ПРЕПРИНТЫ ПОМИ РАН

Учредитель: Санкт-Петербургское отделение Математического института им. В. А. Стеклова Российской академии наук

Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16 октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых коммуникаций

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

телефоны:(812)312-40-58; (812) 571-57-54

e-mail: admin@pdmi.ras.ru

http://www. pdmi.ras.ru /preprint/

Заведующая информационно-издательским сектором Симонова В.Н

PREPRINT POMI-20/2008

EXACTLY SOLVABLE MODEL OF DIPOLAR BOSE CONDENSATE

N.I. Abarenkova, N.M. Bogoliubov

St. Petersburg Department of Steklov Institute of Mathematics,
27, Fontanka, St. Petersburg 191023, Russia

November 2008

ABSTRACT

The model that describes the internal degrees of freedom of the spinor Bose-Einstein condensate with dipole-dipole interaction is solved up to its eigenstates and eigenvalues. The representation of the Hamiltonian of the model in terms of generators of $su(1,1)$ algebra allowed to develop the quantum inverse method for its investigation. The method of solution provides a general framework within which many related problems can similarly be solved.

ПРЕПРИНТЫ
Санкт-Петербургского отделения
Математического института им. В. А. Стеклова
Российской академии наук

PREPRINTS
of the St.Petersburg Department of Steklov Institute of Mathematics

---

1 Introduction

The theory of atomic Bose-Einstein condensates with internal degrees of freedom is an actively studied problem since the spin degree of freedom becomes accessible in an optical traps. The novel dynamical effects such as fragmentation, spin mixing and entanglement that can exhibit such systems have initiated great interest in the physic beyond the mean-field approximation. Because of the complexity of such systems the simplified–"toy" models obtained under certain approximations of the original many-body problem play an important role [1], [2], [3], [4], [5], [6], [7], [8].

In our paper we study the "toy" model of the spinor condensate with the long-range dipole-dipole interactions proposed in [5]. The representation of the dynamical variables of the model as the generators of $su(1,1)$ algebra allows to imbed the obtained Hamiltonian into the well established scheme of the quantum inverse method [9] and to solve the model up to its eigenstates and eigenvalues.

The advantage of the discussed approach is that one can simultaneously solve a number of models with the different atom-atom interactions using proper bosonic realizations of the $su(1,1)$ algebra [10]. The described method allows, as well, to establish connection of the considered models with the ones of the quantum optics [11].

This paper is organized as follows. In section 2 the system of bosonic atoms with the hyperfine spin $F = 1$ possessing the dipole interactions in addition to the contact interactions is considered and the Hamiltonian written in terms of the angular momentum operators is obtained. In section 3 different bosonic realizations of the $su(1,1)$ algebra are used to express the Hamiltonian through the generators of algebra. The quantum inverse scattering approach is applied to the solution of the model in sections 4 and 5. In section 7 the model is considered in the absence of the dipolar interaction.

2 Dipolar spinor condensate

Bosonic atoms with hyperfine spin $F = 1$ and mass M are described by the three component vector field $\psi_\alpha(\vec{x})$ ($\alpha = -1, 0, 1$) with the components subject to the spin manifold. These fields satisfy the bosonic commutation relations $[\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{x}')] = \delta_{\alpha\beta}\delta(\vec{x}-\vec{x}')$. The most general form of the Hamiltonian which describes the dynamics of a dilute gas of trapped bosonic atoms at a very low temperatures including dipolar interaction may be written as [5]:

$$H = H_{spd} + H_{dip}, \tag{1}$$

where the Hamiltonian H_{sp} describes the interaction between atoms with the δ-function interaction in the external spin-independent trap potential $V_{ext}(\vec{x})$

$$\begin{aligned} H_{spd} &= \sum_{\alpha} \int d^3x \psi^{\dagger}_{\alpha}(\vec{x}) \left(-\frac{\hbar^2}{2M} \nabla^2 + V_{ext}(\vec{x}) \right) \psi_{\alpha}(\vec{x}) \\ + \int d^3x &: \left\{ \frac{c_0}{2} \hat{N}^2(\vec{x}) + \frac{c_2}{2} \mathbf{S}(\vec{x}) \cdot \mathbf{S}(\vec{x}) \right\} :, \end{aligned} \tag{2}$$

and the Hamiltonian for the dipole-dipole interactions H_{dip} reads

$$\begin{aligned} H_{dip} &= \frac{c_d}{2} \int d^3x_1 \int d^3x_2 \frac{1}{|\vec{x}_1 - \vec{x}_2|^3} \\ \times &: \{ \mathbf{S}(\vec{x}_1) \cdot \mathbf{S}(\vec{x}_2) - 3S^z(\vec{x}_1) S^z(\vec{x}_2) \} : . \end{aligned} \tag{3}$$

It is supposed that the symmetry axis of the condensate is chosen to be along the quantization axis, z, and c_d is the coefficient of the dipolar interaction. The symbol :: in these expressions denotes a normal ordering which places the annihilation operators to the right of creation ones.

The spin-exchange term in (2) with the interaction parameter c_2 is equal to

$$\mathbf{S}(\vec{x}) \cdot \mathbf{S}(\vec{x}) = S^z(\vec{x}) S^z(\vec{x}) + \frac{1}{2} S^+(\vec{x}) S^-(\vec{x}) + \frac{1}{2} S^-(\vec{x}) S^+(\vec{x}), \tag{4}$$

where the spin density operators $S^m(\vec{x})$ are a pseudo-boson representations of the spin matrix

$$S^m(\vec{x}) = \sum_{\alpha,\beta} \psi^{\dagger}_{\alpha}(\vec{x}) F^m_{\alpha\beta} \psi_{\beta}(\vec{x}), \tag{5}$$

and $F^m_{\alpha\beta}$ being the matrix elements of the 3×3 spin-1 matrices:

$$F^z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad F^+ = \sqrt{2} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad F^+ = \sqrt{2} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The density operator $\hat{N}(\vec{x})$ entering the density-density interaction term in 2 is a pseudo-boson representation of the 3×3 unity matrix

$$N(\vec{x}) = \sum_{\alpha,\beta} \psi^{\dagger}_{\alpha}(\vec{x}) \delta_{\alpha\beta} \psi_{\beta}(\vec{x}). \tag{6}$$

It should be mentioned that the Hamiltonian (1) is invariant with respect to exchange $\psi_1(\vec{x}) \leftrightarrow \psi_{-1}(\vec{x})$, $\psi^{\dagger}_1(\vec{x}) \leftrightarrow \psi^{\dagger}_{-1}(\vec{x})$.

The spin density operators $S^m(\vec{x})$ (5) satisfy commutation relations

$$\begin{aligned} [S^+(\vec{x}), S^-(\vec{x}')] &= 2S^z(\vec{x}) \delta(\vec{x} - \vec{x}'), \\ [S^z(\vec{x}), S^{\pm}(\vec{x}')] &= \pm S^{\pm}(\vec{x}) \delta(\vec{x} - \vec{x}'), \end{aligned} \tag{7}$$

and commute with the density operator (6)

$$[N(\vec{x}), S^z(\vec{x}')] = [N(\vec{x}), S^\pm(\vec{x}')] = 0 \tag{8}$$

The operators of the total spin

$$S^m = \int S^m(\vec{x}) d^3x, \;\; m = z, \pm 1, \tag{9}$$

obey the standard commutation relations of $su(2)$ algebra

$$[S^+, S^-] = 2S^z, \;\; [S^z, S^\pm] = \pm S^\pm. \tag{10}$$

It is easy to check that both the total number of atoms

$$\hat{N} = \int N(\vec{x}) d^3x, \tag{11}$$

and the z-component of the total spin (9) commute with the Hamiltonian (1) and are therefore the integrals of motion.

It is assumed [5] that the density-density interaction part is strong compared with the spin and the dipolar parts $c_0 \gg |c_2| \gg c_d$. This allows to use the single mode approximation [1], [2] representing the field operators as

$$\begin{array}{rcl} \psi_\alpha(\vec{x}) & \sim & a_\alpha \phi(\vec{x}), \\ \psi_\alpha^\dagger(\vec{x}) & \sim & a_\alpha^\dagger \bar{\phi}(\vec{x}) \, , \alpha = \pm 1, 0; \end{array} \tag{12}$$

where $a_\alpha, a_\alpha^\dagger$ are annihilation, creation operators associated with the spin mode satisfying usual commutation relations $[a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}$, and $\phi(\vec{x})$ is the spin-independent ground-state wave function of the symmetric Hamiltonian

$$H_{sym} = \sum_\alpha \int d^3x \psi_\alpha^\dagger(\vec{x}) \left(-\frac{\hbar^2}{2M} \nabla^2 + V_{ext}(\vec{x}) \right) \psi_\alpha(\vec{x}) + \frac{c_0}{2} \int d^3x : N^2(\vec{x}) : . \tag{13}$$

The function $\phi(\vec{x})$, normalized as $\int |\phi(\vec{x})|^2 d^3x = 1$, is determined from the extremum condition

$$\frac{\delta}{\delta \phi(\vec{x})} \langle N | H_{sym} - \mu \int d^3x N(\vec{x}) | N \rangle = 0$$

in the sector with the fixed number of particles. The Fock number state is $|N\rangle \equiv |N_1, N_0, N_{-1}\rangle$, where N_α are the occupation numbers of the correspondent spin modes: $a_\alpha^\dagger a_\alpha |N\rangle = N_\alpha |N\rangle$, and μ is the chemical potential. This condition leads to the Gross-Pitaevskii equation

$$\left(-\frac{\hbar^2}{2M} \nabla^2 + V_{ext}(\vec{x}) \right) \phi(\vec{x}) + c_0 N |\phi(\vec{x})|^2 \phi(\vec{x}) = \mu \phi(\vec{x}). \tag{14}$$

Under the single mode approximation the Hamiltonian (1) will take the form

$$\begin{aligned} H &= \mu\hat{N} - :\left\{g_0\hat{N}^2 - g_2\mathbf{S}^2 + g_d\left(\mathbf{S}^2 - 3(S^z)^2\right)\right\}: \qquad (15)\\ &= \mu\hat{N} - g_0\hat{N}(\hat{N}-1) + g_2(\mathbf{S}^2 - 2\hat{N}) - g_d\left(\mathbf{S}^2 - 3(S^z)^2 + \hat{N} - 3a_0^\dagger a_0\right). \end{aligned}$$

Here g_0, g_2, g_d are renormalized coupling constants $2g_i = c_i \int |\phi(\vec{x})|^4 d^3x,\ (i = 0,2)$; $4g_d = c_d \int d^3x_1 d^3x_2 |\phi(\vec{x}_1)|^2 |\phi(\vec{x}_2)|^2 (1-\cos^2\theta)/|\vec{x}_1 - \vec{x}_2|^3$ with θ being the polar angle of $(\vec{x}_1 - \vec{x}_2)$. The total spin operators (9) are now

$$\begin{aligned} S^+ &= \sqrt{2}\left(a_1^\dagger a_0 + a_0^\dagger a_{-1}\right), S^- = (S^+)^\dagger,\\ S^z &= \left(a_1^\dagger a_1 - a_{-1}^\dagger a_{-1}\right). \qquad (16) \end{aligned}$$

In (15) $\mathbf{S}^2$ is the squared total angular momentum operator

$$\mathbf{S}^2 = (S^z)^2 + \frac{1}{2}(S^+S^- + S^-S^+), \qquad (17)$$

which is the Casimir operator of $su(2)$ commuting with all total spin operators S^m (16).

The number operator

$$\hat{N} = a_1^\dagger a_1 + a_{-1}^\dagger a_{-1} + a_0^\dagger a_0. \qquad (18)$$

commutes with the Hamiltonian (15), so we may drop the terms depending on the number operator in (15) responsible for the density-density interactions and study the spin part of the model only [5]

$$H_{sd} = (g_2 - g_d)\mathbf{S}^2 + 3g_d\left((S^z)^2 + a_0^\dagger a_0\right). \qquad (19)$$

3 Dipolar spinor condensate as the $su(1,1)$ model

To solve the model for its eigenstates and eigenvalues it is convenient to express Hamiltonian (19) in terms of generators of $su(1,1)$ algebra. The generators of this algebra satisfy commutation relations

$$[\mathcal{K}^0, \mathcal{K}^\pm] = \pm\mathcal{K}^\pm, \quad [\mathcal{K}^+, \mathcal{K}^-] = -2\mathcal{K}^0. \qquad (20)$$

The Casimir invariant of $su(1,1)$ is given by

$$\mathcal{K}^2 = (\mathcal{K}^0)^2 - \frac{1}{2}\left(\mathcal{K}^+\mathcal{K}^- + \mathcal{K}^-\mathcal{K}^+\right). \qquad (21)$$

There are several representations of $su(1,1)$. Our interest will be confined to the representations based on the usual bosonic operators.

The two-mode boson realization of this algebra is

$$\begin{aligned} K^0 &= \frac{1}{2}(a_1^\dagger a_1 + a_{-1}^\dagger a_{-1} + 1), \\ K^+ &= a_1^\dagger a_{-1}^\dagger \quad , \quad K^- = a_1 a_{-1}\,. \end{aligned} \tag{22}$$

The Casimir operator for this realization can be written as

$$\begin{aligned} \mathcal{K}_K^2 &= \frac{1}{4}(\Delta^2 - 1), \\ \Delta &= a_1^\dagger a_1 - a_{-1}^\dagger a_{-1}. \end{aligned} \tag{23}$$

From (16) it follows that the operator Δ is equal to the z-component of the total spin $\Delta \equiv S^z$. Obviously, Δ commutes with all the operators in (22) and thus the population difference in the spin modes with $\alpha = \pm 1$ must differ by some fixed amount, the eigenvalue of Δ. We denote this eigenvalue as m and without loss of generality we take m to be a positive integer.

For a single-mode boson field the $su(1,1)$ algebra is realized by the operators

$$\begin{aligned} B^0 &= \frac{1}{2}a_0^\dagger a_0 + \frac{1}{4}, \\ B^+ &= -\frac{1}{2}(a_0^\dagger)^2 \quad , \quad B^- = -\frac{1}{2}(a_0)^2. \end{aligned} \tag{24}$$

The Casimir operator for this realization takes on the value

$$\mathcal{K}_B^2 = -3/16. \tag{25}$$

Expressed in terms of $su(1,1)$ generators the number operator (18) will take the form

$$\hat{N} = 2(B^0 + K^0) - \frac{3}{2}, \tag{26}$$

while the squared momentum operator (17) will be equal to

$$\mathbf{S}^2 = 4\left\{\frac{1}{4}(\Delta^2 - 1) + 2B^0K^0 - B^+K^- - B^-K^+\right\}. \tag{27}$$

The model Hamiltonian (19) finally reads

$$H_{sd} = 4(g_2 - g_d)\left\{2H_o + \frac{\Delta^2 - 1}{4} - (\Delta^2 - \frac{1}{2})\delta\right\} \tag{28}$$

with

$$2H_o = 2B^0K^0 - B^+K^- - B^-K^+ - 2\delta B^0, \tag{29}$$

where $\delta = -\frac{3}{4}\frac{g}{1-g}$, and $g = g_d/g_2$. Comparing equations (27) and (28) one finds that squared momentum operator $\mathbf{S}^2$ is equal to the Hamiltonian of dipole-dipole interaction (19), (28) H_{sd} in the absence of the dipole-dipole interaction $g_d = 0$:

$$\mathbf{S}^2 = \frac{1}{g_2} H_{sd} \mid_{\delta=0} \equiv \frac{1}{g_2} H_{sp}. \tag{30}$$

The most obvious conserved quantities of the Hamiltonian (28), and respectively of (19), are the total number of particles and the population difference in the modes with the opposite spins (the z-component of the total spin):

$$[H_{sd}, \hat{N}] = [H_{sd}, \Delta] = 0. \tag{31}$$

The commutativity of the Hamiltonian with the Casimir operator (25)

$$[H_{sd}, \mathcal{K}_B^2] = 0, \tag{32}$$

means that the parity of the number of particles in the spin mode with $\alpha = 0$ is the conserved quantity as well. These conservation laws follow directly from the fact that the Hamiltonian describes the creation and annihilation of bosonic atoms in pairs.

4 $su(1,1)$ loop algebra and the integrals of motion

To apply algebraic Bethe ansatz to the solution of the model defined by the Hamiltonian (28) let us define the operators which will play the crucial role in our approach:

$$\begin{aligned} X^0(\lambda) &= \frac{B^0}{\delta - \lambda} - \frac{K^0}{\lambda} + 1\,, \\ X^{\pm}(\lambda) &= \frac{B^{\pm}}{\delta - \lambda} - \frac{K^{\pm}}{\lambda}\,. \end{aligned} \tag{33}$$

We shall show that the model Hamiltonian (28) and its eigenvectors may be constructed with the help of these operators. In (33) λ is a complex variable while δ is a constant defined in the previous Section.

The operators (33) satisfy the following commutation relations:

$$
\begin{aligned}
[X^+(\lambda), X^-(\mu)] &= -\frac{2}{\lambda-\mu}\left(X^0(\lambda) - X^0(\mu)\right), \\
[X^0(\lambda), X^\pm(\mu)] &= \pm\frac{1}{\lambda-\mu}\left(X^\pm(\lambda) - X^\pm(\mu)\right), \\
[X^+(\lambda), X^+(\mu)] &= [X^-(\lambda), X^-(\mu)] = [X^0(\lambda), X^0(\mu)] = 0.
\end{aligned} \tag{34}
$$

These equalities are checked by applying the commutation relations of operators (22), (24) and the equality

$$
\frac{1}{(\epsilon-\lambda)(\epsilon-\mu)} = \frac{1}{\lambda-\mu}\left(\frac{1}{\epsilon-\lambda} - \frac{1}{\epsilon-\mu}\right). \tag{35}
$$

Algebra (34) is known as $su(1,1)$ loop algebra.

By the analogy with the Casimir operator (21) we introduce a family of operators depending on the arbitrary complex number λ :

$$
t(\lambda) = \left(X^0(\lambda)\right)^2 - \frac{1}{2}\left(X^+(\lambda)X^-(\lambda) + X^-(\lambda)X^+(\lambda)\right). \tag{36}
$$

The most important property of this operators is that they commute for the arbitrary complex numbers λ, μ:

$$
[t(\lambda), t(\mu)] = 0. \tag{37}
$$

This property is checked by the direct calculation with the help of the commutation relations (34). The operator $t(\lambda)$ may be considered as the generating function of the integrals of motion.

Substituting (33) into (36) we have

$$
\begin{aligned}
t(\lambda) \;=\; & 1 + \left(\frac{B^0}{\delta-\lambda}\right)^2 + \left(\frac{K^0}{\lambda}\right)^2 - \frac{B^+B^- + B^-B^+}{2(\delta-\lambda)^2} - \frac{K^+K^- + K^+K^-}{2\lambda^2} \\
& -\frac{2K^0}{\lambda} + \frac{2B^0}{\delta-\lambda} - \frac{2B^0}{(\delta-\lambda)}\frac{K^0}{\lambda} \\
& +\frac{B^+}{(\delta-\lambda)}\frac{K^-}{\lambda} + \frac{B^-}{(\delta-\lambda)}\frac{K^+}{\lambda}.
\end{aligned} \tag{38}
$$

The coefficient at the simple pole of this expressions when $\lambda = \delta$ is equal to

$$
Res|_{\lambda=\delta} t(\lambda) = 2B^0 - \frac{1}{\delta}\left\{2B^0K^0 - B^+K^+ - B^-K^-\right\}, \tag{39}
$$

and we have the following expression for the Hamiltonian (29)

$$
2H_o = -\delta Res|_{\lambda=\delta} t(\lambda).
$$

The Casimir operators (23) and (25) are the coefficients at the poles of the second order in (38), so for the Hamiltonian (28) we have

$$H_{sd} = 4(g_2 - g_d)\left\{ -\delta Res|_{\lambda=\delta} t(\lambda) + \frac{\Delta^2 - 1}{4} - (\Delta^2 - \frac{1}{2})\delta \right\}, \qquad (40)$$

From (38) and (37) it follows that

$$[t(\lambda), H_{sd}] = [t(\lambda), N] = [t(\lambda), \Delta] = [t(\lambda), \mathcal{K}_B^2] = 0. \qquad (41)$$

Knowing the eigenvectors and eigenvalues of the generating operator $t(\lambda)$ (36) we may find the eigenenergies of the Hamiltonian H_{sd} (28) applying the equation (40).

5 Solution of the model

To develop the algebraic scheme of the diagonalization of the generating function $t(\lambda)$ (36) first we have to remind that the basis of the unitary irreducible representation of the $su(1,1)$ algebra is formed by the eigenvectors $|n\rangle_\nu$ of operator $\mathcal{K}^0$ and Casimir operator $\mathcal{K}^2$:

$$\begin{aligned} \mathcal{K}^0|n\rangle_\nu &= (n+\nu)|n\rangle_\nu, \\ \mathcal{K}^2|n\rangle_\nu &= \nu(\nu-1)|n\rangle_\nu, \end{aligned} \qquad (42)$$

where ν is the so-called Bargmann index. The operators $\mathcal{K}^\pm$ act as the rising and lowering operators, respectively, on the eigenstates of $\mathcal{K}^0$. The non-normalized states $|n\rangle_\nu$ may be constructed by the successive action of operator $\mathcal{K}^+$ from the generating vector $|0\rangle_\nu$ defined by the equation

$$\mathcal{K}^-|0\rangle_\nu = 0. \qquad (43)$$

These states are equal to

$$|n\rangle_\nu = \left(\mathcal{K}^+\right)^n |0\rangle_\nu. \qquad (44)$$

The representation space of the two-mode realization (22,23) of the $su(1,1)$ algebra consists of two-mode Fock states which are the direct product of the number states of spin modes with $\alpha = \pm 1$. The generating vector of this realization is defined by the equation

$$K^-|0\rangle_{\nu_2} = 0. \qquad (45)$$

We may choose $|0\rangle_{\nu_2} \equiv |m\rangle^{(1)} \otimes |0\rangle^{(-1)}$ either $|0\rangle_{\nu_2} \equiv |0\rangle^{(1)} \otimes |m\rangle^{(-1)}$, $(m = 0, 1, ...)$ .The Bargmann index of this realization is $\nu_2 = \frac{m+1}{2}$. It follows from (22) that these states are the eigenstates of the operator Δ:

$$\Delta|0\rangle_{\nu_2} = \alpha m|0\rangle_{\nu_2}. \tag{46}$$

The representation space in the single-mode realization (24) is decomposed into the direct sum of two irreducible components spanned by the states $|2n+s\rangle$ with an even number of particles $(s = 0)$ or by the states with an odd number of particles $(s = 1)$. The Bargmann index of this realization is $\nu_1 = \frac{2s+1}{4}$ and the generating vector (43) is defined by the equation

$$B^-|0\rangle_{\nu_1} = 0. \tag{47}$$

The space with $\nu_1 = \frac{1}{4}$ $(s = 0)$ is built from the Fock vacuum $|0\rangle_{\nu_1} \equiv |0\rangle^{(0)}$, and the space with $\nu_1 = \frac{3}{4}$ $(s = 1)$ is built from the one particle state $|0\rangle_{\nu_1} \equiv |1\rangle^{(0)}$ respectively. The generating space satisfies the relation

$$B^0|0\rangle_{\nu_1} = \frac{2s+1}{4}|0\rangle_{\nu_1} \tag{48}$$

From the definition of the operators $X^{\pm}(\lambda), X^0(\lambda)$ (33) and the number operator $\hat{N}$ (26) it follows that

$$\hat{N}X^{\pm}(\lambda) = X^{\pm}(\lambda)\left(\hat{N} \pm 2\right), \tag{49}$$

and

$$\hat{N}X^0(\lambda) = X^0(\lambda)\hat{N}. \tag{50}$$

So $X^{\pm}(\lambda)$ acts as a creation (annihilation) operator of the pair of boson quasi-particles.

The state which is the direct product of the generating states (47) and (45)

$$|\Omega\rangle = |0\rangle_{\nu_1} \otimes |0\rangle_{\nu_2}, \tag{51}$$

which we shall call the vacuum state, satisfies the following equations

$$\begin{aligned} X^-(\lambda)|\Omega\rangle &= 0, \\ X^0(\lambda)|\Omega\rangle &= x(\lambda)|\Omega\rangle, \end{aligned} \tag{52}$$

where the vacuum eigenvalue of the $X^0(\lambda)$ operator is equal to:

$$x(\lambda) = 1 + \frac{\nu_1}{\delta - \lambda} - \frac{\nu_2}{\lambda}, \tag{53}$$

where ν_1, ν_2 are the Bargmann indices of a single- and two-mode representations respectively. The vacuum state is an eigenstate of the number operator (26)

$$\hat{N}|\Omega\rangle = (s+m)|\Omega\rangle, \tag{54}$$

and of the z-component of the total spin

$$\Delta|\Omega\rangle = \alpha m|\Omega\rangle. \tag{55}$$

It is easy to verify that the vacuum state (51) is an eigenvector of the generating function $t(\lambda)$

$$t(\lambda)|\Omega\rangle = k(\lambda)|\Omega\rangle, \tag{56}$$

with the eigenvalue

$$k(\lambda) = \left(1 + \frac{\nu_1}{\delta - \lambda} - \frac{\nu_2}{\lambda}\right)^2 - \frac{\nu_1}{(\delta-\lambda)^2} - \frac{\nu_2}{\lambda^2}. \tag{57}$$

Due to the conservation laws (41) the eigenvectors of the generating operator $t(\lambda)$ depend on the total number of particles in the system N, the value m of the absolute value of the z-component of the total spin, and the parity s of the $\alpha = 0$ spin mode. We shall look for these eigenvectors in the form of the Bethe vectors

$$|\Phi_{N,m,s}(\lambda_1, \lambda_2, ..., \lambda_{N_p})\rangle = \prod_{j=1}^{N_p} X^+(\lambda_j)|\Omega\rangle. \tag{58}$$

Due to (49), the number of particles in this state is

$$\begin{aligned}\hat{N}|\Phi_{N,m,s}(\lambda_1, \lambda_2, ..., \lambda_{N_p})\rangle &= N|\Phi_{N,m,s}(\lambda_1, \lambda_2, ..., \lambda_{N_p})\rangle, \\ N &\equiv 2N_p + s + m,\end{aligned} \tag{59}$$

and the number of operators $X^+(\lambda)$ in the product (58), which corresponds to a number of pairs of the boson quasi-particles in the system, is equal to $2N_p = N - m - s$. The state (58) clearly satisfies the relations

$$\begin{aligned}\Delta|\Phi_{N,m,s}(\lambda_1, \lambda_2, ..., \lambda_{N_p})\rangle &= \alpha m|\Phi_{N,m,s}(\lambda_1, \lambda_2, ..., \lambda_{N_p})\rangle, \qquad (60)\\ (-1)^{\hat{N}-|\Delta|}|\Phi_{N,m,s}(\lambda_1, \lambda_2, ..., \lambda_{N_p})\rangle &= (-1)^s|\Phi_{N,m,s}(\lambda_1, \lambda_2, ..., \lambda_{N_p})\rangle.\end{aligned}$$

For a given number of particles N, the possible values of quantum numbers m and s are $0 \le m + s \le N$.

The vectors (58) are the eigenvalues of $t(\lambda)$ if parameters λ_j satisfy Bethe equations

$$1 + \frac{\nu_1}{\delta - \lambda_j} - \frac{\nu_2}{\lambda_j} = \sum_{l \neq j}^{N_p} \frac{1}{\lambda_j - \lambda_l}; \ j = 1, ..., N_p. \tag{61}$$

We shall show in the Appendix that there are N_p+1 sets $\{\lambda_j^\sigma\}_{j=1}^{N_p}$ of solutions of these N_p equations ($\sigma = 1, 2, ..., N_p + 1$). They are real, $\lambda_j \in \mathbb{R}$, positive, all different and not equal to δ and 0.

The N-particle eigenvalues $\Theta^\sigma_{N,m,s}(\mu)$ of the generating function $t(\mu)$ (36) are equal to

$$\Theta^\sigma_{N,m,s}(\mu) = k(\mu) - \sum_{j=1}^{N_p} \frac{2\nu_1}{(\delta-\mu)(\delta-\lambda_j^\sigma)} - \sum_{j=1}^{N_p} \frac{2\nu_2}{\mu\lambda_j^\sigma}, \tag{62}$$

with $k(\mu)$ given by the relation (57), and $\lambda_j^\sigma \in \{\lambda_j^\sigma\}_{j=1}^{N_p}$.

From the equation (40) it follows that the N-particle eigenenergies of the Hamiltonian H_{sd} (28) with the fixed value of the third spin component m and parity s are equal to

$$\frac{1}{4(g_2-g_d)} E^\sigma_{N,m,s} = -\delta Res|_{\mu=\delta} \Theta^\sigma_{N,m,s}(\mu) + \frac{m^2-1}{4} - (m^2 - \frac{1}{2})\delta.$$

The substitution of (62) into this expression gives

$$\begin{aligned} \frac{1}{4(g_2-g_d)} E^\sigma_{N,m,s} &= -2\delta\nu_1 + 2\nu_1\nu_2 + \sum_{j=1}^{N_p} \frac{2\nu_1\delta}{\delta-\lambda_j^\sigma} + \frac{m^2-1}{4} - (m^2-\frac{1}{2})\delta \\ &= \frac{1}{4}(m+1)(m+2s) - \delta(m^2+s) + \sum_{j=1}^{N_p} \frac{2\nu_1\delta}{\delta-\lambda_j^\sigma} \end{aligned} \tag{63}$$

where λ_j are the solutions of Bethe equations (61), and the definitions $2\nu_2 = m-1$, $4\nu_1 = 2s+1$ were used. The equality $(m+1)(m+2s) = (N-2N_p+1)(N-2N_p)$ is valid for $s = 0, 1$ and the alternative expression for the eigenenergy is

$$\begin{aligned} \frac{1}{4(g_2-g_d)} E^\sigma_{N,m,s} = &\frac{1}{4}\left[N(N+1) - 2N_p(2N-2N_p+1)\right] \\ &-\delta\left[N - 2N_p + m(m-1)\right] + \sum_{j=1}^{N_p} \frac{2\nu_1\delta}{\delta-\lambda_j^\sigma} \end{aligned}$$

The other expression for the eigenenergies might be obtained in the following way. Multiplying the equation (61) by λ_j and then summing up it by j one obtains

$$\sum_{j=1}^{N_p} \lambda_j + \sum_{j=1}^{N_p} \frac{\nu_1\lambda_j}{\delta-\lambda_j} - N_p\nu_2 = \frac{1}{2}N_p(N_p-1), \tag{64}$$

with the help of the following relation

$$\sum_{j=1}^{N_p}\sum_{i\neq j}^{N_p}\frac{\lambda_j}{\lambda_j-\lambda_i}=\frac{1}{2}N_p(N_p-1).$$

The second term in (64) is expressed as

$$\sum_{j=1}^{N_p}\frac{\nu_1\lambda_j}{\delta-\lambda_j}=\sum_{j=1}^{N_p}\frac{\nu_1\delta}{\delta-\lambda_j}-\nu_1 N_p.$$

And finally

$$\begin{aligned}\sum_{j=1}^{N_p}\frac{2\nu_1\delta}{\delta-\lambda_j} &= N_p\left(N_p-1+2\nu_1+2\nu_2\right)-2\sum_{j=1}^{N_p}\lambda_j\\ &= N_p\left(N_p+s+m+\frac{1}{2}\right)-2\sum_{j=1}^{N_p}\lambda_j\\ &= N_p\left(N-N_p+\frac{1}{2}\right)-2\sum_{j=1}^{N_p}\lambda_j \qquad (65)\end{aligned}$$

So the eigenenergies of the Hamiltonian (28) may be written in the form

$$\begin{aligned}&\frac{1}{4(g_2-g_d)}E^{\sigma}_{N,m,s}=-(s+m^2)\delta+\frac{1}{4}(m+1)(m+2s)\\ +\ &\frac{1}{4}(N-m-s)(N+m+s+\frac{1}{2})-2\sum_{j=1}^{N_p}\lambda_j^{\sigma}\\ =\ &-\delta\left[N-2N_p+m(m-1)\right]+\frac{1}{4}N(N+1)-2\sum_{j=1}^{N_p}\lambda_j^{\sigma}. \qquad (66)\end{aligned}$$

The eigenenergies $E_{N,m,s}$ are real numbers, and the solutions of Bethe equations (61) λ_j may be interpreted as the energies of the boson pairs.

In order to study the behavior of the solutions of Bethe equations (61) it is reasonable to consider the polynomials defined by the solution of this equation

$$P(\lambda)=C\prod_{j=1}^{N_p}(\lambda_j^{\sigma}-\lambda), \qquad (67)$$

where $C^{-1} = \prod_{j=1}^{N_p} \lambda_j^\sigma$. This polynomial satisfies the second order differential equation

$$P''(\lambda) - 2x(\lambda)P'(\lambda) + 2\sum_{j=1}^{N} \frac{x(\lambda) - x(\lambda_j)}{\lambda - \lambda_j} P(\lambda) = 0, \tag{68}$$

where $x(\lambda)$ is the vacuum eigenvalue (53) of the $X^0(\lambda)$ operator. The substitution of (53) into this expression together with the equalities (65) and (66) gives the equation

$$\begin{aligned} \lambda(\delta - \lambda)P''(\lambda) - 2\left[\lambda(\delta - \lambda) + \nu_1\lambda - \nu_2(\delta - \lambda)\right] P'(\lambda) & \\ + \left[\mathcal{E}^\sigma_{N,m,s} + 2(\delta - \lambda)N_p\right] P(\lambda) &= 0 \end{aligned} \tag{69}$$

where

$$\mathcal{E}^\sigma_{N,m,s} = \frac{1}{4(g_2 - g_d)} E^\sigma_{N,m,s} + (s + m^2)\delta - \frac{1}{4}(m+1)(m+2s).$$

The alternative formulation of an eigenvalue problem for the Hamiltonian (28) is the following: the eigenenergies $E_{N,m,s}$ are defined by the condition that the equation (69) has the polynomial solution of the degree N_p without multiple zeros.

6 Spinor condensate

In the absence of the dipolar interaction the Hamiltonian of the model is (30):

$$H_{sp} = g_2\mathbf{S}^2, \tag{70}$$

where $\mathbf{S}^2$ is the squared total angular momentum operator (17).

The operators

$$\begin{aligned} X^\pm &= B^\pm + K^\pm \equiv -\lambda X^\pm(\lambda), \\ X^0 &= B^0 + K^0 \equiv \lambda - \lambda X^0(\lambda), \end{aligned} \tag{71}$$

where $X^\pm(\lambda), X^0(\lambda)$ are (33) with $\delta = 0$, satisfy the $su(1,1)$ algebra commutation relations and generate the tensor product group $SU(1,1) \otimes SU(1,1)$ [12]. The Casimir operator of this group is

$$\begin{aligned} \mathcal{K}^2_X &= (X^0)^2 - \frac{1}{2}\left(X^+X^- + X^-X^+\right) \\ &= H_o + \mathcal{K}^2_B + \mathcal{K}^2_K, \end{aligned} \tag{72}$$

where H_o is (29) with $\delta = 0$. Since Casimir operator commutes with the operators (71) the Hamiltonian

$$H_{sp} = 4g_2(\mathcal{K}_X^2 - \mathcal{K}_B^2) \tag{73}$$

possess the symmetry

$$[H_{sp}, X^{\pm}] = [H_{sp}, X^0] = 0. \tag{74}$$

In the case under consideration the generating function (36) is equal to $t(\lambda) = \lambda^2(\mathcal{K}_X^2 + 1 - 2X^0)$, and thus $t(\lambda)$ does not possess the symmetry relations (74).

The basis of the $SU(1,1) \otimes SU(1,1)$ representation we shall denote as $|n\rangle_\nu$, while the basis states of the two- and single-mode representations as $|n_2\rangle_{\nu_2}$ and $|n_1\rangle_{\nu_1}$ respectively. According to (42) we have

$$\begin{aligned} X^0|n\rangle_\nu &= (n+\nu)|n\rangle_\nu, \\ \mathcal{K}_X^2|n\rangle_\nu &= \nu(\nu-1)|n\rangle_\nu, \end{aligned} \tag{75}$$

and the generating vector of the representation $|0\rangle_\nu$ satisfy

$$\mathcal{K}_X^-|0\rangle_\nu = 0. \tag{76}$$

The tensor product of two representations $D^{(\nu_1)}$ and $D^{(\nu_2)}$ reduces to the sum of irreducible representations according to the $SU(1,1)$ Clebsch-Gordon decomposition

$$D^{(\nu_1)} \otimes D^{(\nu_2)} = \sum_{l=0}^{\infty} D^{(\nu_1+\nu_2+l)}, \tag{77}$$

where $\nu_1 + \nu_2 + l \equiv \nu$ is the Bargmann index of the correspondent representation.

We can look for the non-normalized generating vector $|0\rangle_\nu$ in the form

$$|0\rangle_\nu = \sum_{k=0}^{l} A_k^l (K^+)^k (B^+)^{l-k} |0\rangle_{\nu_2} \otimes |0\rangle_{\nu_1}. \tag{78}$$

From (76) it follows that the coefficients satisfy the recurrent relation

$$A_{k+1}^l (2\nu_2 + k)(k+1) + A_k^l (2\nu_1 + l - k - 1)(l-k) = 0,$$

with $A_0^l = 1$, and are equal to

$$A_k^l = (-1)^k C_l^k \prod_{p=0}^{k-1} \frac{2\nu_1 + l - 1 - p}{2\nu_2 + p}, \tag{79}$$

where the binomial coefficient $C_l^k = \frac{l!}{(l-k)!k!}$.

The states $|n\rangle_\nu$ are generated from the vacuum vector (78) according to

$$(X^+)^n|0\rangle_\nu = |n\rangle_\nu. \tag{80}$$

The number of particles in the state $|n;\nu\rangle$, as it follows from (75), are equal to

$$\begin{aligned}\hat{N}|n\rangle_\nu &= \left(2X^0 - \frac{3}{2}\right)|n\rangle_\nu = \left(2(n+v_1+\nu_2+l) - \frac{3}{2}\right)|n\rangle_\nu \\ &= (2(n+l)+m+s)\,|n\rangle_\nu.\end{aligned} \tag{81}$$

For the fixed number of pairs of particles $N_p = n + l$ the index l takes the values $l = 0, 1, ..., N_p$.

From (73) it follows that the eigenvalues of the Hamiltonian

$$H_{sp}|n\rangle_\nu = E^l_{n,\nu,\nu_1}|n\rangle_\nu \tag{82}$$

are equal to

$$\begin{aligned}E^l_{n,\nu,\nu_1} &= 4g_2\left(\nu(\nu-1) - \nu_1(\nu_1-1)\right) \\ &= 4g_2\left(\nu_2(\nu_2-1) + l(l-1) + 2\nu_1\nu_2 + 2l(\nu_1+\nu_2)\right).\end{aligned} \tag{83}$$

Unlike the model with the dipole interaction with $\delta \neq 0$ the considered case is degenerate and in the sector with the fixed number of particles N we have the set of the vacuum states (78) $|0;\nu_1+\nu_2+l\rangle$ with $l = 0, 1, ..., (N - m - s)/2$.

For the fixed number of particles N in a system ($N = 2(n+l)+m+s$) the energy (83) is equal to:

$$E^l_{n,\nu,\nu_1} = g_2\left[N(N+1) - 2n(2N+1-2n)\right]. \tag{84}$$

The ground state of the model is defined by the sign of the interaction constant g_2. In the antiferromagnetic case $g_2 > 0$ the ground state is defined by the condition $l = 0$ and $2n = N - s$ and is equal to

$$\begin{aligned}|G\rangle_{AF} &= (X^+)^{\frac{N-s}{2}}|0\rangle_\nu, \\ |0\rangle_\nu &= |s\rangle^{(0)} \otimes |0\rangle^{(1)} \otimes |0\rangle^{(-1)}.\end{aligned} \tag{85}$$

where $s = 0, 1$ depending on the parity of N. From (84) it follows that the eigenenergy of the ground state is zero: $E_{AF} = 0$.

In the considered case the solutions of Bethe equations are the energies of the boson pairs in the ground state. When $\delta = 0$ the equation (61) for the ground state has the form

$$1 - \frac{2s+3}{4\lambda_j} = \sum_{l \neq j}^{N_p} \frac{1}{\lambda_j - \lambda_l}; \; j = 1, ..., N_p, \tag{86}$$

and is the equation on the zeros of the Lagerr polynomial $P(\lambda) = L_N(2\lambda; \frac{2s-1}{4})$. The solution of this equation is unique and λ_j are all different and positive: $\lambda_j > 0$.

When $\alpha > -1$ for the big values of N the asymptotics of Lagerr polynomials is

$$L_N(2x; \alpha) \sim \pi^{-\frac{1}{2}} e^x x^{-\frac{\alpha}{2} - \frac{1}{4}} N^{\frac{\alpha}{2} - \frac{1}{4}} \cos \left\{ 2\sqrt{2Nx} - \frac{\pi}{4}(2\alpha + 1) \right\}.$$

From this expression it follows that the energies of the boson pairs in the ground state behave like

$$\lambda_j = \frac{1}{N} \frac{\pi^2}{32} \left\{ 2j + \frac{s-3}{2} \right\}^2, \; j = 1, 2, \ldots, N.$$

For the ferromagnetic case $g_2 < 0$ the ground state is defined by the condition $n = 0$. The eigenenergy of the state is

$$E_F = g_2 N(N+1). \tag{87}$$

The ground state vectors are given by the expression (78) with m and l connected by the equality $2l + m + s = N$, and the state is $2N + 1$ fold degenerate.

7 Conclusion

The model discussed in this paper belongs to a class of the integrable so-called pairing models introduced into the theory of Bose-Einstein condensation by Richardson [13]. The algebraic approach to the solution of such systems was developed by Gaudin [14].

The knowledge of the eigenfunctions of the model allows to study the spin mixing dynamics and the quantum phases of the spinor Bose gas with the dipole-dipole interaction in details. We hope to report further results in this connection elsewhere.

This work was supported by RFBR project 07-01-00358.

8 Appendix

The Bethe equations (61) belong to the type of characteristic equations appearing in the theory of ellipsoidal harmonic functions. To prove that the solutions λ_j of Bethe equations (61)

$$1+\frac{\nu_1}{\delta-\lambda_j}-\frac{\nu_2}{\lambda_j}=\sum_{l\neq j}^{N_p}\frac{1}{\lambda_j-\lambda_l};\ j=1,...,N_p.$$

are real numbers: $\lambda_j \in \mathbb{R}$ let us consider the function

$$F_j(\lambda)=1+\frac{\nu_0}{\delta-\lambda}-\frac{\nu_2}{\lambda}-\sum_{l\neq j}^{N_p}\frac{1}{\lambda-\lambda_l}.$$

The zeros of this function are the solutions of Bethe equations $F_j(\lambda_j)=0$. If $\lambda_j \in \mathbb{C}$, then the complex conjugated solutions satisfy the equation $F_j^*(\lambda_j)=0$. So

$$\sum_{j=1}^{N_p}(\lambda_j-\lambda_j^*)\left(F_j(\lambda_j)-F_j^*(\lambda_j)\right)\equiv 0. \tag{88}$$

From the equality

$$\sum_{j\neq l}u_j(u_j-u_l)=\frac{1}{2}\sum_{j\neq l}(u_j-u_l)^2,$$

it follows that

$$\begin{aligned}\sum_{j\neq l}(\lambda_j-\lambda_j^*)\left\{\frac{1}{\lambda_j-\lambda_l}-\frac{1}{\lambda_j^*-\lambda_l^*}\right\} &= -\sum_{j\neq l}(\lambda_j-\lambda_j^*)\frac{(\lambda_j-\lambda_j^*)-(\lambda_l-\lambda_l^*)}{|\lambda_j-\lambda_l^*|^2}= \\ & -\sum_{j\neq l}\frac{\left((\lambda_j-\lambda_j^*)-(\lambda_l-\lambda_l^*)\right)^2}{|\lambda_j-\lambda_l^*|^2}.\end{aligned}$$

We also have

$$\sum_j(\lambda_j-\lambda_j^*)\left\{\frac{\nu_0}{\delta-\lambda_j}-\frac{\nu_2}{\lambda_j}-\frac{\nu_0}{\delta-\lambda_j^*}+\frac{\nu_2}{\lambda_j^*}\right\}=\sum_j(\lambda_j-\lambda_j^*)^2\left\{\frac{\nu_0}{|\delta-\lambda_j|^2}+\frac{\nu_2}{|\lambda_j|^2}\right\}.$$

Finally, the equality (88) will take the form

$$\sum_j(\lambda_j-\lambda_j^*)^2\left\{\frac{\nu_0}{|\delta-\lambda_j|^2}+\frac{\nu_2}{|\lambda_j|^2}\right\}+\sum_{j\neq l}\frac{\left((\lambda_j-\lambda_j^*)-(\lambda_l-\lambda_l^*)\right)^2}{|\lambda_j-\lambda_l^*|^2}\equiv 0,$$

and hence $\lambda_j = \lambda_j^*$.

The solutions of Bethe equations $\lambda_j \neq 0, \delta$. It may be proved in a standard way [15] with a help of differential equation (69). Really if one of the roots $\lambda_j = 0$ or $\lambda_j = \delta$ then $P'(\lambda_j) = 0$ and from (69) it follows that all higher derivatives must be equal to zero at the same points, but it is not true. In the same way it may be proved that there are no multiple zeros. It may be proved that $0 < \lambda_j < \delta,\ j = 1, ..., N_p$.

To prove that there are N_p+1 sets $\left\{\lambda_j^\sigma\right\}_{j=1}^{N_p}$ of solutions of Bethe equations one should notice that the equation with number m is of order $N_p + 1$ with respect to λ_m, while the rest unknowns $\lambda_l, l \neq m$ have degree equal to 1. In all the rest equations λ_m enters in the first degree. Excluding $\lambda_l, l \neq m$ we shall get the equation of order $N_p(N_p + 1)$ with respect to λ_m. But since in the equation with the arbitrary number $N_p - 1$ unknowns are given, the last unknown is defined uniquely. Hence, the obtained $N_p(N_p + 1)$ roots of the system are divided into $N_p + 1$ sets.

Список литературы

[1] C.K. Law, H. Pu, and N.P. Bigelow, *Quantum spin mixing in spinor Bose-Einstein condensates.*- Phys. Rev. Lett. **24**, 5257 (1998).

[2] H. Pu, C.K. Law, S. Raghavan, J.H. Eberly, and N.P. Bigelow, *Spin-mixing dynamics of a spinor Bose-Einstein condensate,* Phys. Rev. A **60**, 1463 (1999).

[3] E. Goldstein and P. Meystre, *Quantum theory of atomic four-wave mixing in Bose-Einstein condensates*, Phys. Rev. A **59**, 3896 (1999).

[4] Ö.E. Müstecaplioğlu, M. Zhang, S. Yi, L. You, C.P. Sun, *Dynamic fragmentation of a spinor Bose-Einstein condensate.*- Phys. Rev. A **68**, 63616 (2003).

[5] S. Yi, L. You, and H. Pu, *Quantum phases of dipolar spinor condensates*, Phys. Rev. Lett. **93**, 040403 (2004).

[6] M. Takahashi, Sankalpa Ghosh, T. Mizushima, K. Machida, *Spinor dipolar Bose-Einstein condensates; Classical spin approach*, Phys. Rev. Lett. **98**, 260403 (2007).

[7] T. Lahaye, J. Metz, T. Koch, B. Fröhlich, A. Griesmaier and T. Pfau, *A purely dipolar quantum gas*, Atomic Physics 21, Proceeding of the XXI International Conference of Atomic Phiysics (ICAP 2008).

[8] S. Yi and H. Pu, *Dipolar spinor Bose-Einstein condensates*, arXiv:0804.0191v1 (2008).

[9] V.E.Korepin, N.M.Bogoliubov and A.G.Izergin, *Quantum inverse scattering method and correlation functions (*Cambridge: Cambridge University Press, 1993).

[10] N.M. Bogoliubov, *Spinor Bose condensate and* $su(1,1)$ *Richardson model*, Zap. Nauchn. Semin. POMI, **317**, 43 (2004).

[11] A.Rybin, G.Kastelewicz, J.Timonen, N.Bogoliubov, *The* $su(1,1)$ *Tavis-Cummings model*.- Journ. Phys. A **31** (1998), 4705-4723.

[12] C. Gerry and A. Benmoussa, *Two mode coherent states for* $SU(1,1) \otimes SU(1,1)$, Phys. Rev. A **62**, 033812 (2000).

[13] R.W. Richardson, *Exactly solvable many-boson model*.- J. Math. Phys. **9** (1968), 1327-1344.

[14] M.Gauden, *La fonction d'onde de Bethe* (Paris: Masson 1983).

[15] E.W. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge: Cambridge University Press 1931).