ПРЕПРИНТЫ ПОМИ РАН

Учредитель: Санкт-Петербургское отделение Математического института им. В. А. Стеклова Российской академии наук

Свидетельство о регистрации средства массовой информации: ЭЛ №ФС 77-33560 от 16 октября 2008 г. Выдано Федеральной службой по надзору в сфере связи и массовых коммуникаций

Контактные данные: 191023, г. Санкт-Петербург, наб. реки Фонтанки, дом 27

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http://www. pdmi.ras.ru /preprint/

Заведующая информационно-издательским сектором Симонова В.Н

POMI PREPRINT — 14/2008

BLOCH–CARLESON MEASURES AND ALEKSANDROV–RYLL–WOJTASZCZYK POLYNOMIALS

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August 2008

Abstract

Let $\mathcal{H}ol(B_n)$ denote the space of holomorphic functions in the unit ball B_n of $\mathbb{C}^n$, $n \geq 1$. Given $X \subset \mathcal{H}ol(B_n)$ and $0 < q < \infty$, a well-known problem is to characterize the positive measures μ on B_n such that $X \subset L^q(B_n, \mu)$. We obtain such a characterization when X is the Bloch space $\mathcal{B}(B_n)$ and μ is a radial measure. Also, we solve the problem when X is the growth space $\mathcal{A}^{-\log}(B_n)$ or X is the growth space $\mathcal{A}^{-\beta}(B_n)$, $\beta > 0$.

This research was supported by RFBR (grant no. 08-01-00358-a) and by the Russian Science Support Foundation.

1. Introduction

Let $n \in \mathbb{N}$ and let $\mathcal{H}ol(B_n)$ denote the space of holomorphic functions in the unit ball $B_n = \{z \in \mathbb{C}^n : \ |z| < 1\}$.

Carleson measures. Let $X \subset \mathcal{H}ol(B_n)$ and let $0 < q < \infty$. By definition, a positive measure μ on the ball B_n is called q-Carleson for X if $X \subset L^q(B_n, \mu)$. Given $X \subset \mathcal{H}ol(B_n)$, a well-known problem is to characterize the q-Carleson measures for X. Carleson [2] solved the problem when X is the Hardy space $H^q(B_1)$. By now, characterizations of the q-Carleson measures are known for various classical spaces X of holomorphic functions. In the present paper, we study the q-Carleson measures for the Bloch space $\mathcal{B}(B_n)$. Recall that $f \in \mathcal{B}(B_n)$ if and only if $f \in \mathcal{H}ol(B_n)$ and

$$\|f\|_{\mathcal{B}(B_n)} = |f(0)| + \sup_{z\in B_n} (1-|z|)|\mathcal{R}f(z)| < \infty,$$

where

$$\mathcal{R}f(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z), \quad z \in B_n,$$

is the radial derivative of f. Remark that

$$\mathcal{R}f(z) = \sum_{k=0}^{\infty} k f_k(z), \quad z \in B_n,$$

if $f(z) = \sum_{k=0}^{\infty} f_k(z)$, $z \in B_n$, is the homogeneous expansion of $f \in \mathcal{H}ol(B_n)$. The Bloch space $\mathcal{B}(B_n)$ is closely related with the growth space $\mathcal{A}^{-\log}(B_n)$. By definition, $f \in \mathcal{A}^{-\log}(B_n)$ if and only if $f \in \mathcal{H}ol(B_n)$ and

$$\|f\|_{-\log} = \sup_{z\in B_n} \frac{|f(z)|}{\log(e/(1-|z|))} < \infty.$$

It is well-known that $\mathcal{B}(B_n) \subset \mathcal{A}^{-\log}(B_n)$. Also, we consider the growth spaces $\mathcal{A}^{-\beta}(B_n)$, $\beta > 0$. Given $\beta > 0$, the space $\mathcal{A}^{-\beta}(B_n)$ consists of those $f \in \mathcal{H}ol(B_n)$ for which

$$\|f\|_{-\beta} = \sup_{z\in B_n} |f(z)|(1-|z|)^{\beta} < \infty.$$

Remark that $\mathcal{B}(B_n)$, $\mathcal{A}^{-\log}(B_n)$ and $\mathcal{A}^{-\beta}(B_n)$, $\beta > 0$, are Banach spaces with the above norms.

Given $0 < q < \infty$, Girela, Peláez, Pérez-González and Rättyä [5] obtained various results about the q-Carleson measures for $\mathcal{B}(B_1)$. Also, the q-Carleson measures for $\mathcal{A}^{-\log}(B_1)$ are characterized in [5]. In this paper, we focus our attention on the case of arbitrary dimension n. For $n \in \mathbb{N}$, we describe the radial q-Carleson measures for the Bloch space $\mathcal{B}(B_n)$, and we obtain characterizations of the q-Carleson measures for $\mathcal{A}^{-\log}(B_n)$ and for $\mathcal{A}^{-\beta}(B_n)$, $\beta > 0$.

Aleksandrov–Ryll–Wojtaszczyk polynomials. Ryll and Wojtaszczyk [10] constructed holomorphic polynomials which proved to be very useful for many problems of function theory in the unit ball (see, e.g., [9]). The results of the present paper are based on the following improvement of the Ryll–Wojtaszczyk theorem.

Theorem 1.1 (Aleksandrov [1, Theorem 4]). *Let $n \in \mathbb{N}$. Then there exist $\delta = \delta(n) \in (0,1)$ and $J = J(n) \in \mathbb{N}$ with the following property: For every $d \in \mathbb{N}$, there exist holomorphic homogeneous polynomials $W_j[d]$ of degree d, $1 \le j \le J$, such that*

$$\|W_j[d]\|_{L^\infty(\partial B_n)} \le 1 \quad \text{and} \tag{1.1}$$

$$\max_{1\le j\le J} |W_j[d](\zeta)| \ge \delta \text{ for all } \zeta \in \partial B_n. \tag{1.2}$$

2. Carleson measures for the Bloch space

Proposition 2.1. *Let $0 < q < \infty$ and let μ be a q-Carleson measure for $\mathcal{B}(B_n)$. Then*

$$\int_{B_n} \left(\log \frac{e}{1-|z|}\right)^{\frac{q}{2}} d\mu(z) < \infty. \tag{2.1}$$

Proof. Let the constant $\delta(n) \in (0,1)$ and the polynomials $W_j[d]$, $1 \le j \le J(n)$, $d \in \mathbb{N}$, be those provided by Theorem 1.1. For $k \in \mathbb{Z}_+$, let R_k denote the Rademacher function:

$$R_k(t) = \operatorname{sign} \sin(2^{k+1}\pi t), \quad t \in [0,1].$$

For each non-diadic $t \in [0,1]$, consider the functions

$$F_{j,t}(z) = \sum_{k=0}^{\infty} R_k(t) W_j[2^k](z), \quad z \in B_n,\ 1 \le j \le J(n).$$

Estimate (1.1) guarantees that

$$(1-|z|)|(\mathcal{R}F_{j,t})(z)| \le (1-|z|)\sum_{k=0}^{\infty} 2^k |z|^{2^k} \le 2(1-|z|)\sum_{m=1}^{\infty} |z|^m \le 2$$

for all $z \in B_n$. We have $(\mathcal{R}F_{j,t})(0) = 0$, hence, $\|F_{j,t}\|_{\mathcal{B}(B_n)} \le 2$. By assumption, $\mathcal{B}(B_n) \subset L^q(B_n, \mu)$, thus, applying the closed graph theorem, we obtain

$$\int_{B_n} |F_{j,t}(z)|^q \, d\mu(z) \le C \|F_{j,t}\|^q_{\mathcal{B}(B_n)} \le C, \quad 1 \le j \le J(n).$$

Changing the order of integration, we have

$$\int_{B_n} \int_0^1 |F_{j,t}(z)|^q \, dt \, d\mu(z) = \int_0^1 \int_{B_n} |F_{j,t}(z)|^q \, d\mu(z) \, dt \le C, \quad 1 \le j \le J(n).$$

[12, Chapter V, Theorem 8.4] guarantees that

$$\int_{B_n} \left(\sum_{k=0}^{\infty} |W_j[2^k](z)|^2\right)^{\frac{q}{2}} d\mu(z) \le C \int_{B_n} \int_0^1 |F_{j,t}(z)|^q \, dt \, d\mu(z) \le C.$$

Given positive numbers a_j, $1 \le j \le J(n)$, we have

$$\left(\sum_{j=1}^{J(n)} a_j\right)^{\frac{q}{2}} \le C_{q,n} \sum_{j=1}^{J(n)} a_j^{q/2}.$$

Hence,

$$\begin{aligned}\int_{B_n} \left(\sum_{j=1}^{J(n)} \sum_{k=0}^{\infty} |W_j[2^k](z)|^2\right)^{\frac{q}{2}} d\mu(z) &\le C \sum_{j=1}^{J(n)} \int_{B_n} \left(\sum_{k=0}^{\infty} |W_j[2^k](z)|^2\right)^{\frac{q}{2}} d\mu(z) \\ &\le CJ(n) \\ &\le C.\end{aligned}$$

Since $W_j[2^k]$ is a homogeneous polynomial of degree 2^k, estimate (1.2) guarantees that

$$\sum_{k=0}^{\infty} \sum_{j=1}^{J(n)} |W_j[2^k](z)|^2 \ge \delta^2 \sum_{k=0}^{\infty} |z|^{2^{k+1}} \ge \delta^2 \sum_{m=1}^{\infty} \frac{|z|^{2m}}{m} = \delta^2 \log \frac{1}{1-|z|^2}, \quad z \in B_n.$$

So,

$$\int_{B_n} \left(\log \frac{1}{1-|z|^2}\right)^{\frac{q}{2}} d\mu(z) < \infty.$$

Finally, remark that $1 \in \mathcal{B}(B_n)$, thus, μ is a finite measure. So, (2.1) holds. $\square$

Proposition 2.2. *Let $0 < q < \infty$ and let*

$$\int_{B_n} \left(\log \frac{e}{1-|z|}\right)^{q} d\mu(z) < \infty. \tag{2.2}$$

Then μ is a q-Carleson measure for $\mathcal{B}(B_n)$.

Proof. If (2.2) holds, then $\mathcal{A}^{-\log}(B_n) \subset L^q(B_n, \mu)$ by the definition of $\mathcal{A}^{-\log}(B_n)$. It remains to recall that $\mathcal{B}(B_n) \subset \mathcal{A}^{-\log}(B_n)$. $\square$

For $n = 1$, relations between (2.1) and (2.2) are discussed in [5].

Let σ_n denote the normalized Lebesgue measure on the sphere ∂B_n. The following lemma will be used to characterize the radial q-Carleson measures for $\mathcal{B}(B_n)$, $n \in \mathbb{N}$.

Lemma 2.3. *Let $0 < q < \infty$. Then*

$$\int_{\partial B_n} |f(r\zeta)|^q \, d\sigma_n(\zeta) \le C \|f\|_{\mathcal{B}(B_n)} \left(\log \frac{e}{1-r}\right)^{\frac{q}{2}}, \quad 0 \le r < 1, \tag{2.3}$$

for all $f \in \mathcal{B}(B_n)$.

Proof. Let $f \in \mathcal{B}(B_n)$. Given $\zeta \in B_n$, put $f_\zeta(\lambda) = f(\lambda\zeta)$ for $\lambda \in B_1$. So, $f_\zeta \in \mathcal{H}ol(B_1)$. Remark that $(\mathcal{R}f)(\lambda\zeta) = \lambda f'_\zeta(\lambda)$, hence,

$$\max_{|\lambda|\le 1/2} |f'_\zeta(\lambda)| \le 4\|f\|_{\mathcal{B}(B_n)}$$

by the maximum principle. Also, we have

$$\sup_{1/2<|\lambda|<1} (1-|\lambda|)|f'_\zeta(\lambda)| \le 2 \sup_{1/2<|\lambda|<1} (1-|\lambda\zeta|)|(\mathcal{R}f)(\lambda\zeta)| \le 2\|f\|_{\mathcal{B}(B_n)}.$$

Since $f_\zeta(0) = f(0)$, we obtain $\|f_\zeta\|_{\mathcal{B}(B_1)} \le C\|f\|_{\mathcal{B}(B_n)}$ for all $\zeta \in \partial B_n$.

Now, remark that Clunie and MacGregor [3] and Makarov [6] proved (2.3) for $n = 1$. So, applying [8, Proposition 1.4.7], we obtain

$$\begin{aligned}
\int_{\partial B_n} |f(r\zeta)|^q \, d\sigma_n(\zeta) &= \int_{\partial B_n}\int_{\partial B_1} |f_\zeta(rw)|^q \, d\sigma_1(w)\, d\sigma_n(\zeta) \\
&\le C \int_{\partial B_n} \|f_\zeta\|_{\mathcal{B}(B_1)} \left(\log\frac{e}{1-r}\right)^{\frac{q}{2}} d\sigma_n(\zeta) \\
&\le C\|f\|_{\mathcal{B}(B_n)} \left(\log\frac{e}{1-r}\right)^{\frac{q}{2}}
\end{aligned}$$

for $0 \le r < 1$, as required. □

Theorem 2.4. *Let $0 < q < \infty$ and let ρ be a positive measure on $[0,1)$. Then the following properties are equivalent:*

$$\int_0^1 \int_{\partial B_n} |f(r\zeta)|^q \, d\sigma_n(\zeta)\, d\rho(r) < \infty \quad \textit{for all } f \in \mathcal{B}(B_n); \tag{2.4}$$

$$\int_0^1 \left(\log\frac{e}{1-r}\right)^{\frac{q}{2}} d\rho(r) < \infty. \tag{2.5}$$

Proof. Let (2.5) holds. Assume that $f \in \mathcal{B}(B_n)$, then

$$\int_0^1 \int_{\partial B_n} |f(r\zeta)|^q \, d\sigma_n(\zeta)\, d\rho(r) \le C\|f\|_{\mathcal{B}(B_n)} \int_0^1 \left(\log\frac{e}{1-|z|}\right)^{\frac{q}{2}} d\rho(r) < \infty$$

by Lemma 2.3. So, (2.5) implies (2.4). It remains to remark that the converse implication holds by Proposition 2.1. □

3. Carleson measures for growth spaces

The following assertion is the key technical tool for the study of growth spaces. If $n = 1$, then the first part of Lemma 3.1 is obtained in [7] and [4] for $\beta = 1$ and for all $\beta > 0$, respectively. The second part of Lemma 3.1 is obtained in [5] for $n = 1$.

Lemma 3.1. *Let $n \in \mathbb{N}$. Then there exists $M = M(n)$ such that the following properties hold.*

(i) *Let* $\beta > 0$. *Then there exist functions* $f_m \in \mathcal{A}^{-\beta}(B_n)$, $0 \le m \le M$, *such that*

$$\sum_{m=0}^{M} |f_m(z)| \ge \frac{1}{(1-|z|)^\beta}, \quad z \in B_n. \tag{3.1}$$

(ii) *There exist functions* $g_m \in \mathcal{A}^{-\log}(B_n)$, $0 \le m \le M$, *such that*

$$\sum_{m=0}^{M} |g_m(z)| \ge \log \frac{e}{1-|z|}, \quad z \in B_n.$$

The proof of Lemma 3.1 is given in Section 4.

Theorem 3.2. *Let* $0 < q < \infty$ *and let* μ *be a positive measure on* B_n.

(i) *Let* $\beta > 0$. *Then* μ *is a* q*-Carleson measure for* $\mathcal{A}^{-\beta}(B_n)$ *if and only if*

$$\int_{B_n} \frac{d\mu(z)}{(1-|z|)^{\beta q}} < \infty. \tag{3.2}$$

(ii) μ *is a* q*-Carleson measure for* $\mathcal{A}^{-\log}(B_n)$ *if and only if*

$$\int_{B_n} \left(\log \frac{e}{1-|z|}\right)^q d\mu(z) < \infty.$$

Proof. Assume that (3.2) holds. If $f \in \mathcal{A}^{-\beta}(B_n)$, then

$$\int_{B_n} |f(z)|^q \, d\mu(z) \le \|f\|^q_{-\beta} \int_{B_n} \frac{d\mu(z)}{(1-|z|)^{\beta q}} < \infty.$$

To prove the converse implication, suppose that $\mathcal{A}^{-\beta}(B_n) \subset L^q(\mu)$. Let the number $M = M(n)$ and the functions f_m, $0 \le m \le M$, be those provided by Lemma 3.1. Given positive numbers a_m, $0 \le m \le M$, we have

$$\left(\sum_{m=0}^{M} a_m\right)^q \le C_{q,n} \sum_{m=0}^{M} a_m^q. \tag{3.3}$$

Using (3.1) and (3.3), we obtain

$$\begin{aligned} \int_{B_n} \frac{d\mu(z)}{(1-|z|)^{\beta q}} &\le \int_{B_n} \left(\sum_{m=0}^{M} |f_m(z)|\right)^q d\mu(z) \\ &\le C_{q,n} \sum_{m=0}^{M} \int_{B_n} |f_m(z)|^q \, d\mu(z) \\ &< \infty \end{aligned}$$

because $f_m \in \mathcal{A}^{-\beta}(B_n) \subset L^q(\mu)$. So, (i) holds. The proof of (ii) is analogous, so, we omit it. $\square$

4. Proof of Lemma 3.1

Proof of Lemma 3.1(i). Let the constant $\delta(n) \in (0,1)$ and the polynomials $W_j[d]$, $1 \le j \le J(n)$, $d \in \mathbb{N}$, be those provided by Theorem 1.1. Put

$$F_j(z) = \sum_{k=0}^{\infty} Q^{(\beta-1)k} W_j[Q^k](z), \quad z \in B_n,\ 1 \le j \le J(n),$$

where $Q \in \mathbb{N}$ is sufficiently large. Considering the slice functions $(F_j)_\zeta \in \mathcal{H}ol(B_1)$, $\zeta \in \partial B_n$, and applying (1.1) and the main result in [11], we infer that

$$|\mathcal{R}F_j(z)|(1-|z|)^\beta \le C, \quad z \in B_n.$$

In other words, $\mathcal{R}F_j \in \mathcal{A}^{-\beta}(B_n)$, $1 \le j \le J(n)$. Now, put

$$f_j(z) = \mathcal{R}F_j(z) = \sum_{k=0}^{\infty} Q^{\beta k} W_j[Q^k](z), \quad z \in B_n,\ 1 \le j \le J(n).$$

Claim. For all $Q \in \mathbb{N}$ large enough, we have

$$\sum_{j=1}^{J(n)} |f_j(z)| \ge \frac{C}{(1-|z|)^\beta} \quad \text{for} \tag{4.1}$$

$$1 - Q^{-k} \le |z| \le 1 - Q^{-(k+1/2)}, \quad k \in \mathbb{N}. \tag{4.2}$$

Proof of the claim. The argument below is similar to that used in the proof of [7, Proposition 5.4].

For any $z \in B_n$, we have

$$\begin{aligned} \sum_{j=1}^{J(n)} |f_j(z)| &\ge Q^{\beta k} \sum_{j=1}^{J(n)} |W_j[Q^k](z)| - J(n) \sum_{m=0}^{k-1} Q^{\beta m} |z|^{Q^m} - J(n) \sum_{m=k+1}^{\infty} Q^{\beta m} |z|^{Q^m} \\ &= \Sigma_0 - \Sigma_- - \Sigma_+, \quad k \in \mathbb{N}. \end{aligned}$$

Remark that, by (1.2),

$$\Sigma_0 \ge \delta Q^{\beta k} |z|^{Q^k}, \quad k \in \mathbb{N}. \tag{4.3}$$

Below we assume that (4.2) holds. So, we have

$$(1 - Q^{-k})^{Q^k} \le |z|^{Q^k} \le \left(\left(1 - Q^{-(k+1/2)}\right)^{Q^{k+1/2}} \right)^{Q^{-1/2}}, \quad k \in \mathbb{N}.$$

Thus, if Q is large enough, then

$$1/3 \le |z|^{Q^k} \le 2^{-Q^{-1/2}}, \quad k \in \mathbb{N}. \tag{4.4}$$

Therefore, $\Sigma_0 \geq \delta Q^{\beta k}/3$ by (4.3). Also, we have

$$\Sigma_- \leq J(n) \sum_{m=0}^{k-1} Q^{\beta m} \leq \frac{J(n)Q^{\beta k}}{Q-1}.$$

Now, consider the third term. Remark that

$$|z|^{Q^m(Q-1)} \leq |z|^{Q^{k+1}(Q-1)} \quad \text{for } m \geq k+1,\ z \in B_n.$$

So, the ratio of two successive terms in Σ_+ is not greater that the ratio of the first two terms. Hence, the series Σ_+ is dominated by the geometric series having the same first two terms. Thus, putting $x = |z|^{Q^k}$, we obtain

$$\begin{aligned}
\Sigma_+/J(n) &\leq Q^{(k+1)\beta}|z|^{Q^{k+1}} \sum_{m=0}^{\infty} \left(Q^\beta |z|^{(Q^{k+2}-Q^{k+1})} \right)^m \\
&= \frac{Q^{(k+1)\beta}|z|^{Q^{k+1}}}{1 - Q^\beta |z|^{(Q^{k+2}-Q^{k+1})}} \\
&= Q^{k\beta} \frac{Q^\beta x^Q}{1 - Q^\beta x^{(Q^2-Q)}} \\
&\leq Q^{k\beta} \frac{Q^\beta 2^{-Q^{1/2}}}{1 - Q^\beta 2^{(Q^{1/2}-Q^{3/2})}}
\end{aligned}$$

by (4.4). In sum, we have

$$\sum_{j=1}^{J(n)} |f_j(z)| \geq \frac{\delta}{4} Q^{\beta k} = \frac{\delta}{4Q^{\beta/2}} Q^{\beta(k+1/2)} \geq \frac{\delta}{4Q^{\beta/2}} \frac{1}{(1-|z|)^\beta}$$

if Q is sufficiently large and z satisfies (4.2). The proof of the claim is complete. $\square$

Similarly, let

$$f_{J(n)+j}(z) = \sum_{k=0}^{\infty} Q^{\beta(k+1/2)} W_j[Q^{k+1/2}](z), \quad z \in B_n,\ 1 \leq j \leq J(n).$$

where $Q = q^2$ and $q \in \mathbb{N}$. If q is sufficiently large, then $f_{J(n)+j} \in \mathcal{A}^{-\beta}(B_n)$ and

$$\sum_{j=1}^{J(n)} |f_{J(n)+j}(z)| \geq \frac{C}{(1-|z|)^\beta} \quad \text{for} \tag{4.5}$$

$$1 - Q^{-(k+1/2)} \leq |z| \leq 1 - Q^{-(k+1)}, \qquad k \in \mathbb{N}. \tag{4.6}$$

The proof of the above estimate is analogous to that of the claim; so, we omit it.

Now, fix Q so large that (4.1) and (4.5) hold under assumptions (4.2) and (4.6), respectively. Put $M = 2J(n)$ and multiply the functions f_m, $1 \leq m \leq M$, by a sufficiently large constant. Then

$$\sum_{m=1}^{M} |f_m(z)| \geq \frac{1}{(1-|z|)^{\beta}} \quad \text{for } 1 - Q^{-1} \leq |z| < 1.$$

It remains to define $f_0 \equiv Q$. The proof of Lemma 3.1(i) is complete. □

Proof of Lemma 3.1(ii). Put

$$g_j(z) = \sum_{k=0}^{\infty} Q^k W_j[Q^{Q^k}](z), \quad z \in B_n,\ 1 \leq j \leq J(n),$$

where the notation from the proof of Lemma 3.1(i) is used. Then $g_j \in \mathcal{A}^{-\log}(B_n)$, $1 \leq j \leq J(n)$, by Theorem 12 from [5]. The argument used in the proof of Theorem 2 from [5] guarantees that

$$\sum_{j=1}^{J(n)} |g_j(z)| \geq C \log \frac{1}{1-|z|} \quad \text{for } 1 - Q^{-Q^k} \leq |z| \leq 1 - Q^{-Q^{(k+1/2)}},\ k \in \mathbb{N},$$

if $Q \in \mathbb{N}$ is large enough (see also the proof of Lemma 3.1(i)).

Similarly, let

$$g_{J(n)+j}(z) = \sum_{k=0}^{\infty} Q^{(k+1/2)} W_j[Q^{Q^{(k+1/2)}}](z), \quad z \in B_n,\ 1 \leq j \leq J(n).$$

where $Q = q^2$ and $q \in \mathbb{N}$. If q is large enough, then $g_{J(n)+j} \in \mathcal{A}^{-\log}(B_n)$ and

$$\sum_{j=1}^{J(n)} |g_{J(n)+j}(z)| \geq C \log \frac{1}{1-|z|} \quad \text{for } 1 - Q^{-Q^{(k+1/2)}} \leq |z| \leq 1 - Q^{-Q^{(k+1)}},\ k \in \mathbb{N}.$$

To finish the proof, put $M = 2J(n)$ and $f_0 \equiv C > 0$, where the constant C is sufficiently large. □

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