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Determinantal representation of the stationary correlation function for the totally asymmetric exclusion model

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Using the Quantum Inverse Scattering Method we obtain the exact expression for the stationary correlation function of the totally asymmetric exclusion process.

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I. TOTALLY ASYMMETRIC EXCLUSION MODEL

The totally asymmetric exclusion process (TASEP) is one of the most studied systems in non-equilibrium low dimensional physics (see [1] and refs.). It describes a system of N particles on a periodic ring with M sites labelled $i = M, M-1, ..., 2, 1$. The particles move randomly in one direction from right to left. The exclusion rule forbids to have more than one particle per site.

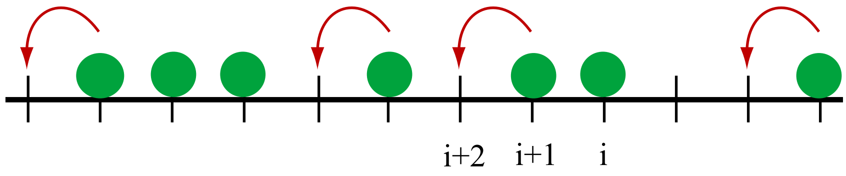

FIG. 1: Schematic representation of the TASEP on a lattice.

This process can be conveniently represented using the spin description - in each lattice site the spin-up state corresponds to the empty site $|0\rangle$, the spin-down state corresponds to the occupied one $|1\rangle$. The configuration of N particles ($2N \leq M$) located in the sites $M \geq m_1 > m_2 > ... > m_N \geq 1$ is associated with the basis vector

$$|m_1, m_2, ..., m_N\rangle = \sigma^-_{m_1}\sigma^-_{m_2}...\sigma^-_{m_N}|\Omega\rangle,$$

where the generating state $|\Omega\rangle$ is the state with all spins up

$$|\Omega\rangle = \otimes_{i=1}^{M}|0\rangle_i.$$

The generator of the described exclusion process is a non-Hermitian Hamiltonian

$$H = -\sum_{j=1}^{M}\left\{\sigma^-_{j+1}\sigma^+_j + \frac{1}{4}(\sigma^z_{j+1}\sigma^z_j - 1)\right\}. \tag{1}$$

Here $\sigma^{\pm,z}$- are the Pauli matrices, the matrix with subindex j acts nontrivially only in the j-th spin space of the total space of states of the chain $(\mathcal{C}^2)^{\otimes M}$: $s_j = I\otimes\ldots\otimes I\otimes s\otimes I\otimes\ldots\otimes I$, and the periodic boundary conditions are assumed: $\sigma_n = \sigma_{n+M}$.

The first term of the Hamiltonian describes the hoppings of particles with hard-core repulsion

$$\sigma^-_{j+1}\sigma^+_j|0\rangle_{j+1}|1\rangle_j = |1\rangle_{j+1}|0\rangle_j.$$

The second term counts the number of the allowed jumps of the particles.

The correlation function of interest [2] is the stationary correlation function:

$$Z_N^{-1}\langle S_N|s_1 e^{-|t|H} s_m|S_N\rangle, \tag{2}$$

where the projection operator $s_k = \frac{1}{2}(1+\sigma_k^z)$ has value 1 if there is no particle at site k and 0 otherwise. The state $|S_N\rangle$ is a ground state of Hamiltonian with the eigenvalue zero:

$$H|S_N\rangle = 0. \tag{3}$$

In the steady state $|S_N\rangle$ every spin configuration with N spins down has the equal weight:

$$|S_N\rangle = \sum_{M\geq m_1>m_2>\ldots>m_N\geq 1} |m_1,\ldots,m_N\rangle. \tag{4}$$

The left eigenvector of H with the eigenvalue zero is

$$\langle S_N| = \sum_{M\geq m_1>m_2>\ldots>m_N\geq 1} \langle m_1,\ldots,m_N|. \tag{5}$$

The total number of configurations in the ground state

$$Z_N = \langle S_N|S_N\rangle = \frac{M!}{N!(M-N)!}. \tag{6}$$

The right steady state can be expressed as

$$|S_N\rangle = \mathcal{P}_R^N|\Omega\rangle, \tag{7}$$

where

$$\mathcal{P}_R = \sum_{k=1}^{M} s_M \ldots s_{k+1}\sigma_k^- . \tag{8}$$

For the left state we have respectively

$$\langle S_N| = \langle\Omega|\mathcal{P}_L^N \tag{9}$$

with

$$\mathcal{P}_L = \sum_{k=1}^{M} \sigma_k^+ s_{k-1}\ldots s_1. \tag{10}$$

II. SOLUTION OF THE MODEL

To the solution of the model we shall apply the Quantum Inverse Scattering Method [3–5]. The L-operator of the considered model [6] is a 2×2 matrix with the entries acting on the space of states of an M-site spin-$\frac{1}{2}$ chain:

$$\begin{aligned} L(n|u) \;&=\; \begin{pmatrix} us_n & \sigma_n^- \\ \sigma_n^+ & uI - u^{-1}s_n \end{pmatrix} \\ &=\; ss_n + (I-s)(uI - u^{-1}s_n) + \sigma^-\sigma_n^+ + \sigma^+\sigma_n^-, \end{aligned} \tag{11}$$

where the parameter $u \in \mathcal{C}$.

The L-operator (11) satisfies the intertwining relation

$$R(u,v)\,(L(n|u) \otimes L(n|v)) = (L(n|v) \otimes L(n|u))\,R(u,v), \tag{12}$$

in which $R(u,v)$ is the 4×4 matrix

$$R(u,v) = \begin{pmatrix} f(v,u) & 0 & 0 & 0 \\ 0 & g(v,u) & 1 & 0 \\ 0 & 0 & g(v,u) & 0 \\ 0 & 0 & 0 & f(v,u) \end{pmatrix} \tag{13}$$

with

$$f(v,u) = \frac{u^2}{u^2 - v^2}, \;\; g(v,u) = \frac{uv}{u^2 - v^2}. \tag{14}$$

The monodromy matrix is the product of L-operators

$$T(u) = L(M|u)L(M-1|u)...L(1|u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{15}$$

The commutation relations of the matrix elements of the monodromy matrix are given by the same $R-$matrix (13)

$$R(u,v)\,(T(u) \otimes T(v)) = (T(v) \otimes T(u))\,R(u,v). \tag{16}$$

The most important relations are

$$\begin{aligned} C(u)B(v) \;&=\; g(u,v)\,\{A(u)D(v) - A(v)D(u)\} \\ A(u)B(v) \;&=\; f(u,v)B(v)A(u) + g(v,u)B(u)A(v), \\ D(u)B(v) \;&=\; f(v,u)B(v)D(u) + g(u,v)B(u)D(v), \\ [B(u),B(v)] \;&=\; [C(u),C(v)] = 0. \end{aligned} \tag{17}$$

The transfer matrix $\tau(u)$ is the trace of the monodromy matrix in the auxiliary space

$$\tau(u) = u^{-M} trT(u) = u^{-M}\left(A(u) + D(u)\right). \tag{18}$$

The relation (16) means that $[\tau(u), \tau(v)] = 0$ for arbitrary values of parameters u, v.

The cyclic shift operator in the total spin space $(\mathcal{C}^2)^{\otimes M}$ is expressed through the transfer matrix:

$$\tau \equiv \tau(1) = \Pi_{12}\Pi_{23}\dots\Pi_{M-1M}. \tag{19}$$

Here

$$\Pi_{mn} = s_m s_n + (I - s_m)(I - s_n) + \sigma_m^- \sigma_n^+ + \sigma_m^+ \sigma_n^-,$$

is the permutation operator: $\Pi_{mn}\sigma_m = \sigma_n \Pi_{mn}$. The shift operator shifts the site indices

$$\tau^{n-1}\sigma_1\tau^{1-n} = \sigma_n \tag{20}$$

and possesses the property $\tau^M = I$.

The Hamiltonian (1) is expressed through the transfer matrix:

$$H = -\frac{1}{2}\tau^{-1}(1)\frac{\partial}{\partial u}\tau(u)|_{u=1}. \tag{21}$$

The right state vector of the model is the vector generated by the multiple action of operators $\tilde{B}(u) \equiv u^{-(M-1)}B(u)$ on the generating state $|\Omega\rangle = \otimes_{i=1}^M |0\rangle_i$

$$|\Psi(u_1, u_2, ..., u_N)\rangle = \prod_{i=1}^N \tilde{B}(u_i)|\Omega\rangle. \tag{22}$$

The generating state is annihilated by the operator $C(u)$

$$C(u)|\Omega\rangle = 0, \tag{23}$$

and it is an eigenvector of operators $A(u)$ and $D(u)$,

$$A(u)|\Omega\rangle = \alpha(u)|\Omega\rangle; \;\; D(u)|\Omega\rangle = \delta(u)|\Omega\rangle \tag{24}$$

with the eigenvalues

$$\alpha(u) = u^M, \;\; \delta(u) = (u - u^{-1})^M. \tag{25}$$

The left state vector is equal to

$$\langle\Psi(u_1, u_2, ..., u_N)| = \langle\Omega| \prod_{i=1}^N \tilde{C}(u_i), \tag{26}$$

where $\tilde{C}(u_i) = u^{-(M-1)}C(u)$, and $\langle\Omega|B(u) = 0$.

From definitions (11) and (15) it follows that

$$\begin{aligned} u^{M-1}B(u) &= u^{2(M-1)}\mathcal{P}_R + \ldots + (-1)^{M-1}\sigma_M^- s_{M-1}...s_1, \\ u^{M-1}C(u) &= u^{2(M-1)}\mathcal{P}_L + \ldots + (-1)^{M-1}s_M...s_2\sigma_1^+, \end{aligned} \tag{27}$$

and hence the steady states may be represented as

$$\begin{aligned} |S_N\rangle &= \lim_{\{u\}\to\infty} \prod_{i=1}^{N} \tilde{B}(u_i)|\Omega\rangle, \\ \langle S_N| &= \lim_{\{u\}\to\infty} \langle\Omega| \prod_{i=1}^{N} \tilde{C}(u_i). \end{aligned} \tag{28}$$

The scalar product of the state vectors (22) and (26) is evaluated by means of the commutation relations (17) and for the arbitrary variables $u, v \in \mathcal{C}$ is given by the following expression

$$\begin{aligned} &\langle\Psi(v_1, v_2, ..., v_N)|\Psi(u_1, u_2, ..., u_N)\rangle \\ &= \left\{ \prod_{j=1}^{N} \frac{1}{(v_j u_j)^{M-1}} \prod_{N\geq j>k\geq 1} \frac{v_j v_k}{v_k^2 - v_j^2} \prod_{N\geq l>n\geq 1} \frac{u_l u_n}{u_l^2 - u_n^2} \right\} \det Q. \end{aligned} \tag{29}$$

The matrix elements of the $N \times N$ matrix Q are

$$\begin{aligned} Q_{jk} &= \left\{ v_j^M (u_k - u_k^{-1})^M \left(\frac{u_k}{v_j}\right)^{N-1} - u_k^M (v_j - v_j^{-1})^M \left(\frac{u_k}{v_j}\right)^{-N+1} \right\} \\ &\quad \times \frac{1}{\frac{u_k}{v_j} - \left(\frac{u_k}{v_j}\right)^{-1}}. \end{aligned} \tag{30}$$

Using this representation and equalities (28) that represent the steady states through the state vectors of the model we can calculate the projection of the state vectors (22) and (26) on the steady states. For the left steady state we have

$$\langle S_N|\Psi(u_1, u_2, ..., u_N)\rangle = \lim_{\{v\}\to\infty} \prod_{i=1}^{N} \langle\Psi(v_1, v_2, ..., v_N)|\Psi(u_1, u_2, ..., u_N)\rangle.$$

Taking the limit we obtain

$$\langle S_N|\Psi(u_1, u_2, ..., u_N)\rangle = \prod_{k=1}^{N} u_k^2 \prod_{N\geq l>n\geq 1} \frac{1}{u_l^2 - u_n^2} \det V^{(M)}, \tag{31}$$

where $V^{(M)}$ is a $N \times N$ matrix with the entries equal to

$$
\begin{aligned}
V_{jk}^{(M)} &= \sum_{n=0}^{j-1} (-1)^n \binom{M}{n} u_k^{2(j-1-n)}, \;\; 1 \le j \le N-1; \\
V_{Nk}^{(M)} &= -\sum_{n=N-1}^{M} (-1)^n \binom{M}{n} u_k^{-2(n-N+1)}.
\end{aligned}
\tag{32}
$$

The projection $\langle \Psi(u_1, u_2, ..., u_N)|S_N\rangle$ is given by the similar expression

$$
\begin{aligned}
\langle \Psi(u_1, u_2, ..., u_N)|S_N\rangle &= \prod_{k=1}^{N} u_k^2 \prod_{N \ge n > l \ge 1} \frac{1}{u_l^2 - u_n^2} \det \tilde{V}^{(M)} \\
\tilde{V}_{jk}^{(M)} &= \sum_{n=0}^{N-j} (-1)^n \binom{M}{n} u_k^{2(N-j-n)}, \;\; 2 \le j \le N; \\
\tilde{V}_{1k}^{(M)} &= -\sum_{n=N}^{M} (-1)^n \binom{M}{n} u_k^{-2(n-N+1)}.
\end{aligned}
\tag{33}
$$

III. FORM-FACTORS

To calculate the correlation function (2) we need to calculate the form-factor of the projection operator:

$$\langle \Psi(v_1, v_2, ..., v_N)|s_1|\Psi(u_1, u_2, ..., u_N)\rangle.$$

The monodromy matrix (15) may be represented as

$$
T(u) = \begin{pmatrix} A_{M-1}(u) & B_{M-1}(u) \\ C_{M-1}(u) & D_{M-1}(u) \end{pmatrix} \begin{pmatrix} us_1 & \sigma_1^- \\ \sigma_1^+ & uI - u^{-1}s_1 \end{pmatrix}.
\tag{34}
$$

From this representation it follows that

$$
\begin{aligned}
A(u) &= uA_{M-1}(u)s_1 + B_{M-1}(u)\sigma_1^+, \\
B(u) &= A_{M-1}(u)\sigma_1^- + uB_{M-1}(u) - u^{-1}B_{M-1}(u)s_1, \\
C(u) &= uC_{M-1}(u)s_1 + D_{M-1}(u)\sigma_1^+, \\
D(u) &= C_{M-1}(u)\sigma_1^- + uD_{M-1}(u) - u^{-1}D_{M-1}(u)s_1.
\end{aligned}
$$

In particular we have

$$
\begin{aligned}
s_1 B(u) &= (u - u^{-1})B_{M-1}(u)s_1, \\
C(u)s_1 &= us_1 C_{M-1}(u).
\end{aligned}
\tag{35}
$$

These commutation relations allow us to calculate the matrix element

$$\langle \Psi(v_1, v_2, ..., v_N)|s_1|\Psi(u_1, u_2, ..., u_N)\rangle$$

for the arbitrary values of parameters u, v. We have

$$\begin{aligned} & \langle \Psi(v_1, v_2, ..., v_N)|s_1|\Psi(u_1, u_2, ..., u_N)\rangle \\ = & \langle\Omega|\prod_{j=1}^{N}\tilde{C}(v_j)s_1\prod_{i=1}^{N}\tilde{B}(u_i)|\Omega\rangle = \langle\Omega|\prod_{j=1}^{N}\tilde{C}(v_j)s_1^2\prod_{i=1}^{N}\tilde{B}(u_i)|\Omega\rangle \\ = & \prod_{k=1}^{N}(1-u_k^{-2})\langle\Omega|\prod_{j=1}^{N}\tilde{C}_{M-1}(v_j)\prod_{i=1}^{N}\tilde{B}_{M-1}(u_i)|\Omega\rangle. \end{aligned}$$

From this formula we see that the form-factor is proportional to the scalar product (29) of the state vectors on a lattice with $M-1$ sites. Taking the limit $\{v\} \to \infty$ we obtain

$$\begin{aligned} & \langle S_L|s_1|\Psi(u_1, u_2, ..., u_N)\rangle \\ = & \prod_{k=1}^{N}(1-u_k^{-2})\lim_{\{v\}\to\infty}\langle\Omega|\prod_{j=1}^{N}\tilde{C}_{M-1}(v_j)\prod_{i=1}^{N}\tilde{B}_{M-1}(u_i)|\Omega\rangle \\ = & \prod_{k=1}^{N}(u_k^2-1)\prod_{N\geq l>n\geq 1}\frac{1}{u_l^2-u_n^2}\det V^{(M-1)}, \end{aligned} \tag{36}$$

where the entries of $N \times N$ matrix $V^{(M-1)}$ are (32) with M replaced on $M-1$. Respectively we have

$$\langle \Psi(u_1, u_2, ..., u_N)|s_1|S_N\rangle = \prod_{k=1}^{N}u_k^2\prod_{N\geq n>l\geq 1}\frac{1}{u_l^2-u_n^2}\det\tilde{V}^{(M-1)}. \tag{37}$$

IV. BETHE ANSATZ SOLUTION

According to the algebraic Bethe ansatz method the state vectors (22) and (26) are the right and the left eigenstates of the transfer matrix (18) with the same eigenvalues

$$\begin{aligned} \tau(v)|\Psi(u_1, u_2, ..., u_N)\rangle &= \Theta_N(v, \{u\})|\Psi(u_1, u_2, ..., u_N)\rangle, \\ \langle\Psi(u_1, u_2, ..., u_N)|\tau(v) &= \langle\Psi(u_1, u_2, ..., u_N)|\Theta_N(v, \{u\}) \end{aligned} \tag{38}$$

if parameters $u_1, u_2, ..., u_N$ satisfy Bethe equations

$$(1-u_n^{-2})^{-M}u_n^{-2N} = (-1)^{N-1}\prod_{j=1}^{N}u_j^{-2} \equiv (-1)^{N-1}U^{-2}. \tag{39}$$

The eigenvalues $\Theta_N(v,\{u\})$ are equal to

$$\Theta_N(v,\{u\}) = \prod_{j=1}^{N} \frac{u_j^2}{u_j^2 - v^2} + (1-v^{-2})^M \prod_{j=1}^{N} \frac{v^2}{v^2 - u_j^2}. \tag{40}$$

The transfer matrix (18) $\tau(u)$ is stochastic:

$$\tau(u)|S_N\rangle = \Theta_N(v,\{\infty\})|S_N\rangle = |S_N\rangle. \tag{41}$$

From the definition of the cyclic shift operator (19) it follows that its eigenvalues are equal to

$$\Theta_N(1,\{u\}) = \prod_{j=1}^{N} \frac{1}{1-u_j^{-2}}. \tag{42}$$

From the definition (21) it follows that the eigenenergies of the Hamiltonian (1) are

$$E_N = -\frac{1}{2}\Theta_N^{-1}(1,\{u\})\frac{\partial}{\partial v}\Theta_N(v,\{u\})|_{v=1} = -\sum_{j=1}^{N} \frac{1}{u_j^2 - 1}. \tag{43}$$

It is known that there is Z_N independent solutions of Bethe equations (39) [7]. The obvious solution $u_1 = u_2 = ... = u_N = \infty$ provides the stationary solution with the eigenvalue $E_N = 0$.

The scalar product of eigenvectors (22), (26) is found from formula (29) with $v_j = u_j$ satisfying Bethe equations (39). Understanding the diagonal elements of the matrix Q in the sense of L'Hôpitall rule one obtains the following expression for the norm of any eigenvector:

$$\begin{aligned}\mathcal{N}^2(u_1,u_2,...,u_N) &= \langle\Psi(u_1,u_2,...,u_N)|\Psi(u_1,u_2,...,u_N)\rangle \\ &= U^{2N}\prod_{l\neq n}\frac{1}{u_l^2 - u_n^2}\det\tilde{Q}\end{aligned} \tag{44}$$

with the entries of the matrix $\tilde{Q}$ equal to

$$\tilde{Q}_{jk} = \frac{N-1+(M-N+1)u_j^{-2}}{1-u_j^{-2}}\delta_{jk} - (1-\delta_{jk}).$$

For the special solution $u_1 = u_2 = ... = u_N = \infty$ the norm of eigenvectors is equal to Z_N.

It may be proved that the state vectors belonging to the different sets of the solutions of the Bethe equations are orthogonal, and they provide a complete basis of eigenvectors. It means that the eigenvectors (38) provide the resolution of the identity operator

$$I = \sum_{\{u\}} \frac{|\Psi(u_1,u_2,...,u_N)\rangle\langle\Psi(u_1,u_2,...,u_N)|}{\mathcal{N}^2(u_1,u_2,...,u_N)}, \tag{45}$$

where the summation is over all different solutions of Bethe equations (39).

V. STATIONARY CORRELATION FUNCTION

For the transitionally invariant system the form-factor of the projection operator s_m is expressed through the form-factor of the operator s_1:

$$\begin{aligned}\langle\Psi(u_1,u_2,...,u_N)|s_m|S_N\rangle &= \langle\Psi(u_1,u_2,...,u_N)|\tau^{m-1}s_1\tau^{1-m}|S_N\rangle \\ &= \prod_{j=1}^{N}\left(1-u_j^{-2}\right)^{1-m}\langle\Psi(u_1,u_2,...,u_N)|s_1|S_N\rangle,\end{aligned}$$

where the property (42) was used.

The substitution of the resolution of the identity into $\langle S_N|s_1e^{-|t|H}s_m|S_N\rangle$ gives

$$\begin{aligned}&\langle S_N|s_1e^{-|t|H}s_m|S_N\rangle \\ &= \sum_{\{u\}}\frac{\langle S_N|s_1e^{-|t|H}|\Psi(u_1,u_2,...,u_N)\rangle\langle\Psi(u_1,u_2,...,u_N)|s_m|S_N\rangle}{\mathcal{N}^2(u_1,u_2,...,u_N)} \\ &= \sum_{\{u\}}e^{-|t|E_N}\frac{\langle S_N|s_1|\Psi(u_1,u_2,...,u_N)\rangle\langle\Psi(u_1,u_2,...,u_N)|s_1|S_N\rangle}{\mathcal{N}^2(u_1,u_2,...,u_N)\prod_{j=1}^{N}\left(1-u_j^{-2}\right)^{m-1}}.\end{aligned} \tag{46}$$

With the help of the determinantal representations for the form-factors (36), (37) and for the norm (44) we finally obtain

$$\begin{aligned}&\frac{1}{Z_N}\langle S_N|s_1e^{-|t|H}s_m|S_N\rangle \\ &= \left(\frac{M-N}{M}\right)^2 \\ &+\frac{1}{Z_N}\sum_{\{u\}}\frac{e^{-|t|E_N}}{U^{2N}}\prod_{j=1}^{N}\frac{(u_j^2-1)u_j^2}{\left(1-u_j^{-2}\right)^{m-1}}\frac{\det\tilde{V}^{(M-1)}\det V^{(M-1)}}{\det\tilde{Q}},\end{aligned} \tag{47}$$

where the summation is over all different solutions of Bethe equations (39) except the special one. The first term on the r.h.s. is the contribution of the stationary state.

VI. CONCLUSION

The integrable models connected with the "crystal base" R-matrix (13) are important in the enumerative combinatorics. These models naturally appear in the theory of the boxed plane partitions – three-dimensional Young diagrams placed into a box of a finite size, and in the theory of random walkers [8, 9]. Some aspects of these connections were discussed in the papers [10–13].

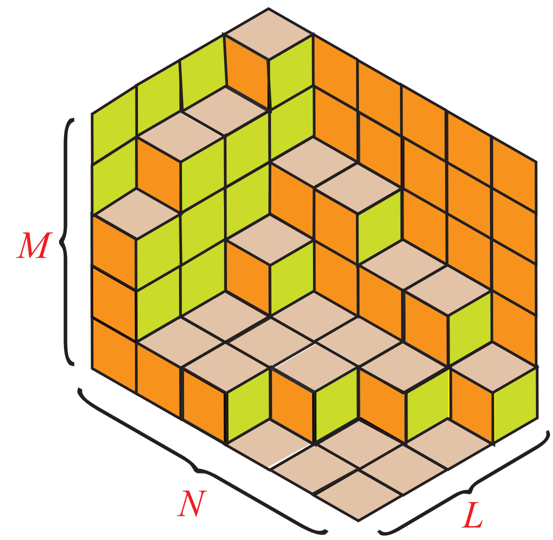

FIG. 2: Boxed plane partition.

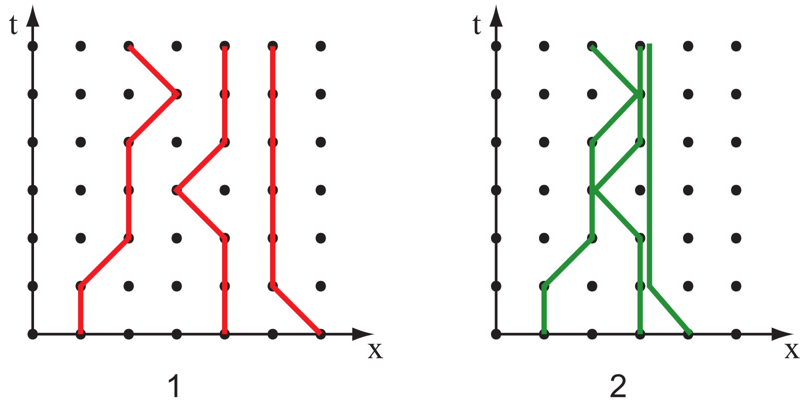

FIG. 3: The typical lattice paths of vicious (1) and friendly (2) walkers.

In its turn plane partitions and random walkers are employed in analyzes of the models of statistical physics describing faceted crystals [14], direct percolation [15], one-dimensional growth processes [16]. It emphasizes the importance of the mentioned integrable models in the theory of the non-equilibrium processes.

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