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ERGODICITY OF MULTIPLICATIVE STATISTICS

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ABSTRACT

For a subfamily of multiplicative measures on integer partitions we give conditions for properly rescaled associated Young diagrams to converge in probability to a certain deterministic curve named the limit shape of partitions. We provide explicit formulas for the scaling function and the limit shape covering some known and some new examples.

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Introduction

It is now widely known that a random partition of a large integer taken with equal probability among all partitions of that integer has the Young diagram which looks (after rescaling) close to a deterministic object called a limit shape of random partition. The discovery of a phenomenon of limit shape formation for the random partition of a large integer has a long history. First it was mentioned in the paper by H. N. V. Temperley [15] in 1952 with heuristic arguments. Much later but independently, the principal calculations leading to this result was made by M. Szalay and P. Turán [14] in 1977, however they did not state their result in the modern way. It was done by A. Vershik and stated in his joint paper with S. Kerov [16] in 1985.

Later a new proof of the same result was found based on the fact that the uniform measures on partitions of n are just the product measures restricted to some linear subspace. (The probabilist would say that a random partition of n is just a sequence of independent random variables with specific distributions conditioned to have some weighted sum equal n.) This technique seems to be first applied to random partitions by B. Fristedt [7]; now it is usually referred to as Fristedt's conditional device. Later it has been frequently used by various authors in the related problems, see [11, 6, 8], to name just a few references. A. Vershik [17] noted that the similar technique is applicable to a wider range of problems with the same property that the measure is a product measure restricted to a certain affine subset. Vershik called such measures on partitions *multiplicative*; we give the precise definition in Section 1.

Similar "limit shape type" results have appeared in diverse contexts including some probability measures on partitions, both multiplicative and not. One of the first results of this type goes back to a seminal paper by P. Erdős and J. Lehner [10]: it can be read from their paper that rescaled Young diagrams of strict partitions of a large integer concentrate around a certain limit shape. This is also a multilplicative case, although Erdős and Lehner did not use the related technique. A. Comtet et al. [5] recently found the limit shape for the generalization of this case, namely for partitions such that the difference between parts exceed some fixed number p. For $p \geq 2$ this family is not multiplicative. R. Cerf and R. Kenyon [4] confirmed Vershik's conjecture that the limit shape exists also for plane partitions.

In the paper [17] Vershik introduced several families of multiplicative measures on partitions and stated that for these measures limit shapes also appear in the proper scaling. He called such examples *ergodic* (see Definition 2 below) and asked a general question about conditions for ergodicity of multiplicative measures. In this note we give a partial answer to this question. The whole family of multiplicative measures is naturally parameterized by a sequence of functions f_k analytic in some neighborhood of zero, as explained in Section 1. We restrict ourself to a subfamily of multiplicative measures such that each member of this sequence is a power of some fixed function: $f_k(z) = f(z)^{b_k}$, $b_k \geq 0$, with some additional constraints

on f and sequence b_k. This restriction looks quite confining at a first glance but it allows us to find exact formulas for the limit shape and to catch what happens in a more general case. This family includes, for instance, the measures on partitions which arise in connection with Meinardus' Theorem ($f(z) = 1/(1-z)$), see [1, Ch. 6]. Granovsky et al. [9] applied recently a probabilistic technique going back to Khinchin to this problem and improved Meinardus' results; our approach partly overlaps with one used in [9].

The lack of natural nonergodic multiplicative examples makes it harder to answer Vershik's question. Actually, essentially the only well studied example is the so-called Ewens measure on partitions. We briefly describe its construction, see [2] (where the term Ewens sampling formula is used) for a more detailed exposition. Take a random permutation π from the symmetric group S_n with probability proportional to $\theta^{l(\pi)}$ where $\theta > 0$ is a parameter and $l(\pi)$ is the number of cycles in permutation π; $\theta = 1$ corresponds to the uniform measure on S_n. Given π, consider the partition of n on cycle lengths of π. The induced probability measure on partitions is called Ewens measure. Mapping partition $\lambda = (\lambda_1, \lambda_2, \dots)$ with the usual order $\lambda_1 \geq \lambda_2 \geq \dots$ to a simplex $\nabla = \{(x_1, x_2, \dots) : \sum x_i \leq 1 \text{ and } x_1 \geq x_2 \geq \dots\}$ by dividing parts of λ by its weight $\sum \lambda_i$ induces the sequence of discrete measures on ∇; taking their weak limit in the standard topology leads to the Poisson–Dirichlet measure $\mathcal{PD}(\theta)$ on ∇. This measure is not concentrated on the unique element of ∇, so the Ewens measure is not ergodic.

In this note we give another examples of nonergodic behavior (Proposition 2) in a slightly weaker sense. However all these examples are degenerate.

The rest of the paper is organized as follows. In the next section we give a precise definition of multiplicative measures on partitions and present some basic facts about them. In particular, we introduce the notions of grand and small canonical ensembles of partitions. In the end of the section we formulate further assumptions we impose on the multiplicative measure. In Section 2 we present a definition of ergodicity and discuss its basic consequence. Section 3 is devoted to ergodicity in the grand canonical ensemble. We give necessary and sufficient conditions for ergodicity in the considered class of multiplicative measures (Theorem 6) and provide an explicit formula for the limit shape in the ergodic case. In Section 4 we give sufficient conditions for ergodicity in the small canonical ensemble. We conclude the paper with examples in Section 5.

1 Multiplicative families

The multiplicative families of measures on partitions were defined in [17] in the following way. For each $n = 1, 2, \dots$ let $\mu^{(n)}$ be a probability measure defined on a set $\mathcal{P}(n)$ of integer partitions of n. For partition $\lambda \in \mathcal{P}(n)$ define *completion numbers* or *counts* $R_k(\lambda) = \#\{i : \lambda_i = k\}$, $k = 1, 2, \dots$, that is the number of parts k in partition λ. Measure $\mu^{(n)}$ makes R_k random variables: $\mathbb{P}[R_k = j] = \mu^{(n)}\{\lambda \in \mathcal{P}(n) : R_k(\lambda) = j\}$. These random variables are obviously dependent since the

relation

$$N(\lambda) := \sum_{k=1}^{\infty} kR_k(\lambda) = n$$

holds for all $\lambda \in \mathcal{P}(n)$, i.e. $\mu^{(n)}$-almost sure. We also introduce a set $\mathcal{P}(0) = \{\varnothing\}$ and the trivial probability measure $\mu^{(0)}$ on it.

Definition 1. The family of probability measures on partitions $\mu^{(n)}$ is called *multiplicative* if there exists a sequence of positive numbers $\{\bar{a}_n\}_{n\geq 0}$ such that $\sum_n \bar{a}_n = 1$ and counts R_k are mutually independent with respect to the convex combination $\bar{\mu} := \sum_n \bar{a}_n \mu^{(n)}$.

Independence of R_k with respect to $\bar{\mu}$ means that there exists a rectangular array of nonnegative numbers $\bar{g}_{k,j}$, $k \geq 1$, $j \geq 0$ and $\sum_{j=0}^{\infty} \bar{g}_{k,j} = 1$ for any k, such that for any partition $\lambda \in \mathcal{P} = \cup_{n=0}^{\infty} \mathcal{P}(n)$

$$\bar{\mu}\{\lambda\} = \prod_{k=1}^{\infty} \bar{g}_{k,R_k(\lambda)} \,. \tag{1}$$

Introduce normalized coefficients $g_{k,j} = \bar{g}_{k,j}/\bar{g}_{k,0}$ (division by $\bar{g}_{k,0}$ is possible since $\prod_k \bar{g}_{k,0} = \bar{a}_0 > 0$ by definition) and consider functions $f_k(x) = \sum_{j=0}^{\infty} g_{k,j} x^j$; these are analytic functions at least in the unit disk. Let us define a family of measures μ_x on the set of all integer partitions $\mathcal{P}$ by

$$\mu_x\{\lambda\} = \prod_{k=1}^{\infty} \frac{g_{k,R_k(\lambda)} x^{kR_k(\lambda)}}{f_k(x^k)} = \frac{x^{N(\lambda)}}{F(x)} \prod_{k=1}^{\infty} g_{k,R_k(\lambda)}$$

where

$$F(x) = \prod_{k=1}^{\infty} f_k(x^k) \,. \tag{2}$$

Inequalities $1 \leq f_k(x^k) \leq 1/\bar{g}_{k,0}$ valid for $0 \leq x \leq 1$ ensure that the products above converge at least for these x. Moreover, summation over all $\lambda \in \mathcal{P}$ yields that μ_x are probability measures:

$$\mu_x \mathcal{P} = \sum_{\lambda \in \mathcal{P}} \mu_x\{\lambda\} = \sum_{(r_1, r_2, \dots)} \prod_{k=1}^{\infty} \frac{g_{k,r_k} x^{kr_k}}{f_k(x^k)} = \prod_{k=1}^{\infty} \sum_{r_k=0}^{\infty} \frac{g_{k,r_k} x^{kr_k}}{f_k(x^k)} = 1$$

where the middle sum is taken over all sequences $(r_1, r_2, \dots)$ of nonnegative integers with finitely many nonzero terms (such sequences are in one-to-one correspondence with partitions via $r_k = R_k(\lambda)$).

Equation (1) and definition of $\bar{\mu}$ imply that for any n if $\lambda \in \mathcal{P}(n)$ then

$$\mu_x\{\lambda\} = \frac{x^n}{\bar{a}_0 F(x)} \bar{\mu}\{\lambda\} = \frac{\bar{a}_n x^n}{\bar{a}_0 F(x)} \mu^{(n)}\{\lambda\} \,.$$

Thus measures μ_x are convex combinations of $\mu^{(n)}$. Introducing $a_n = \bar{a}_n/\bar{a}_0$ makes it possible to write down expression for measures of partition $\lambda \in \mathcal{P}(n)$ as

$$\mu^{(n)}\{\lambda\} = \frac{F(x)}{a_n x^n}\mu_x\{\lambda\} = \frac{1}{a_n}\prod_{k=1}^{\infty} g_{k,R_k(\lambda)}$$

and the Taylor decomposition of $F(\cdot)$ as

$$F(x) = \sum_{n=0}^{\infty} a_n x^n \,.$$

Since $a_n > 0$, this is the analytic function in the unit disk, however the actual radius of convergence ρ can be greater (and even infinite).

Function F plays a rôle of normalization factor, so a man with background in statistical mechanics would call it a *partition function*. We utilize this terminology and extend the analogy with statistical mechanics by using terms *grand canonical ensemble* of partitions for the set $\mathcal{P}$ equipped with measure μ_x and *small canonical ensemble* for the pair $(\mathcal{P}(n), \mu^{(n)})$. Further discussion of these analogy and terms can be found in [12, 17].

Partition function F is also closely related to the probabilistic notion of a probability generating function. Moments of N can be expressed in terms of F as

$$\mathbb{E}_x N^m = \frac{1}{F(x)}\left(x\frac{d}{dx}\right)^m F(x)\,, \qquad m = 0, 1, 2, \ldots, \tag{3}$$

where $\mathbb{E}_x$ is an expectation operator with respect to measure μ_x. Similar formula expresses moments of R_k in terms of f_k:

$$\mathbb{E}_x R_k^m = \frac{1}{f_k(z)}\left(z\frac{d}{dz}\right)^m f_k(z)\bigg|_{z=x^k}\,.$$

Since $N = \sum kR_k$, mean and variance of N can be also easily expressed in terms of functions f_k and get a particularly simple form in terms of its logarithmic derivative $h_k(z) = f_k'(z)/f_k(z)$:

$$\mathbb{E}_x N = \sum_{k=1}^{\infty}\frac{kx^k f_k'(x^k)}{f_k(x^k)} = \sum_{k=1}^{\infty} kx^k h_k(x^k)\,, \tag{4}$$

$$\begin{aligned}\mathrm{Var}_x\, N &= \sum_{k=1}^{\infty}\frac{k^2\left(x^k\left(f_k'(x^k) + x^k f_k''(x^k)\right)f_k(x^k) - x^{2k}\left(f_k'(x^k)\right)^2\right)}{\left(f_k(x^k)\right)^2} \\ &= \sum_{k=1}^{\infty} k^2\big(x^k h_k(x^k) + x^{2k}h_k'(x^k)\big)\,. \end{aligned} \tag{5}$$

Note that F itself does not define measures μ_x or $\mu^{(n)}$ but F along with its decomposition (2) does. However this decomposition is not unique. Indeed, given

$F(\cdot)$ we could have taken $\bar{a}_n = a_n x_0^n / F(x_0)$ in Definition 1, for some $x_0 \in (0, \rho)$, and constructed a new function $\hat{F}(\cdot)$ in the similar way. However it would satisfy $\hat{F}(x) = F(xx_0)$ and $\hat{f}_k(z) = \hat{f}_k(zx_0^k)$, as can be easily checked. Up to this change of variable F and its decomposition is uniquely defined.

The case considered in this note

Although multiplicativity is a rather restrictive requirement on measures $\mu^{(n)}$ the range of multiplicative measures is quite big. In this note we consider only measures $\mu^{(n)}$ such that after some appropriate change of variables described in the previous paragraph $f_k(z) = f(z)^{b_k}$ for some function $f(\cdot)$ and sequence of nonnegative numbers $\{b_k\}$, i. e.

$$F(x) = \prod_{k=1}^{\infty} f(x^k)^{b_k} \,. \tag{6}$$

The natural requirement on the Taylor coefficients of $f(z)^{b_k}$ to be positive may imply certain restrictions on b_k. We impose more conditions on the sequence $\{b_k\}$, namely we assume that partial sums

$$B_k = \sum_{j=1}^{k} b_k = k^{\beta} \ell(k), \qquad \beta > 0, \tag{7}$$

where $\ell(\cdot)$ is a regularly varying function in the sense of Karamata, i. e. it is measurable and for each fixed $y \in (0, \infty)$ there exists $\lim_{x\to\infty} \ell(xy)/\ell(x) = 1$, see [3]. Note that the requirement $a_1 > 0$ implies that $b_1 > 0$, otherwise the Taylor series for $F(x)$ would have no x term.

For certain statements below these assumptions on behavior of b_k are not enough and additional conditions are required. In order to formulate the first of them we introduce for a positive real s the set K_s of integers behaving similar to the arithmetic progression with the difference s. More formally, define

$$K_s = \{k \in \mathbb{Z}_+ : \exists j \text{ such that } |k - sj| < 1/2\} \,. \tag{8}$$

Obviously, for $s \le 1$ these sets coincide with $\mathbb{Z}_+$ but for $s > 1$ holes in K_s occur. In some statements we shall need the following regularity assumption in addition to (7): there exists $\chi \in (0, 1)$ such that for any $s \ge 2$

$$\sum_{\substack{j \le k \\ j \in K_s}} b_j \le \chi B_k \,. \tag{9}$$

If $\beta > 2$ assumption (7) is strong enough for all our purposes. However for $0 < \beta \le 2$ we need more detailed asymptotics of partial sums B_k:

$$B_k = \theta k^{\beta} + O\left(k^{\beta - \zeta}\right), \qquad k \to \infty \tag{10}$$

for some constants $\beta, \theta > 0$ and $\zeta > 1 - \beta/2$.

We also suppose that for some $\rho_1 \in (0, \infty]$, $f(x)$ is finite for $x \in (0, \rho_1)$ and has a nonremovable singularity at $x = \rho_1$; the nonnegativity of the Taylor coefficients implies that $f(x^k)$ is an analytic function in a disk of radius $\rho_k = \rho_1^{1/k}$ ($\rho_k = \infty$ if $\rho_1 = \infty$). If the singularity happens at $\rho_1 \le 1$ we shall often require that it is a pole. If ρ_1 is finite the change of variables $x \mapsto \rho_1 x$ can make the radius of convergence of all function $f_k(x^k)$ equal 1 however functions $f_k(\cdot)$ won't be equal after it. The following simple statement holds.

Proposition 1. *Let F be defined by* (6) *and condition* (7) *holds. If $\rho_1 < 1$ then F is analytic in the disk $|z| < \rho_1$ and has a singularity at $\rho = \rho_1$. If $\rho_1 \ge 1$ (in particular $\rho_1 = \infty$) then $F(\cdot)$ is analytic in the unit disk and has a singularity at $\rho = 1$.*

Proof. If $x < \rho_1 < 1$ then for all $0 < y < x$ inequality $f(y) \le 1 + (f(x) - 1)y$ holds since function f is convex. Consequently the product (2) evaluated at point y is dominated by the converging product $\prod_{k=1}^{\infty}\big(1 + (f(x) - 1)y^k\big)^{b_k}$. On the other hand, $F(x) \to \infty$ as $x \to \rho_1$ since so does the first factor in (2).

If $x < 1 \le \rho_1$ the same argument shows the convergence of the infinite product evaluated at x, but $F(1) = \prod_k f(1)^{b_k} = \infty$ since $B_k \to \infty$ by (7). □

2 Ergodicity

Given a partition λ of n we consider its Young diagram which can be defined as a subgraph of function

$$\varphi_\lambda(t) = \sum_{k>t} R_k(\lambda), \qquad t \ge 0.$$

For a sequence of positive numbers $\alpha^{(n)}$ we consider its scaled version

$$\widetilde{\varphi}_\lambda^{(n)}(t) = \frac{\alpha^{(n)}}{n}\varphi_\lambda(t\alpha^{(n)}) = \frac{\alpha^{(n)}}{n}\sum_{k>t\alpha^{(n)}} R_k(\lambda)\,.$$

Taking $\lambda \in \mathcal{P}(n)$ at random with probability $\mu^{(n)}\{\lambda\}$ makes these random functions.

Definition 2. We call a family of measures $\mu^{(n)}$ *ergodic* if there exists a sequence $\alpha^{(n)}$ and a piecewise continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\int_0^\infty \varphi(t)dt = 1$ and for any finite collection $0 < t_1 < \cdots < t_\ell$ of its continuity points values $\widetilde{\varphi}_\lambda^{(n)}(t_j)$, $j = 1, \ldots, \ell$, converge to $\varphi(t_j)$ in probability, that is for any $\varepsilon > 0$

$$\lim_{n\to\infty} \mu^{(n)}\big\{\lambda\colon \big|\widetilde{\varphi}_\lambda^{(n)}(t_j) - \varphi(t_j)\big| < \varepsilon \text{ for all } j = 1, \ldots, \ell\big\} = 1\,.$$

The function φ is called a *limit shape* of partitions.

Remarks. 1. If a sequence $\alpha^{(n)}$ exists it is essentially unique meaning that for $\alpha_1^{(n)}$ and $\alpha_2^{(n)}$ two such sequences $\alpha_1^{(n)}/\alpha_2^{(n)} \to c \in (0,\infty)$ and function φ is appropriately transformed.

2. If a function φ exists it is nonincreasing since all $\widetilde{\varphi}_\lambda^{(n)}$ do not increase.

The notion of ergodicity can be also defined in the grand canonical ensemble. However it should be done in slightly different way to keep the main advantage of measures μ_x that $\varphi_\lambda(t)$ is a sum of independent random variables. Given a positive function α_x defined for $x \in (0,\rho)$ define the scaled Young diagram as

$$\widetilde{\varphi}_{x;\lambda}(t) = \frac{\alpha_x}{\mathbb{E}_x N}\,\varphi_\lambda(\alpha_x t) = \frac{\alpha_x}{\mathbb{E}_x N}\sum_{k>\alpha_x t} R_k(\lambda)\,.$$

Scaling here depends on x so $\int_0^\infty \widetilde{\varphi}_{x;\lambda}(t)dt = 1$ does *not* hold for all λ however the mean value of this integral is 1. A family of measures μ_x is called ergodic if there exist a scaling function α_x and a limit shape φ, $\int_0^\infty \varphi(t)dt = 1$, such that for any $\varepsilon > 0$ and $(t_1,\dots,t_\ell)$ a set of continuity points of φ

$$\lim_{x\nearrow\rho} \mu_x\big\{\lambda\colon \big|\widetilde{\varphi}_{x;\lambda}(t_j) - \varphi(t_j)\big| < \varepsilon \text{ for all } j = 1,\dots,\ell\big\} = 1\,.$$

Ergodicity of $\mu^{(n)}$ and μ_x are closely related however not equivalent. The subject of this note is to establish conditions for ergodicity of the first family but we shall investigate properties of the second one as well. We start with a simple criterion for the case when measures μ_x can not be ergodic.

Proposition 2. *If function F has a pole in point ρ then measures μ_x are not ergodic.*

Proof. Suppose that F has a pole of order $m \geq 1$ at ρ. Then in some neighborhood of ρ it can be decomposed into the Laurent series

$$F(x) = \sum_{j=-m}^{\infty} c_j (x-\rho)^j\,, \qquad\qquad c_{-m} \neq 0.$$

Using formula (3) we see that as $x \nearrow \rho$

$$\mathbb{E}_x N = \frac{-xmc_{-m}(x-\rho)^{-m-1} + \dots}{c_{-m}(x-\rho)^{-m} + \dots} \sim \frac{m\rho}{\rho - x},$$
$$\mathbb{E}_x N^2 = \frac{m(m+1)xc_{-m}(x-\rho)^{-m-2} + \dots}{c_{-m}(x-\rho)^{-m} + \dots} \sim \frac{m(m+1)\rho^2}{(\rho-x)^2}$$

where dots denote lower order terms. Hence the variance of $N/\mathbb{E}_x N$ is bounded away from zero. Consequently there exists $\tau > 0$ and $c > 0$ such that $\mu_x\{\lambda : |N(\lambda)/\mathbb{E}_x N - 1| > \tau\} > c$ for all x close to ρ. Moreover, since the mean of $N/\mathbb{E}_x N$ is one, the one-sided inequality should also take place with positive probability: $\mu_x\{\lambda : N(\lambda)/\mathbb{E}_x N < 1 - \tau\} > c$.

Suppose that measures μ_x are ergodic with scaling a_x and limit shape φ. Recall that φ is a weakly decreasing piecewise continuous function with unit integral. Take the φ-continuity point $\varepsilon > 0$ small enough so that

$$\int_0^\varepsilon (\varphi(t) - \varphi(\varepsilon))dt + \int_0^\infty \min\{\varepsilon, \varphi(t)\}dt < \tau/3.$$

Geometrically it means that the area of the limit shape φ lying lower than ε or higher than $\varphi(\varepsilon)$ is less than $\tau/3$. Let $T = \inf\{t : \varphi(t) < \varepsilon\}$. Given ε and $\delta \in (0, \varepsilon)$ define a finite collection of points recursively by the following procedure: take $t_0 = \varepsilon$ and let $t_i = \inf\{t : \varphi(t) < \varphi(t_{i-1} + 0) - \delta\}$ until on some step $t_d \geq T$. (Here $\varphi(t + 0)$ is the right limit at t.) The procedure ends in final number of steps since $\varphi(t_i + 0) \leq \varphi(t_{i-1}) - \delta$. Define now a function $\varphi^* : [0, T] \to \mathbb{R}$ to be equal $\varphi(\varepsilon) - 2\delta$ on $[0, t_1]$ and for all $i = 2, \dots d$ let $\varphi^*(t) = \varphi(t_{i-1} + 0) - 2\delta$ on $(t_{i-1}, t_i]$. Thus φ^* is a piecewise constant function with discontinuities at $\{t_i\}$ and by construction it satisfies $\varphi^*(t) \leq \varphi(t) - \delta$ for all $t \in [\varepsilon, T]$. Consequently ergodicity implies that $\mu_x\{\lambda : \widetilde{\varphi}_{x;\lambda}(t) > \varphi^*(t), t \in [\varepsilon, T]\} \to 1$ as $x \to \rho$. For all such λ

$$N(\lambda) = \sum_{k=1}^{\infty} kR_k(\lambda) \geq \sum_{k=\lfloor \varepsilon\alpha_x \rfloor}^{\lfloor T\alpha_x \rfloor} kR_k(\lambda) \geq \mathbb{E}_x N \int_0^T (\varphi^*(t) - \varepsilon)dt.$$

Taking $\delta > 0$ and $\varepsilon > 0$ small enough the last integral can be made greater than $1 - 2\tau/3$, thus providing the contradiction which finishes the proof. □

Remark. Note that the proof does not use the specific form (6) of decomposition (2) and thus the result holds for any multiplicative measure.

3 Ergodicity in the grand canonical ensemble

Independence of R_k in grand canonical ensemble allows establishing sufficient conditions for ergodicity in the grand canonical ensemble. We start with finding asymptotics of mean value of N with respect to measure μ_x.

Lemma 3. *Let measure μ_x be defined by decomposition* (6) *and b_k satisfy* (7)*. If $\rho_1 < 1$ then $\mathbb{E}_x N \sim b_1\rho_1 f'(x)/f(x)$ as $x \nearrow \rho_1$. If $\rho_1 \geq 1$ (if $\rho_1 = 1$ in addition f has a pole in* 1*) then as $x \nearrow 1$*

$$\mathbb{E}_x N \sim \Omega \frac{\ell(1/(1-x))}{(1-x)^{\beta+1}}, \quad \Omega = \int_0^1 \Big(|\log u|^{\beta+1}(h(u) + uh'(u)) - |\log u|^\beta h(u) \Big) du \tag{11}$$

where $h(u) = f'(u)/f(u)$ is the logarithmic derivative of f, and ℓ, β are defined in (7)*.*

Proof. If F satisfies (6) then $h_k(u) = b_k h(u)$ in (4) hence it can be rewritten as

$$\mathbb{E}_x N = \sum_{k=1}^{\infty} kb_k x^k h(x^k).$$

If $\rho_1 < 1$ then the only summand above which goes to infinity as $x \to \rho_1$ is the first one, and the sum of all remaining summands is dominated by the convergent series $h(\rho_1^2)\sum_{k=2}^{\infty} kb_k\rho_1^k$, so the statement holds.

Suppose $\rho_1 \geq 1$. Taking partial sum and using summation by parts yields

$$\sum_{k=1}^{m} kb_k x^k h(x^k) = (m+1)B_m x^{m+1} h(x^{m+1}) + \sum_{k=1}^{m} B_k \left(kx^k h(x^k) - (k+1)x^{k+1}h(x^{k+1}) \right) .$$

The first summand vanishes as $m \to \infty$ for fixed x, so taking m large enough it can be hold bounded. Since h is an analytic function in some disk including points x^{k+1} and x^k the mean value theorem allows to conclude that

$$\begin{aligned} kx^k h(x^k) &- (k+1)x^{k+1}h(x^{k+1}) \\ &= kx^k h(x^k) - (k+1)x^{k+1}\left(h(x^k) + h'(x^{\varkappa})(x^{k+1} - x^k) \right) \\ &= -x^{k+1}h(x^k) + k(1-x)x^k h(x^k) + (1-x)(k+1)x^{2k+1}h'(x^{\varkappa}) \end{aligned}$$

where $\varkappa \in [k, k+1]$. Combining the above formulas we obtain

$$\sum_{k=1}^{m} kb_k x^k h(x^k) = (m+1)B_m x^{m+1} h(x^{m+1}) - \sum_{k=1}^{m} B_k x^{k+1} h(x^k) + (1-x)\sum_{k=1}^{m} B_k \left(kh(x^k)x^k + (k+1)h'(x^{\varkappa})x^{2k+1} \right) .$$

Take $T > 0$ and put $m = [T/(1-x)]$. Since $B_{[t/(1-x)]}/B_{[1/(1-x)]} \to t^{\beta}$ uniformly in $t \in (0, T]$ (see [3, Th. 1.5.2]), the sums above should be multiplied by $(1-x)^{1+\beta}/\ell(1/(1-x))$ to become the Riemann sums for the convergent integrals $\int_0^T t^{\beta} h(e^{-t})e^{-t}dt$ and $\int_0^T t^{\beta+1}\left(h(e^{-t})e^{-t} + h'(e^{-t})e^{-2t} \right) dt$, correspondingly.[1] Letting $T \to \infty$ and changing variable $u = e^{-t}$ finish the proof. □

Proposition 2 shows that ergodicity in the grand canonical ensemble can not take place for the case $\rho_1 < 1$ if f has a pole in this point. Indeed, since $\sum_{k\geq 2} \mathbb{E}_x R_k$ is bounded as $x \nearrow \rho$ while $\mathbb{E}_x N \sim \mathbb{E}_x R_1$ is not one can take scaling $\alpha_x \equiv 1$ to get a limit of scaled Young diagrams $\varphi_x(t) = 1_{[0,1)}(t)R$ with R a (nondegenerate) limit of $R_1/\mathbb{E}_x R_1$. Thus in this case "almost all" partitions consist mostly of ones, and all larger parts constitute a vanishing fraction of a sum. Further questions can be asked about the distribution of larger parts etc. however they are beyond the scope of this note.

[1] The integrals are convergent even if $\rho_1 = 1$: since f has a pole at 1, $h(e^{-t})$ has a simple pole and $h'(e^{-t})$ has a pole of order 2 at 0, so multiplication by t^{β} and $t^{\beta+1}$ kills both singularities. To be completely rigorous, one should take integral from $1/T$ to T in order to speak about the Riemann sum, and then use uniformness in T to exchange limits.

Suppose $\rho_1 \geq 1$. The change of variables $k \leftrightarrow t/(1-x)$ made implicitly in the proof of Lemma 3 suggests the choice of the scaling function $\alpha_x = 1/(1-x)$. The following lemma states that the mean is not degenerate with this scaling.

Lemma 4. *Suppose that $\rho_1 \geq 1$ and for $\rho_1 = 1$ additionally the singularity of f in 1 is a pole. For the scaling function $\alpha_x = 1/(1-x)$ the mean value of scaled Young diagram at point $t > 0$ is*

$$\varphi(t) := \lim_{x \nearrow 1} \mathbb{E}_x \widetilde{\varphi}_x(t) = \frac{1}{\Omega} \left(\int_0^{e^{-t}} \big(h(u) + uh'(u)\big) \, |\log u|^\beta du - t^\beta h(e^{-t}) e^{-t} \right) \quad (12)$$

where $h(u) = f'(u)/f(u)$ is the logarithmic derivative of f, Ω is defined in (11) and ℓ, β are defined in (7). If $\rho_1 = 1$ and $\beta \in (0,1]$ then $\varphi(0) = \infty$, otherwise it is finite and the convergence takes place also for $t = 0$.

Proof. The proof is similar to that of Lemma 3 so we present only a sketch. The partial sum is

$$\sum_{k=m_1}^{m_2} \mathbb{E}_x R_k = B_{m_2} x^{m_2+1} h(x^{m_2+1}) - B_{m_1-1} x^{m_1} h(x^{m_1}) + \sum_{k=m_1}^{m_2} B_k \left(x^k h(x^k) - x^{k+1} h(x^{k+1}) \right) .$$

Expression in brackets under the summation sign can be represented as

$$x^k h(x^k) - x^{k+1} h(x^{k+1}) = (1-x)(h(x^k)x^k + h'(x^k)x^{2k+1}) + \tfrac{1}{2}(1-x)^2 h''(x^\varkappa) x^{3k+1}$$

for $\varkappa \in [k, k+1]$, so taking $m_1 = t/(1-x)$, $m_2 = T/(1-x)$ and representing sums by integrals yields

$$\sum_{k=m_1}^{m_2} \mathbb{E}_x R_k \sim \frac{\ell(1/(1-x))}{(1-x)^\beta} \Big(T^\beta e^{-T} h(e^{-T}) - t^\beta e^{-t} h(e^{-t}) + \int_t^T v^\beta \left(h(e^{-v}) + h'(e^{-v}) e^{-v} \right) e^{-v} dv \Big)$$

for $t > 0$. If $\beta > 1$ then the integral converges also for $t = 0$. To finish the proof it remains ot change variable, divide by the asymptotic expression (11) for $\mathbb{E}_x N$ and let $T \to \infty$. □

The function φ defined by (12) is a candidate for the limit shape. To show that the definition of ergodicity really holds we give a bound for probability of deviation at a fixed point.

Lemma 5. *Suppose that $\rho_1 \geq 1$, f has a pole in 1 if $\rho_1 = 1$, and fix $t, \varepsilon > 0$. Then, for x close to 1*

$$\mu_x \big\{ \lambda \colon \big| \widetilde{\varphi}_{x;\lambda}(t) - \varphi(t) \big| > \varepsilon \big\} \leq e^{-(1-x)^{-\beta/2}} .$$

Proof. For fixed $t > 0$ and $k > t/(1-x)$ there exist exponential moments $\mathbb{E}_x e^{uR_k} = f(x^k e^u)^{b_k}/f(x^k)^{b_k}$ at least for $u \in [0, t + \log \rho_1)$. Moreover, convexity of f implies that exponential moments of $\sum_{k>t/(1-x)} R_k$ also exist for such u: argument follows the lines of the proof of Proposition 1. Hence for fixed $\varepsilon > 0$, $u \in (0, t + \log \rho_1)$ and x close enough to 1

$$\mu_x\big\{\lambda\colon \widetilde{\varphi}_{x;\lambda}(t) - \varphi(t) \geq \varepsilon\big\} = \mu_x\big\{\lambda\colon \ : e^{u\widetilde{\varphi}_{x;\lambda}(t)} \geq e^{u(\varphi(t)+\varepsilon)}\big\} \leq \frac{\mathbb{E}_x e^{u\widetilde{\varphi}_x(t)}}{e^{u(\varphi(t)+\varepsilon)}}$$

$$= \frac{\mathbb{E}_x e^{u(\widetilde{\varphi}_x(t) - \mathbb{E}_x \widetilde{\varphi}_x(t))}}{e^{u(\varphi(t) - \mathbb{E}_x \widetilde{\varphi}_x(t) + \varepsilon)}} = e^{-u\varepsilon/2} \prod_{k>t/(1-x)} \mathbb{E}_x \exp\left(\frac{u(1-x)^\beta(1+o(1))}{\Omega\ell(1/(1-x))}(R_k - \mathbb{E}_x R_k)\right)$$

$$\leq e^{-u\varepsilon/2} \prod_{k>t/(1-x)} \mathbb{E}_x \exp\left(\frac{2u(1-x)^\beta}{\Omega\ell(1/(1-x))}(R_k - \mathbb{E}_x R_k)\right) \quad (13)$$

where we have used Markov's inequality, Lemmas 3 and 4 and independence of R_k. Denote for short $\delta = \delta(x) = \frac{2(1-x)^\beta}{\Omega\ell(1/(1-x))}$, note that $\delta(x) \to 0$ as $x \nearrow 1$. Each factor in the right-hand side of (13) is defined at least for $u < (t + \log \rho_1)\delta^{-1}$, and the product converges. Since

$$\mathbb{E}_x \exp\left(\frac{2u(1-x)^\beta}{\Omega\ell(1/(1-x))}(R_k - \mathbb{E}_x R_k)\right) = \exp b_k \left(\log \frac{f(x^k e^{u\delta(x)})}{f(x^k)} - u\delta(x)\frac{x^k f'(x^k)}{f(x^k)}\right)$$

the logarithm of k'th factor in (13) divided by b_k can be bounded as follows:

$$\log \frac{f(x^k e^{u\delta})}{f(x^k)} - u\delta\frac{x^k f'(x^k)}{f(x^k)} \leq \log\left(1 + \frac{f(x^k e^{u\delta}) - f(x^k)}{f(x^k)}\right) - u\delta\frac{x^k f'(x^k)}{f(x^k)}$$

$$\leq \frac{f(x^k e^{u\delta}) - f(x^k)}{f(x^k)} - u\delta\frac{x^k f'(x^k)}{f(x^k)}$$

$$\leq \frac{x^k(e^{u\delta} - 1)f'(x^k)}{f(x^k)} - u\delta\frac{x^k f'(x^k)}{f(x^k)} \leq (u\delta)^2\frac{x^k f'(x^k)}{f(x^k)}$$

for $u \leq 1/\delta$ since $f'(x)$ is nondecreasing function and $e^v - v - 1 \leq v^2$ for $v \in [0, 1]$. Hence continuing (13) we obtain

$$\mu_x\big\{\lambda\colon \widetilde{\varphi}_{x;\lambda}(t) - \varphi(t) \geq \varepsilon\big\} \leq \exp\left((u\delta)^2\left(\sum_{k>t/(1-x)} b_k\frac{x^k f'(x^k)}{f(x^k)}\right) - u\varepsilon/2\right).$$

We have already found the asymptotics of the sum above in the proof of Lemma 4: it is asymptotically equivalent to $\frac{\ell(1/(1-x))}{(1-x)^\beta}\Omega\varphi(t)$. Hence the upper bound takes form $\exp(c_1\delta(x)u^2 - u\varepsilon/2)$ for some $c_1 > 0$. Now we can choose u such that it provides the best estimate. It is achieved at $u = \varepsilon/(4c_1\delta(x))$ which gives an upper bound $\exp(-\varepsilon^2/(16c_1\delta(x)))$ and at least for small ε all the calculations above are valid. An observation that $\exp(-\varepsilon^2/(16c_1\delta(x))) < \exp(-(1-x)^{\beta/2})$ for x close enough to 1 finishes the proof. □

We combine the results about ergodicity in the grand canonical ensemble in the next statement.

Theorem 6. *Measures μ_x defined by* (6) *with b_k satisfying* (7) *are ergodic if either the radius of convergence ρ_1 of function f is greater than 1 or if it is equal to 1 and f has a pole at 1. The possible choice of scaling function in this case is $\alpha_x = 1/(1-x)$ which leads to the limit shape φ defined by* (12).

If $\rho_1 < 1$ and f has a pole at ρ_1 then ergodicity does not hold.

Proof. Lemma 5 gives the exponential upper bound for the probability of deviation greater than $\varepsilon > 0$ of $\widetilde{\varphi}_{x;\lambda}(t)$ from $\varphi(t)$. Consequently the probability of deviation greater than ε in finite number of points still decays exponentially as $x \to 1$.

The last statement follows from Proposition 2. □

Remark. If $\rho_1 < 1$ and f has an essential singularity at ρ_1 then ergodic case is possible: take, say, $f(x) = e^{1/(1-2x)}$ and $b_k = 1$. We do not know examples of nonergodic behavior in this case under conditions (6) and (7).

4 Ergodicity in the small canonical ensemble

In order to approximate measures $\mu^{(n)}$ by measures μ_x we want to choose x depending on n to maximize $\mu_x\mathcal{P}(n) = a_n x^n/F(x)$. Differentiation with respect to x shows that it is achieved at $x = x_n$, a solution of equation

$$n = \mathbb{E}_{x_n} N = \frac{x_n F'(x_n)}{F(x_n)}\,. \tag{14}$$

Note that this solution always exists and is unique since $\mathbb{E}_x N$ strictly increase in x. Lemma 3 and [3, Prop. 1.5.15] shows that there exists a slowly varying function ℓ_1 such that

$$\tau_n := 1 - x_n = \frac{\ell_1(n)}{n^{1/(\beta+1)}}\,. \tag{15}$$

In the most simple case when $\ell(k) \equiv 1$ this simplifies to $1-x_n \sim \Omega^{1/(\beta+1)} n^{-1/(\beta+1)}$, in the general case ℓ_1 is connected to the de Bruijn conjugate of ℓ, see [3].

The next lemma gives a lower bound for $\mu_{x_n}\mathcal{P}(n)$.

Lemma 7. *Let measures μ_x be defined by decomposition* (6) *where either the radius of convergence $\rho_1 > 1$ or $\rho_1 = 1$ and additionally f has a pole at* 1. *Let the sequence b_k satisfy both conditions* (7) *and* (9) *and additionally either $\beta > 2$ or $0 < \beta \le 2$ and* (10) *holds for some $\zeta > 1 - \beta/2$. Then for n large enough*

$$\mu_{x_n}\mathcal{P}(n) \ge n^{-\gamma} \qquad \textit{for any } \gamma > \frac{\beta+2}{2\beta+2}\,. \tag{16}$$

Proof. We start with the Cauchy formula where we take the circle of radius x_n as the integration path:

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(x_n e^{\mathrm{i}t}) x_n^{-n} e^{-\mathrm{i}nt} dt\,.$$

Using product representation (6) and expressing n in terms of f combining (14) and (4) gives

$$\begin{aligned}\mu_{x_n}\mathcal{P}(n) &= \frac{a_n x_n^n}{F(x_n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\mathrm{i}nt}\prod_{k=1}^{\infty}\frac{f(x_n^k e^{\mathrm{i}kt})^{b_k}}{f(x_n^k)^{b_k}}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\exp\left(\sum_{k=1}^{\infty} b_k\left(\log\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)} - \mathrm{i}kt\frac{x_n^k f'(x_n^k)}{f(x_n^k)}\right)\right)dt\,.\end{aligned} \tag{17}$$

The function under the integral sign in (17) sends negative values of t to complex conjugates. Thus taking the real part does not change the value of the integral and changing the integration interval to $[0,\pi]$ halves its value. So we can write

$$\mu_{x_n}\mathcal{P}(n) = I_1 + I_2 + I_3 + I_4 \geq I_1 - |I_2| - |I_3| - |I_4|$$

where for some $0 = \delta_0(n) \leq \delta_1(n) \leq \delta_2(n) \leq \delta_3(n) \leq \delta_4(n) = \pi$ we denote

$$I_j = \frac{1}{\pi}\int_{\delta_{j-1}(n)}^{\delta_j(n)} \mathrm{Re}\exp\left(\sum_{k=1}^{\infty} b_k\left(\log\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)} - \mathrm{i}kt\frac{x_n^k f'(x_n^k)}{f(x_n^k)}\right)\right)dt\,, \quad j = 1,2,3,4. \tag{18}$$

Before specifying the exact values of $\delta_j(n)$, $j = 1,2,3$, we investigate asymptotics of $\mathrm{Var}_x N$ as $x \to 1$. Recalling formula (5) yields that

$$\mathrm{Var}_x N = \sum_{k=1}^{\infty} b_k k^2 (x^k h(x^k) + x^{2k} h'(x^k)) \sim \frac{\ell(1/(1-x))}{(1-x)^{\beta+2}}\sigma^2 \tag{19}$$

where

$$\sigma^2 = \int_0^1 \Big(2|\log u|^{\beta+1}\big(h(u) + uh'(u)\big) - |\log u|^{\beta+2}\big(h(u) + 3uh'(u) + u^2 h''(u)\big)\Big)\,du.$$

The argument repeats that of proof of Lemma 3; note that the integral is convergent even if $\rho_1 = 1$ because in this case $h'(u)$ has a pole of order 2 and $h''(u)$ has a pole of order 3 at $u = 1$ and both $\int^1 |\log u|^{\beta+1} h'(u)du$ and $\int^1 |\log u|^{\beta+2} h''(u)du$ converge. Consequently

$$\mathrm{Var}_{x_n} N = \ell_2(n) n^{-\frac{\beta+2}{\beta+1}} \tag{20}$$

where $\ell_2(n) = \sigma^2 \ell\big(n^{1/(\beta+1)}/\ell_1(n)\big)/\ell_1(n)^{\beta+2}$ is slowly varying.

We define $\delta_i(n)$ in terms of τ_n (defined by (15)). Namely we take

$$\delta_1(n) = \tau_n^{1+\beta/2-\alpha_1}, \qquad \delta_2(n) = \tau_n, \qquad \delta_3(n) = \tau_n^{\alpha_3}$$

where the obvious inequalities which provide the right order for δ_i are $0 < \alpha_1 \le \beta/2$ and $0 < \alpha_3 \le 1$.

First we show that α_1 can be chosen so that for any $\gamma > \frac{\beta+2}{2\beta+2}$ and large n inequality $I_1 \ge n^{-\gamma}$ holds. Since $0 < t < \delta_1(n) \searrow 0$ as $n \to \infty$, in order to estimate the sum in the exponent in equation (18) for $j = 1$ we are going to find $k_0 = k_0(n)$ such that for $k \le k_0$ each summand can be estimated using the Taylor formula and the sum over $k > k_0$ is small. To this end we take $k_0(n) = \lfloor \tau_n^{-1-\varepsilon_1} \rfloor$ for some $\varepsilon_1 > 0$ which exact value will be specified later. Then for all $k \ge k_0$

$$x_n^k = (1-\tau_n)^k = (1-\tau_n)^{\tau_n^{-1} k \tau_n} \le e^{-k\tau_n} \le e^{-\tau_n^{-\varepsilon_1}} \to 0$$

in view of inequality $(1-y)^{1/y} \le e^{-1}$ valid for $y \in (0,1]$. Consequently for all such k and for all t inequality $|f(x_n^k e^{\mathrm{i}kt}) - 1| < c_1 x_n^k$ holds uniformly in $k \ge k_0$ and t. Here and below c_j, $j = 1, 2, \dots$, are some positive constants. It follows that the absolute value of the k-th summand in (17) is bounded by $c_2 b_k x_n^k$, hence the whole sum over $k \ge k_0$ is bounded by $c_3 x_n^{k_0}/(1-x_n)^{\beta+1} \le e^{-n^{\varepsilon_1/2}} \to 0$.

On the other hand, if $0 \le t \le \delta_1(n)$ and $1 \le k \le k_0(n)$ then $tk_0 \le \tau_n^{\beta/2-\alpha_1-\varepsilon_1}$ vanish if $\alpha_1 + \varepsilon_1 < \beta/2$ so each summand can be estimated by its Taylor expansion. Utilizing again notation $h(z) = f'(z)/f(z)$ for the logarithmic derivative of f gives

$$\begin{aligned}\log \frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)} &- \mathrm{i}kt \frac{x_n^k f'(x_n^k)}{f(x_n^k)} \\ &= -(x_n^k h(x_n^k) + x_n^{2k} h'(x_n^k)) \frac{k^2 t^2}{2} - \frac{\mathrm{i}}{2} \int_0^t h_1(x_n^k e^{\mathrm{i}ks}) k^3 (t-s)^2 ds\end{aligned}$$

where $h_1(z) = zh(z) + 3z^2 h'(z) + z^3 h''(z)$. Multiplying $(x_n^k h(x_n^k) + x_n^{2k} h'(x_n^k))k^2$ by b_k and summing over all k one gets $\mathrm{Var}_{x_n} N$ as can be seen from (5); cutting off the series at k_0 gives exponentially small error as it was shown in the previous paragraph. Hence

$$-\sum_{k=1}^{k_0} b_k (x_n^k h(x_n^k) + x_n^{2k} h'(x_n^k)) \frac{k^2 t^2}{2} \sim -\frac{t^2}{2} \mathrm{Var}_{x_n} N$$

as $n \to \infty$.

The error term can be bounded as follows. Since

$$\left| \sum_{k=1}^{\infty} \frac{\mathrm{i} b_k}{2} \int_0^t h_1(x_n^k e^{\mathrm{i}ks}) k^3 (t-s)^3 ds \right| \le \int_0^t \sum_{k=1}^{\infty} k^3 b_k |h_1(x_n^k e^{\mathrm{i}ks})| (t-s)^2 ds$$

and for large n the sum in the right-hand side is uniformly bounded by

$$\sum_{k=1}^{\infty} k^3 b_k |h_1(x_n^k e^{\mathrm{i}ks})| \le c_4 \ell(\tau_n^{-1}) \tau_n^{-\beta-3} \le \tau_n^{-\beta-3+\varepsilon_2}$$

for any $\varepsilon_2 > 0$ (this bound is valid even if $\rho_1 = 1$ because $h_1(x)$ has a pole of order 3 and the corresponding integral converges), the inequality

$$\left|\sum_{k=1}^{\infty} \frac{\mathrm{i}b_k}{2} \int_0^t h_1(x_n^k e^{\mathrm{i}ks}) k^3 (t-s)^3 ds\right| \le \frac{t^3}{3} \tau_n^{-\beta-3-\varepsilon_2}$$
$$\le \frac{t^2}{2} \delta_1(n) \tau_n^{-\beta-3-\varepsilon_2} = \frac{t^2}{2} \tau_n^{-\beta/2-2-\varepsilon_2-\alpha_1}$$

holds for all $t \in [0, \delta_1(n)]$. Consequently, for any $\varepsilon_2 \in (0, 4\beta/(9\beta+9))$ and $\gamma > \frac{\beta+2}{2\beta+2}$

$$\begin{aligned}
I_1 &= \frac{1}{\pi} \int_0^{\delta_1(n)} \operatorname{Re} \exp\left(\sum_{k=1}^{\infty} b_k \left(\log \frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)} - \mathrm{i}kt \frac{x_n^k f'(x_n^k)}{f(x_n^k)}\right)\right) dt \\
&= \frac{1}{\pi} \int_0^{\delta_1(n)} \operatorname{Re} \exp\left(-\frac{t^2 \operatorname{Var}_{x_n} N}{2} \left(1 + O\left(\frac{\tau_n^{-\beta/2-2-\varepsilon_2-\alpha_1}}{\operatorname{Var}_{x_n} N}\right)\right)\right) dt \\
&= \frac{1}{\pi \sqrt{\operatorname{Var}_{x_n} N}} \int_0^{\delta_1(n)\sqrt{\operatorname{Var}_{x_n} N}} \operatorname{Re} \exp\left(-\frac{s^2}{2}\left(1 + o\left(\tau_n^{\beta/2-2\varepsilon_2-\alpha_1}\right)\right)\right) dt \\
&\sim \frac{1}{n^{(\beta+2)/(2\beta+2)} \sqrt{2\pi \ell_2(n)}} \\
&\ge n^{-\gamma}
\end{aligned}$$

for large enough n if $4\varepsilon_2 + 2\alpha_1 < \beta$.

The bounds for I_2, I_3 and I_4 are based on the same estimate:

$$\begin{aligned}
|I_j| &\le \frac{1}{\pi} \int_{\delta_{j-1}(n)}^{\delta_j(n)} \left|\exp \sum_{k=1}^{\infty} b_k \log \frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)} - \mathrm{i}kt \frac{x_n^k f'(x_n^k)}{f(x_n^k)}\right| dt \\
&\le \frac{1}{\pi} \int_{\delta_{j-1}(n)}^{\delta_j(n)} \exp\left(\sum_{k=1}^{\infty} b_k \operatorname{Re} \log \frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)}\right) dt \qquad (21) \\
&= \frac{1}{\pi} \int_{\delta_{j-1}(n)}^{\delta_j(n)} \exp\left(\sum_{k=1}^{\infty} b_k \log \left|\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)}\right|\right) dt\,.
\end{aligned}$$

Since the Taylor coefficients of f are nonnegative, each summand in the exponent is nonpositive. Moreover, since the first Taylor coefficient $g_1 > 0$ and thus for all $y \in (0,1)$ and real t

$$\begin{aligned}
-\log\left|\frac{f(ye^{\mathrm{i}s})}{f(y)}\right| &= -\log\left(1 - \frac{f(y) - |f(ye^{\mathrm{i}s})|}{f(y)}\right) \\
&\ge \frac{f(y) - |f(ye^{\mathrm{i}s})|}{f(y)} = \frac{f(y)^2 - |f(ye^{\mathrm{i}s})|^2}{f(y)(f(y) + |f(ye^{\mathrm{i}s})|)} \ge \frac{g_1 y(1-\cos s)}{f(y)^2} \qquad (22)
\end{aligned}$$

because

$$f(y)^2 - |f(ye^{\mathrm{i}s})|^2 = 2\sum_{j>k\geq 0} g_j g_k y^{j+k}(1-\cos((j-k)s)) \geq 2g_0 g_1 y(1-\cos s)\,.$$

If $\rho_1 > 1$ then $f(y)$ is bounded in the left neighborhood of 1; if $\rho_1 = 1$ and $f(y)$ has a pole of order m then $f(y) \leq c_6(1-y)^{-m}$. Taking $m = 0$ if $\rho_1 > 1$ implies that for some $\tilde{x} \in (0,1)$ and all $y \in [\tilde{x}, 1)$

$$\frac{g_1 y}{f(y)^2} \geq c_7(1-y)^{2m}. \tag{23}$$

On the other hand, for small y there exists a constant $c_8 > 0$ such that

$$\frac{g_1 y}{f(y)^2} \geq c_8 y \tag{24}$$

It can be shown that this inequality can be extended to the set $y \in (0, \tilde{x}]$ with the same $\tilde{x}$ as above (but with smaller c_8, possibly), however we do not rely on this fact.

In order to find constraints on $|I_2|$, $|I_3|$ and $|I_4|$ it suffices to take just some summands in the sum in the right-hand part of (21) (and bound all other summands by zero). The right choice differs for these integrals, and we start with the simpler case of $|I_2|$.

Inequality $1 - \cos y \geq 2y^2/\pi^2$ holds for $y \in [-\pi, \pi]$. Since $t \leq \delta_2(n)$ is small, for large n a lot of values kt get into this interval and it suffices to sum only over these k. Namely, we take $k \leq k_1(n) = \lfloor \eta \tau_n^{-1} \rfloor$ where $\eta = \min\{|\log \tilde{x}|/(2\log 2), \pi\}$. For these k and n large enough, on the one hand

$$x_n^k = e^{k\log(1-\tau_n)} \geq e^{-2k\tau_n \log 2} \geq e^{-|\log \tilde{x}|} = \tilde{x}$$

since $\log(1-\tau) \geq -2\tau \log 2$ for $\tau \in [0, 1/2]$, and on the other hand $kt \leq \pi$ for $t \leq \delta_2(n)$. Hence the bound (23) applies and taking $y = x_n^k$ and $s = kt$ in (22) yields

$$-\log\left|\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)}\right| \geq c_7(1-x_n^k)^{2m}\,\frac{2(kt)^2}{\pi^2}\,.$$

It is easier to utilize the Stieltjes integrals instead of summation by parts in this case and we introduce $B(u) = B_{\lfloor u \rfloor}$ for this purpose. Integration by parts gives

$$\begin{aligned}\sum_{k=1}^{k_1} b_k(1-x_n^k)^{2m}k^2 &= \int_{\frac{1}{2}}^{k_1+\frac{1}{2}} (1-x_n^u)^{2m}\,u^2 dB(u)\\ &= (1-x_n^u)^{2m}\left(u^2B(u) - 2B_1(u)\right)\Big|_{u=\frac{1}{2}}^{k_1+\frac{1}{2}}\\ &\quad - 2m|\log x_n| \int_{\frac{1}{2}}^{k_1+\frac{1}{2}} \left(u^2B(u) - 2B_1(u)\right) x_n^u\,(1-x_n^u)^{2m-1}\,du\end{aligned} \tag{25}$$

where

$$B_1(u) = \int_{\frac{1}{2}}^{u} vB(v)\,dv \sim \frac{\ell(u)u^{\beta+2}}{\beta+2}\,, \qquad\qquad u \to \infty\,.$$

The first summand

$$(1-x_n^u)^{2m}\left(u^2B(u)-2B_1(u)\right)\Big|_{u=\frac{1}{2}}^{k_1+\frac{1}{2}} \sim \frac{\beta}{\beta+2}\left(1-e^{-\eta}\right)^{2m}\ell(k_1)k^{\beta+2}$$

and for $m=0$ (25) gives the desired asymptotics. For positive m more work is needed to show that the difference in the right-hand part of (25) does not vanish and lower the growth rate. To this end we use the following observation: for any $\varepsilon>0$ and for large enough k_1 inequality $u^2B(u)-2B_1(u) < (1+\varepsilon)\beta\ell(k_1)k_1^{2+\beta}/(2+\beta)$ holds for all $u \le k_1$ hence

$$\begin{aligned}
&2m|\log x_n| \int_{\frac{1}{2}}^{k_1+\frac{1}{2}} \left(u^2B(u)-2B_1(u)\right) x_n^u\,(1-x_n^u)^{2m-1}\,du \\
&\quad = 2m|\log x_n| \left(\int_{\frac{1}{2}}^{\frac{k_1+1}{2}} + \int_{\frac{k_1+1}{2}}^{k_1+\frac{1}{2}}\right)\left(u^2B(u)-2B_1(u)\right) x_n^u\,(1-x_n^u)^{2m-1}\,du \\
&\quad \le (1+\varepsilon)2m|\log x_n|\ell(k_1) \\
&\qquad \times \left(\frac{(k_1+1)^{\beta+2}}{2^{\beta+2}}\int_{\frac{1}{2}}^{\frac{k_1+1}{2}} + (k_1+\tfrac{1}{2})^{\beta+2}\int_{\frac{k_1+1}{2}}^{k_1+\frac{1}{2}}\right) x_n^u\,(1-x_n^u)^{2m-1}\,du \\
&\quad \le (1-\varepsilon_1)\frac{\beta}{\beta+2}\left(1-\max\{\tilde{x}, e^{-\pi}\}\right)^{2m}\ell(k_1)k^{\beta+2}
\end{aligned}$$

for the suitable choice of $\varepsilon_1>0$. Consequently for $t \le \delta_2(n)$

$$-\sum_{k=1}^{\infty} b_k \log\left|\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)}\right| \ge c_9 t^2 \ell(\tau_n^{-1})\tau_n^{-\beta-2}\,.$$

It follows that for all $t \in [\delta_1(n), \delta_2(n)]$ the exponential bound from below on $|I_2|$ holds.

Let us proceed now with I_4. Suppose that $\delta_3(n) \le t \le \pi$. For all $k \ge k_2 = \lfloor \tau_n^{-1}|\log\tilde{x}|+1\rfloor$ the inequality $x_n^k < \tilde{x}$ holds and hence bound (24) applies. Thus for any $\varepsilon=\varepsilon(n)>0$

$$-\sum_{k=1}^{\infty} b_k \log\left|\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)}\right| \ge c_{10}\varepsilon \sum_{\substack{k=k_2\\ 1-\cos kt>\varepsilon}}^{\infty} b_k x_n^k.$$

The strategy is to add also summands for which $1-\cos kt \le \varepsilon$ and to choose ε small enough so that condition (9) guarantees that additional summands do not change the asymptotics too much. The sum over all k starting from k_2 can be bounded from below by $c_{11}\tau_n^{-\beta}$. Take $\varepsilon=\delta_3(n)^2/4$, if $1-\cos kt \le \varepsilon$ then there exists $j \in \mathbb{Z}$

such that $kt - 2\pi j \in [-\arccos(1-\varepsilon), \arccos(1-\varepsilon)]$ and consequently $k \in K_{2\pi/t}$ (recall definition (8)). Now apply the assumption (9) to see that

$$c_{10}\varepsilon \sum_{\substack{k=k_2 \\ 1-\cos kt>\varepsilon}}^{\infty} b_k x_n^k \geq c_{12}(1-\chi)\varepsilon\tau_n^{-\beta}\,.$$

If $\alpha_3 < \beta/2$ the right-hand side of the above inequality grows to ∞ providing a proper bound for $|I_4|$.

If $\beta > 2$ we can take $\alpha_3 = 1$ and still have an exponential bound for $|I_4|$. But this choice of α_3 implies $\delta_2(n) = \delta_3(n)$ and $I_3 = 0$. Hence the theorem is proved for the case $\beta > 2$.

If $0 < \beta \leq 2$ we still have to estimate I_3 and we can manage to do it under additional assumption (10) on ℓ, that is $\ell(k) = \theta + O(k^{-\zeta})$, $\zeta > 1 - \beta/2$. Choose α_3 such that $1 - \zeta < \alpha_3 < \beta/2$ and suppose $t \in [\delta_2(n), \delta_3(n)]$. In order to estimate $|I_3|$ we consider a sum over k for which $1 - \cos kt$ is large enough but x_n^k is still not too small.

To be more precise, let us introduce intervals $\mathfrak{I}_j = [\pi(6j+1)/(3t), \pi(6j+5)/(3t)]$; $k \in \mathfrak{I}_j$ implies $1 - \cos kt > 1/2$ for any j. If $j > j_0 = \lfloor |\log \tilde{x}|t/(2\pi\tau_n)\rfloor + 1$ then for any $k \in \mathfrak{I}_j$ one has $x_n^k < \tilde{x}$ and thus inequality (24) applies with $y = x_n^k$. Take $j_1 = 2j_0$. Then

$$\begin{aligned} -\sum_{k=1}^{\infty} b_k \log\left|\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)}\right| &\geq \sum_{j=j_0}^{j_1} \sum_{k\in\mathfrak{I}_j} 2c_8 x_n^k (1-\cos kt) \\ &\geq c_8 \sum_{j=j_0}^{j_1} \left(B_{m_1(j)} - B_{m_0(j)}\right) x^{m_1(j)} \end{aligned}$$

where $m_0(j) = \lfloor \pi(6j+1)/(3t)\rfloor$ and $m_1(j) = \lfloor \pi(6j+5)/(3t)\rfloor$. Detailed asymptotics (10) for B_k implies that

$$\begin{aligned} B_{m_1(j)} - B_{m_0(j)} &\geq \theta\left(\lfloor \pi(6j+5)/(3t)\rfloor^{\beta} - \lfloor \pi(6j+1)/(3t)\rfloor^{\beta}\right) - c_{13}\frac{j^{\beta-\zeta}}{t^{\beta-\zeta}} \\ &\geq c_{14}\frac{j^{\beta-1}}{t^{\beta}} - c_{13}\frac{j^{\beta-\zeta}}{t^{\beta-\zeta}} = c_{14}\frac{j^{\beta-1}}{t^{\beta}}\left(1 - \frac{c_{13}}{c_{14}} j^{1-\zeta} t^{\zeta}\right). \end{aligned}$$

Since $j < j_1 \leq |\log \tilde{x}|t/(\pi\tau_n) + 2$ and $t < \tau_n^{\alpha_3}$ the expression in brackets above is bounded from below by $1 - c_{15}\tau_n^{\alpha_3+\zeta-1}$ and tends to 1 as $n \to \infty$ by the choice of α_3. At the same time $x_n^{m_1(j)} \geq \tilde{x}^4$ for $j < j_1$. Thus $B_{m_1(j)} - B_{m_0(j)} \geq c_{16} j^{\beta-1} t^{-\beta}$ and

$$-\sum_{k=1}^{\infty} b_k \log\left|\frac{f(x_n^k e^{\mathrm{i}kt})}{f(x_n^k)}\right| \geq c_{17} j_0^{\beta} t^{-\beta} \geq c_{18}\tau_n^{-\beta}$$

grows to infinity. This observation finishes the proof. □

Remark. A little more effort allows to prove the local limit theorem for N in the settings of Lemma 7. It remains to find the exact asymptotics for I_1 instead of providing a lower bound. However we do not need it.

Theorem 8. *Let measures $\mu^{(n)}$ induce measures in the grand canonical ensemble such that the decomposition of F in product can be written in form* (6) *with b_k satisfying* (7)*. Suppose also that for some $\gamma > 0$ inequality* (16) *holds (in particular, if assumptions of Lemma 7 hold).*

In these settings if $\rho_1 \geq 1$ then measures $\mu^{(n)}$ are ergodic with the scaling function $\alpha^{(n)} = 1/(1 - x_n)$ where x_n is the solution of equation (14)*. This choice of scaling function leads to the limit shape φ defined by* (12)*.*

Conjecture. *If $\rho < 1$ then the possible choice of the scaling function could be $\alpha^{(n)} \equiv 1$ and it leads to the degenerate limit shape $\varphi(t) = \mathbf{1}(t \in [0, 1])$.*

Proof. Lemma 5 gives the exponential bound for $\mu_x\big\{\lambda : \big|\widetilde{\varphi}_{x;\lambda}(t) - \varphi(t)\big| > \varepsilon\big\}$. Evaluation at $x = x_n$ taking (15) into account gives

$$\mu_{x_n}\big\{\lambda\colon \big|\widetilde{\varphi}_{x_n;\lambda}(t) - \varphi(t)\big| > \varepsilon\big\} \leq e^{-n^{-\beta/(3(\beta+1))}}\,. \tag{26}$$

Let $\alpha^{(n)} = \alpha_{x_n}$. Then for $\lambda \in \mathcal{P}(n)$ $\widetilde{\varphi}^{(n)}_\lambda(t) \equiv \widetilde{\varphi}_{x_n;\lambda}(t)$ and

$$\begin{aligned}\mu^{(n)}\left\{\lambda : |\widetilde{\varphi}^{(n)}_\lambda(t) - \varphi(t)| > \varepsilon\right\} &= \frac{\mu_{x_n}\left(\left\{\lambda : |\widetilde{\varphi}_{x_n;\lambda}(t) - \varphi(t)| > \varepsilon\right\} \cap \mathcal{P}(n)\right)}{\mu_{x_n}\mathcal{P}(n)} \\ &\leq \frac{\mu_{x_n}\left\{\lambda : |\widetilde{\varphi}_{x_n;\lambda}(t) - \varphi(t)| > \varepsilon\right\}}{\mu_{x_n}\mathcal{P}(n)}\,.\end{aligned}$$

Inequalities (26) and (16) imply that this probability tends to 0 as $n \to \infty$. Probability of deviations greater than ε in finite number of points is bounded by the number of points times the maximal probability of deviation greater than ε and also tends to 0, proving the ergodicity. □

5 Examples

In this section we introduce three examples of families of multiplicative measures. They are obtained from the well-known measures by distinct deformations. These deformations can be combined to produce another examples, and also a different measure can be taken as a starting point.

Weighted partitions

Let us consider the measures $\mu^{(n)}$ which are proportional to some constant $y > 0$ to the power of the number of summands in partition. It corresponds to the following decomposition (2) of F:

$$F(x) = \prod_{k=1}^{\infty} \frac{1}{1 - yx^k}\,.$$

Similar measures were considered in [19]. If $y \le 1$ the convergence radius $\rho_1 \ge 1$ and the limit shape is defined by

$$\varphi(t) = \frac{-\log(1 - ye^{-t})}{\operatorname{Li}_2 y}$$

with scaling $\alpha^{(n)} = \sqrt{n/\operatorname{Li}_2 y}$ where the dilogarithm $\operatorname{Li}_2 y$ is the normalizing factor. Taking $y = 1$ makes all weights equal and leads to the uniform measures on partitions. In this case it is more natural to take symmetric scaling $\alpha^{(n)} = \sqrt{n}$ which leads to the celebrated limit shape for the uniform measure on partitions defined by

$$e^{-c\varphi(t)} + e^{-ct} = 1, \qquad c = \frac{\pi}{\sqrt{6}},$$

mentioned in the Introduction.

If $y > 1$ there is no limit shape in the grand canonical ensemble of partitions: the distribution of N is asymptotically equivalent to that of R_1, so taking scaling $\alpha_x = 1$ leads to the scaled Young diagram close to the rectangle of unit width and random (asymptotically exponentially distributed) height. In the small canonical ensemble, however, there is a degenerate ergodicity, as can be easily shown combinatorially. With the same scaling $\alpha^{(n)} = 1$ the scaled Young diagram looks like the unit square, i.e. "almost all" parts in "almost all" partitions are ones, and larger parts do not comprise notable ratio to the weight, in the asymptotic sense.

Partitions with restricted part sizes

Another possibility is to take $b_k = \mathbf{1}(k \in \mathcal{S})$ for a certain set $\mathcal{S}$ of positive integers. This choice of b_k makes $\mu^{(n)}$ the uniform measure on partitions of n with all parts from $\mathcal{S}$.

The distribution of the number of parts in such partitions has been studied recently in [8] under some assumptions on growth of B_k. Namely its is shown that if $B_k - ck^\beta$, $\beta \in (0, 1)$, satisfies some additional condition then the number of parts in a random partition of n behaves like a nondegenerate random variable (explicitly specified in [8]) multiplied by $n^{1/(1+\beta)}$. Theorem 6 shows that if just summands greater than $tn^{1/(\beta+1)}$ are counted then their number is much less: it is proportional to $n^{\beta/(1+\beta)}$ and the coefficient converges in probability to a constant (depending on t). It means that in this case a generic partition has plenty of small summands which do not contribute a notable part to the whole sum. This is related to a physical effect known as Bose–Einstein condensation, see [18].

Permutations with marked cycles

As it was mentioned in the Introduction, the uniform measure on permutations induces a multiplicative measure on partition by considering partition on cycle

lengths. It is defined by decomposition

$$F(x) = \frac{1}{1-x} = \prod_{k=1}^{\infty} e^{x^k/k} = \prod_{k=1}^{\infty} \left(e^{x^k}\right)^{1/k}$$

and hence satisfies (6) with $b_k = 1/k$ but not (7) since $\beta = 0$. Taking different b_k in a form $b_k = c_k/k$ with integer c_k corresponds to marking cycles of length k in one of c_k ways. In particular, taking $c_k = k$ can be interpreted as choosing the first element in each cycle, or, in the other words, making a set of ordered lists from a permutation. The numbers of such objects is sequence A000262 in [13]. If one does not insist on a combinatorial interpretation, it is possible to take real c_k, say, $c_k = \theta k^\beta$ for $\beta, \theta > 0$. It leads to the fulfillment of the condition (7) (and even (10)) with $\ell(k) = \theta/\beta + O(k^{-1})$. Under this assumptions taking the scaling function $\alpha^{(n)} = (\theta\Gamma(\beta+1))^{-1/(\beta+1)} n^{1/(\beta+1)}$ leads to the limit shape

$$\varphi(t) = \frac{\Gamma(\beta+1, t) - te^{-t}}{\beta\Gamma(\beta+1)}$$

where $\Gamma(\beta, t) = \int_t^\infty u^{\beta-1} e^{-u} du$ is the incomplete Gamma function.

If $\beta = \theta = 1$ (i.e. the measure is induced by the uniform measure on partitions of the set $\{1, \ldots, n\}$ into ordered lists) the limit shape is the exponent function ($\varphi(t) = e^{-t}$) in the scaling $\alpha^{(n)} = \sqrt{n}$. Farther, formally letting $\beta \searrow 0$ and keeping θ fixed we approach the Poisson–Dirichlet distribution $\mathcal{PD}(\theta)$. However the limit shape becomes degenerate (infinity at 0 and zero at $t > 0$). It reflects the nonergodicity of the limiting distribution.

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