neighborhood) of \mathcal{G}_θ and it possesses the property

$$\nabla R_\theta(x) = \mathbf{N}_\theta(\bar{x}^\theta), \quad (2.2)$$

where \bar{x}^θ is the closest point of \mathcal{G}_θ to x . We have $x = \bar{x}^\theta + \mathbf{N}(\bar{x}^\theta)R_\theta(x)$, i.e.,

$$\bar{x}^\theta = x - R_\theta(x) \nabla R_\theta(x) \equiv \mathfrak{R}_\theta(x). \quad (2.3)$$

The function \mathfrak{R}_θ is also smooth in the δ_1 -neighborhood of \mathcal{G}_θ . In the case $\theta = 0$ the index 0 is sometimes omitted, in particular, $R_0(x) = R(x)$.

It is easily seen that $R(y) = R_\theta(\mathcal{Z}(\theta)y)$, i.e., $R_\theta(z) = R(\mathcal{Z}(-\theta)z)$, and $\mathcal{Z}(\theta)\mathbf{N}_0(y) = \mathbf{N}_\theta(z)$. It is also easily verified that $h_\theta(\mathcal{Z}(\theta)y) = h_0(y)$, $y \in \mathcal{G}_0$, and that $b_\theta(z) = b_0(y)$, where $b_\theta(z) = -\omega^2 z' \cdot \mathbf{N}_\theta(z) - \kappa \frac{\partial \mathcal{U}_\theta(z)}{\partial N_\theta}$, $\mathcal{U}_\theta(z) = \int_{\mathcal{F}_\theta} \frac{d\zeta}{|z-\zeta|}$. It follows that the quadratic form (1.4) is invariant under the rotation about the x_3 -axis.

Let us consider the family of surfaces Γ_t given by the equation (1.13) with $\xi \in \Gamma_0$. In the case of small ρ_0 and \mathbf{u} these surfaces are close to a certain \mathcal{G} (say, \mathcal{G}_0) - see [1], Proposition 4.5. We want to construct the function $\theta(t)$ such that Γ_t can be given by (1.21) with $\widehat{\rho}$ satisfying the condition similar to (1.8).

Let $\Gamma_{t,\lambda}$ be a surface obtained by rotation of Γ_t through the angle λ about the x_3 -axis: $\Gamma_{t,\lambda} = \mathcal{Z}(\lambda)\Gamma_t$, where

$$\mathcal{Z}(\lambda) = \begin{pmatrix} \cos \lambda & -\sin \lambda & 0 \\ \sin \lambda & \cos \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For small λ , $\Gamma_{t,\lambda}$ is also located in a certain small neighborhood of \mathcal{G}_0 , and can be defined by the equation

$$x = y + \mathbf{N}_0(y)\widetilde{\rho}(y, t, \lambda), \quad y \in \mathcal{G}_0. \quad (2.4)$$

It follows that

$$\widetilde{\rho}(y, t, \lambda) = R(\mathcal{Z}(\lambda)X(\xi, t))$$

and $y = \overline{\mathcal{Z}(\lambda)X}$.

We look for the function $\lambda(t)$ such that

$$\int_{\Gamma_0} R(\mathcal{Z}(\lambda(t))X(\xi, t))h_0(\overline{\mathcal{Z}X})dS_\xi = 0, \quad (2.5)$$

which is equivalent to (1.25) with $r(\xi, t) = R(\mathcal{Z}(\lambda(t))X(\xi, t))$, $\theta(t) = -\lambda(t)$. Moreover, by Proposition 4.2 in [1], (2.5) can be written in the form

$$\int_{\mathcal{G}_0} \widetilde{\rho}(y, t, \lambda(t))h_0(y)\Psi^{-1}dS = 0, \quad (2.6)$$

where

$$\Psi = \frac{|A(\xi, t)\mathbf{n}_0(\xi)|}{|\widehat{\mathcal{L}}^T(y, \widetilde{\rho})\mathbf{N}_0(y)|}, \quad y = \overline{\mathcal{Z}(\lambda(t))X(\xi, t)}. \quad (2.7)$$

By $\widehat{\mathcal{L}}^T(y, \widetilde{\rho})$ we mean the co-factors matrix of the matrix of Jacobi of the transformation (2.4), and the sign "T" means transposition. If ρ_0 and \mathbf{u} are small, then Ψ^{-1} is close to 1.

In the paper [7] where the stability of the rotating capillary liquid was analyzed, we were looking for $\lambda = \lambda(t)$ such that

$$\int_{\mathcal{G}_0} \widetilde{\rho}(y, t, \lambda(t))h_0(y)dS = 0,$$

but when the surface tension is neglected, then the equation (2.5) is more convenient for technical reasons.

Let us compute the partial derivative of the function

$$f(t, \lambda) = \int_{\Gamma_0} R(\mathcal{Z}(\lambda)X(\xi, t))h_0(\overline{\mathcal{Z}X})dS_\xi \quad (2.8)$$

with respect to λ . Since

$$\begin{aligned} \frac{\partial R(\mathcal{Z}(\lambda)X(\xi, t))}{\partial \lambda} &= \mathbf{N}_0(\overline{\mathcal{Z}X}) \cdot \mathcal{Z}'(\lambda)X = \mathbf{N}_0(\overline{\mathcal{Z}X}) \cdot \mathcal{Z}(e_3 \times X) \\ &= \mathbf{N}_\theta(\bar{X}^\theta)(e_3 \times \bar{X}^\theta) = h_\theta(\bar{X}^\theta) = h_0(\overline{\mathcal{Z}(\lambda)X}), \end{aligned}$$

we have

$$f_\lambda(t, \lambda) = \int_{\mathcal{G}_0} h_0^2(y)\Psi^{-1}dS_y + \int_{\Gamma_0} R(\mathcal{Z}X)\nabla h_0(\overline{\mathcal{Z}X}) \cdot (\nabla \mathfrak{R}(\mathcal{Z}(\lambda)X)\mathcal{Z}(e_3 \times X))dS_\xi. \quad (2.9)$$

If $\mathbf{u} \in \widetilde{W}_2^{2+l, 1+l/2}(Q_T)$ is small, then, by Proposition 5.4 in [9], $X(\xi, t)$ is bounded by a constant independent of t and $|\Psi^{-1}| \geq k > 0$. This implies

$$f_\lambda(t, \lambda) \geq k \int_{\mathcal{G}_0} h_0^2(y)dS - c_0\delta_1 \geq \frac{k}{2} \int_{\mathcal{G}_0} h_0^2(y)dS, \quad (2.10)$$

provided $c_0\delta_1 \leq \frac{k}{2} \int_{\mathcal{G}_0} h_0^2(y)dS$. For $\lambda = 0$ we have

$$f(t, 0) = \int_{\Gamma_0} R(X)h_0(\bar{X})dS_\xi,$$

hence in the interval

$$|\lambda| \leq 2k^{-1}|f(t, 0)| \left(\int_{\mathcal{G}_0} h_0^2(y)dS \right)^{-1} = 2k^{-1} \left| \int_{\Gamma_0} R(X)h_0(\bar{X})dS \right| \left(\int_{\mathcal{G}_0} h_0^2(y)dS \right)^{-1}$$

there exists the number $\lambda(t)$ that is sought.

We set $\tilde{\rho}(y, t) = \tilde{\rho}(y, t, \lambda(t))$, $y \in \mathcal{G}_0$, $\theta(t) = -\lambda(t)$ and

$$\widehat{\rho}(z, t) = \tilde{\rho}(\mathcal{Z}(\lambda(t))z, t), \quad z \in \mathcal{G}_{\theta(t)}. \quad (2.11)$$

It is clear that the equation (2.4) for the surface $\mathcal{Z}(\lambda)\Gamma_t$ is equivalent to the equation (1.21) for Γ_t . Condition (1.25) is a consequence of (2.5); it is equivalent to

$$\int_{\mathcal{G}_{\theta(t)}} \widehat{\rho}(z, t)h_{\theta(t)}(z)\Psi_{\theta(t)}^{-1}dS = 0, \quad (2.12)$$

where

$$\Psi_\theta = \frac{|A(\xi, t)\mathbf{n}_0(\xi)|}{|\widehat{\mathcal{L}}^T(z, \widehat{\rho})\mathbf{N}_\theta(z)|}, \quad z = \bar{X}^\theta(\xi, t),$$

and $\widehat{\mathcal{L}}^T(z, \widehat{\rho})$ is a co-factors matrix corresponding to the transformation (1.21). It can be verified that $\Psi_\theta = \Psi$.

In particular, if Γ_0 is sufficiently close to a certain \mathcal{G}' , then there exists such θ_0 that Γ_0 is representable in the form (1.11) with $y \in \mathcal{Z}(\theta_0)\mathcal{G}' \equiv \mathcal{G}_0$ and with ρ_0 satisfying (1.18). This defines the choice of \mathcal{G}_0 ; we also have $\lambda(0) = 0$.

By the implicit function theorem, $\lambda(t)$ possesses the derivative

$$\lambda'(t) = - \frac{f_t(t, \lambda)}{f_\lambda(t, \lambda)} \Big|_{\lambda=\lambda(t)}, \quad (2.13)$$

where

$$\begin{aligned} f_t(t, \lambda) &= \int_{\Gamma_0} \mathbf{N}_0(\overline{\mathcal{Z}X}) \cdot \mathcal{Z}(\lambda) \mathbf{u}(\xi, t) h_0(\overline{\mathcal{Z}X}) dS \\ &+ \int_{\Gamma_0} R(\mathcal{Z}X) \nabla h_0(\overline{\mathcal{Z}X}) \cdot \nabla \mathfrak{R}(\mathcal{Z}X) \mathcal{Z}(\lambda) \mathbf{u}(\xi, t) dS, \end{aligned} \quad (2.14)$$

and f_λ is defined in (2.9). It is easily seen that

$$|\lambda'(t)| \leq \frac{|f_t(t, \lambda)|}{|f_\lambda(t, \lambda)|} \Big|_{\lambda=\lambda(t)} \leq c \int_{\Gamma_0} |\mathbf{u}(\xi, t)| dS_\xi, \quad (2.15)$$

hence for $\mathbf{u} \in \widetilde{W}_2^{l+2, 1+1/2}(Q_\infty)$

$$\lambda(t) = \int_0^t \lambda'(\tau) d\tau \rightarrow \lambda_\infty, \quad \text{as } t \rightarrow \infty. \quad (2.16)$$

Thus we have proved the following proposition.

Proposition 2.1. *If Γ_t and is defined by (1.13) and the norms $\|\rho_0\|_{W_2^{l+3/2}(\Gamma_0)}$ and $\|\mathbf{u}\|_{\widetilde{W}_2^{2+l, 1+1/2}(Q_T)}$ are sufficiently small, then there exists a function $\lambda(t)$ satisfying (2.15), (2.16) such that Γ_t can be given by (1.21), and $\widehat{\rho}$ satisfies (2.12) with $\theta(t) = -\lambda(t)$.*

Moreover, the following proposition holds.

Proposition 2.2. *If $\mathbf{u} \in \widetilde{W}_2^{2+l, 1+1/2}(Q_T)$, then*

$$\|\lambda'\|_{\widetilde{W}_2^{l/2+3/4}(0, T)} + \sup_{t < T} |\lambda(t)| \leq c \|\mathbf{u}\|_{\widetilde{W}_2^{0, l/2+3/4}(G_T)} \leq c \|\mathbf{u}\|_{\widetilde{W}_2^{2+l, 1+1/2}(G_T)} \quad (2.17)$$

with the constant independent of $T \leq \infty$.

We observe in conclusion that $\lambda'(t)$ can be represented in the form

$$\lambda'(t) = - \frac{\int_{\Gamma_0} \mathbf{N}_0(\bar{\xi}) \cdot \mathbf{u}(\xi, t) h_0(\bar{\xi}) dS_\xi}{\int_{\Gamma_0} h_0^2(\bar{\xi}) dS_\xi} + m(t), \quad (2.18)$$

where

$$\begin{aligned} m(t) &= \frac{\int_{\Gamma_0} \mathbf{N}_0(\bar{\xi}) \cdot \mathbf{u}(\xi, t) h_0(\bar{\xi}) dS_\xi - f_t(t, \lambda(t))}{f_\lambda(t, \lambda(t))} \\ &+ \int_{\Gamma_0} \mathbf{N}_0(\bar{\xi}) \cdot \mathbf{u}(\xi, t) h_0(\bar{\xi}) dS_\xi \left(\frac{1}{\int_{\Gamma_0} h_0^2(\bar{\xi}) dS_\xi} - \frac{1}{f_\lambda(t, \lambda(t))} \right). \end{aligned} \quad (2.19)$$

The first term in (2.18) is a linear part of $\lambda'(t)$ with respect to \mathbf{u} and $m(t)$ is a nonlinear remainder. The estimate of $m(t)$ and the proof of Proposition 2.2 is given below in Sec.4.

3. Scheme of the proof of Theorem 1.1.

As the first step, we reproduce (with necessary modifications) the transformation of the problem (1.14) made in [1] in the symmetric case. We introduce the projection $\Pi \mathbf{f} = \mathbf{f} - \mathbf{n}(\mathbf{n} \cdot \mathbf{f})$ and write the boundary condition $T_u(\mathbf{u}, q)\mathbf{n} = M\mathbf{n}$, where $M = \frac{\omega^2}{2}|x'|^2 + \kappa U(x, t) + p_0$, in an equivalent way as follows:

$$\Pi_0 \Pi S_u(\mathbf{u})\mathbf{n} = 0, \quad -q + \nu \mathbf{n} \cdot S_u(\mathbf{u})\mathbf{n} = M.$$

Next, we make use of (1.2) and write M in the form

$$M = \frac{\omega^2}{2}|X'|^2 + \kappa U(X, t) + p_0 = \frac{\omega^2}{2}(|x'|^2 - |z'|^2) + \kappa(U(x, t) - \mathcal{U}(z)), \quad (3.1)$$

where $x = X(\xi, t) \in \Gamma_t$ and $z = \bar{X}^\theta \in \mathcal{G}_{\theta(t)}$. Let $y = \mathcal{Z}(\lambda(t))z$. As in [1], we have

$$M = -B_0(z)\widehat{\rho}(z, t) + \frac{\omega^2}{2}|\mathbf{N}'_\theta(z)|^2\widehat{\rho}^2(z, t) + \kappa \int_0^1 (1-s) \frac{\partial^2 U_s}{\partial s^2} ds, \quad (3.1)$$

where

$$B_0(z)\widehat{\rho}(z, t) = b(z)\widehat{\rho}(z, t) - \kappa \int_{\mathcal{G}_{\theta(t)}} \frac{\widehat{\rho}(\zeta, t) dS}{|z - \zeta|} = b(y)\widetilde{\rho}(y, t) - \kappa \int_{\mathcal{G}_0} \frac{\widetilde{\rho}(\eta, t) dS}{|y - \eta|}, \quad (3.2)$$

$$U_s(z, t) = \int_{\mathcal{F}_\theta} \frac{L_s(\zeta, t) d\zeta}{|\mathbf{e}_{s\widehat{\rho}}(z) - \mathbf{e}_{s\widehat{\rho}}(\zeta)|}, \quad (3.3)$$

$$\mathbf{e}_{s\widehat{\rho}}(z) = z + \mathbf{N}_\theta^*(z)\widehat{\rho}^*(z, t), \quad (3.4)$$

\mathbf{N}^* and $\widehat{\rho}^*$ are extensions of \mathbf{N}_θ and $\widehat{\rho}$ from \mathcal{G}_θ in \mathcal{F}_θ , and $L_s(z, t)$ is the Jacobian of the transformation (3.4). When we pass in (3.2) to the variables $\xi \in \Gamma_0$, according to the formula $y = \overline{\mathcal{Z}(\lambda(t))X}(\xi, t)$, we obtain

$$\begin{aligned} B_0\widehat{\rho} &= b(\bar{X}^\theta)r(\xi, t) - \kappa \int_{\Gamma_0} \frac{r(\eta, t)\Psi(\eta, t) dS}{|\bar{X}^\theta(\xi, t) - \bar{X}^\theta(\eta, t)|} \\ &= b(\overline{\mathcal{Z}(\lambda(t))X})r - \kappa \int_{\Gamma_0} \frac{r(\eta, t)\Psi(\eta, t) dS}{|\overline{\mathcal{Z}X}(\xi, t) - \overline{\mathcal{Z}X}(\eta, t)|}, \end{aligned}$$

where r is the function (1.24), i.e.,

$$r(\xi, t) = R(\mathcal{Z}(\lambda(t))X(\xi, t)) = \widetilde{\rho}(\overline{\mathcal{Z}X}, t) = \widehat{\rho}(\bar{X}^\theta, t).$$

It follows that

$$B_0\widehat{\rho} = B_0^l(\xi)r + B_1(r, \mathbf{u}),$$

where

$$\begin{aligned}
B'_0(\xi)r &= b(\bar{\xi})r(\xi, t) - \kappa \int_{\Gamma_0} \frac{r(\eta, t)dS}{|\bar{\xi} - \bar{\eta}|}, \\
B_1(r, \mathbf{u}) &= (b(\overline{\mathcal{Z}(\lambda(t))X}) - b(\bar{\xi}))r(\xi, t) \\
&\quad - \kappa \int_{\Gamma_0} \frac{r(\eta, t)\Psi(\eta, t)dS}{|\overline{\mathcal{Z}X}(\xi, t) - \overline{\mathcal{Z}X}(\eta, t)|} + \kappa \int_{\Gamma_0} \frac{r(\eta, t)dS}{|\bar{\xi} - \bar{\eta}|}, \\
M &= -B'_0r + B_1(r, \mathbf{u}) + \frac{\omega^2}{2}|\mathbf{N}'_0(y)|^2\tilde{\rho}^2(y, t) + \kappa \int_0^1 (1-s)\frac{\partial^2 U_s}{\partial s^2}ds.
\end{aligned} \tag{3.5}$$

Next, we make one more modification of the problem (1.14) by inserting the function r into it. We note that $r(\xi, 0) = R(\xi) = \rho_0(\bar{\xi})$ and

$$\begin{aligned}
r_t(\xi, t) &= \mathbf{N}_0(\overline{\mathcal{Z}(\lambda(t))X}) \cdot \mathcal{Z}(\lambda(t)(\mathbf{u}(\xi, t) + \lambda'(t)(\mathbf{e}_3 \times X(\xi, t))) \\
&= \mathbf{N}_0(\overline{\mathcal{Z}X}) \cdot \mathcal{Z}\mathbf{u} + h_0(\overline{\mathcal{Z}X})\lambda'(t),
\end{aligned} \tag{3.6}$$

because

$$\begin{aligned}
\mathbf{N}_0(\overline{\mathcal{Z}X}) \cdot \mathcal{Z}(\mathbf{e}_3 \times X) &= \mathbf{N}_\theta(\bar{X}^\theta) \cdot (\mathbf{e}_3 \times \bar{X}^\theta + \mathbf{N}_\theta(\bar{X}^\theta)\hat{\rho}) \\
&= \mathbf{N}_\theta(\bar{X}^\theta) \cdot (\mathbf{e}_3 \times \bar{X}^\theta) = h_\theta(\bar{X}^\theta) = h_0(\overline{\mathcal{Z}(\lambda)X}).
\end{aligned}$$

Thus, (\mathbf{u}, q, r) can be regarded as a solution to the problem

$$\begin{aligned}
\mathbf{u}_t + 2\omega(\mathbf{e}_3 \times \mathbf{u}) - \nu\nabla^2\mathbf{u} + \nabla q &= l_1(\mathbf{u}, q), \\
\nabla \cdot \mathbf{u} &= l_2(\mathbf{u}), \quad \xi \in \Omega_0, \quad t > 0, \\
\Pi_0 S(\mathbf{u})\mathbf{n}_0 &= l_3(\mathbf{u}), \\
-q + \nu\mathbf{n}_0 \cdot S(\mathbf{u})\mathbf{n}_0 + B'_0(\xi)r &= l_4(\mathbf{u}) + l_5(\mathbf{u}, r), \\
r_t(\xi, t) &= \mathbf{N}_0(\bar{\xi}) \cdot \mathbf{u} - \frac{h_0(\bar{\xi})}{\|h_0(\bar{\xi})\|_{L^2(\Gamma_0)}^2} \int_{\Gamma_0} \mathbf{u}(\eta, t) \cdot \mathbf{N}_0(\bar{\eta})h_0(\bar{\eta})dS + l_6(\mathbf{u}), \quad \xi \in \Gamma_0, \\
\mathbf{u}(\xi, 0) &= \mathbf{w}_0(\xi), \quad \xi \in \Omega_0, \quad r(\xi, 0) = \rho_0(\bar{\xi}), \quad \xi \in \Gamma_0.
\end{aligned} \tag{3.7}$$

The expressions $l_1, l_2, l_3, l_4, l_5, l_6$ are nonlinear (at least quadratic) with respect to \mathbf{u}, q, r ; they are given by the formulas

$$\begin{aligned}
l_1(\mathbf{u}, q) &= \nu(\nabla_{\mathbf{u}}^2\mathbf{u} - \nabla^2\mathbf{u}) + \nabla q - \nabla_{\mathbf{u}}q, \\
l_2(\mathbf{u}) &= (\nabla - \nabla_{\mathbf{u}}) \cdot \mathbf{u}, \\
l_3(\mathbf{u}) &= \Pi_0(\Pi_0 S(\mathbf{u})\mathbf{n}_0 - \Pi S_{\mathbf{u}}(\mathbf{u})\mathbf{n}), \\
l_4(\mathbf{u}) &= \nu(\mathbf{n}_0 \cdot S(\mathbf{u})\mathbf{n}_0 - \mathbf{n} \cdot S_{\mathbf{u}}(\mathbf{u})\mathbf{n}),
\end{aligned} \tag{3.8}$$

$$l_5(\mathbf{u}, R) = \frac{\omega^2}{2}\mathbf{N}'^2(\bar{X})r^2(\xi, t) + \int_0^1 (1-s)\frac{d^2 U_s}{ds^2}ds + B_1(r, \mathbf{u}), \tag{3.9}$$

$$l_6(\mathbf{u}) = (\mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})) \cdot \mathbf{u}(\xi, t) + (h_0(\overline{\mathcal{Z}X}) - h_0(\bar{\xi}))\lambda'(t) + h_0(\bar{\xi})m(t). \quad (3.10)$$

Owing to the Piola identity $\nabla \cdot A^T = \left(\sum_{j=1}^3 \frac{\partial}{\partial x_j} A_{ij} \right)_{i=1,2,3} = 0$, where A^T means the transposed matrix A , we have

$$l_2(\mathbf{u}) = \nabla \cdot \mathbf{L}(\mathbf{u}), \quad \mathbf{L}(\mathbf{u}) = (I - A^T)\mathbf{u}. \quad (3.11)$$

Now we outline the proof of Theorem 1.1. As in [1], we use maximum regularity estimates for the solutions of the linear problem

$$\begin{aligned} \mathbf{v}_t + 2\omega(\mathbf{e}_3 \times \mathbf{v}) - \nu \nabla^2 \mathbf{v} + \nabla p &= \mathbf{f}(\xi, t), \\ \nabla \cdot \mathbf{v} &= f(\xi, t), \quad \xi \in \Omega_0, \quad t \in (0, T), \\ \Pi_0 S(\mathbf{v})\mathbf{n}_0 &= \Pi_0 \mathbf{d}(\xi, t), \\ -q + \nu \mathbf{n}_0 \cdot S(\mathbf{v})\mathbf{n}_0 + B'_0(\xi)r &= d(\xi, t), \\ r_t(\xi, t) &= \mathbf{N}_0(\bar{\xi}) \cdot \mathbf{v} - \frac{h_0(\bar{\xi})}{\|h_0(\bar{\xi})\|_{L_2(\Gamma_0)}^2} \int_{\Gamma_0} \mathbf{v}(\eta, t) \cdot \mathbf{N}_0(\bar{\eta}) h_0(\bar{\eta}) dS + g(\xi, t), \quad \xi \in \Gamma_0, \\ \mathbf{v}(\xi, 0) &= \mathbf{v}_0(\xi), \quad \xi \in \Omega_0, \quad r(\xi, 0) = r_0(\xi), \quad \xi \in \Gamma_0. \end{aligned} \quad (3.12)$$

and of a similar problem in \mathcal{F}_0 :

$$\begin{aligned} \mathbf{v}'_t + 2\omega(\mathbf{e}_3 \times \mathbf{v}') - \nu \nabla^2 \mathbf{v}' + \nabla p' &= \mathbf{f}'(x, t), \quad \nabla \cdot \mathbf{v}' = f'(x, t) \quad x \in \mathcal{F}_0, \\ \Pi_{\mathcal{G}} S(\mathbf{v}')\mathbf{N}_0(x) &= \Pi_{\mathcal{G}} \mathbf{d}'(x, t), \\ -p' + \nu \mathbf{N}_0 \cdot S(\mathbf{v}')\mathbf{N}_0 + B_0(x)r'(x, t) &= d'(x, t), \\ r'_t &= \mathbf{N}_0(x) \cdot \mathbf{v}'(x, t) - \frac{h_0(x)}{\|h_0\|_{L_2(\mathcal{G}_0)}^2} \int_{\mathcal{G}_0} \mathbf{v}'(y, t) \cdot \mathbf{N}_0(y) h_0(y) dS + g'(x, t), \quad x \in \mathcal{G}_0, \\ \mathbf{v}'(x, 0) &= \mathbf{v}'_0(x), \quad x \in \mathcal{F}_0, \quad r'(x, 0) = r'_0(x), \quad x \in \mathcal{G}_0, \end{aligned} \quad (3.13)$$

where $\Pi_{\mathcal{G}} \mathbf{f} = \mathbf{f} - \mathbf{N}_0(\mathbf{N}_0 \cdot \mathbf{f})$. In comparison with the case of axially symmetric \mathcal{F} , these problems contain an extra integral term in the boundary conditions.

We consider at first the problem (3.13).

Theorem 3.1. *Let $l \in (1, 3/2)$, $\Omega_T = \mathcal{F}_0 \times (0, T)$, $\mathfrak{G}_T = \mathcal{G}_0 \times (0, T)$ and let the data of the problem (3.13) possess the following regularity properties: $\mathbf{f}' \in W_2^{l, l/2}(\Omega_T)$, $f' \in W_2^{1+l, 0}(\Omega_T)$, $\mathbf{f}' = \nabla \cdot \mathbf{F}'$, $\mathbf{F}' \in W_2^{0, 1+l/2}(\Omega_T)$, $\mathbf{v}'_0 \in W_2^{1+l}(\mathcal{F})$, $r'_0 \in W_2^{l+1}(\mathcal{G}_0)$, $\mathbf{d}' \in W_2^{l+1/2, l/2+1/4}(\mathfrak{G}_T)$, $d' \in W_2^{l+1/2, l/2+1/4}(\mathfrak{G}_T)$, $g' \in W_2^{l+3/2, l/2+3/4}(\mathfrak{G}_T)$. Assume also that the compatibility conditions*

$$\nabla \cdot \mathbf{v}'_0 = f'(x, 0), \quad x \in \mathcal{F}_0, \quad \Pi_{\mathcal{G}} S(\mathbf{v}'_0)\mathbf{N}_0 = \Pi_{\mathcal{G}} \mathbf{d}'(x, 0), \quad x \in \mathcal{G}_0$$

are satisfied. Then the problem (3.13) has a unique solution $\mathbf{v}' \in W_2^{2+l, 1+l/2}(\Omega_T)$, $\nabla p' \in W_2^{l, l/2}(\Omega_T)$, $r' \in W_2^{l+1/2, 0}(\mathfrak{G}_T)$, such that $p'|_{\mathfrak{G}_T} \in W_2^{l+1/2, l/2+1/4}(\mathfrak{G}_T)$, $r'(\cdot, t) \in W_2^{l+1}(\mathcal{G}_0)$ for arbitrary $t \in (0, T)$, and

$$Y(T) \equiv \|\mathbf{v}'\|_{W_2^{2+l, 1+l/2}(\Omega_T)} + \|\nabla p'\|_{W_2^{l, l/2}(\Omega_T)} + \|p'\|_{W_2^{l+1/2, l/2+1/4}(\mathfrak{G}_T)} + \|r'\|_{W_2^{l+1/2, 0}(\mathfrak{G}_T)}$$

$$+ \sup_{t < T} \|r'(\cdot, t)\|_{W_2^{l+1}(\mathcal{G}_0)} \leq c \left(N(T) + \left(\int_0^T (\|\mathbf{v}'\|_{L_2(\mathcal{F}_0)}^2 + \|r'\|_{W_2^{-1/2}(\mathcal{G}_0)}^2) dt \right)^{1/2} \right), \quad (3.14)$$

where

$$N(T) = \|\mathbf{f}'\|_{W_2^{l,1/2}(\mathcal{Q}_T)} + \|\mathbf{f}'\|_{W_2^{l+1,0}(\mathcal{Q}_T)} + \|\mathbf{F}'\|_{W_2^{0,1+l/2}(\mathcal{Q}_T)} + \|r'_0\|_{W_2^{l+1}(\mathcal{G}_0)} \\ + \|\mathbf{v}'_0\|_{W_2^{1+l}(\mathcal{F}_0)} + \|\mathbf{d}'\|_{W_2^{l+1/2, l/2+1/4}(\mathcal{G}_T)} + \|d'\|_{W_2^{l+1/2, l/2+1/4}(\mathcal{G}_T)} + \|g'\|_{W_2^{l+3/2, l/2+3/4}(\mathcal{G}_T)}.$$

Moreover, if $\mathbf{f}' \in \widetilde{W}_2^{l, l/2}(\mathcal{Q}_T)$, $\mathbf{d}' \in \widetilde{W}_2^{l+1/2, l/2+1/4}(\mathcal{G}_T)$, $d' \in \widetilde{W}_2^{l+1/2, l/2+1/4}(\mathcal{G}_T)$, $g' \in \widetilde{W}_2^{l+3/2, l/2+3/4}(\mathcal{G}_T)$, $\mathbf{f}' \in \widetilde{W}_2^{1+l, 0}(\mathcal{Q}_T)$, $\mathbf{F}' \in \widetilde{W}_2^{0, 1+l/2}(\mathcal{Q}_T)$ (this means that $\mathbf{f}' \in W_2^{1+l, 0}(\mathcal{Q}_T)$, $t\mathbf{f}' \in W_2^{l, 0}(\mathcal{Q}_T)$), $\mathbf{F}' \in W_2^{0, 1+l/2}(\mathcal{Q}_T)$, $t\mathbf{F}' \in W_2^{0, (l+1)/2}(\mathcal{Q}_T)$), then

$$\widetilde{Y}(T) \equiv \|\mathbf{v}'\|_{\widetilde{W}_2^{2+l, 1+l/2}(\mathcal{Q}_T)} + \|\nabla p'\|_{\widetilde{W}_2^{l, l/2}(\mathcal{Q}_T)} + \|p'\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(\mathcal{G}_T)} + \|r'\|_{\widetilde{W}_2^{l+1/2, 0}(\mathcal{G}_T)} \\ + \sup_{t < T} \|r'(\cdot, t)\|_{W_2^{l+1}(\mathcal{G}_0)} + \sup_{t < T} t \|r'(\cdot, t)\|_{W_2^l(\mathcal{G}_0)} \\ \leq c \left(\widetilde{\mathcal{N}}(T) + \left(\int_0^T (1+t^2) (\|\mathbf{v}'\|_{L_2(\mathcal{F}_0)}^2 + \|r'\|_{W_2^{-1/2}(\mathcal{G}_0)}^2) dt \right)^{1/2} \right), \quad (3.15)$$

where

$$\widetilde{\mathcal{N}}(T) = \|\mathbf{f}'\|_{\widetilde{W}_2^{l, l/2}(\mathcal{Q}_T)} + \|\mathbf{f}'\|_{\widetilde{W}_2^{l+1, 0}(\mathcal{Q}_T)} + \|\mathbf{F}'\|_{\widetilde{W}_2^{0, 1+l/2}(\mathcal{Q}_T)} + \|r'_0\|_{W_2^{l+1}(\mathcal{G}_0)} \\ + \|\mathbf{v}'_0\|_{W_2^{1+l}(\mathcal{F}_0)} + \|\mathbf{d}'\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(\mathcal{G}_T)} + \|d'\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(\mathcal{G}_T)} + \|g'\|_{\widetilde{W}_2^{l+3/2, l/2+3/4}(\mathcal{G}_T)}.$$

The constants in (3.14), (3.15) are independent of T .

In fact, the theorem is valid for $l \in (0, 5/2)$, and the inequality (3.15) is obtained by combination of (3.14) with l and $l-1$. The proof is given in [10, 9]. The problem (3.12) reduces to (3.13) by the transformation

$$\xi = x + \mathbf{N}_0^*(x) \rho_0^*(x) \equiv e_{\rho_0}(x), \quad x \in \mathcal{F}_0, \quad (3.16)$$

where \mathbf{N}_0^* and ρ_0^* are extensions of \mathbf{N}_0 and ρ_0 from \mathcal{G}_0 into \mathcal{F}_0 such that \mathbf{N}_0^* is sufficiently regular and

$$\|\rho_0^*\|_{W_2^{l+2}(\mathcal{F}_0)} \leq c \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G}_0)}. \quad (3.17)$$

This transformation converts (3.12) to

$$\mathbf{v}'_t + 2\omega(\mathbf{e}_3 \times \mathbf{v}') - \nu \nabla^2 \mathbf{v}' + \nabla p' = \mathbf{f}'(x, t) + \mathbf{m}_1(\mathbf{v}', p'), \\ \nabla \cdot \mathbf{v}' = L_0 f'(x, t) + m_2(\mathbf{v}'), \quad x \in \mathcal{F}, \\ \Pi_{\mathcal{G}} S(\mathbf{v}') \mathbf{N}_0 = \Pi_{\mathcal{G}} \Pi_0 \mathbf{d}' + \mathbf{m}_3(\mathbf{v}'), \\ -p' + \nu \mathbf{N}_0 \cdot S(\mathbf{v}') \mathbf{N}_0 + B_0(x) r'(x, t) = d'(x, t) + m_4(\mathbf{v}'), \quad (3.18)$$

$$r'_t = \mathbf{N}_0(x) \cdot \mathbf{v}'(x, t) - \frac{h_0(x)}{\|h_0\|_{L_2(\mathcal{G}_0)}^2} \int_{\mathcal{G}_0} \mathbf{v}'(y, t) \cdot \mathbf{N}_0(y) h_0(y) dS + g'(x, t) + m_5(\mathbf{v}', r'), \quad x \in \mathcal{G},$$

$$\mathbf{v}'(x, 0) = \mathbf{v}'_0(x), \quad x \in \mathcal{F}, \quad r'(x, 0) = r'_0(x), \quad x \in \mathcal{G},$$

where "'' denotes the change of variables (3.16): $f'(x, t) = f(e_{\rho_0}^{-1}(\xi), t)$. The expressions m_i are given by

$$\mathbf{m}_1(\mathbf{v}', p') = \nu(\widehat{\nabla}^2 - \nabla^2)\mathbf{v}'(y, t) + (\nabla - \widehat{\nabla})p'(y, t),$$

$$m_2(\mathbf{v}') = (\nabla - L_0 \widehat{\nabla}) \cdot \mathbf{v}',$$

$$\mathbf{m}_3(\mathbf{v}') = \Pi_{\mathcal{G}}(\Pi_{\mathcal{G}} S(\mathbf{v}') \mathbf{N}_0 - \Pi_0 \widehat{S}(\mathbf{v}') \mathbf{n}_0), \quad (3.19)$$

$$m_4(\mathbf{v}', r') = \nu(\mathbf{N}_0 \cdot S(\mathbf{v}') \mathbf{N}_0 - \mathbf{n}_0 \cdot \widehat{S}(\mathbf{v}') \mathbf{n}_0) + \kappa \int_{\mathcal{G}_0} \frac{r'(z, t)}{|y - z|} (|\widehat{\mathcal{L}}_0^T(z, \rho_0) \mathbf{N}_0(z)| - 1) dS,$$

$$m_5(\mathbf{v}', r')$$

$$= h_0(x) \left(\left(\int_{\mathcal{G}_0} h_0^2(y) dS \right)^{-1} - \left(\int_{\mathcal{G}_0} h_0^2(y) |\widehat{\mathcal{L}}_0^T(y, \rho_0) \mathbf{N}_0(y)| dS \right)^{-1} \right) \int_{\mathcal{G}_0} \mathbf{v}' \cdot \mathbf{N}_0(y) h_0(y) dS$$

$$+ h_0(x) \left(\int_{\mathcal{G}_0} h_0^2(y) |\widehat{\mathcal{L}}_0^T(y, \rho_0) \mathbf{N}_0(y)| dS \right)^{-1} \int_{\mathcal{G}_0} \mathbf{v}' \cdot \mathbf{N}_0(y) h_0(y) (1 - |\widehat{\mathcal{L}}_0^T(y, \rho_0) \mathbf{N}_0(y)|) dS.$$

By $L_0 = \det \mathcal{L}_0$ we mean the Jacobian of the transformation e_{ρ_0} , \mathcal{L}_0 is its Jacobian matrix, $\widehat{\mathcal{L}}_0 = L_0 \mathcal{L}_0^{-1}$, $\widehat{\nabla} = \mathcal{L}_0^{-T} \nabla$ is a transformed gradient with respect to ξ , $\nabla = \left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3} \right)$, $\widehat{S}(\mathbf{v}) = \widehat{\nabla} \mathbf{v} + (\widehat{\nabla} \mathbf{v})^T$ is a transformed rate-of-strain tensor. The normals \mathbf{N}_0 and \mathbf{n}_0 are connected with each other by

$$\mathbf{n}_0(e_{\rho_0}(y)) = \frac{\widehat{\mathcal{L}}_0^T \mathbf{N}_0(y)}{|\widehat{\mathcal{L}}_0^T \mathbf{N}_0(y)|}.$$

We notice that $m_2(\mathbf{v}')$ is representable in the divergence form:

$$(\nabla - L_0 \widehat{\nabla}) \cdot \mathbf{v}' = (\nabla - \widehat{\mathcal{L}}^T \nabla) \cdot \mathbf{v}' = \nabla \cdot (I - \widehat{\mathcal{L}}^T) \mathbf{v}' \equiv \nabla \cdot \mathbf{M},$$

where $\mathbf{M} = (I - \widehat{\mathcal{L}}) \mathbf{v}'$.

The expressions (3.19) are linear functions of their arguments with small coefficients proportional to the derivatives of ρ_0 . Under the assumption (1.19) they satisfy the inequality

$$\begin{aligned} & \|\mathbf{m}_1\|_{\widetilde{W}_2^{t, t/2}(Q_T)} + \|m_2\|_{\widetilde{W}_2^{t+1, 0}(Q_T)} + \|\mathbf{M}\|_{\widetilde{W}_2^{0, 1+t/2}(Q_T)} \\ & + \|\mathbf{m}_3(\mathbf{v})\|_{\widetilde{W}_2^{t+1/2, t/2+1/4}(G_T)} + \|m_4(\mathbf{v}, r)\|_{\widetilde{W}_2^{t+1/2, t/2+1/4}(G_T)} \\ & + \|m_5(\mathbf{v}, r)\|_{\widetilde{W}_2^{t+3/2, t/2+3/4}(G_T)} \leq c\epsilon \widetilde{Y}_T(\mathbf{v}, p, r) \end{aligned} \quad (3.20)$$

that can be obtained with the help of Proposition 4.1 in [1]. The estimate of m_5 follows from

$$\left| 1 - \left| \widehat{\mathcal{L}}_0^T(z, \rho_0) \mathbf{N}_0(z) \right| \right| \leq c\epsilon. \quad (3.18)$$

Using (3.14) and (3.20), it is possible to prove the solvability of the problem (3.12) and estimate the solution in a standard way, provided ϵ is sufficiently small (the details are omitted). We obtain the following result.

Theorem 3.2. *Let $l \in (1, 3/2)$, $Q_T = \Omega_0 \times (0, T)$, $G_T = \Gamma_0 \times (0, T)$ and let the data of the problem (3.12) possess the following regularity properties: $\mathbf{f} \in W_2^{l, l/2}(Q_T)$, $f \in W_2^{1+l, 0}(Q_T)$, $\mathbf{f} = \nabla \cdot \mathbf{F}$, $\mathbf{F} \in W_2^{0, 1+1/2}(Q_T)$, $\mathbf{v}_0 \in W_2^{1+l}(\Omega_0)$, $r_0 \in W_2^{l+1}(\Gamma_0)$, $\mathbf{d} \in W_2^{l+1/2, l/2+1/4}(G_T)$, $d \in W_2^{l+1/2, l/2+1/4}(G_T)$, $g \in W_2^{l+3/2, l/2+3/4}(G_T)$. Assume also that the compatibility conditions*

$$\nabla \cdot \mathbf{v}_0 = f(\xi, 0), \quad \xi \in \Omega_0, \quad \Pi_0 S(\mathbf{v}_0) \mathbf{n}_0 = \Pi_0 \mathbf{d}(\xi, 0), \quad \xi \in \Gamma_0$$

are satisfied. Then the problem (3.12) has a unique solution $\mathbf{v} \in W_2^{2+l, 1+1/2}(Q_T)$, $\nabla p \in W_2^{l, l/2}(Q_T)$, $r \in W_2^{l+1/2, 0}(G_T)$, such that $p|_{G_T} \in W_2^{l+1/2, l/2+1/4}(G_T)$, $r(\cdot, t) \in W_2^{l+1}(\Gamma_0)$ for arbitrary $t \in (0, T)$, and

$$\begin{aligned} Y(T) &\equiv \|\mathbf{v}\|_{W_2^{2+l, 1+1/2}(Q_T)} + \|\nabla p\|_{W_2^{l, l/2}(Q_T)} + \|p\|_{W_2^{l+1/2, l/2+1/4}(G_T)} + \|r\|_{W_2^{l+1/2, 0}(G_T)} \\ &\quad + \sup_{t < T} \|r(\cdot, t)\|_{W_2^{l+1}(\Gamma_0)} \leq c \left(N(T) + \left(\int_0^T (\|\mathbf{v}\|_{L_2(\Omega_0)}^2 + \|r\|_{W_2^{-1/2}(\Gamma_0)}^2) dt \right)^{1/2} \right), \end{aligned} \quad (3.21)$$

where

$$\begin{aligned} N(T) &= \|\mathbf{f}\|_{W_2^{l, l/2}(Q_T)} + \|f\|_{W_2^{1+l, 0}(Q_T)} + \|\mathbf{F}\|_{W_2^{0, 1+1/2}(Q_T)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)} \\ &\quad + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega_0)} + \|\mathbf{d}\|_{W_2^{l+1/2, l/2+1/4}(G_T)} + \|d\|_{W_2^{l+1/2, l/2+1/4}(G_T)} + \|g\|_{W_2^{l+3/2, l/2+3/4}(G_T)}. \end{aligned}$$

Moreover, if $\mathbf{f} \in \widetilde{W}_2^{l, l/2}(Q_T)$, $\mathbf{d} \in \widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)$, $d \in \widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)$, $g \in \widetilde{W}_2^{l+3/2, l/2+3/4}(G_T)$, $f \in \widetilde{W}_2^{1+l, 0}(Q_T)$, $\mathbf{F} \in \widetilde{W}_2^{0, 1+1/2}(Q_T)$ (this means that $\mathbf{f} \in W_2^{1+l, 0}(Q_T)$, $tf \in W_2^{1, 0}(Q_T)$), $\mathbf{F} \in W_2^{0, 1+1/2}(Q_T)$, $t\mathbf{F} \in W_2^{0, (l+1)/2}(Q_T)$), then

$$\begin{aligned} \widetilde{Y}(T) &\equiv \|\mathbf{v}\|_{\widetilde{W}_2^{2+l, 1+1/2}(Q_T)} + \|\nabla p\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|p\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|r\|_{\widetilde{W}_2^{l+1/2, 0}(G_T)} \\ &\quad + \sup_{t < T} \|r(\cdot, t)\|_{W_2^{l+1}(\Gamma_0)} + \sup_{t < T} t \|r(\cdot, t)\|_{W_2^l(\Gamma_0)} \\ &\leq c \left(\widetilde{N}(T) + \left(\int_0^T (1+t^2) (\|\mathbf{v}\|_{L_2(\Omega_0)}^2 + \|r\|_{W_2^{-1/2}(\Gamma_0)}^2) dt \right)^{1/2} \right), \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} \widetilde{N}(T) &= \|\mathbf{f}\|_{\widetilde{W}_2^{l, l/2}(Q_T)} + \|f\|_{\widetilde{W}_2^{1+l, 0}(Q_T)} + \|\mathbf{F}\|_{\widetilde{W}_2^{0, 1+1/2}(Q_T)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)} \\ &\quad + \|\mathbf{v}_0\|_{W_2^{1+l}(\Omega_0)} + \|\mathbf{d}\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|d\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|g\|_{\widetilde{W}_2^{l+3/2, l/2+3/4}(G_T)}. \end{aligned}$$

The constants in (3.21), (3.22) are independent of T .

The norm $\|r\|_{W_2^{-1/2}(\Gamma_0)}$ is defined in a standard way:

$$\|r\|_{W_2^{-1/2}(\Gamma_0)} = \sup_{\varphi \in W_2^{1/2}(\Gamma_0)} \frac{\left| \int_{\Gamma_0} r(x) \varphi(x) dx \right|}{\|\varphi\|_{W_2^{1/2}(\Gamma_0)}}.$$

In order to be able to apply the inequality (3.22) to the problem (3.7), we need to estimate the nonlinear terms in (3.7) and the lower order norms in (3.21), (3.22).

Theorem 3.3. *If (\mathbf{u}, q, r) satisfy the inequality*

$$\tilde{Y}(T) \leq \delta \ll 1, \quad (3.23)$$

where $\tilde{Y}(T)$ is defined in (3.22), then

$$\begin{aligned} & \|l_1\|_{\tilde{W}_2^{l, l/2}(Q_T)} + \|l_2\|_{\tilde{W}_2^{1+l, 0}(Q_T)} + \|\mathbf{L}\|_{\tilde{W}_2^{0, (l+1)/2}(Q_T)} \\ & + \|l_3\|_{\tilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|l_4\|_{\tilde{W}_2^{l+1/2, l/2+1/4}(G_T)} \\ & + \|l_5\|_{\tilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|l_6\|_{\tilde{W}_2^{l+3/2, l/2+3/4}(G_T)} \\ & \leq c \left(\|\mathbf{u}\|_{\tilde{W}_2^{2+l, 1+l/2}(Q_T)}^2 + \|\nabla q\|_{\tilde{W}_2^{l, l/2}(Q_T)}^2 + \|r\|_{\tilde{W}_2^{l+1/2, l/2+1/4}(G_T)}^2 \right) \end{aligned} \quad (3.24)$$

with the constant c independent of $T \geq 1$.

Theorem 3.4. *If the solution of the problem (3.7) is defined for $t \in (0, T)$ and (3.23) holds, then \mathbf{w} and $\hat{\rho}$ satisfy the inequality*

$$\begin{aligned} & \|\mathbf{w}(\cdot, t)\|_{L_2(\Omega_t)}^2 + \|\hat{\rho}(\cdot, t)\|_{L_2(\mathcal{G}_{\theta(t)})}^2 + \int_0^t \left(\|\mathbf{w}(\cdot, \tau)\|_{L_2(\Omega_\tau)}^2 + \|\hat{\rho}(\cdot, \tau)\|_{W_2^{-1/2}(\mathcal{G}_{\theta(\tau)})}^2 \right) d\tau \\ & \leq c \left(\|\mathbf{w}_0\|_{L_2(\Omega_0)}^2 + \|\rho_0\|_{L_2(\mathcal{G}_0)}^2 \right) \end{aligned} \quad (3.25)$$

with the constant independent of T .

The proof of Theorem 3.4 is given in Sec.5. By Proposition 4.6 in [1], (3.25) implies

$$\begin{aligned} & \|\mathbf{u}(\cdot, t)\|_{L_2(\Omega_0)}^2 + \|r(\cdot, t)\|_{L_2(\Gamma_0)}^2 + \int_0^t \left(\|\mathbf{u}(\cdot, \tau)\|_{L_2(\Omega_0)}^2 + \|r(\cdot, \tau)\|_{W_2^{-1/2}(\Gamma_0)}^2 \right) d\tau \\ & \leq c \left(\|\mathbf{w}_0\|_{L_2(\Omega_0)}^2 + \|\rho_0\|_{L_2(\mathcal{G}_0)}^2 \right). \end{aligned} \quad (3.26)$$

As in the case of axially symmetric \mathcal{F} , inequalities (3.22), (3.24), (3.26) allow us to obtain the following uniform estimate of the solution of (3.7) playing a crucial role in the analysis of the problem (1.9) (cf. [1], Theorem 2.3).

Theorem 3.5. *Assume that the assumptions of Theorem 1.1 are satisfied. If the solution of (3.7) is defined for $t \in (0, T)$ and (3.23) holds, then*

$$\tilde{Y}(T) \leq c \left(\|\mathbf{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G}_0)} \right). \quad (3.27)$$

Inequality (3.23) is verified in the process of the proof of the solvability of the problem (1.14). As in [1], the proof is carried out in two steps. First, using the maximum regularity estimates for the Neumann problem

$$\begin{aligned} \mathbf{v}_t - \nu \nabla^2 \mathbf{v} + \nabla p &= \mathbf{f}(x, t), \quad \nabla \cdot \mathbf{v} = f(x, t) \quad x \in \Omega_0, \\ T(\mathbf{v}, p) \mathbf{n}_0 &= \mathbf{d}(x, t), \quad x \in \Gamma_0, \\ \mathbf{v}(x, 0) &= \mathbf{v}_0(x), \quad x \in \Omega_0, \end{aligned}$$

and the estimate (3.24) of the nonlinear terms, we prove the solvability of the problem (1.14) in the interval $t \in (0, 1)$, and the estimate

$$\begin{aligned} \|\mathbf{u}\|_{W_2^{2+l, 1+l/2}(Q_1)} + \|\nabla q\|_{W_2^{l, l/2}(Q_1)} + \|q\|_{W_2^{l+1, (l+1)/2}(G_1)} \\ \leq c \left(\|\mathbf{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G}_0)} \right) \end{aligned}$$

for the solution (cf. [1], Theorem 3.1). Then we construct $\theta(t) = -\lambda(t)$, as made in Propositions 2.1, 2.2, and estimate the function

$$r(\xi, t) = \rho_0(\bar{\xi}) + \int_0^t \left(N_0(\overline{\mathcal{Z}(\lambda(\tau)X(\xi, \tau))}) \cdot \mathcal{Z}\mathbf{u} + h_0(\overline{\mathcal{Z}X})\lambda'(\tau) \right) d\tau.$$

If ϵ in (1.19) is small, then we arrive at (3.23) and, by Theorem 3.5, at (3.27) for $t \in (0, 1)$. Now we can make one more step and define the solution for $t \in (T, 2T)$. Assume that the solution of (3.7), as well as the function $\theta(t)$, is defined for $t \in (0, T)$ and inequalities (3.23) and (2.17) are satisfied. Then it is possible to extend the solution in the time interval $t \in (0, T + 1)$. As in [1] (see Theorem 3.2), this reduces to the problem (3.5) in [1], slightly more complicated than (1.14). It is essential that in the proof of Theorems 3.1 and 3.2 in [1] the symmetry properties of \mathcal{F} are not used. If \mathbf{u} and q are constructed for $t \in (0, T + 1)$, then it is possible to define $\theta(t)$, $t \in (0, T + 1)$, satisfying (2.17), and estimate

$$r(\xi, t) = r(\xi, T) + \int_T^t \left(N_0(\overline{\mathcal{Z}X}) \cdot \mathcal{Z}\mathbf{u} + h_0(\overline{\mathcal{Z}X})\lambda'(\tau) \right) d\tau,$$

$t \in (T, T + 1)$. By Theorem 3.5, the extended functions satisfy (3.27), (2.17) with constants independent of T , as in the symmetric case. In this way we construct the solution in the infinite time interval and conclude the proof of Theorem 1.1.

4. Proof of Proposition 2.2 and of the estimate (3.24)

This section is devoted to some estimates presented in Sec. 2 and 3.

Proof of Proposition 2.2. We consider the function $f(t, \lambda)$ defined in (2.8). When we extend h_0 from \mathcal{G}_0 in the δ_1 -neighborhood of \mathcal{G}_0 so that this function remains smooth (which reduces to the extension of N_0 , as it has been done

above) and take account of the relation $h_0(\bar{y}) = h_0(\mathfrak{A}(y))$, then we can write $f(t, \lambda)$ as

$$f(t, \lambda) = \int_{\Gamma_0} F(\mathcal{Z}(\lambda)X(\xi, t))dS_\xi, \quad (4.1)$$

where F is a smooth function in a certain neighborhood of Γ_0 . The partial derivatives of f with respect to λ are given by

$$\begin{aligned} f_\lambda(t, \lambda) &= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X(\xi, t)) \cdot \mathcal{Z}'X dS_\xi = \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X) \cdot \mathcal{Z}'\mathcal{Z}^{-1}\mathcal{Z}X dS_\xi \\ &= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X(\xi, t)) \cdot (\mathbf{e}_3 \times \mathcal{Z}X) dS_\xi \equiv \int_{\Gamma_0} F_1(\mathcal{Z}X) dS, \\ f_{\lambda\lambda}(t, \lambda) &= \int_{\Gamma_0} \nabla F_1(\mathcal{Z}(\lambda)X) \cdot (\mathbf{e}_3 \times \mathcal{Z}X) dS \equiv \int_{\Gamma_0} F_2(\mathcal{Z}X) dS_\xi, \end{aligned} \quad (4.2)$$

where F_1 and F_2 are also smooth functions. Moreover,

$$\begin{aligned} f_t(t, \lambda) &= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X) \cdot \mathcal{Z}\mathbf{u}(\xi, t) dS_\xi, \\ f_{t\lambda}(t, \lambda) &= \int_{\Gamma_0} \nabla F_1(\mathcal{Z}(\lambda)X) \cdot \mathcal{Z}\mathbf{u}(\xi, t) dS_\xi, \\ f_{t\lambda\lambda}(t, \lambda) &= \int_{\Gamma_0} \nabla F_2(\mathcal{Z}(\lambda)X) \cdot \mathcal{Z}\mathbf{u}(\xi, t) dS_\xi, \\ f_{tt}(t, \lambda) &= \int_{\Gamma_0} \nabla F(\mathcal{Z}(\lambda)X) \cdot \mathcal{Z}\mathbf{u}_t dS_\xi + \int_{\Gamma_0} \mathcal{Z}\mathbf{u} \cdot \nabla \nabla F(\mathcal{Z}X) \cdot \mathcal{Z}\mathbf{u} dS_\xi \\ &\equiv \phi_1(t) + \phi_2(t). \end{aligned} \quad (4.4)$$

Differentiating (2.13) with respect to t , we obtain

$$\lambda''(t) = -\left(\frac{\partial f_t}{\partial t f_\lambda}\right)_{\lambda=\lambda(t)} - \left(\frac{\partial f_t}{\partial \lambda f_\lambda}\right)_{\lambda=\lambda(t)} \lambda'(t) = \lambda_1(t) + \lambda_2(t). \quad (4.5)$$

Since $X(\xi, t)$ and $\mathbf{u}(\xi, t)$ are bounded uniformly with respect to t and f_λ satisfies (2.10), we have

$$\begin{aligned} \left|\left(\frac{\partial f_t}{\partial \lambda f_\lambda}\right)_{\lambda=\lambda(t)}\right| &\leq c, \\ \left|\left(\frac{\partial f_t}{\partial t f_\lambda}\right)_{\lambda=\lambda(t)}\right| &\leq c \int_{\Gamma_0} (|\mathbf{u}_t(\xi, t)| + |\mathbf{u}(\xi, t)|) dS, \end{aligned}$$

and in view of (2.15)

$$\begin{aligned} |\lambda''(t)| &\leq c \int_{\Gamma_0} (|\mathbf{u}_t(\xi, t)| + |\mathbf{u}(\xi, t)|) dS, \\ \|\lambda''\|_{L_2(0, T)} &\leq c \|\mathbf{u}\|_{W_2^{0,1}(G_T)}. \end{aligned} \quad (4.6)$$

Now we estimate

$$\left(\int_0^{\min(T,1)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)\lambda''(t)|^2 dt \right)^{1/2},$$

where $\mu = l/2 - 1/4$, $\Delta_t(-h)\lambda''(t) = \lambda''(t-h) - \lambda''(t)$. Using (4.2) it is not difficult to verify that

$$\left(\int_0^{\min(T,1)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)\lambda_2(t)|^2 dt \right)^{1/2} \leq \|\lambda_2\|_{W_2^1(0,T)} \leq c\|\mathbf{u}\|_{W_2^{0,1}(G_T)}. \quad (4.7)$$

The function $\lambda_1(t)$ is given by

$$\lambda_1(t) = -\left(\frac{f_{tt}}{f_\lambda} - \frac{f_t f_{\lambda t}}{f_\lambda^2} \right)_{\lambda=\lambda(t)} \equiv \lambda_3(t) + \lambda_4(t).$$

with λ_4 also satisfying (4.7). Now we consider the difference

$$\begin{aligned} & \Delta_t(-h)\lambda_3(t) \\ &= -\frac{1}{f_\lambda(t-h, \lambda(t-h))} \Delta_t(-h)f_{tt}(t, \lambda(t)) - f_{tt}(t, \lambda(t)) \Delta_t(-h) \frac{1}{f_\lambda(t, \lambda(t))}. \end{aligned}$$

Since $|f_\lambda(t, \lambda(t))| \geq c > 0$ and $|\frac{\partial}{\partial t} \frac{1}{f_\lambda(t, \lambda(t))}| \leq c$, we have

$$\begin{aligned} & \left(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu}} \int_h^T |f_{tt}|^2 \left| \Delta_t(-h) \frac{1}{f_\lambda} \right|^2 dt \right)^{1/2} \leq c\|f_{tt}\|_{L_2(0,T)}, \\ & \left(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)f_{tt}|^2 \frac{1}{|f_\lambda|^2} dt \right)^{1/2} \\ & \leq c \left(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)f_{tt}|^2 dt \right)^{1/2}, \end{aligned}$$

which implies

$$\begin{aligned} \|\lambda_3\|_{W_2^\mu(0,T)} &\leq c\|f_{tt}\|_{W_2^\mu(0,T)}, \\ \|\lambda''\|_{W_2^\mu(0,T)} &\leq c \left(\|f_{tt}\|_{W_2^\mu(0,T)} + \|\mathbf{u}\|_{W_2^{0,1}(Q_T)} \right). \end{aligned}$$

The function $f_{tt}(t)$ is representable in the form (4.4) with ϕ_2 satisfying

$$\|\phi_2\|_{W_2^\mu(0,T)} \leq c\|\phi_2\|_{W_2^1(0,T)} \leq c\|\mathbf{u}\|_{W_2^{0,1}(G_T)} \quad (4.8)$$

and

$$\phi_1(t) = \int_{\Gamma_0} \mathbf{b}(\xi, t) \cdot \mathbf{u}_t(\xi, t) dS,$$

where $\mathbf{b} = \mathcal{Z}^{-1}(\lambda(t)) \nabla F(\mathcal{Z}(\lambda(t))X(\xi, t))$ is the function such that

$$\sup_{G_T} |\mathbf{b}(\xi, t)| + \sup_{G_T} |\mathbf{b}_t(\xi, t)| \leq c.$$

Hence

$$\begin{aligned}\|\phi_{1t}\|_{W_2^\mu(0,T)} &\leq c\|\mathbf{u}_t\|_{W_2^{0,\mu}(G_T)}, \\ \|\mathbf{f}_{tt}\|_{W_2^\mu(0,T)} &\leq c\|\mathbf{u}\|_{W_2^{0,1+\mu}(G_T)},\end{aligned}$$

Together with (4.8), this inequality implies

$$\|\lambda'\|_{W_2^{l/2+3/4}(0,T)} \leq c\|\mathbf{u}\|_{W_2^{0,l/2+3/4}(G_T)}. \quad (4.9)$$

In order to conclude the proof of (2.17), we need to estimate the norm

$$\|(1+t)\lambda'\|_{W_2^{\mu_1}(0,T)}$$

with $\mu_1 = l/2 + 1/4$. This can be done by repeating the above arguments. In view of (2.15), we have

$$\begin{aligned}\|(1+t)\lambda'\|_{L_2(0,T)} &\leq c\|(1+t)\mathbf{u}\|_{L_2(G_T)}, \\ &\left(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu_1}} \int_h^T (1+t)^2 |\Delta_t(-h)\lambda'(t)|^2 dt\right)^{1/2} \\ &\leq c\left(\int_0^{\min(1,T)} \frac{dh}{h^{1+2\mu_1}} \int_h^T (1+t)^2 |\Delta_t(-h)f_t(t)|^2 dt\right)^{1/2} + c\|(1+t)f_t\|_{L_2(0,T)} \\ &\leq c\|(1+t)\mathbf{u}\|_{W_2^{0,\mu_1}(G_T)},\end{aligned}$$

which concludes the proof of (2.17) and of Proposition 2.2.

On the estimate (3.24). The expressions l_1, l_2, l_3, l_4 are the same as in the symmetric case, and they have been estimated in [9], Propositions 5.5 and 5.6, but l_5 and l_6 are somewhat different. As in the symmetric case, main technical difficulties arise in the estimate of l_5 , in particular, of the second derivative $\frac{\partial^2 U_s}{\partial s^2}$ of the potential (3.3). It has the same form as the analogous function in [1], only the role of ρ is played by $\widehat{\rho}$ or $\widetilde{\rho}$. We have:

$$\frac{\partial^2 U_s(z, t)}{\partial s^2} = V_1(z, t) + V_2(z, t) - \mathbf{W}_1(z, t) \cdot \mathbf{N}_{\theta(t)}(z) \widehat{\rho}(z, t) - \mathbf{W}_2(z, t) \cdot \mathbf{N}_\theta(z) \widetilde{\rho}(z, t), \quad (4.10)$$

$$V_1(z, t) = \int_{\mathcal{G}_{\theta(t)}} \widehat{\rho}(\zeta, t) \frac{\partial \Lambda(\zeta, s\widehat{\rho})}{\partial s} \frac{dS_\zeta}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|},$$

$$V_2(z, t) = \int_{\mathcal{G}_\theta} \widetilde{\rho}(\zeta, t) \Lambda(\zeta, s\widetilde{\rho}) \frac{\partial}{\partial s} \frac{1}{|e_{s\widetilde{\rho}}(z) - e_{s\widetilde{\rho}}(\zeta)|} dS,$$

$$\mathbf{W}_1(z, t) = \int_{\mathcal{F}_\theta} \frac{\partial L(\zeta, s\widehat{\rho}^*)}{\partial s} \frac{e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|^3} d\zeta,$$

$$\mathbf{W}_2(z, t) = \int_{\mathcal{F}_\theta} L(\zeta, s\widetilde{\rho}^*) \frac{\partial}{\partial s} \frac{e_{s\widetilde{\rho}}(z) - e_{s\widetilde{\rho}}(\zeta)}{|e_{s\widetilde{\rho}}(z) - e_{s\widetilde{\rho}}(\zeta)|^3} d\zeta.$$

Since $\tilde{\rho}(y, t) = \hat{\rho}(\mathcal{Z}(\theta(t))y, t)$, the formula (4.10) is equivalent to

$$\frac{\partial^2 U_s(\mathcal{Z}(\theta(t))y, t)}{\partial s^2}$$

$$= V_1^{(0)}(y) + V_2^{(0)}(y, t) - \mathbf{W}_1^{(0)}(y, t) \cdot \mathbf{N}_0(y) \tilde{\rho}(y, t) - \mathbf{W}_2^{(0)}(y, t) \cdot \mathbf{N}_0(y) \tilde{\rho}(y, t),$$

where $y \in \mathcal{G}_0$,

$$\begin{aligned} V_1^{(0)}(y, t) &= \int_{\mathcal{G}_0} \tilde{\rho}(\eta, t) \frac{\partial \Lambda(\eta, s\tilde{\rho})}{\partial s} \frac{dS_\eta}{|e_{s\tilde{\rho}}(y) - e_{s\tilde{\rho}}(\eta)|}, \\ V_2^{(0)}(y, t) &= \int_{\mathcal{G}_0} \tilde{\rho}(\eta, t) \Lambda(\eta, s\tilde{\rho}) \frac{\partial}{\partial s} \frac{1}{|e_{s\tilde{\rho}}(y) - e_{s\tilde{\rho}}(\eta)|} dS, \\ \mathbf{W}_1^{(0)}(y, t) &= \int_{\mathcal{F}_0} \frac{\partial L(\eta, s\tilde{\rho}^*)}{\partial s} \frac{e_{s\tilde{\rho}}(y) - e_{s\tilde{\rho}}(\eta)}{|e_{s\tilde{\rho}}(y) - e_{s\tilde{\rho}}(\eta)|^3} d\eta, \\ \mathbf{W}_2^{(0)}(y, t) &= \int_{\mathcal{F}_0} L(\eta, s\tilde{\rho}^*) \frac{\partial}{\partial s} \frac{e_{s\tilde{\rho}}(y) - e_{s\tilde{\rho}}(\eta)}{|e_{s\tilde{\rho}}(y) - e_{s\tilde{\rho}}(\eta)|^3} d\eta. \end{aligned}$$

Estimates of these potentials are made exactly as in [11] and they lead to the inequality analogous to (3.20) in [11], namely,

$$\begin{aligned} \left\| \frac{\partial^2 U_s}{\partial s^2} \right\|_{y=\overline{\mathcal{Z}X}} \|\tilde{w}_2^{l+1/2, l/2+1/4}(G_T)\| &\leq c \sup_{t < T} \|r(\cdot, t)\|_{W_2^{l+1}(\Gamma_0)} \\ &\left(\|r\|_{W_2^{l+1/2, 0}(G_T)} + \|(1+t)\mathbf{u}\|_{W_2^{1/2, 0}(G_T)} \right). \end{aligned} \quad (4.11)$$

The proof is based on the estimates of the Newtonian and single layer potentials obtained in [12]. We also make use of the estimate of the time derivative $\tilde{\rho}_t$. Let V'_n be the velocity of the evolution of the surface $\mathcal{Z}(\lambda(t))X(\xi, t)$ in the direction of the exterior normal \mathbf{n}' . We have

$$\begin{aligned} V'_n &= \frac{\partial}{\partial t} \mathcal{Z}(\lambda(t))X(\xi, t) \cdot \mathbf{n}' = \mathcal{Z}\mathbf{u} \cdot \mathbf{n}' + \mathcal{Z}'X\mathbf{n}'\lambda'(t) \\ &= \mathbf{u} \cdot \mathbf{n} + (\mathbf{e}_3 \times X) \cdot \mathbf{n}\lambda'(t). \end{aligned}$$

Hence

$$\begin{aligned} \tilde{\rho}_t(y, t) &= \frac{V'_n}{\mathbf{n}' \cdot \mathbf{N}_0} = \frac{\mathbf{u} \cdot \mathbf{n}}{\mathbf{n}' \cdot \mathbf{N}_0} + \frac{(\mathbf{e}_3 \times X) \cdot \mathbf{n}}{\mathbf{n}' \cdot \mathbf{N}_0} \lambda'(t) \\ &= \frac{\mathbf{u}(\xi, t) \cdot \hat{\mathcal{L}}^T(z) \mathbf{N}_\theta(z)}{\Lambda(z, \hat{\rho})} + \frac{(\mathbf{e}_3 \times X) \cdot \hat{\mathcal{L}}^T(z) \mathbf{N}_\theta(z)}{\Lambda(z, \hat{\rho})} \lambda'(t). \end{aligned} \quad (4.12)$$

Here, as usual, the points y and z are connected with $\mathcal{Z}(\theta(t))y = z$ and $\Lambda(z, \hat{\rho}) = 1 - \hat{\rho} \mathcal{H}_\theta(z) + \hat{\rho}^2 \mathcal{K}_\theta(z)$, where \mathcal{H}_θ is the doubled mean curvature and \mathcal{K}_θ is the Gaussian curvature of \mathcal{G}_θ . From (4.12) and (1.22) it follows that

$$\|\tilde{\rho}_t(\cdot, t)\|_{W_2^{1/2}(\mathcal{G}_0)} \leq c \|\mathbf{u}(\cdot, t)\|_{W_2^{1/2}(\Gamma_0)},$$

which is analogous to the estimate (3.19) in [11] for ρ_t . This allows us to obtain (4.11).

Now we turn our attention to $B_1(r, \mathbf{u})$. According to (3.5),

$$B_1(r, \mathbf{u}) = B_2(r, \mathbf{u}) - \kappa B_3(r, \mathbf{u}) - \kappa B_4(r, \mathbf{u}) - \kappa B_5(r, \mathbf{u}),$$

where

$$\begin{aligned} B_2(r, \mathbf{u}) &= (b(\overline{\mathcal{Z}(\lambda(t)\bar{X})}) - b(\bar{\xi}))r(\xi, t), \\ B_3(r, \mathbf{u}) &= \int_{\Gamma_0} \frac{r(\eta, t)(\Psi(\eta, t) - 1)dS_\eta}{|\overline{\mathcal{Z}X(\xi, t)} - \overline{\mathcal{Z}X(\eta, t)}|}, \\ B_4(r, \mathbf{u}) &= \int_0^1 ds \int_{\Gamma_0} r(\eta, t) \frac{\partial}{\partial s} \frac{1}{|\bar{X}_s(\xi, t) - \bar{X}_s(\eta, t)|} dS, \\ B_5(r, \mathbf{u}) &= \int_0^1 ds \int_{\Gamma_0} r(\eta, t) \frac{\partial}{\partial s} \frac{1}{|\overline{\mathcal{Z}(s\lambda(t))X(\xi, t)} - \overline{\mathcal{Z}(s\lambda)X(\eta, t)}|} dS. \end{aligned}$$

In the case of axially symmetric \mathcal{F} we have $\mathcal{Z} = I$ and the term B_5 drops out.

We start with the estimate of B_2 and show that

$$\begin{aligned} &\|B_2\|_{\widetilde{W}_2^{t+1/2, t/2+3/4}(G_T)} \\ &\leq c \left(\|\mathbf{u}\|_{\widetilde{W}_2^{t+2, 0}(Q_T)} + \|r\|_{\widetilde{W}_2^{t+1/2, 0}(G_T)} \right) \|\mathbf{u}\|_{\widetilde{W}_2^{t+2, 0}(Q_T)}. \end{aligned} \quad (4.13)$$

Following the arguments in the proof of Proposition 5.7 in [9], we estimate the difference

$$b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi}). \quad (4.14)$$

In Proposition 5.7 it is proved that

$$\|b_0(\bar{X})(\xi, t) - b_0(\bar{\xi})\|_{W_2^{t+1/2}(\Gamma_0)} \leq c \|\mathbf{u}\|_{\widetilde{W}_2^{2+t, 0}(Q_t)}, \quad \forall t \in (0, T). \quad (4.15)$$

The difference (4.14) satisfies the same inequality; indeed,

$$b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi}) = (b_0(\overline{\mathcal{Z}X}) - b_0(\bar{X})) + (b_0(\bar{X}) - b_0(\bar{\xi})),$$

$$\begin{aligned} b_0(\overline{\mathcal{Z}X}) - b_0(\bar{X}) &= \int_0^1 \frac{\partial}{\partial s} b_0(\mathcal{Z}(s\lambda)X) ds \\ &= \int_0^1 \nabla b_0(\mathcal{Z}(s\lambda)X) ds \lambda(t), \end{aligned}$$

where $b_0(\cdot) = b_0(\mathfrak{A}(\cdot))$. Hence, by (2.15),

$$\begin{aligned} &\|b_0(\overline{\mathcal{Z}X}) - b_0(\bar{X})\|_{W_2^{t+1/2}(\Gamma_0)} \\ &\leq c \left(1 + \|\mathbf{u}\|_{\widetilde{W}_2^{t+2, 0}(Q_t)} \right) |\lambda(t)| \leq c \int_0^t \int_{\Gamma_0} |\mathbf{u}(\xi, \tau)| dS d\tau \leq c \|\mathbf{u}\|_{\widetilde{W}_2^{t+2, 0}(Q_t)}. \end{aligned} \quad (4.16)$$

We also need to estimate the time derivative

$$\begin{aligned} & \frac{\partial}{\partial t} \left(b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi}) \right) r \\ &= \left(b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi}) \right) r_t + r \nabla \mathbf{b}_0(\mathcal{Z}X) \left(\mathcal{Z}(\lambda) \mathbf{u}(\xi, t) + \mathcal{Z}'(\lambda) X \lambda'(t) \right). \end{aligned}$$

Using the inequalities (4.16), (2.15), we obtain

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} \left(b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi}) \right) r \right\|_{L_2(\Gamma_0)} \leq \sup_{\Gamma_0} |b_0(\overline{\mathcal{Z}X}) - b_0(\bar{\xi})| \|r_t(\cdot, t)\|_{L_2(\Gamma_0)} \\ & + \left\| \frac{\partial b_0(\overline{\mathcal{Z}X})}{\partial t} \right\|_{L_2(\Gamma_0)} \sup_{\Gamma_0} |r(\xi, t)| \leq c \|\mathbf{u}(\cdot, t)\|_{L_2(\Gamma_0)} \left(\|\mathbf{u}\|_{\widetilde{W}_2^{l+2,0}(Q_t)} + \sup_{\Gamma_0} |r(\xi, t)| \right). \end{aligned}$$

Together with (4.15), (4.16), this estimate implies (4.13).

For the estimate of B_3, B_4, B_5 we can use Proposition 2.10 in [11]. It concerns the surface integrals of the form

$$\begin{aligned} v(y, t) &= \int_{\Gamma_0} |\mathcal{T}(y, t) - \mathcal{T}(\eta, t)|^{-1} g(\eta, t) dS, \\ v_1(y, t) &= \int_{\Gamma_0} \frac{\mathcal{T}(y, t) - \mathcal{T}(\eta, t)}{|\mathcal{T}(y, t) - \mathcal{T}(\eta, t)|^3} \cdot (\mathbf{a}(y, t) - \mathbf{a}(\eta, t)) g(\eta, t) dS, \end{aligned}$$

where $\mathcal{T}(y, t)$ is an invertible mapping of class $W_2^{l+3/2-\epsilon}(\Omega_0)$, $\epsilon \in (0, l-1)$, with $\mathcal{T}_t \in W_2^1(\Omega_0)$ and \mathbf{a} is as regular as $\tilde{\rho}$. It is easily seen that the transformation $\mathcal{T} = \mathcal{Z}(\lambda(t))X(\xi, t)$ possesses these properties and $\overline{\mathcal{Z}X} = e_{\tilde{\rho}}^{-1}X$. Therefore the application of Proposition 2.10 leads to the same estimates for B_1 (and for l_5) as in [11], namely,

$$\begin{aligned} & \|l_5\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} \\ & \leq c \left(\sup_{t < T} \|r(\cdot, t)\|_{W_2^{l+1}(\Gamma_0)} + \|r(\cdot, t)\|_{\widetilde{W}_2^{l+1/2, l/2+1/4}(G_T)} + \|\mathbf{u}\|_{\widetilde{W}_2^{l+2, l/2+1}(Q_T)} \right)^2. \end{aligned} \quad (4.17)$$

Now we pass to the estimate of

$$\begin{aligned} l_6(\mathbf{u}) &= (\mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})) \cdot \mathbf{u}(\xi, t) + (h_0(\overline{\mathcal{Z}X}) - h_0(\bar{\xi}))\lambda'(t) + h_0(\bar{\xi})m(t) \\ & \equiv l_6^{(1)}(\mathbf{u}) + l_6^{(2)}(\mathbf{u}) + l_6^{(3)}(\mathbf{u}) \end{aligned}$$

where $m(t)$ is defined in (2.19). We have already seen above that

$$\begin{aligned} & \|h_0(\overline{\mathcal{Z}X}) - h_0(\bar{\xi})\|_{W_2^{l+1/2}(\Gamma_0)} \leq c \|\mathbf{u}\|_{\widehat{W}_2^{l+2,0}(Q_t)}, \\ & \|\mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})\|_{W_2^{l+1/2}(\Gamma_0)} \leq c |\lambda(t)| \|\mathbf{N}_0(\overline{\mathcal{Z}X})\|_{W_2^{l+1/2}(\Gamma_0)} \\ & + \|\mathbf{N}_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})\|_{W_2^{l+1/2}(\Gamma_0)} \leq c \|\mathbf{u}\|_{\widehat{W}_2^{l+2,0}(Q_t)}. \end{aligned} \quad (4.18)$$

Exactly in the same way we obtain

$$\begin{aligned} & \|h_0(\overline{\mathcal{Z}X}) - h_0(\bar{\xi})\|_{\widetilde{W}_2^{l+3/2}(\Gamma_0)} + \|\mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})\|_{\widetilde{W}_2^{l+1/2}(\Gamma_0)} \\ & \leq c(1 + \sqrt{t})\|\mathbf{u}\|_{\widetilde{W}_2^{l+2,0}(Q_t)}. \end{aligned}$$

In addition, we have

$$\begin{aligned} & \left\| \frac{\partial}{\partial t} h_0(\overline{\mathcal{Z}X}) \right\|_{L_2(\Gamma_0)} + \left\| \frac{\partial}{\partial t} \mathcal{Z}^{-1} \mathbf{N}_0(\overline{\mathcal{Z}X}) \right\|_{L_2(\Gamma_0)} \leq c \|\mathbf{u}\|_{L_2(\Gamma_0)}, \\ & \left\| \frac{\partial^2}{\partial t^2} h_0(\overline{\mathcal{Z}X}) \right\|_{L_2(\Gamma_0)} + \left\| \frac{\partial^2}{\partial t^2} \mathcal{Z}^{-1} \mathbf{N}_0(\overline{\mathcal{Z}X}) \right\|_{L_2(\Gamma_0)} \leq c \left(\|\mathbf{u}_t\|_{L_2(\Gamma_0)} + \|\mathbf{u}\|_{L_2(\Gamma_0)} \right). \end{aligned}$$

These inequalities allow us to estimate $l_6^{(1)}$ and $l_6^{(2)}$ exactly in the same way as l_6 has been estimated in [9], Proposition 5.8:

$$\begin{aligned} & \|l_6^{(1)}(\mathbf{u})\|_{\widetilde{W}_2^{3/2+l,3/4+l/2}(G_T)} + \|l_6^{(2)}(\mathbf{u})\|_{\widetilde{W}_2^{3/2+l,3/4+l/2}(G_T)} \\ & \leq c\|\mathbf{u}\|_{\widetilde{W}_2^{l+2,1/2+1}(Q_T)} \left(\|\mathbf{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)} + \sup_{Q_T} |\mathbf{u}(\xi, t)| \right). \end{aligned}$$

The proof reduces to repeating the arguments in this Proposition. Finally, it is easily seen that

$$\|l_6^{(3)}(\mathbf{u})\|_{\widetilde{W}_2^{3/2+l,3/4+l/2}(G_T)} \leq c\|m\|_{\widetilde{W}_2^{1/2+3/4}(0,T)}.$$

According to (2.19), $m(t) = m_1(t) + m_2(t)$, where

$$\begin{aligned} m_1(t) &= -\frac{1}{f_\lambda(t, \lambda(t))} \int_{\Gamma_0} \left(\mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})h_0(\bar{\xi}) \right) \cdot \mathbf{u}(\xi, t) dS_\xi, \\ & \qquad \qquad \qquad m_2(t) \\ &= \frac{\int_{\Gamma_0} \mathbf{N}_0(\bar{\xi}) \cdot \mathbf{u}(\xi, t) h_0(\bar{\xi}) dS}{f_\lambda(t, \lambda(t)) \int_{\Gamma_0} h_0^2(\bar{\xi}) dS} \int_{\Gamma_0} r(\xi, t) \nabla h_0(\overline{\mathcal{Z}X}) \cdot \left(\nabla \mathfrak{R}(\mathcal{Z}X)(\mathbf{e}_3 \times \mathcal{Z}X) \right) dS \\ & \quad - \left(\int_{\Gamma_0} h_0^2(\bar{\xi}) dS \right)^{-1} \int_{\Gamma_0} r(\xi, t) \nabla h_0(\overline{\mathcal{Z}X}) \nabla \mathfrak{R}(\mathcal{Z}X) \mathcal{Z}(\lambda(t)) \mathbf{u}(\xi, t) dS. \quad (4.19) \end{aligned}$$

In view of (4.18), (2.15),

$$\begin{aligned} & \left| \mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})h_0(\bar{\xi}) \right| \leq c\|\mathbf{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}, \\ & \left\| \frac{\partial}{\partial t} \left(\mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})h_0(\bar{\xi}) \right) \right\|_{L_2(\Gamma_0)} \leq c\|\mathbf{u}(\cdot, t)\|_{L_2(\Gamma_0)}, \\ & \left\| \frac{\partial^2}{\partial t^2} \left(\mathcal{Z}^{-1}(\lambda(t))\mathbf{N}_0(\overline{\mathcal{Z}X})h_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi})h_0(\bar{\xi}) \right) \right\|_{L_2(\Gamma_0)} \\ & \leq c \left(\|\mathbf{u}_t(\cdot, t)\|_{L_2(\Gamma_0)} + \|\mathbf{u}(\cdot, t)\|_{L_2(\Gamma_0)} \right), \end{aligned}$$

hence

$$\begin{aligned}
& m_1'(t) \\
&= \frac{f_{t\lambda}(t, \lambda(t))}{f_\lambda^2(t, \lambda(t))} \Big|_{\lambda=\lambda(t)} \int_{\Gamma_0} \left(\mathcal{Z}^{-1}(\lambda(t)) \mathbf{N}_0(\overline{\mathcal{Z}X}) h_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi}) h_0(\bar{\xi}) \right) \cdot \mathbf{u}(\xi, t) dS_\xi \\
&\quad - \frac{1}{f_\lambda(t, \lambda(t))} \left(\int_{\Gamma_0} \frac{\partial}{\partial t} \left(\mathcal{Z}^{-1}(\lambda(t)) \mathbf{N}_0(\overline{\mathcal{Z}X}) h_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi}) h_0(\bar{\xi}) \right) \cdot \mathbf{u}(\xi, t) dS_\xi \right. \\
&\quad \left. + \int_{\Gamma_0} \left(\mathcal{Z}^{-1}(\lambda(t)) \mathbf{N}_0(\overline{\mathcal{Z}X}) h_0(\overline{\mathcal{Z}X}) - \mathbf{N}_0(\bar{\xi}) h_0(\bar{\xi}) \right) \cdot \mathbf{u}_t(\xi, t) dS_\xi \right).
\end{aligned}$$

satisfies the inequalities

$$\begin{aligned}
& \|m_1'\|_{L_2(0,T)} \leq c \|\mathbf{u}\|_{W_2^{0,1}(G_T)} \|\mathbf{u}\|_{\widetilde{W}_2^{2+t,0}(Q_T)}, \\
& \left(\int_0^1 \frac{dh}{h^{1+2\mu}} \int_h^T |\Delta_t(-h)m_1'(t)|^2 dt \right)^{1/2} \leq c \|\mathbf{u}\|_{W_2^{0,1+\mu}(G_T)} \|\mathbf{u}\|_{\widetilde{W}_2^{t+2,0}(Q_T)}
\end{aligned}$$

and

$$\begin{aligned}
& \left(\int_0^1 \frac{dh}{h^{1+2\mu_1}} \int_h^T (1+t)^2 |\Delta_t(-h)m_1(t)|^2 dt \right)^{1/2} + \|(1+t)m_1\|_{L_2(0,T)} \\
& \leq c \|(1+t)\mathbf{u}\|_{W_2^{0,\mu_1}(G_T)} \|\mathbf{u}\|_{\widetilde{W}_2^{t+2,0}(Q_T)},
\end{aligned}$$

established in the same way as (2.17). This implies

$$\|m_1\|_{\widetilde{W}_2^{t/2+3/4}(0,T)} \leq c \|\mathbf{u}\|_{\widetilde{W}_2^{0,t/2+3/4}(G_T)} \|\mathbf{u}\|_{\widetilde{W}_2^{2+t,0}(Q_T)}. \quad (4.20)$$

The function $m_2(t)$ is estimated by similar arguments. Taking (3.6) into account we obtain

$$\|m_2\|_{\widetilde{W}_2^{t/2+3/4}(0,T)} \leq c \|\mathbf{u}\|_{\widetilde{W}_2^{0,t/2+3/4}(G_T)} \left(\|\mathbf{u}\|_{L_2(\Gamma_0)} + \|r\|_{L_2(\Gamma_0)} \right).$$

This implies

$$\begin{aligned}
& \|l_6(\mathbf{u})\|_{\widetilde{W}_2^{t+3/2,t/2+1/2}(G_T)} \\
& \leq c \|\mathbf{u}\|_{\widetilde{W}_2^{t+2,t/2+1}(Q_T)} \left(\|\mathbf{u}\|_{\widetilde{W}_2^{t+2,0}(Q_T)} + \sup_{Q_T} |\mathbf{u}(\xi, t)| + \|r\|_{L_2(\Gamma_0)} \right),
\end{aligned}$$

Thus (3.24) is proved.

5. Proof of Theorem 3.4

We start by obtaining some auxiliary relations and estimates. Let \mathbf{w} be a solution of the problem (1.9) and let \mathbf{w}^\perp be a projection of \mathbf{w} on the subspace of vector fields orthogonal to all rigid displacements. In view of (1.11),

$$\mathbf{w}(x, t) = \mathbf{w}^\perp(x, t) + \sum_{k=1}^3 g_k(t) \boldsymbol{\eta}_k(x), \quad (5.1)$$

where $\boldsymbol{\eta}_k(x) = \mathbf{e}_k \times x$, $\mathbf{e}_k = (\delta_{ik})_{i=1,2,3}$, and $g_k(t)$ are functions defined as a solution of a linear algebraic system

$$\sum_{k=1}^3 S_{ik}(t)g_k(t) = \int_{\Omega_t} \boldsymbol{\omega}(x, t) \cdot \boldsymbol{\eta}_i(x) dx = I_i(t) \quad (5.2)$$

with

$$\begin{aligned} S_{ik}(t) &= \int_{\Omega_t} \boldsymbol{\eta}_i(x) \cdot \boldsymbol{\eta}_k(x) dx, \\ I_i(t) &= -\omega \left(\int_{\Omega_t} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx - \int_{\mathcal{F}} \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_i(x) dx \right); \end{aligned} \quad (5.3)$$

by \mathcal{F} we mean arbitrary \mathcal{F}_θ . Since

$$-\int_{\mathcal{F}} \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j dx = \int_{\mathcal{F}} x_j x_3 dx = 0, \quad j = 1, 2 \quad (5.4)$$

(see [2]), we have

$$I_i(t) = \beta \delta_{i3} - S_{i3}(t)\omega, \quad (5.5)$$

where $\beta = \omega \int_{\mathcal{F}} |x'|^2 dx$ is the magnitude of the angular momentum of the rotating liquid. The matrix $\mathcal{S} = (S_{ik}(t))_{i,k=1,2,3}$ is symmetric and positive definite, because for arbitrary real ξ_k

$$\sum_{i,k=1}^3 S_{ik}(t)\xi_i\xi_k = \int_{\Omega_t} \left| \sum_{i=1}^3 \xi_i \boldsymbol{\eta}_i(x) \right|^2 dx = \int_{\Omega_t} |\boldsymbol{\xi} \times x|^2 dx \geq c|\boldsymbol{\xi}|^2.$$

Hence there exists the inverse matrix $\mathcal{S}^{-1} = (S^{ik}(t))_{i,k=1,2,3}$, and

$$g_k(t) = \sum_{m=1}^3 S^{km}(t)(\beta \delta_{m3} - S_{m3}(t)\omega) = \beta S^{k3}(t) - \delta_{k3}\omega. \quad (5.6)$$

We recall that Γ_t is given by the equation (1.21) with $\widehat{\rho}$ satisfying (1.25). We compute the projection $\widehat{\rho}^\perp$ of $\widehat{\rho}$ on the subspace of $L_2(\mathcal{G}_\theta)$ orthogonal to the functions $(1, x_1, x_2, x_3, h_{\theta(t)}(x))$. It is clear that $(1, x_1, x_2, x_3)$ are linearly independent functions of $x \in \mathcal{G}$ and $h_\theta(x) = \mathbf{N}_\theta(x) \cdot \boldsymbol{\eta}_3(x)$ is orthogonal to them, because

$$\int_{\mathcal{G}_\theta} \mathbf{N}_\theta(z) \cdot \boldsymbol{\eta}_3(z) dS = \int_{\mathcal{F}_\theta} \nabla \cdot \boldsymbol{\eta}_3(x) dx = 0, \quad (5.7)$$

$$\int_{\mathcal{G}_\theta} z_i \mathbf{N}_\theta(z) \cdot \boldsymbol{\eta}_3(z) dS = \int_{\mathcal{F}_\theta} \nabla \cdot x_i \boldsymbol{\eta}_3(x) dx = 0. \quad (5.8)$$

We have

$$\widehat{\rho} = \widehat{\rho}^\perp + \sum_{k=0}^4 c_k(t)\varphi_k,$$

where $\varphi_0(x) = 0$, $\varphi_i(x) = x_i$, $i = 1, 2, 3$, $\varphi_4(x) = h_\theta(x)$. By (5.7), (5.8),

$$\int_{\mathcal{G}_\theta} \widehat{\rho} \varphi_a dS = \sum_{b=0}^3 c_b(t) \int_{\mathcal{G}_\theta} \varphi_a(x) \varphi_b(x) dS, \quad a = 0, 1, 2, 3,$$

and

$$c_a(t) = \sum_{b=0}^3 \phi^{ab}(t) \int_{\mathcal{G}_\theta} \widehat{\rho} \varphi_b dS,$$

where $\phi^{ab}(t)$ are elements of the matrix inverse to $\Phi = \left(\int_{\mathcal{G}_\theta} \varphi_a \varphi_b dS \right)_{a,b=0,1,2,3}$. It follows that

$$\rho = \rho^\perp + \sum_{a,b=0}^3 \phi^{ab} \int_{\mathcal{G}_\theta} \rho \varphi_b dS \varphi_a(x) + h_\theta(x) \|h_\theta\|_{L^2(\mathcal{G}_\theta)}^{-2} \int_{\mathcal{G}_\theta} \rho h(y) dS.$$

Conditions (1.28) for $\widehat{\rho}$ imply

$$\begin{aligned} \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) dS &= \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) (1 - \varphi(x, \widehat{\rho})) dS, \\ \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) x_i dS &= \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) (x_i - \psi_i(x, \widehat{\rho})) dS, \\ \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) h_\theta(x) dS &= \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) h_\theta(x) (1 - \Psi) dS, \end{aligned}$$

and, as a consequence,

$$\left| \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) dS \right| \leq \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \|1 - \varphi\|_{W_2^{1/2}(\mathcal{G}_\theta)} \leq c\delta \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)},$$

$$\left| \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) x_i dS \right| \leq c\delta \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)};$$

moreover, since

$$\begin{aligned} |1 - \Psi| &\leq c \left(|1 - |An_0|| + |1 - |\widehat{\mathcal{L}}^T(z, \widehat{\rho}) \mathbf{N}_\theta|| \right) \\ &\leq c \left(\|\mathbf{u}\|_{\widetilde{W}^{2+l,0}(\mathcal{G}_t)} + \|\widehat{\rho}\|_{W_2^{l+1-\epsilon}(\mathcal{G}_\theta)} \right) \leq c\delta, \end{aligned}$$

we have

$$\left| \int_{\mathcal{G}_\theta} \widehat{\rho}(x, t) h_\theta(x) dS \right| \leq c\delta \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}.$$

Hence

$$\|\widehat{\rho} - \widehat{\rho}^\perp\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \leq c\delta \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)},$$

which means that for small δ the norms $\|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}$ and $\|\widehat{\rho}^\perp\|_{W_2^{-1/2}(\mathcal{G}_\theta)}$ are equivalent to each other:

$$c_1 \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \leq \|\widehat{\rho}^\perp\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \leq c_2 \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}. \quad (5.9)$$

Now we proceed to the proof of Theorem 3.4, assuming that the solution of (1.9), (1.14) is constructed for $t \in (0, T)$ and the condition (3.23) is satisfied. The following propositions play an important role in the proof.

Proposition 5.1. *Given the function $f_0 \in W_2^{1/2}(\mathcal{G}_0)$ such that $\int_{\mathcal{G}_0} f_0 dS = 0$, there exists a divergence free vector field $\mathbf{W} \in W_2^1(\Omega_t)$ satisfying the condition*

$$\int_{\Omega_t} \mathbf{W}(x, t) \cdot \boldsymbol{\eta}_j(x) dx = 0, \quad j = 1, 2, 3, \quad (5.10)$$

and the inequalities

$$\begin{aligned} \|\mathbf{W}\|_{W_2^1(\Omega_t)} &\leq c \|f_0\|_{W_2^{1/2}(\mathcal{G}_0)}, \\ \|\mathbf{W}\|_{L_2(\Omega_t)} &\leq c \|f_0\|_{L_2(\mathcal{G}_0)}, \\ \|\mathbf{W}_t\|_{L_2(\Omega_t)} &\leq c \left(\|f_0\|_{W_2^{1/2}(\mathcal{G}_0)} + \|f_{0t}\|_{L_2(\mathcal{G}_0)} \right). \end{aligned} \quad (5.11)$$

Sketch of the proof. At first we construct a divergence free $\widetilde{\mathbf{W}}$ in the domain $\widetilde{\Omega}_t = \mathcal{Z}(\lambda(t))\Omega_t$ with the normal component on $\partial\widetilde{\Omega}_t$ equal to $f_0(e_{\widetilde{\rho}}(y))|\widehat{\mathcal{L}}^T(y, \widetilde{\rho})\mathbf{N}_0|^{-1}$ that satisfies inequalities (5.11) and the condition (5.10) in $\widetilde{\Omega}_t$. The construction (for a particular f_0) is given in [13], Lemma 4.1, and it is valid for arbitrary $f_0 \in W_2^{1/2}(\mathcal{G}_0)$. The vector field \mathbf{W} is defined by

$$\mathbf{W}(x, t) = \mathcal{Z}^{-1}(\lambda(t))\widetilde{\mathbf{W}}(\mathcal{Z}(\lambda(t))x, t), \quad x \in \Omega_t.$$

Direct computation shows that \mathbf{W} satisfies (5.10). Inequalities (5.11) follow from similar inequalities for $\widetilde{\mathbf{W}}$.

Proposition 5.2. *Let U_s be a potential defined in (3.3). For arbitrary $f_1 \in W_2^{1/2}(\mathcal{G}_0)$ the following inequality holds:*

$$\left| \int_{\mathcal{G}_\theta} \frac{\partial^2 U_s}{\partial s^2} f_1(z) dS \right| \leq c\delta \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)}. \quad (5.12)$$

Proof. According to (4.10),

$$\begin{aligned} &\int_{\mathcal{G}_\theta} \frac{\partial^2 U_s}{\partial s^2} f_1(z) dS \\ &= \int_{\mathcal{G}_\theta} \left(V_1(z, t) + V_2(z, t) - \mathbf{W}_1(z, t) \cdot \mathbf{N}_{\theta(t)}(z) \widehat{\rho}(z, t) - \mathbf{W}_2(z, t) \cdot \mathbf{N}_\theta(z) \widehat{\rho}(z, t) \right) f_1 dS \\ &= \int_{\mathcal{G}_\theta} \widehat{\rho}(z, t) \left(\frac{\partial \Lambda(z, s\widehat{\rho})}{\partial s} V_3[f_1] + \Lambda(z, s\widehat{\rho}) V_4[f_1] \right) dS \end{aligned}$$

$$- \int_{\mathcal{G}_\theta} \widehat{\rho}(z, t) f_1(z) \mathbf{N}_\theta(z) \cdot (\mathbf{W}_1(z, t) + \mathbf{W}_2(z, t)) dS, \quad (5.13)$$

where

$$V_3[f_1] = \int_{\mathcal{G}_\theta} \frac{f_1(\zeta) dS}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|}, \quad V_4[f_1] = \int_{\mathcal{G}_\theta} f_1(\zeta) \frac{\partial}{\partial s} \frac{1}{|e_{s\widehat{\rho}}(z) - e_{s\widehat{\rho}}(\zeta)|} dS.$$

The right hand side of (5.13) does not exceed

$$\begin{aligned} & \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \left(\left\| \frac{\partial \Lambda}{\partial s} V_3[f_1] \right\|_{W_2^{1/2}(\mathcal{G}_\theta)} + \|\Lambda V_4[f_1]\|_{W_2^{1/2}(\mathcal{G}_\theta)} \right. \\ & \left. + \|f_1 \mathbf{N}_\theta \cdot (\mathbf{W}_1 + \mathbf{W}_2)\|_{W_2^{1/2}(\mathcal{G}_\theta)} \right). \end{aligned}$$

Applying Proposition 4.1 in [1] that concerns the estimate of the product of two functions, we obtain

$$\begin{aligned} & \left| \int_{\mathcal{G}_\theta} \frac{\partial^2 U_s}{\partial s^2} f_1(z) dS \right| \leq c \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \left(\left\| \frac{\partial \Lambda}{\partial s} \right\|_{W_2^{t+1/2}(\mathcal{G}_\theta)} \|V_3[f_1]\|_{W_2^{1/2}(\mathcal{G}_\theta)} \right. \\ & \left. + \|\Lambda\|_{W_2^{t+1/2}(\mathcal{G}_\theta)} \|V_4[f_1]\|_{W_2^{1/2}(\mathcal{G}_\theta)} + \|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)} (\|\mathbf{W}_1\|_{W_2^{t+1/2}(\mathcal{G}_\theta)} + \|\mathbf{W}_2\|_{W_2^{t+1/2}(\mathcal{G}_\theta)}) \right). \end{aligned}$$

In view of the estimates of the volume and surface potentials obtained in [11], Sec.3, this inequality implies (5.12). The proposition is proved.

Inequality (3.25) follows from the estimate of a "generalized energy". We multiply the first equation in (1.9) by \mathbf{w} and integrate over Ω_t . Making use of the transport theorem and of the boundary conditions, we arrive at the energy relation

$$\frac{1}{2} \frac{d}{dt} \left(\|\mathbf{w}\|_{L_2(\Omega_t)}^2 - \omega^2 \int_{\Omega_t} |x'|^2 dx - \kappa \int_{\Omega_t} U(x, t) dx \right) + \frac{\nu}{2} \int_{\Omega_t} |S(\mathbf{w})|^2 dx = 0. \quad (5.14)$$

By (5.1) and (5.5),

$$\begin{aligned} \|\mathbf{w}\|_{L_2(\Omega_t)}^2 &= \|\mathbf{w}^\perp\|_{L_2(\Omega_t)}^2 + \sum_{k,j=1}^3 S_{kj}(t) g_k(t) g_j(t) = \|\mathbf{w}^\perp\|_{L_2(\Omega_t)}^2 \\ &+ \sum_{k,j=1}^3 S_{kj} (\beta S^{k3}(t) - \delta_{k3} \omega) (\beta S^{j3}(t) - \delta_{j3} \omega) = \|\mathbf{w}^\perp\|_{L_2(\Omega_t)}^2 \\ &+ S^{33}(t) \beta^2 + S_{33}(t) \omega^2 - 2\beta \omega = \|\mathbf{w}^\perp\|_{L_2(\Omega_t)}^2 + \frac{\beta^2}{\int_{\Omega_t} |x'|^2 dx} \\ &+ \beta^2 (S_{33} - S_{33}^{-1}) + \omega^2 \int_{\Omega_t} |x'|^2 dx - 2\beta \omega. \end{aligned}$$

The expression

$$\beta^2 (S_{33} - S_{33}^{-1}) = -\beta^2 S_{33}^{-1} \sum_{j=1}^2 S^{3j} S_{j3}$$

$$= \frac{\beta^2}{S_{33} \det \mathcal{S}} (S_{11} S_{23}^2 + S_{22} S_{13}^2 - 2S_{12} S_{13} S_{23}) \equiv Q(t)$$

is a positive definite quadratic form with respect to S_{13}, S_{23} , since $2S_{12} \leq \sqrt{S_{11}}\sqrt{S_{22}}$. Hence (5.14) may be written in the form

$$\frac{d}{dt} \left(\frac{1}{2} \|\mathbf{w}\|_{L_2(\Omega_t)}^2 + Q(t) + \mathcal{R}(t) - \mathcal{R}_0 \right) + \frac{\nu}{2} \int_{\Omega_t} |S(\mathbf{w})|^2 dx = 0, \quad (5.15)$$

where

$$\begin{aligned} \mathcal{R}(t) &\equiv \beta^2 \left(2 \int_{\Omega_t} |x'|^2 dx \right)^{-1} - \frac{\kappa}{2} \int_{\Omega_t} U(x, t) dx - p_0 |\Omega_t|, \\ \mathcal{R}_0 &= \beta^2 \left(2 \int_{\mathcal{F}_\theta} |x'|^2 dx \right)^{-1} - \frac{\kappa}{2} \int_{\mathcal{F}_\theta} \mathcal{U}(x) dx - p_0 |\mathcal{F}_\theta|. \end{aligned}$$

Now we use the relations

$$2(\mathbf{e}_3 \times \boldsymbol{\eta}_i) = -\nabla(\boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_3) + \boldsymbol{\eta}^i, \quad i = 1, 2, 3,$$

where $\boldsymbol{\eta}^1 = \boldsymbol{\eta}_2$, $\boldsymbol{\eta}^2 = -\boldsymbol{\eta}_1$, $\boldsymbol{\eta}^3 = 0$, and write the first equation in (1.9) in the form

$$\begin{aligned} \mathbf{w}_t^\perp + (\mathbf{w} \cdot \nabla) \mathbf{w}^\perp + (\mathbf{w} \cdot \nabla) \mathbf{w}' + 2\omega(\mathbf{e}_3 \times \mathbf{w}^\perp) \\ - \nu \nabla^2 \mathbf{w}^\perp + \nabla(p - \omega \sum_{j=1}^3 g_j(t) \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j) = -\mathbf{w}'_t - \omega \sum_{\alpha=1}^2 g_\alpha \boldsymbol{\eta}^\alpha(x), \end{aligned} \quad (5.16)$$

where $\mathbf{w}' = \sum_{j=1}^3 g_j(t) \boldsymbol{\eta}_j(x)$. Since $(\mathbf{w}' \cdot \nabla) \mathbf{w}' = -\frac{1}{2} \nabla |\mathbf{w}'|^2$, (5.16) is equivalent to

$$\begin{aligned} \mathbf{w}_t^\perp + (\mathbf{w} \cdot \nabla) \mathbf{w}^\perp + (\mathbf{w}^\perp \cdot \nabla) \mathbf{w}' + 2\omega(\mathbf{e}_3 \times \mathbf{w}^\perp) \\ - \nu \nabla^2 \mathbf{w}^\perp + \nabla(p - \omega \sum_{j=1}^3 g_j(t) \boldsymbol{\eta}_3 \cdot \boldsymbol{\eta}_j - \frac{1}{2} |\mathbf{w}'|^2) = -\mathbf{w}'_t - \omega \sum_{\alpha=1}^2 g_\alpha \boldsymbol{\eta}^\alpha(x), \end{aligned} \quad (5.17)$$

We multiply (5.17) by the auxiliary vector field \mathbf{W} constructed in Proposition 5.2 leaving for the moment the function f_0 indefinite. Then we integrate the product over Ω_t . Elementary calculations lead to

$$\begin{aligned} \frac{d}{dt} \int_{\Omega_t} \mathbf{w}^\perp \cdot \mathbf{W} dx - \int_{\Omega_t} \mathbf{w}^\perp \cdot (\mathbf{W}_t + (\mathbf{w} \cdot \nabla) \mathbf{W}) dx + 2\omega \int_{\Omega_t} (\mathbf{e}_3 \times \mathbf{w}^\perp) \cdot \mathbf{W} dx \\ + \frac{\nu}{2} \int_{\Omega_t} S(\mathbf{w}^\perp) \cdot S(\mathbf{W}) dx + \int_{\Omega_t} (\mathbf{w}^\perp \cdot \nabla) \mathbf{w}' \cdot \mathbf{W} dx \\ - \int_{\Gamma_t} (M + \omega \sum_{j=1}^3 g_j(t) \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) + \frac{1}{2} |\mathbf{w}'|^2) \mathbf{W} \cdot \mathbf{n} dS = 0. \end{aligned} \quad (5.18)$$

We multiply (5.18) by a small positive γ and add to (5.15). As a result we obtain

$$\frac{dE(t)}{dt} + E_1(t) = 0 \quad (5.19)$$

with

$$E(t) = \frac{1}{2} \|\mathbf{w}\|_{L_2(\Omega_t)}^2 + Q + (\mathcal{R} - \mathcal{R}_0) + \gamma \int_{\Omega_t} \mathbf{w}^\perp \cdot \mathbf{W} dx, \quad (5.20)$$

$$\begin{aligned} E_1(t) &= \frac{\nu}{2} \|S(\mathbf{w}^\perp)\|_{L_2(\Omega_t)}^2 - \gamma \int_{\Omega_t} \mathbf{w}^\perp \cdot (\mathbf{W}_t + (\mathbf{w} \cdot \nabla) \mathbf{W}) dx + 2\omega\gamma \int_{\Omega_t} (\mathbf{e}_3 \times \mathbf{w}^\perp) \cdot \mathbf{W} dx \\ &\quad + \frac{\nu\gamma}{2} \int_{\Omega_t} S(\mathbf{w}^\perp) \cdot S(\mathbf{W}) dx + \gamma \int_{\Omega_t} (\mathbf{w}^\perp \cdot \nabla) \mathbf{w}' \cdot \mathbf{W} dx - \gamma \mathcal{J}, \end{aligned} \quad (5.21)$$

where \mathcal{J} is the surface integral in (5.18).

We pass to the estimates of E and E_1 . At first we consider the integral \mathcal{J} . It can be written in the form

$$\mathcal{J} = - \int_{\mathcal{G}_{\theta(t)}} (M + \omega \sum_{j=1}^3 g_j(t) \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) + \frac{1}{2} |\mathbf{w}'|^2) \Big|_{x=e_{\hat{\rho}^{-1}}(z)} f_1 dS_z$$

where $f_1 = \mathbf{W} \cdot \mathbf{n}|_{x=e_{\hat{\rho}}(z)} |\widehat{\mathcal{L}}^T(z, \hat{\rho}) \mathbf{N}_\theta(z)|$. We introduce the matrix $\mathcal{S}_0 = (S_{jk}^0)_{j,k=1,2,3}$ with the elements

$$S_{jk}^0 = \int_{\mathcal{G}_{\theta(t)}} \boldsymbol{\eta}_j(x) \cdot \boldsymbol{\eta}_k(x) dx.$$

In view of (5.4), $S_{\alpha 3}^0$ and $S_{3\alpha}^0$ vanish, $S_{33}^0 = \int_{\mathcal{G}_\theta} |x'|^2 dx$ and the matrix $(S_{\alpha\beta}^0)_{\alpha,\beta=1,2}$ is positive definite. We make use of the relation

$$\begin{aligned} &M + \omega \sum_{j=1}^3 g_j(t) \boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) + \frac{1}{2} |\mathbf{w}'|^2 \Big|_{x=e_{\hat{\rho}}(z)} \\ &= -B_0(z) \hat{\rho}(z, t) + \omega \sum_{k,j=1}^3 S_0^{jk} d_k(t) \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) + M', \end{aligned} \quad (5.22)$$

where

$$\begin{aligned} B_0(z) &= b_0(z) \hat{\rho} - \kappa \int_{\mathcal{G}_\theta} \frac{\hat{\rho}(\zeta, t) dS}{|z - \zeta|}, \\ d_k(t) &= -\omega \int_{\mathcal{G}_\theta} \hat{\rho}(z, t) \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_k(z) dS, \\ M' &= \omega^2 |\mathbf{N}'(z, t)|^2 \hat{\rho}^2(z, t) + \kappa \int_0^1 (1-s) \frac{\partial^2 U_s}{\partial s^2} ds + \frac{1}{2} |\mathbf{w}'|^2 \\ &\quad + \omega \sum_{j,k=1}^3 (S^{jk}(t) I_k(t) - S_0^{jk}(t) d_k(t)) \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) \\ &\quad + \sum_{j,k=1}^3 S^{jk}(t) I_k(t) (\boldsymbol{\eta}_3(x) \cdot \boldsymbol{\eta}_j(x) - \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z)), \quad x = e_{\hat{\rho}}(z) \end{aligned} \quad (5.23)$$

is the sum of nonlinear terms with respect to $\widehat{\rho}$ in (5.22). Let

$$\begin{aligned}
B(z)\widehat{\rho}(z, t) &= B_0(z)\widehat{\rho}(z, t) - \omega S_0^{33} d_3(t) |\boldsymbol{\eta}_3(z)|^2 \\
&= B_0(z)\widehat{\rho}(z, t) + \frac{\omega^2 |z'|^2}{\int_{\mathcal{G}_\theta} |\zeta'|^2 dS} \int_{\mathcal{G}_\theta} \widehat{\rho}(\zeta, t) |\zeta'|^2 dS, \\
B_1(z)\widehat{\rho}(z, t) &= B_0(z)\widehat{\rho}(z, t) - \omega \sum_{k,j=1}^3 S_0^{jk} d_k(t) \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) \\
&= B(z)\widehat{\rho}(z, t) + \omega^2 \sum_{\alpha,\beta=1}^2 S_0^{\alpha\beta} z_\alpha z_\beta \int_{\mathcal{G}_\theta} \widehat{\rho}(\zeta, t) \zeta_\beta \zeta_\alpha dS, \\
\mathcal{B}_1 \widehat{\rho} &= P B_1 P \widehat{\rho} + \sum_{k=1}^4 \varphi_k(z) \int_{\mathcal{G}_\theta} \widehat{\rho}(\zeta, t) \varphi_k(\zeta) dS,
\end{aligned}$$

where P is the projection on the subspace of $L_2(\mathcal{G}_\theta)$ orthogonal to the functions φ_k , i.e., to $(1, z_1, z_2, z_3, h_\theta(t)(z))$ defined on \mathcal{G}_θ . The quadratic form $\int_{\mathcal{G}_\theta} \rho B \rho dS$ of the operator B coincides with the form (1.4), hence, for ρ satisfying (1.7), (1.8) we have $\int_{\mathcal{G}_\theta} B_1(\rho) \rho dS \geq c \|\rho\|_{L_2(\mathcal{G}_\theta)}^2$. It follows that $\int_{\mathcal{G}_\theta} \rho \mathcal{B}_1(\rho) dS \geq c \|\rho\|_{L_2(\mathcal{G}_\theta)}^2$ for arbitrary $\rho \in L_2(\mathcal{G}_\theta)$. The integral equation

$$\mathcal{B}_1 f = g$$

of the Fredholm type is uniquely solvable for arbitrary $g \in L_2(\mathcal{G}_\theta)$; moreover, if $g = P g$, then the equation $B_1 f = g$ holds. Finally, if $g \in W_2^{1/2}(\mathcal{G}_\theta)$, then $f \in W_2^{1/2}(\mathcal{G}_\theta)$, and

$$\|f\|_{W_2^{1/2}(\mathcal{G}_\theta)} \leq c \|g\|_{W_2^{1/2}(\mathcal{G}_\theta)}. \quad (5.24)$$

Now we define f_1 as the solution of the equation

$$\mathcal{B}_1 f_1 = P(-\Delta_\theta)^{-1} P \widehat{\rho} = P(-\Delta_\theta)^{-1} \widehat{\rho}^\perp,$$

where Δ_θ is the Laplace-Beltrami operator on \mathcal{G}_θ (in fact, the equation $B_1 f_1 = P(-\Delta_\theta)^{-1} \widehat{\rho}^\perp$ is satisfied). By virtue of (5.24) and (5.9),

$$\begin{aligned}
\|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)} &\leq c \|(-\Delta_\theta)^{-1} \widehat{\rho}^\perp\|_{W_2^{1/2}(\mathcal{G}_\theta)} \\
&\leq c \|\widehat{\rho}^\perp\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \leq c \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}.
\end{aligned}$$

The function $f_0(y) = f_1(\mathcal{Z}(\theta(t))y, t)$, $y \in \mathcal{G}_0$, is a solution of the equation

$$B_1(y) f_0 = P_0(-\Delta_0)^{-1/2} \widetilde{\rho}^\perp, \quad y \in \mathcal{G}_0,$$

where Δ_0 is the Laplace-Beltrami operator on \mathcal{G}_0 and P_0 is a projection on the subspace of functions orthogonal to $(1, y_1, y_2, y_3, h_0(y))$. Hence

$$\|\widetilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)} = \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}.$$

We set $\mathbf{W} \cdot \mathbf{n} \Big|_{x=e_{\hat{\rho}}(z)} = f_1(z) |\widehat{\mathcal{L}}^T(z, \widehat{\rho}) \mathbf{N}_\theta|^{-1}$. By the definition of the operator B_1 ,

$$\begin{aligned} -\mathcal{J}' &\equiv \int_{\mathcal{G}_\theta(t)} \left(B_0(z) \widehat{\rho}(z, t) - \omega \sum_{j=1}^3 S_0^{jk} d_k(t) \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) \right) f_1(z, t) dS \\ &= \int_{\mathcal{G}_\theta} B_1(z) \widehat{\rho} f_1(z, t) dS = \int_{\mathcal{G}_\theta} \widehat{\rho} B_1 f_1 dS = \int_{\mathcal{G}_\theta} \widehat{\rho}^\perp (-\Delta_\theta)^{-1} \widehat{\rho}^\perp dS \geq c \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}^2. \end{aligned}$$

Now we consider the contribution of the nonlinear terms (5.23) into $-\mathcal{J}$, i.e., the integral

$$-\mathcal{J}'' = \int_{\mathcal{G}_\theta} M' f_1(z, t) dS.$$

We have

$$\begin{aligned} \left| \int_{\mathcal{G}_\theta} |\mathbf{N}'(z)|^2 \widehat{\rho}^2(z, t) f_1(z, t) dS \right| &\leq \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \|\mathbf{N}'(z)|^2 \widehat{\rho} f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)} \\ &\leq c\delta \|\rho\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \|f_1\|_{W_2^{1/2}(\mathcal{G}_\theta)} \leq c\delta \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}^2, \end{aligned}$$

From the formula (2.9) in [14] it follows that (5.3) can be written in the form

$$I_k(t) = -\omega \int_0^1 ds \int_{\mathcal{G}_\theta} \widehat{\rho}(z, t) \boldsymbol{\eta}_3(e_{s\widehat{\rho}}(z)) \cdot \boldsymbol{\eta}_k(e_{s\widehat{\rho}}(z)) \Lambda(s\widehat{\rho}) dS,$$

which implies

$$\begin{aligned} I_k(t) - d_k(t) &= -\omega \int_0^1 ds \int_{\mathcal{G}_\theta} \widehat{\rho}(z, t) (\boldsymbol{\eta}_3(e_{s\widehat{\rho}}(z)) \cdot \boldsymbol{\eta}_k(e_{s\widehat{\rho}}(z)) \Lambda(s\widehat{\rho}) - \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_k(z)) dS, \\ |d_k(t)| + |I_k(t)| &\leq c \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}, \\ |I_k(t) - d_k(t)| &\leq c\delta \|\widehat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}. \end{aligned}$$

For the estimate of $S^{jk}(t) - S_0^{jk}(t)$ we use the relations $\mathcal{S}^{-1} - \mathcal{S}_0^{-1} = \mathcal{S}_0^{-1}(\mathcal{S}_0 - \mathcal{S})\mathcal{S}^{-1}$ and

$$S_{jk}(t) - S_{jk}^0 = \int_0^1 ds \int_{\mathcal{G}_\theta} \widehat{\rho}(z, t) \left(\boldsymbol{\eta}_j(e_{s\widehat{\rho}}(z)) \cdot \boldsymbol{\eta}_k(e_{s\widehat{\rho}}(z)) \Lambda(s\widehat{\rho}) - \boldsymbol{\eta}_j(z) \cdot \boldsymbol{\eta}_k(z) \right) dS.$$

It follows that

$$|S_{jk}(t) - S_{jk}^0| \leq c\delta \|\rho\|_{W_2^{-1/2}(\mathcal{G}_\theta)}. \quad (5.25)$$

From the above inequalities it is easy to conclude that

$$\left| \int_{\mathcal{G}_\theta} \left(\frac{1}{2} |\mathbf{w}'|^2 + \omega \sum_{j,k=1}^3 (S^{jk} I_k - S_0^{jk} d_k(t)) \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_j(z) \right) \right|$$

$$\begin{aligned}
& + \sum_{j,k=1}^3 S^{jk} I_k(t) (\boldsymbol{\eta}_3(e_{s\hat{\rho}}(z)) \cdot \boldsymbol{\eta}_k(e_{s\hat{\rho}}(z)) - \boldsymbol{\eta}_3(z) \cdot \boldsymbol{\eta}_k(z)) f_1(z, t) dS \Big| \\
& \leq c\delta \|\hat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}^2.
\end{aligned}$$

Finally, by Proposition 5.2,

$$\left| \int_0^1 (1-s) ds \int_{\mathcal{G}_\theta} \frac{\partial^2 U_s}{\partial s^2} f_1 dS \right| \leq c\delta \|\hat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}^2.$$

Putting all the estimates together we see that for small δ

$$-\gamma \mathcal{J} \geq c\gamma \|\hat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)}^2 = c\gamma \|\tilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)}^2.$$

We pass to the estimates of the volume integrals in (5.21). By Proposition 5.1,

$$\begin{aligned}
\left| \int_{\Omega_t} \mathbf{w}^\perp \cdot \mathbf{W}_t dx \right| & \leq c \|\mathbf{w}^\perp\|_{L_2(\Omega_t)} \left(\|f_0\|_{W_2^{1/2}(\mathcal{G}_0)} + \|f_{0t}\|_{L_2(\mathcal{G})} \right) \\
& \leq c \|\mathbf{w}^\perp\|_{L_2(\Omega_t)} \left(\|\tilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)} + \|\tilde{\rho}_t\|_{L_2(\mathcal{G})} \right),
\end{aligned}$$

and since

$$\begin{aligned}
\|\tilde{\rho}_t\|_{L_2(\mathcal{G}_0)} & \leq c \|\mathbf{w}\|_{L_2(\Gamma_t)} \leq c \|\mathbf{w}^\perp\|_{L_2(\Gamma_t)} + c \sum_{k=1}^3 |I_k(t)| \\
& \leq c \left(\|\mathbf{w}^\perp\|_{W_2(\Omega_t)} + \|\tilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)} \right),
\end{aligned}$$

we have

$$\gamma \left| \int_{\Omega_t} \mathbf{w}^\perp \cdot \mathbf{W}_t dx \right| \leq c\gamma \|\mathbf{w}^\perp\|_{W_2^1(\Omega_t)} \left(\|\mathbf{w}^\perp\|_{W_2^1(\Omega_t)} + \|\tilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)} \right).$$

In view of Proposition 5.1, other integrals in (5.21) do not exceed

$$c\gamma \|\mathbf{w}^\perp\|_{W_2^1(\Omega_t)} \|f_0\|_{W_2^{1/2}(\mathcal{G}_0)} \leq c\gamma \|\mathbf{w}^\perp\|_{W_2^1(\Omega_t)} \|\tilde{\rho}\|_{W_2^{-1/2}(\mathcal{G}_0)} = c\gamma \|\mathbf{w}^\perp\|_{W_2^1(\Omega_t)} \|\hat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)},$$

which allows us to conclude, taking the Korn inequality into account, that for small γ

$$E_1(t) \geq c \left(\nu \|\mathbf{w}^\perp\|_{W_2^1(\Omega_t)} + \gamma \|\hat{\rho}\|_{W_2^{-1/2}(\mathcal{G}_\theta)} \right).$$

As for $E(t)$, this function satisfies (also for small γ) the inequality

$$c_1 \left(\|\mathbf{w}^\perp\|_{L_2(\Omega_t)}^2 + \|\hat{\rho}\|_{L_2(\mathcal{G}_\theta)}^2 \right) \leq E(t) \leq c_2 \left(\|\mathbf{w}^\perp\|_{L_2(\Omega_t)}^2 + \|\hat{\rho}\|_{L_2(\mathcal{G}_\theta)}^2 \right).$$

that is a consequence of the estimate

$$c_3 \|\tilde{\rho}\|_{L_2(\mathcal{G}_0)}^2 = c_1 \|\hat{\rho}\|_{L_2(\mathcal{G}_\theta)}^2 \leq \mathcal{R} - \mathcal{R}_0 \leq c_4 \|\hat{\rho}\|_{L_2(\mathcal{G}_\theta)}^2 = c_2 \|\tilde{\rho}\|_{L_2(\mathcal{G}_0)}^2$$

(see the remark at the end of Sec.1). When we integrate (5.19), we arrive at (3.25). Thus Theorem 3.4 is proved.

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