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ON THE NONSTATIONARY MOTION OF A VISCOUS INCOMPRESSIBLE LIQUID OVER A ROTATING BALL

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Abstract

The paper is concerned with the evolution free boundary problem for the Navier-Stokes equations governing the motion of a viscous incompressible liquid that covers the surface of a ball rotating with a constant angular velocity ω. It is assumed that the liquid is subject to the gravitational forces generated by the mass of the ball. We consider the problem of stability of the solution corresponding to the rigid rotation of the liquid with the same angular velocity ω. It is shown that this solution is stable, if $|\omega| \leq \omega_0$ where ω_0 depends on the parameters of the problem.

Contents

1 Introduction

We consider the free boundary problem for the Navier-Stokes equations governing the non-stationary motion of a viscous incompressible liquid that covers the surface of a ball rotating with the constant angular velocity ω around a fixed axis (x_3-axis). The liquid is contained in a domain $\Omega_t \subset \mathbb{R}^n$ bounded by the surface $S = \{|x| = d\}$ of the ball $K = \{|x| \leq d\}$ and by a free surface Γ_t that has no points of intersection with S. It is assumed that the liquid is subject to the gravitation force $k\nabla|x|^{-1}$ from the part of the ball but the self-gravitation forces between the liquid particles and capillary forces on Γ_t are not taken into account. So the problem consists of determination of Ω_t, of the vector field of velocities $\boldsymbol{v}(x,t) = (v_1, v_2, v_3)$, and of the scalar pressure $p(x,t)$, $x \in \Omega_t$, satisfying the following relations:

$$\begin{gathered}
\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = k\nabla|x|^{-1}, \\
\nabla\cdot\boldsymbol{v} = 0, \qquad x \in \Omega_t,\ \ t > 0, \\
\boldsymbol{v}(x,t) = \omega\boldsymbol{\eta}_3, \quad x \in S, \\
T(\boldsymbol{v},p)\boldsymbol{n} = 0, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \quad x \in \Gamma_t, \\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_0.
\end{gathered}$$

Here $\nu, k = const > 0$, $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor, $S(\boldsymbol{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\right)_{j,k=1,2,3}$ is the doubled rate-of-strain tensor, V_n is the velocity of the evolution of Γ_t in the direction of the exterior normal $\boldsymbol{n}$, $\boldsymbol{\eta}_3 = \boldsymbol{e}_3 \times x$, $\boldsymbol{e}_3$ is the unit vector parallel to the x_3-axis. The function $k/|x|$ is a gravitational potential of the ball K at the point $x \in \mathbb{R}^3 \setminus K$. The density of the liquid is assumed to be equal to one. The domain Ω_0 is given.

Introducing a new pressure $p - k/|x|$ instead of p we can write the above problem in the form

$$\boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \qquad x \in \Omega_t,\ \ t > 0,$$

$$T(\boldsymbol{v},p)\boldsymbol{n} = \boldsymbol{n}k/|x|, \quad V_n = \boldsymbol{v}\cdot\boldsymbol{n}, \qquad x \in \Gamma_t, \tag{1.1}$$
$$\boldsymbol{v}(x,t) = \omega\boldsymbol{\eta}_3, \quad x \in S,$$
$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad x \in \Omega_0.$$

The problem (1.1) has a solution corresponding to a rigid rotation of the liquid around the x_3-axis with the angular velocity ω. The corresponding velocity and pressure are given by

$$\boldsymbol{V}(x) = \omega(\boldsymbol{e}_3 \times \boldsymbol{x}) = \omega(-x_2, x_1, 0), \qquad P(x) = \frac{\omega^2}{2}|x'|^2 + p_0 \tag{1.2}$$

where $x' = (x_1, x_2, 0)$, $p_0 = const.$ These functions satisfy the Navier-Stokes equations. When we plug them into the boundary condition $T(\boldsymbol{v},p)\boldsymbol{n} = \boldsymbol{n}k/|x|$ we obtain the equation for the free surface $\mathcal{G}$ of the rotating liquid:

$$\frac{k}{|x|} + \frac{\omega^2}{2}|x'|^2 = -p_0 \equiv a^2, \quad x \in \mathcal{G}. \tag{1.3}$$

It is natural to call the domain bounded by S and $\mathcal{G}$ *the equilibrium figure.* It is easily seen that $\mathcal{G}$ is smooth and axially symmetric. The kinematic boundary condition $V_n = \boldsymbol{V}\cdot\boldsymbol{n}$ on $\mathcal{G}$ and the adherence condition on S are satisfied, hence the functions (1.2) given in $\mathcal{F}$ represent a stationary solution of (1.1).

We assume that the equilibrium figure $\mathcal{F}$ is given. Some properties of $\mathcal{F}$ will be considered in Sec.2.

We analyze the problem of stability of the solution (1.2) and consider the problem for the perturbations of the velocity and pressure

$$\boldsymbol{v}_r(x,t) = \boldsymbol{v}(x,t) - \boldsymbol{V}(x), \quad p_r(x,t) = p(x,t) - P(x)$$

written in the coordinate system rotating around the x_3-axis with the same angular velocity ω. We introduce new variables

$$x = \mathcal{Z}(\omega t)y$$

where

$$\mathcal{Z}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and new unknown functions

$$\boldsymbol{w}(y,t) = \mathcal{Z}^{-1}(\omega t)\boldsymbol{v}_r(\mathcal{Z}(\omega t)y,t), \quad s(y,t) = p_r(\mathcal{Z}(\omega t)y,t)$$

and arrive at the free boundary problem

$$\boldsymbol{w}_t + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}) - \nu\nabla^2\boldsymbol{w} + \nabla s = 0,$$

$$\nabla\cdot\boldsymbol{w} = 0, \qquad y \in \Omega_t', \quad t > 0,$$

$$T(\boldsymbol{w}, s)\boldsymbol{n}' = (\frac{k}{|y|} + \frac{\omega^2}{2}|y'|^2 - a^2)\boldsymbol{n}', \tag{1.4}$$
$$V_n' = \boldsymbol{w} \cdot \boldsymbol{n}', \quad y \in \Gamma_t',$$
$$\boldsymbol{w} = 0, \quad y \in S,$$
$$\boldsymbol{w}(y, 0) = \boldsymbol{v}_0(y) - \boldsymbol{V}(y) \equiv \boldsymbol{w}_0(y), \quad y \in \Omega_0,$$

where $\Omega_t' = \mathcal{Z}^{-1}(\omega t)\Omega_t$, $\Gamma_t' = \partial\Omega_t'$, $\boldsymbol{n}'$ is the exterior normal to Γ_t'. To the solution (1.2) of (1.1) corresponds the solution $\boldsymbol{w} = 0, s = 0$ of (1.4).

Our objective is to prove that, under a certain restriction on ω, problem (1.4) with a small initial value of the velocity $\boldsymbol{w}_0$ given in the domain Ω_0 close to $\mathcal{G}$ has a unique solution in an infinite time interval $t > 0$ and $\boldsymbol{w}, s \to 0$ as $t \to \infty$. This means the stability of the solution (1.2).

We write the problem (1.4) as a nonlinear problem in a fixed domain. We return to original notations Ω_t, Γ_t, x (instead of Ω_t', Γ_t', y) and introduce the Lagrangian coordinates $\xi \in \Omega_0$ related to the Eulerian coordinates $x \in \Omega_t$ by

$$x = \xi + \int_0^t \boldsymbol{u}(\xi, \tau)d\tau \equiv X(\xi, t), \tag{1.5}$$

where

$$\boldsymbol{u}(\xi, t) = \boldsymbol{w}(X(\xi, t), t).$$

We notice that the transformation (1.5) is invertible and its Jacobian equals one. In the Lagrangian coordinates, relations (1.4) take the form

$$\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla_u^2\boldsymbol{u} + \nabla_u q = 0,$$
$$\nabla_u \cdot \boldsymbol{u} = 0, \qquad \xi \in \Omega_0, \ \ t > 0,$$
$$T_u(\boldsymbol{u}, q)\boldsymbol{n} = \Big(\frac{k}{|X|} + \frac{\omega^2}{2}|X'|^2 - a^2\Big)\boldsymbol{n}, \quad \xi \in \Gamma_0, \tag{1.6}$$
$$\boldsymbol{u} = 0, \quad \xi \in S,$$
$$\boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \qquad \xi \in \Omega_0,$$

where $q(\xi, t) = s(X(\xi, t), t))$, ∇_u is a transformed gradient with respect to x, T_u is a transformed stress tensor, i.e.,

$$\nabla_u = A\nabla, \quad T_u(\boldsymbol{u}, q) = -qI + \nu S_u, \quad S_u(\boldsymbol{u}) = (\nabla_u\boldsymbol{u}) + (\nabla_u\boldsymbol{u})^T,$$

$\nabla = \Big(\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_3}\Big) \equiv \nabla_\xi$, $A = (A_{ij})_{i,j=1,2,3}$, A_{ij} is a co-factor of the element $a_{ij} = \delta_{ij} + \int_0^t \frac{\partial u_i(\xi,\tau)}{\partial\xi_j}d\tau$ of the Jacobi matrix of the transformation (1.5), and finally $\boldsymbol{n} = \boldsymbol{n}(X)$. This vector is connected with the normal $\boldsymbol{n}_0$ to Γ_0 by

$$\boldsymbol{n}(X) = \frac{A(\xi, t)\boldsymbol{n}_0(\xi)}{|A(\xi, t)\boldsymbol{n}_0(\xi)|}. \tag{1.7}$$

Let Π and Π_0 be projections on the tangential planes to Γ_t and Γ_0, respectively, i.e.,

$$\Pi \boldsymbol{f}(x) = \boldsymbol{f}(x) - \boldsymbol{n}(\boldsymbol{f}\cdot\boldsymbol{n}), \quad x\in\Gamma_t,$$

$$\Pi_0 \boldsymbol{f}(\xi) = \boldsymbol{f}(\xi) - \boldsymbol{n}_0(\boldsymbol{f}\cdot\boldsymbol{n}_0), \quad \xi\in\Gamma_0.$$

Then $T_u(\boldsymbol{u},q)\boldsymbol{n} = \nu\Pi S_u(\boldsymbol{u})\boldsymbol{n} + \boldsymbol{n}(\boldsymbol{n}\cdot T_u(\boldsymbol{u},q)\boldsymbol{n})$, and it is easily verified that in the case $\boldsymbol{n}(X)\cdot\boldsymbol{n}_0(\xi) > 0$ the equation $T_u(\boldsymbol{u},q)\boldsymbol{n} = M(X)\boldsymbol{n}$ is equivalent to the system

$$\Pi_0\Pi S_u(\boldsymbol{u})\boldsymbol{n} = 0, \quad -q + \nu\boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n} = M(X)$$

where

$$M(X) = \frac{k}{|X|} + \frac{\omega^2}{2}|X'|^2 - a^2. \tag{1.8}$$

Problem (1.6) can be written in the form

$$\begin{gathered}
\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3\times\boldsymbol{u}) - \nu\nabla_u^2\boldsymbol{u} + \nabla_u q = 0,\\
\nabla_u\cdot\boldsymbol{u} = 0, \qquad \xi\in\Omega_0,\ \ t>0,\\
\Pi_0\Pi S_u(\boldsymbol{u})\boldsymbol{n} = 0,\\
-q + \nu\boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n} = M(X), \quad \xi\in\Gamma_0,\\
\boldsymbol{u} = 0, \quad \xi\in S,\\
\boldsymbol{u}(\xi,0) = \boldsymbol{w}_0(\xi), \qquad \xi\in\Omega_0.
\end{gathered} \tag{1.9}$$

We assume that Γ_t is close to $\mathcal{G}$ and can be defined by the equation

$$x = z + \boldsymbol{N}(z)\rho(z,t) \equiv e_\rho(z), \tag{1.10}$$

where $\boldsymbol{N}$ is a unit normal to $\mathcal{G}$, $\rho(z,t)$ is a small function and $z=\bar{x}$ is the point of $\mathcal{G}$ closest to x. It is clear that

$$\rho = \pm\mathrm{dist}(x,\mathcal{G}) \equiv R(x),$$

the sign " $-$ " corresponds to the case $x\in\mathcal{F}$ and the sign " $+$ " corresponds to the case $x\in\mathbb{R}^3\setminus\mathcal{F}$. The function $R(x)$ is smooth in a certain neighborhood of $\mathcal{G}$ and

$$\nabla R(x) = \boldsymbol{N}(\bar{x}).$$

If $\xi\in\Gamma_0$, then $X(\xi,t)\in\Gamma_t$, and

$$X(\xi,t) = \bar{X}(\xi,t) + R(X)\boldsymbol{N}(\bar{X}) = \bar{X}(\xi,t) + R(X)\nabla_X R(X).$$

The time derivative of $R(X)$ is given by

$$\frac{\partial R(X)}{\partial t} = \nabla_X R(X)\cdot X_t = \boldsymbol{N}(\bar{X})\cdot\boldsymbol{u}(\xi,t).$$

We include the function $r(\xi,t) = R(X)$ into the problem (1.9) and separate out linear terms with respect to $\boldsymbol{u}, q, r$ in the relations (1.9). Owing to (1.3),

$$M(x) = m(x) - m(\bar{x}) = \int_0^1 \frac{\partial}{\partial s} m(e_{s\rho}(z))ds =$$

$$= \frac{\partial m(e_{s\rho})}{\partial s}\Big|_{s=0} + \int_0^1 (1-s)\frac{\partial^2 m(e_{s\rho})}{\partial s^2}ds,$$

where $m(x) = \frac{k}{|x|} + \frac{\omega^2}{2}|x'|^2$. Hence

$$\frac{\partial m(e_{s\rho})}{\partial s}\Big|_{s=0} = \sum_{j=1}^3 m_j(z)N_j(z)\rho(z,t) = \frac{\partial m(z)}{\partial N}\rho(z,t),$$

$$\frac{\partial^2 m(e_{s\rho})}{\partial s^2} = \sum_{j,k=1}^3 m_{jk}(e_{s\rho})N_j(z)N_k(z)\rho^2(z,t),$$

where $m_j(z) = \frac{\partial m(z)}{\partial z_j}$, $m_{jk}(z) = \frac{\partial^2 m(z)}{\partial z_j \partial z_k}$. It follows that

$$M(X) = -b(\bar{\xi})r(\xi,t) + (b(\bar{\xi}) - b(\bar{X}))r(\xi,t) + b_1(\boldsymbol{u}, r)r^2(\xi,t),$$

where $b(z) = -\sum_{j=1}^3 m_j(z)N_j(z) = -\frac{\partial m(z)}{\partial N}$,

$$b_1(\boldsymbol{u}, r) = \int_0^1 (1-s)\sum_{j,k=1}^3 m_{jk}(e_{s\rho})dsN_j(z)N_k(z)\Big|_{z=\bar{X}}$$

$$= \int_0^1 (1-s)\sum_{j,k=1}^3 m_{jk}(\bar{X} + s\boldsymbol{N}(\bar{X})r(\xi,t))dsN_j(\bar{X})N_k(\bar{X}). \tag{1.11}$$

The expression $-b(\bar{\xi})r(\xi,t)$ is the first variation of M that was sought for, and other two terms represent a nonlinear (quadratic) remainder.

Thus, $\boldsymbol{u}, q, r$ can be regarded as a solution of the problem

$$\begin{gathered}
\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q),\\
\nabla\cdot\boldsymbol{u} = l_2(\boldsymbol{u}), \qquad \xi\in\Omega_0, \quad t>0,\\
\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}),\\
-q + \nu\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0 + b(\bar{\xi})r = l_4(\boldsymbol{u}) + l_5(\boldsymbol{u}, r),\\
r_t(\xi,t) = \boldsymbol{N}(\bar{\xi})\cdot\boldsymbol{u} + l_6(\boldsymbol{u}), \quad \xi\in\Gamma_0,\\
\boldsymbol{u}(\xi,t) = 0, \quad \xi\in S,\\
\boldsymbol{u}(\xi,0) = \boldsymbol{w}_0(\xi), \quad \xi\in\Omega_0, \quad r(\xi,0) = R(\xi) \equiv \rho_0(\bar{\xi}), \quad \xi\in\Gamma_0.
\end{gathered} \tag{1.12}$$

In these relations

$$
\begin{aligned}
\boldsymbol{l}_1(\boldsymbol{u},q) &= \nu(\nabla_u^2\boldsymbol{u} - \nabla^2\boldsymbol{u}) + \nabla q - \nabla_u q,\\
l_2(\boldsymbol{u}) &= (\nabla - \nabla_u)\cdot\boldsymbol{u},\\
\boldsymbol{l}_3(\boldsymbol{u}) &= \Pi_0(\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 - \Pi S_u(\boldsymbol{u})\boldsymbol{n}),\\
l_4(\boldsymbol{u}) &= \nu(\boldsymbol{n}_0\cdot S(\boldsymbol{u})\boldsymbol{n}_0 - \boldsymbol{n}\cdot S_u(\boldsymbol{u})\boldsymbol{n}),\\
l_5(\boldsymbol{u},r) &= (b(\bar\xi) - b(\bar X))r + b_1(\boldsymbol{u},r)r^2,\\
l_6(\boldsymbol{u}) &= (\boldsymbol{N}(\bar X) - \boldsymbol{N}(\bar\xi))\cdot\boldsymbol{u}(\xi,t).
\end{aligned}
\tag{1.13}
$$

By the Piola identity $\nabla\cdot A^T = \left(\sum_{j=1}^3 \frac{\partial}{\partial x_j}A_{ij}\right)_{i=1,2,3} = 0$, where A^T means the transposed matrix A, we have

$$
l_2(\boldsymbol{u}) = \nabla\cdot\boldsymbol{L}(\boldsymbol{u}), \quad \boldsymbol{L}(\boldsymbol{u}) = (I - A^T)\boldsymbol{u}.
$$

We assume that $|\Omega_0| = |\Omega_t| = |\mathcal{F}|$. This condition is equivalent to

$$
\int_{\mathcal{G}} \varphi(z,\rho)dz = 0, \tag{1.14}
$$

where

$$
\varphi(z,\rho) = \rho - \frac{\rho^2}{2}\mathcal{H}(z) + \frac{\rho^3}{3}\mathcal{K}(z), \tag{1.15}
$$

$\mathcal{H}$ and $\mathcal{K}$ are the doubled mean curvature and the Gaussian curvature of $\mathcal{G}$, respectively.

In conclusion, we linearize the problem (1.12). For this it is necessary to map the domain Ω_0 onto $\mathcal{F}$ by the transformation inverse to

$$
x = y + \boldsymbol{N}^*(y)\rho_0^*(y) \equiv e_{\rho_0}(y), \quad y\in\mathcal{F}, \tag{1.16}
$$

where $\boldsymbol{N}^*$ and ρ_0^* are extensions of $\boldsymbol{N}$ and ρ_0 in $\mathcal{F}$, and to omit all the nonlinear terms with respect to $\boldsymbol{u}, q, r$. This provides the following linear problem for $\boldsymbol{v}(y,t)$, $p(y,t)$, $y\in\mathcal{F}$, and $\rho(y,t)$, $y\in\mathcal{G}$:

$$
\begin{gathered}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3\times\boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = 0,\\
\nabla\cdot\boldsymbol{v}(y,t) = 0, \quad y\in\mathcal{F}, \ \ t>0,\\
S(\boldsymbol{v})\boldsymbol{N} - \boldsymbol{N}(y)(\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N}(y)) = 0,\\
-p + \nu\boldsymbol{N}\cdot S(\boldsymbol{v})\boldsymbol{N} + b(y)\rho = 0,\\
\rho_t = \boldsymbol{N}(y)\cdot\boldsymbol{v}(y,t), \quad y\in\mathcal{G},\\
\boldsymbol{v}(y,t) = 0, \quad y\in S,\\
\boldsymbol{v}(y,0) = \boldsymbol{v}_0(y), \quad y\in\mathcal{F}, \qquad \rho(y,0) = \rho_0(y), \quad y\in\mathcal{G}.
\end{gathered}
\tag{1.17}
$$

Linearization of (1.14) gives

$$
\int_{\mathcal{G}} \rho(y,t)dS = 0. \tag{1.18}
$$

If (1.18) holds for $t=0, \rho=\rho_0$, then it is satisfied for all $t>0$.

2 On the equilibrium figure.

We consider the curve on the plane (y_1, y_2) given by the equation

$$m(y) - a^2 = 0, \tag{2.1}$$

where

$$m(y) = \frac{k}{|y|} + \frac{\omega^2}{2} y_1^2$$

(this curve is a meridional section of the axially symmetric surface $\mathcal{G}$). It intersects the y_2-axis at the points $y_1 = 0, y_2 = \pm\frac{k}{a^2}$ and the y_1-axis at the points $y_1 = \pm t_1,\ y_2 = 0$, where $t_1 > 0$ is a minimal positive root of the equation

$$f(t) \equiv a^2 t - \frac{\omega^2}{2} t^3 - k = 0. \tag{2.2}$$

The function $f(t)$ takes a negative value $-k$ at the point $t = 0$ and grows in the interval $(0, t_0)$ where $t_0 = \sqrt{\frac{2}{3}\frac{a}{\omega}}$ (we assume that $\omega > 0$). The maximal value of $f(t)$ equals $f(t_0) = (\frac{2}{3})^{3/2}\frac{a^3}{\omega} - k$, and the equation (2.1) defines a closed curve if and only if $f(t_0) > 0$, which is equivalent to

$$\omega < \left(\frac{2}{3}\right)^{3/2} \frac{a^3}{k}. \tag{2.3}$$

Under this condition, the equation (2.2) has a minimal positive root $t_1 < t_0$, which can be expressed by Cardano's formula but we do not give it here.

The exterior normal to the curve (2.1) is parallel to

$$-\nabla m(y) = \Big((k/|y|^3 - \omega^2) y_1, k y_2/|y|^3 \Big).$$

When the point y moves along the curve (2.1) from the point $y^{(1)} = (0, k/a^2)$ to $y^{(2)} = (t_1, 0)$, the function $k/|y|^3 - \omega^2$ decreases from the value $a^6/k^2 - \omega^2 > 0$ to $k/t_1^3 - \omega^2 \geq 0$. It can be shown that $k/t_1^3 - \omega^2 > 0$. This follows from the inequality

$$f\left(\left(\frac{k}{\omega^2}\right)^{1/3}\right) = a^2 \left(\frac{k}{\omega^2}\right)^{1/3} - \frac{\omega^2}{2}\frac{k}{\omega^2} - k = a^2 \frac{k^{1/3}}{\omega^{2/3}} - \frac{3}{2} k > 0,$$

which is equivalent to (2.3). It follows that the point $t_4 = \left(\frac{k}{\omega^2}\right)^{1/3}$ possessing the property $k/t_4^3 - \omega^2 = 0$ is located to the right of t_1 on the y_1-axis, which implies $k/t_1^3 - \omega^2 > 0$.

We summarize our results: under the assumption (2.3) equation (2.1) defines a curve γ possessing the property $\nabla m(y)|_\gamma \neq 0$; moreover,

$$-\frac{\partial m(y)}{\partial N}\Big|_\gamma = \frac{k}{|y|^3} y \cdot \boldsymbol{N} - \omega^2 y_1 N_1 \Big|_\gamma \geq b_0 > 0, \tag{2.4}$$

where $\boldsymbol{N}$ is the unit exterior normal to γ. The corresponding surface $\mathcal{G}$ is the boundary of an oblate spheroid $\mathcal{F}_1$. If $k/a^2 > d$, then $K \subset \mathcal{F}_1$.

It will be shown below that condition (2.4) guarantees the stability of the equilibrium figure $\mathcal{F}$.

In conclusion, we give a formula for the volume of the domain $\mathcal{F}_1$. The equation (2.1) can be resolved with respect to y_2:

$$y_2 = \pm\sqrt{g(y_1^2)}$$

where

$$g(y_1^2) = \frac{k^2}{(a^2 - \frac{\omega^2}{2}y_1^2)^2} - y_1^2.$$

By the symmetry of $\mathcal{G}$,

$$|\mathcal{F}_1| = 4\pi \int_0^{t_1} y_1\sqrt{g(y_1^2)}dy_1 = 2\pi \int_0^{t_1^2} \sqrt{g(\xi)}d\xi.$$

Since $g(t_1^2) = 0$, we have

$$\frac{\partial|\mathcal{F}_1|}{\partial a^2} = -2\pi \int_0^{t_1^2} \frac{1}{\sqrt{g(\xi)}} \frac{k^2}{(a^2 - \frac{\omega^2}{2}\xi)^3} d\xi < 0$$

for a fixed ω and

$$\frac{\partial|\mathcal{F}_1|}{\partial \omega^2} = 2\pi \int_0^{t_1^2} \frac{1}{2\sqrt{g(\xi)}} \frac{k^2\xi}{(a^2 - \frac{\omega^2}{2}\xi)^3} d\xi > 0$$

for a fixed a.

3 Main result

We consider the problem (1.12) in weighted anisotropic Sobolev-Slobodetskii spaces [1]. By $W_2^l(\Omega)$ where $l > 0$ and Ω is a domain in $\mathbb{R}^n$ we mean the space of functions $u(x)$, $x \in \Omega$, with the norm

$$\|u\|^2_{W_2^l(\Omega)} = \sum_{0\le|j|\le l} \|D^j u\|^2_{L_2(\Omega)} \equiv \sum_{0\le|j|\le l} \int_\Omega |D^j u(x)|^2 dx,$$

if $l = [l]$, i.e., if l is an integral number, and

$$\|u\|^2_{W_2^l(\Omega)} = \|u\|^2_{W_2^{[l]}(\Omega)} + \sum_{|j|=[l]} \int_\Omega\int_\Omega |D^j u(x) - D^j u(y)|^2 \frac{dxdy}{|x-y|^{n+2\lambda}},$$

if $l = [l]+\lambda$, $\lambda \in (0,1)$. As usual, $D^j u$ denotes a partial derivative $\frac{\partial^{|j|}u}{\partial^{j_1}x_1 \ldots \partial^{j_n}x_n}$ where $j = (j_1, j_2, \ldots j_n)$ and $|j| = j_1 + \ldots + j_n$. The space $W_2^{l,l/2}(Q_T)$, $Q_T = \Omega \times (0,T)$, can be defined as

$$L_2((0,T), W_2^l(\Omega)) \cap W_2^{l/2}((0,T), L_2(\Omega))$$

and supplied with the norm

$$\|u\|^2_{W_2^{l,l/2}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^l(\Omega)}dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{l/2}(0,T)}dx. \tag{3.1}$$

There exist many other equivalent norms in $W_2^{l,l/2}(Q_T)$; some of them will be used below. Sobolev spaces of functions given on smooth surfaces, in particular, on $\mathcal{G}$ and on $\mathcal{G}\times(0,T)$, are introduced in a standard way, with the help of local maps and partition of unity. We also find it convenient to introduce the spaces

$$W_2^{l,0}(Q_T) = L_2((0,T), W_2^l(\Omega)), \quad W_2^{0,l/2}(Q_T) = W_2^{l/2}((0,T), L_2(\Omega));$$

the squares of norms in these spaces coincide, respectively, with the first and the second term in (3.1). Finally, following Y.Hataya [2], we introduce weighted spaces $\widetilde{W}_2^{l,l/2}(Q_T)$, $l>1$, $T\le\infty$, as the sets of functions $u\in W_2^{l,l/2}(Q_T)$ such that $tu\in W_2^{l-1,(l-1)/2}(Q_T)$, and we define the norm in $\widetilde{W}_2^{l,l/2}(Q_T)$ by

$$\|u\|^2_{\widetilde{W}_2^{l,l/2}(Q_T)} = \|u\|^2_{W_2^{l,l/2}(Q_T)} + \|tu\|^2_{W_2^{l-1,(l-1)/2}(Q_T)}.$$

In the same way the spaces $\widetilde{W}_2^{l,l/2}(G_T)$ are introduced.

By the interpolation inequality, we have

$$\|(1+t)^\gamma u\|_{W_2^{l-\gamma,(l-\gamma)/2}(Q_T)} \le c\|u\|_{\widetilde{W}_2^{l,l/2}(Q_T)} \tag{3.2}$$

for arbitrary $\gamma\in[0,1]$.

We recall the theorem on mixed derivatives and the trace theorem for the spaces $W_2^{l,l/2}(Q_T)$. If $u\in W_2^{l,l/2}(Q_T)$ and $l_1 = l-2k-|j|>0$, then $D_t^kD^ju(x,t)\in W_2^{l_1,l_1/2}(Q_T)$, and

$$\|D_t^kD^ju(x,t)\|_{W_2^{l_1,l_1/2}(Q_T)} \le c\|u\|_{W_2^{l,l/2}(Q_T)}.$$

Moreover, the functions from $W_2^{l,l/2}(Q_T)$ have "traces" $u|_{x\in\partial\Omega}\in W_2^{l-1/2,l/2-1/4}(G_T)$, $G_T=\partial\Omega\times(0,T)$, and $u|_{t=t_0}\in W_2^{l-1}(\Omega)$, provided that $l>1/2$ and $l>1$, respectively, and the norms of the traces can be estimated by $c\|u\|_{W_2^{l,l/2}(Q_T)}$. The converse statements are also true: every function $\varphi\in W_2^{l-1/2,l/2-1/4}(G_T)$ can be extended into Q_T so that

$$\|\varphi\|_{W_2^{l,l/2}(Q_T)} \le c\|\varphi\|_{W_2^{l-1/2,l/2-1/4}(G_T)}.$$

An analogous proposition holds for the functions $\psi\in W_2^{l-1}(\Omega)$.

Our main result is the following.

Theorem 3.1. *Let $\boldsymbol{w}_0\in W_2^{l+1}(\Omega_0)$ with $l\in(1,3/2)$, and let the surface Γ_0 be given by*

$$x = z + \boldsymbol{N}(z)\rho_0(z), \quad z\in\mathcal{G},$$

with $\rho_0\in W_2^{l+3/2}(\mathcal{G})$. Assume also that $\boldsymbol{w}_0$ satisfies the compatibility conditions

$$\nabla\cdot\boldsymbol{w}_0 = 0, \quad \Pi_0S(\boldsymbol{w}_0)\boldsymbol{n}_0|_{\Gamma_0} = 0, \quad \boldsymbol{w}_0|_S = 0 \tag{3.3}$$

and that

$$\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \varepsilon \tag{3.4}$$

where ε is a sufficiently small positive number. Moreover, let the condition (2.4) be satisfied. Then the problem (1.4)-(1.9)-(1.12) has a unique solution $\boldsymbol{u} \in \widetilde{W}_2^{l+2,l/2+1}(Q_\infty)$, $\nabla q \in \widetilde{W}_2^{l,l/2}(Q_\infty)$, $r \in \widetilde{W}_2^{l+1/2,0}(G_\infty)$, where $Q_\infty = \Omega_0 \times \mathbb{R}_+$, $G_\infty = \Gamma_0 \times \mathbb{R}_+$ such that $p|_{G_\infty} \in \widetilde{W}_2^{l+1/2,l/2+1/4}(G_\infty)$, $r \in W_2^{l+1}(\Gamma_0)$, $\forall t > 0$. The surface Γ_t is representable in the form (1.10), where ρ is connected with $r(\xi, t) = R(X)$ by

$$\rho(\bar{X}) = r(\xi, t). \tag{3.5}$$

The solution satisfies the inequality

$$\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_\infty)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_\infty)} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_\infty)}$$

$$+\|r\|_{\widetilde{W}_2^{l+1/2,0}(G_\infty)} + \sup_{t>0} \|r(\cdot, t)\|_{W_2^{l+1}(\Gamma_0)}$$

$$+\sup_{t>0} \|tr(\cdot, t)\|_{W_2^{l}(\Gamma_0)} \leq c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big). \tag{3.6}$$

A similar theorem is proved in [3] for the problem governing the evolution of an isolated mass of a viscous incompressible self- gravitating liquid bounded only by a free surface Γ_t. This problem is more complicated technically, in particular, in view of the presence of the non-local terms in the form of the Newtonian and single layer potentials. However, the scheme of the proof of the main result is the same as in the present paper. We can say that (1.1) is a model problem for the problem studied in [3].

The proof of Theorem 3.1 is based, in particular, on the estimates of solutions of a non-homogeneous problem (1.17), i.e.,

$$\begin{gathered}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\
\nabla \cdot \boldsymbol{v}(x,t) = f(x,t), \quad x \in \mathcal{F}, \quad t > 0, \\
T(\boldsymbol{v}, p)\boldsymbol{N} + \boldsymbol{N} b(x)\rho = \boldsymbol{d}(x,t), \\
\rho_t - \boldsymbol{N}(x) \cdot \boldsymbol{v}(x,t) = g(x,t), \quad x \in \mathcal{G}, \\
\boldsymbol{v}(x,t) = \boldsymbol{a}(x,t), \quad x \in S, \\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \mathcal{F}, \qquad \rho(x,0) = \rho_0(x), \quad x \in \mathcal{G},
\end{gathered} \tag{3.7}$$

and of the evolution problem for the Stokes equations with the Dirichlet boundary condition on S and the Neumann condition on $\mathcal{G}$:

$$\begin{gathered}
\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\
\nabla \cdot \boldsymbol{v}(x,t) = f(x,t), \quad x \in \mathcal{F}, \quad t > 0, \\
T(\boldsymbol{v}, p)\boldsymbol{N} = \boldsymbol{d}(x,t), \quad x \in \mathcal{G},
\end{gathered} \tag{3.8}$$

$$\boldsymbol{v}(x,t)=\boldsymbol{a}(x,t),\quad x\in S,$$

$$\boldsymbol{v}(x,0)=\boldsymbol{v}_0(x),\quad x\in\mathcal{F}.$$

We assume for simplicity that $\boldsymbol{a}\cdot\boldsymbol{n}=0$, where $\boldsymbol{n}(x)$ is the normal to S exterior with respect to $\mathcal{F}$.

We prove theorems on the solvability of these problems in the Sobolev spaces of functions and obtain the corresponding coercive estimates. Our main auxiliary result concerns the problem (3.7).

Theorem 3.2 *Let $l\in(1,3/2)$, $\mathfrak{Q}_T=\mathcal{F}\times(0,T)$, $\mathfrak{G}_T=\mathcal{G}\times(0,T)$ and let the data of the problem (3.7) possess the following regularity properties: $\boldsymbol{f}\in W_2^{l,l/2}(\mathfrak{Q}_T)$, $f\in W_2^{1+l,0}(\mathfrak{Q}_T)$, $f=\nabla\cdot\boldsymbol{F}$, $\boldsymbol{F}\in W_2^{0,1+l/2}(\mathfrak{Q}_T)$, $\boldsymbol{v}_0\in W_2^{1+l}(\mathcal{F})$, $\boldsymbol{d}\in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)$, $g\in W_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)$, $\boldsymbol{a}\in W_2^{l+3/2,l/2+3/4}(\Sigma_T)$, $\rho_0\in W_2^{l+1}(\mathcal{G})$, where $\Sigma_T=S\times(0,T)$. Assume also that $\boldsymbol{a}\cdot\boldsymbol{n}=0$, $\boldsymbol{F}\cdot\boldsymbol{n}|_{x\in S}=0$ and that the compatibility conditions*

$$\nabla\cdot\boldsymbol{v}_0=f(x,0),\quad x\in\mathcal{F},\quad \nu\Pi_{\mathcal{G}}S(\boldsymbol{v})\boldsymbol{N}=\Pi_{\mathcal{G}}\boldsymbol{d}(x,0),\quad x\in\mathcal{G},$$

$$\boldsymbol{v}_0(x)=\boldsymbol{a}(x,0),\quad x\in S, \tag{3.9}$$

are satisfied, where $\Pi_{\mathcal{G}}\boldsymbol{d}=\boldsymbol{d}-\boldsymbol{N}(\boldsymbol{N}\cdot\boldsymbol{d})$. Then the problem (3.7) has a unique solution $\boldsymbol{v}\in W_2^{2+l,1+l/2}(\mathfrak{Q}_T)$, $\nabla p\in W_2^{l,l/2}(\mathfrak{Q}_T)$, $\rho\in W_2^{l+1/2,0}(\mathfrak{G}_T)$, such that $p|_{\mathfrak{G}_T}\in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)$, $\rho(\cdot,t)\in W_2^{l+1}(\mathcal{G})$ for arbitrary $t\in(0,T)$, and

$$\mathcal{Y}(T)\equiv\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}+\|\nabla p\|_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|p\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}+\|\rho\|_{W_2^{l+1/2,0}(\mathfrak{G}_T)}$$

$$+\sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})}\le c\Big(\mathcal{N}(T)+\Big(\int_0^T(\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})}+\|\rho\|^2_{W_2^{-1/2}(\mathcal{G})})dt\Big)^{1/2}\Big), \tag{3.10}$$

where

$$\mathcal{N}(T)=\|\boldsymbol{f}\|_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|f\|_{W_2^{l+1,0}(\mathfrak{Q}_T)}+\|\boldsymbol{F}\|_{W_2^{0,1+l/2}(\mathfrak{Q}_T)}+\|\rho_0\|_{W_2^{l+1}(\mathcal{G})}$$

$$+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{F})}+\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}+\|g\|_{W_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)}+\|\boldsymbol{a}\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)}.$$

Moreover, if $\boldsymbol{f}\in\widetilde{W}_2^{l,l/2}(\mathfrak{Q}_T)$, $\boldsymbol{d}\in\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)$, $g\in\widetilde{W}_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)$, $\boldsymbol{a}\in\widetilde{W}_2^{l+3/2,l/2+3/4}(\Sigma_T)$, $f\in\widetilde{W}_2^{1+l,0}(\mathfrak{Q}_T)$, $\boldsymbol{F}\in\widetilde{W}_2^{0,1+l/2}(\mathfrak{Q}_T)$ (this means that $f\in W_2^{1+l,0}(\mathfrak{Q}_T)$, $tf\in W_2^{l,0}(\mathfrak{Q}_T)$), $\boldsymbol{F}\in W_2^{0,1+l/2}(\mathfrak{Q}_T)$, $t\boldsymbol{F}\in W_2^{0,(l+1)/2}(\mathfrak{Q}_T)$), then

$$\widetilde{\mathcal{Y}}(T)\equiv\|\boldsymbol{v}\|_{\widetilde{W}_2^{2+l,1+l/2}(\mathfrak{Q}_T)}+\|\nabla p\|_{\widetilde{W}_2^{l,l/2}(\mathfrak{Q}_T)}+\|p\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}+\|\rho\|_{\widetilde{W}_2^{l+1/2,0}(\mathfrak{G}_T)}$$

$$+\sup_{t<T}\|\rho(\cdot,t)\|_{W_2^{l+1}(\mathcal{G})}+\sup_{t<T}t\|\rho(\cdot,t)\|_{W_2^{l}(\mathcal{G})}$$

$$\le c\Big(\widetilde{\mathcal{N}}(T)+\Big(\int_0^T(1+t^2)(\|\boldsymbol{v}\|^2_{L_2(\mathcal{F})}+\|\rho\|^2_{W_2^{-1/2}(\mathcal{G})})dt\Big)^{1/2}\Big), \tag{3.11}$$

where

$$\widetilde{\mathcal{N}}(T)=\|\boldsymbol{f}\|_{\widetilde{W}_2^{l,l/2}(\mathfrak{Q}_T)}+\|f\|_{\widetilde{W}_2^{l+1,0}(\mathfrak{Q}_T)}+\|\boldsymbol{F}\|_{\widetilde{W}_2^{0,1+l/2}(\mathfrak{Q}_T)}+\|\rho_0\|_{W_2^{l+1}(\mathcal{G})}$$

$$+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{F})} + \|\boldsymbol{d}\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|g\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)} + \|\boldsymbol{a}\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(\Sigma_T)}.$$

The constants in (3.10), (3.11) are independent of T.

In the proof of this theorem we use a similar result for the problem (3.8).

Theorem 3.3 *Let* $l \in (1, 3/2)$, *and let the data of the problem (3.8) possess the following regularity properties:* $\boldsymbol{f} \in W_2^{l,l/2}(\mathfrak{Q}_T)$, $f \in W_2^{1+l,0}(\mathfrak{Q}_T)$, $f = \nabla \cdot \boldsymbol{F}$, $\boldsymbol{F} \in W_2^{0,1+l/2}(\mathfrak{Q}_T)$, $\boldsymbol{v}_0 \in W_2^{1+l}(\mathcal{F})$, $\boldsymbol{d} \in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)$, $g \in W_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)$. *Assume also that the compatibility conditions (3.9) are satisfied and that* $\boldsymbol{a} \cdot \boldsymbol{n} = 0$, $\boldsymbol{F} \cdot \boldsymbol{n}|_{x \in S}$. *Then the problem (3.8) has a unique solution* $\boldsymbol{v} \in W_2^{2+l,1+l/2}(\mathfrak{Q}_T)$, $\nabla p \in W_2^{l,l/2}(\mathfrak{Q}_T)$ *such that* $p|_{\mathfrak{G}_T} \in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)$, *and*

$$\mathfrak{V}(T) \equiv \|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + \|\nabla p\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \leq c\mathfrak{F}(T), \tag{3.12}$$

where

$$\mathfrak{F}(T) = \|\boldsymbol{f}\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|f\|_{W_2^{l+1,0}(\mathfrak{Q}_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(\mathfrak{Q}_T)}$$
$$+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\mathcal{F})} + \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|\boldsymbol{a}\|_{W_2^{l+3/2,3/4}(\Sigma_T)},$$

and c *is a constant independent of* T.

Similar theorems hold for the problems (3.7) and (3.8) in the domain Ω_0 bounded by the surfaces $S = \{|x| = d\}$ and

$$\Gamma_0 = \{x = z + \boldsymbol{N}(z)\rho_0(z), \quad z \in \mathcal{G}, \} \tag{3.13}$$

with a small $\rho_0 \in W_2^{l+3/2}(\mathcal{G})$ (we restrict ourselves with the case $\boldsymbol{a}(x,t) = 0$):

$$\begin{gathered}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\
\nabla \cdot \boldsymbol{v}(x,t) = f(x,t), \quad x \in \Omega_0, \quad t > 0, \\
T(\boldsymbol{v},p)\boldsymbol{n}_0 + \boldsymbol{n}_0 b(\bar{x}) r = \boldsymbol{d}(x,t), \\
r_t - \boldsymbol{N}(\bar{x}) \cdot \boldsymbol{v}(x,t) = g(x,t), \quad x \in \Gamma_0, \\
\boldsymbol{v}(x,t) = 0, \quad x \in S, \\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_0, \qquad r(x,0) = r_0(x), \quad x \in \Gamma_0,
\end{gathered} \tag{3.14}$$

$$\begin{gathered}
\boldsymbol{v}_t - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{f}(x,t), \\
\nabla \cdot \boldsymbol{v}(x,t) = f(x,t), \quad x \in \Omega_0, \quad t > 0, \\
T(\boldsymbol{v},p)\boldsymbol{n}_0 = \boldsymbol{d}(x,t), \quad x \in \Gamma_0, \\
\boldsymbol{v}(x,t) = 0, \quad x \in S, \\
\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad x \in \Omega_0.
\end{gathered} \tag{3.15}$$

Theorem 3.4 *Let* $Q_T = \Omega_0 \times (0,T)$, $G_T = \Gamma_0 \times (0,T)$, *and let* Γ_0 *be given by the equation (3.13) with a small* $\rho_0 \in W_2^{l+3/2}(\mathcal{G})$, $l \in (1, 3/2)$. *Assume that the data of the problem (3.14) possess the* *following regularity properties:* $\boldsymbol{f} \in W_2^{l,l/2}(Q_T)$, $f \in W_2^{1+l,0}(Q_T)$, $f = \nabla \cdot \boldsymbol{F}$, $\boldsymbol{F} \in W_2^{0,1+l/2}(Q_T)$ $\boldsymbol{v}_0 \in W_2^{1+l}(\Omega_0)$, $\boldsymbol{d} \in W_2^{l+1/2,l/2+1/4}(G_T)$, $g \in W_2^{l+3/2,l/2+3/4}(G_T)$, $r_0 \in W_2^{l+1}(\Gamma_0)$. *Assume also that the compatibility conditions*

$$\nabla \cdot \boldsymbol{v}_0(x) = f(x,0), \quad x \in \Omega_0, \quad \nu \Pi_0 S(\boldsymbol{v}_0)\boldsymbol{n}_0 = \Pi_0 \boldsymbol{d}(x,0), \quad x \in \Gamma_0,$$

$$\boldsymbol{v}_0(x) = 0, \quad x \in S,$$

are satisfied and that $\boldsymbol{F} \cdot \boldsymbol{n}|_{x \in S} = 0$. *Then the problem (3.14) has a unique solution* $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_T)$, $\nabla p \in W_2^{l,l/2}(Q_T)$, $r \in W_2^{l+1/2,0}(G_T)$, *such that* $p|_{G_T} \in W_2^{l+1/2,l/2+1/4}(G_T)$, $r(\cdot,t) \in W_2^{l+1}(\Gamma_0)$ *for arbitrary* $t \in (0,T)$, *and*

$$Y(T) \equiv \|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|r\|_{W_2^{l+1/2,0}(G_T)}$$

$$+ \sup_{t<T} \|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} \leq c\Big(N(T) + \Big(\int_0^T (\|\boldsymbol{v}\|^2_{L_2(\Omega_0)} + \|r\|^2_{W_2^{-1/2}(\Gamma_0)})dt\Big)^{1/2}\Big), \tag{3.16}$$

where

$$N(T) = \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)} + \|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_0)}$$
$$+\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)} + \|g\|_{W_2^{l+3/2,l/2+3/4}(G_T)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)}.$$

Moreover, if $\boldsymbol{f} \in \widetilde{W}_2^{l,l/2}(Q_T)$, $\boldsymbol{d} \in \widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)$, $g \in \widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)$, $f \in \widetilde{W}_2^{1+l,0}(Q_T)$, $\boldsymbol{F} \in \widetilde{W}_2^{0,1+l/2}(Q_T)$, *then*

$$\widetilde{Y}(T) \equiv \|\boldsymbol{v}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|p\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|r\|_{\widetilde{W}_2^{l+1/2,0}(G_T)}$$

$$+ \sup_{t<T} \|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} + \sup_{t<T} t\|r(\cdot,t)\|_{W_2^{l}(\Gamma_0)}$$

$$\leq c\Big(\widetilde{N}(T) + \Big(\int_0^T (1+t^2)(\|\boldsymbol{v}\|^2_{L_2(\Omega_0)} + \|r\|^2_{W_2^{-1/2}(\Gamma_0)})dt\Big)^{1/2}\Big), \tag{3.17}$$

where

$$\widetilde{N}(T) = \|\boldsymbol{f}\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|f\|_{\widetilde{W}_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)} + \|r_0\|_{W_2^{l+1}(\Gamma_0)}$$
$$+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_0)} + \|\boldsymbol{d}\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|g\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)}.$$

The constants in (3.16), (3.17) are independent of T.

Theorem 3.5 *Let* $l \in (1,3/2)$, *and let the data of the problem (3.15),* $\boldsymbol{f}$, f, $\boldsymbol{d}$, $\boldsymbol{v}_0$, *satisfy the assumptions of theorem 3.4. Then this problem has a unique solution* $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_T)$, $\nabla p \in W_2^{l,l/2}(Q_T)$ *such that* $p|_{G_T} \in W_2^{l+1/2,l/2+1/4}(G_T)$, *and*

$$\mathcal{V}(T) \equiv \|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_T)} + \|\nabla p\|_{W_2^{l,l/2}(Q_T)} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_T)} \leq c\mathcal{F}(T), \tag{3.18}$$

where

$$\mathcal{F}(T) = \|\boldsymbol{f}\|_{W_2^{l,l/2}(Q_T)} + \|f\|_{W_2^{l+1,0}(Q_T)} + \|\boldsymbol{F}\|_{W_2^{0,1+l/2}(Q_T)}$$

$$+\|\boldsymbol{v}_0\|_{W_2^{1+l}(\Omega_0)} + \|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(G_T)},$$

with the constant independent of T.

The proof of Theorems 3.4 and 3.5 consists of the reduction of the problems (3.14), (3.15) to (3.7) and (3.8) by means of the mapping

$$\xi = y + \boldsymbol{N}^*\rho_0^*(y) \equiv e_{\rho_0}(y)$$

of the domain $\mathcal{F}$ onto Ω_0, where $\boldsymbol{N}^*$ and ρ_0^* are extensions of $\boldsymbol{N}$ and ρ_0 into $\mathcal{F}$ such that $\boldsymbol{N}^*$ is sufficiently regular and ρ_0^* satisfies the conditions $\rho_0^*|_S = 0$ and

$$\|\rho_0^*\|_{W_2^{l+2}(\mathcal{F})} \leq c\|\rho_0\|_{W_2^{l+3/2}(\mathcal{G})} \leq \epsilon \ll 1. \tag{3.19}$$

By this mapping, the problems (3.14) and (3.15) are converted into the problems (3.7), (3.8) with small additional linear terms, and the solvability of (3.14), (3.15) follows from the contraction mapping principle (see [3] for more detail).

A general scheme of the proof of Theorem 3.1 is the following: using (3.17), we obtain a uniform with respect to T estimate of the solution given in the time interval $(0,T)$ with an arbitrary $T > 0$ and satisfying a certain smallness condition, which can be guaranteed by the choice of sufficiently small initial data. For this we need to estimate the nonlinear terms in (1.12) and the weak norms of the solution (the L_2-norm of $\boldsymbol{u}$ and the $W_2^{-1/2}$-norm of r). Then we prove a local in time theorem on the solvability of the problem (1.9) and extend the solution step by step to the infinite time interval $t > 0$. We obtain the estimate (3.6) that guarantees a power-like decay of $(\boldsymbol{u}, q, r)$ as $t \to \infty$ which means the stability of the zero solution of the problem (1.9).

Proposition 3.1. *Let* $(\boldsymbol{u}, q, r)$ *be a solution of (1.12) with the finite norm* $\widetilde{Y}(T)$, $T \geq 1$, *defined in (3.17), and let*

$$\widetilde{Y}(T) \leq \delta \tag{3.21}$$

with a certain small $\delta > 0$. *Then*

$$\begin{aligned}
&\|\boldsymbol{l}_1\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|l_2\|_{\widetilde{W}_2^{1+l,0}(Q_T)} + \|\boldsymbol{L}\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)} \\
&+\|\boldsymbol{l}_3\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} \\
&+\|l_5\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|l_6\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)} \\
\leq c\Big(&\|\boldsymbol{u}\|^2_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|^2_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|r\|^2_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}\Big)
\end{aligned} \tag{3.22}$$

with a constant c *independent of* $T \geq 1$.

Proposition 3.2. *Assume that the solution* $\boldsymbol{w}, s$ *of the problem (1.4) is given for* $t \in (0,T)$, *is square integrable together with the derivatives that occur in (1.4), and* Γ_t *is representable in the form (1.10) with* $\rho \in W_2^{l+1-\epsilon}(\mathcal{G})$, $\epsilon \in (0, l-1)$, *satisfying the inequality*

$$\sup_{t<T} |\rho(\cdot,t)|_{C^1(\mathcal{G})} \leq \delta \ll 1. \tag{3.23}$$

Then there exist two positive functions, $E(t)$ and $E_1(t)$, such that

$$\frac{dE(t)}{dt} + E_1(t) = 0, \tag{3.24}$$

$$c_1\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big) \le E(t) \le c_2\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big), \tag{3.25}$$

and

$$\begin{aligned} E_1(t) &\ge c_3\Big(\|S(\boldsymbol{w}(\cdot,t))\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G})}\Big) \\ &\ge c_4\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{W_2^1(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^{-1/2}(\mathcal{G})}\Big) \end{aligned} \tag{3.26}$$

with the constants independent of T.

When we integrate (3.24) between zero and t, we obtain

$$\begin{aligned} c_1\Big(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{L_2(\mathcal{G})}\Big) + c_4\int_0^t \Big(\|\boldsymbol{w}(\cdot,\tau)\|^2_{W_2^1(\Omega_\tau)} + \|\rho(\cdot,\tau)\|^2_{W_2^{-1/2}(\mathcal{G})}\Big)d\tau \\ \le c_2\Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_t)} + \|\rho_0\|^2_{L_2(\mathcal{G})}\Big). \end{aligned}$$

By Proposition 4.6 in [3], this implies

$$\begin{aligned} \|\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega_0)} + \|r(\cdot,t)\|^2_{L_2(\Gamma_0)} + \int_0^t \Big(\|\boldsymbol{u}(\cdot,\tau)\|^2_{L_2(\Omega_0)} + \|r(\cdot,\tau)\|^2_{W_2^{-1/2}(\Gamma_0)}\Big)d\tau \\ \le c\Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_0)} + \|\rho_0\|^2_{L_2(\mathcal{G})}\Big). \end{aligned} \tag{3.27}$$

Propositions 3.1 and 3.2 are proved in Sec. 5 and 6.

Theorem 3.2 and Propositions 3.1, 3.2 enable us to obtain the following uniform estimate of the solution of the problem (1.12).

Theorem 3.6. *Assume that the solution of (1.12) is given for $t \in (0,T)$ and satisfies (3.21). Then*

$$\widetilde{Y}(T) \le cN_0, \tag{3.28}$$

where

$$N_0 = \|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}.$$

Proof. Making use of (3.16), (3.21), (3.27), we obtain

$$Y(T) \le c(\widetilde{Y}^2(T) + N_0) \le c(\delta\widetilde{Y}(T) + N_0).$$

Now, we multiply (3.24) by t^2 which leads to

$$\frac{dt^2E(t)}{dt} + t^2E_1(t) = 2tE(t),$$

and, as a consequence, to

$$t^2E(t)+\int_0^t\tau^2E_1(\tau)d\tau=2\int_0^t\tau E(\tau)d\tau\leq 2\sqrt{\int_0^t\tau^2E(\tau)d\tau}\sqrt{\int_0^t E(\tau)d\tau}$$

$$\leq c\sqrt{\widetilde{Y}(t)}\sqrt{Y(t)}.$$

By (3.17), (3.27), we have

$$\widetilde{Y}(T)\leq c(\delta\widetilde{Y}(T)+N_0)+c\sqrt{\widetilde{Y}(T)}\sqrt{Y(T)},$$

which implies

$$\widetilde{Y}(T)\leq c_1\sqrt{\delta}\widetilde{Y}(T)+c_2N_0,$$

and if

$$\sqrt{\delta}\leq\frac{1}{2}c_1,$$

then

$$\widetilde{Y}(T)\leq 2c_2N_0,$$

q.e.d.

Now, we turn to the question of solvability of the problem (1.12). We follow the arguments in [3] and consider the problem (1.9). By (1.3), we can represent $M(X)$ as

$$M(X)=m(X)-m(\bar{\xi})=m(X)-m(\xi)+m(\xi)-m(\bar{\xi}).$$

We express the difference $m(X)-m(\xi)=m(\xi+\int_0^t\boldsymbol{u}(\xi,\tau)d\tau)-m(\xi)$ in the form

$$m(X)-m(\xi)=\int_0^1\frac{\partial m(X_s)}{\partial s}ds=\frac{\partial m(X_s)}{\partial s}\Big|_{s=0}+\int_0^1(1-s)\frac{\partial^2m(X_s)}{\partial s^2}ds,$$

where $X_s=\xi+s\int_0^t\boldsymbol{u}(\xi,\tau)d\tau$, and we set

$$\ell_1(\boldsymbol{u})=\frac{\partial m(X_s)}{\partial s}\Big|_{s=0}=\sum_{j=1}^3 m_j(\xi)\int_0^t u_j(\xi,\tau)d\tau=\nabla m(\xi)\cdot\int_0^t\boldsymbol{u}(\xi,\tau)d\tau,$$

$$\ell_2(\boldsymbol{u})=\int_0^1(1-s)\frac{\partial^2m(X_s)}{\partial s^2}ds$$

$$=\int_0^s(1-s)\sum_{k,j=1}^3 m_{kj}(X_s)ds\int_0^t u_k(\xi,\tau)d\tau\int_0^t u_m(\xi,\tau)d\tau,$$

where $m_j(\xi)=\frac{\partial m(\xi)}{\partial\xi_j}$, $m_{kj}(\xi)=\frac{\partial^2m(\xi)}{\partial\xi_k\partial\xi_j}$; finally, we put

$$M_1(\rho_0)=m(\xi)-m(\bar{\xi})=m(\bar{\xi}+\boldsymbol{N}(\bar{\xi})\rho_0(\bar{\xi}))-m(\bar{\xi}).$$

It follows that the problem (1.9) can be written in the form

$$
\begin{gathered}
\boldsymbol{u}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{u}) - \nu\nabla^2\boldsymbol{u} + \nabla q = \boldsymbol{l}_1(\boldsymbol{u}, q), \\
\nabla \cdot \boldsymbol{u} = l_2(\boldsymbol{u}), \quad \xi \in \Omega_0, \ \ t > 0, \\
\Pi_0 S(\boldsymbol{u})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}), \\
-q + \nu\boldsymbol{n}_0 \cdot S(\boldsymbol{u})\boldsymbol{n}_0 - \ell_1(\boldsymbol{u}) = l_4(\boldsymbol{u}) + \ell_2(\boldsymbol{u}) + M_1(\rho_0), \quad \xi \in \Gamma_0, \\
\boldsymbol{u}(\xi, t) = 0, \quad \xi \in S, \\
\boldsymbol{u}(\xi, 0) = \boldsymbol{w}_0(\xi), \quad \xi \in \Omega_0,
\end{gathered}
\tag{3.29}
$$

We should also consider the problem of extending the solution of (1.12) given in the time interval $[0, T]$ into a larger interval $[0, T+1]$. It reduces to the construction of the solution of the problem (1.9) with the initial condition $\boldsymbol{u}(\xi, T) = \boldsymbol{u}(\xi, T-0)$ in the interval $t \in (T, T+1)$; here, as usual, $\boldsymbol{u}(\xi, T-0) = \lim_{\tau\to 0} \boldsymbol{u}(\xi, T-\tau)$ with $\tau > 0$. As in [3], we introduce the functions $\boldsymbol{u}_0$ and q_0 that coincide with $\boldsymbol{u}$ and q for $t < T$ and are defined by

$$
\begin{aligned}
\boldsymbol{u}_0(\xi, t) &= -3\boldsymbol{u}(\xi, 2T - t) + 4\boldsymbol{u}(\xi, 3T/2 - t/2), \\
q_0(\xi, t) &= -3q(\xi, 2T - t) + 4q(\xi, 3T/2 - t/2),
\end{aligned}
\tag{3.30}
$$

for $t > T$ (this extension guarantees preservation of class) and we set

$$
\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_0, \quad p = q - q_0,
$$

so that $\boldsymbol{v}(\xi, t) = 0$, $p(\xi, t) = 0$ for $t < T$. We represent the difference $M(X[\boldsymbol{u}_0 + \boldsymbol{v}]) - M(X[\boldsymbol{u}_0])$, where $X[\boldsymbol{w}] = \xi + \int_0^t \boldsymbol{w}(\xi, \tau)d\tau$, in the form

$$
\begin{aligned}
M(X[\boldsymbol{u}_0 + \boldsymbol{v}]) - M(X[\boldsymbol{u}_0]) &= m(X[\boldsymbol{u}_0 + \boldsymbol{v}]) - m(X[\boldsymbol{u}_0]) \\
&= \int_0^1 \frac{\partial}{\partial s} m(X[\boldsymbol{u}_0 + s\boldsymbol{v}])ds = \ell_3(\boldsymbol{v}) + \ell_4(\boldsymbol{v}),
\end{aligned}
$$

where

$$
\ell_3(\boldsymbol{v}) = \nabla m(X[\boldsymbol{u}_0]) \cdot \int_T^t \boldsymbol{v}(\xi, \tau)d\tau,
$$

$$
\ell_4(\boldsymbol{v}) = \int_0^1 (1-s) \sum_{j,k=1}^{3} m_{jk}(X[\boldsymbol{u}_0 + s\boldsymbol{v}])ds \int_T^t v_j(\xi, \tau)d\tau \int_T^t v_k(\xi, \tau)d\tau.
$$

It is easily seen that the problem mentioned above is equivalent to

$$
\begin{gathered}
\boldsymbol{v}_t + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}) - \nu\nabla^2\boldsymbol{v} + \nabla p = \boldsymbol{l}_1(\boldsymbol{u}_0 + \boldsymbol{v}, q_0 + p) - \boldsymbol{l}_1(\boldsymbol{u}_0, q_0) + \boldsymbol{f}(\xi, t), \\
\nabla \cdot \boldsymbol{v} = l_2(\boldsymbol{u}_0 + \boldsymbol{v}) - l_2(\boldsymbol{u}_0) + f(\xi, t), \qquad \xi \in \Omega_0, \\
\Pi_0 S(\boldsymbol{v})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}_0 + \boldsymbol{v}) - \boldsymbol{l}_3(\boldsymbol{u}_0) + \boldsymbol{d}(\xi, t),
\end{gathered}
\tag{3.31}
$$

$$-p+\nu \boldsymbol{n}_0\cdot S(\boldsymbol{v})\boldsymbol{n}_0-\ell_3(\boldsymbol{v})=l_4(\boldsymbol{u}_0+\boldsymbol{v})-l_4(\boldsymbol{u}_0)+\ell_4(\boldsymbol{v})+d(\xi,t),\quad \xi\in\Gamma_0,$$
$$\boldsymbol{v}(\xi,t)=0,\quad \xi\in S,$$
$$\boldsymbol{v}(\xi,T)=0,\quad \xi\in\Omega_0,$$

where

$$\begin{aligned}
\boldsymbol{f}&=\boldsymbol{l}_1(\boldsymbol{u}_0,q_0)-\boldsymbol{l}_1^{(0)}(\boldsymbol{u}_0,q_0)+\boldsymbol{u}_t^*,\\
f(\xi,t)&=l_2(\boldsymbol{u}_0)-l_2^{(0)}(\boldsymbol{u}_0)=\nabla\cdot\boldsymbol{F}(\xi,t),\\
\boldsymbol{F}(\xi,t)&=\boldsymbol{L}(\boldsymbol{u}_0)-\boldsymbol{L}^{(0)}(\boldsymbol{u}_0),\\
\boldsymbol{d}(\xi,t)&=\boldsymbol{l}_3(\boldsymbol{u}_0)-\boldsymbol{l}_3^{(0)}(\boldsymbol{u}_0),\\
d(\xi,t)&=(l_4(\boldsymbol{u}_0)-l_4^{(0)}(\boldsymbol{u}_0))+m(X[\boldsymbol{u}_0])-m^{(0)}(X[\boldsymbol{u}_0]),
\end{aligned}\tag{3.32}$$

$l_i^{(0)}$ and $m^{(0)}$ are extensions of l_i and m, $t<T$, constructed according to the rule (3.30),

$$\boldsymbol{u}_t^*=(\boldsymbol{u}_t)^0-\boldsymbol{u}_{0t}=-6\boldsymbol{u}_t(\xi,2T-t)+6\boldsymbol{u}_t(\xi,3T/2-t/2),\quad t\in(T,T+1),$$
$$\boldsymbol{u}_t^*=0,\quad t\le T \tag{3.33}$$

.

Theorem 3.7. *There exists $\epsilon>0$ such that if $\boldsymbol{w}_0$ and ρ_0 satisfy (3.4), then the problem (3.29) has a unique solution $\boldsymbol{u}\in W_2^{2+l,1+l/2}(Q_1)$, $\nabla q\in W_2^{l,l/2}(Q_1)$, and*

$$\|\boldsymbol{u}\|_{W_2^{2+l,1+l/2}(Q_1)}+\|\nabla q\|_{W_2^{l,l/2}(Q_1)}+\|q\|_{W_2^{l+1,(l+1)/2}(G_1)}$$
$$\le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)}+\|\rho_0\|_{W_2^{l+1/2}(\mathcal{G})}\Big). \tag{3.34}$$

Theorem 3.8. *Assume that the solution of the problem (1.12) is given for $t\in(0,T)$. There exists $\delta>0$ such that if (3.21) holds, then the problem (3.31) is uniquely solvable in the interval $(T,T+1)$ and*

$$\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_{T,T+1})}+\|\nabla p\|_{W_2^{l,l/2}(Q_{T,T+1})}+\|p\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}$$
$$+T\Big(\|\boldsymbol{v}\|_{W_2^{1+l,1/2+l/2}(Q_{T,T+1})}+\|\nabla p\|_{W_2^{l-1,l/2-1/2}(Q_{T,T+1})}+\|p\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}\Big)$$
$$\le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)}+\|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)}\Big), \tag{3.35}$$

where $Q_{T,T+1}=\Omega_0\times(T,T+1)$, $G_{T,T+1}=\Gamma_0\times(T,T+1)$.

Theorem 3.1 is a consequence of Theorems 3.6-3.8. We reproduce here the proof of this statement outlined in [3]. By Theorem 3.7, the solution $\boldsymbol{u},q$ of (1.9) exists for $t\in[0,1]$ and satisfies (3.34); moreover, it satisfies (1.12) together with the function

$$r(\xi,t)=R(X)=R(\xi)+\int_0^t \boldsymbol{N}(\bar X)\cdot\boldsymbol{u}(\xi,\tau)d\tau.$$

By Proposition 4.5 in [3],

$$\|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} \le c\Big(\|\rho_0\|_{W_2^{l+1}(\Gamma_0)} + \int_0^t \|\boldsymbol{u}(\cdot,\tau)\|_{W_2^{l+1}(\Gamma_0)} d\tau\Big)$$

$$\le c\Big(\|\boldsymbol{w}_0\|_{W_2^{l+1}(\Omega_0)} + \|\rho_0\|_{W_2^{l+1}(\mathcal{G})}\Big), \quad t \le 1.$$

Hence $\widetilde{Y}(1) \le c_1 N_0$ with a certain c_1 independent of T. If $c_1\epsilon \le \delta$, then, by Theorem 3.6, inequality (3.28) with $T = 1$ is satisfied, so we can assume that c_1 coincides with the constant in (3.28). Suppose that the solution of (1.12) is defined for $t \in (0,T)$ and satisfies (3.28). Then $\widetilde{Y}(T) \le \delta$, and, by Theorem 3.8, the problem (3.31) has a solution $\boldsymbol{v} \in W_2^{2+l,1+l/2}(Q_{T,T+1})$, $\nabla p \in W_2^{l,l/2}(Q_{T,T+1})$. For $t = T$ we have $\boldsymbol{v} = 0$, $\boldsymbol{v}_t + \nabla p = 0$, $\frac{\partial p}{\partial n}\Big|_S = 0$, $p|_{\mathcal{G}} = 0$, hence, $p = 0$, $\boldsymbol{v}_t = 0$. As a consequence, $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{v} \in W_2^{l+2,l/2+1}(Q_{T+1})$, $\nabla q = \nabla q_0 + \nabla p \in W_2^{l,l/2}(Q_{T+1})$. By virtue of (3.35),

$$\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_{T+1})} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_{T+1})} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_{T+1})}$$

$$\le \|\boldsymbol{u}_0\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_{T+1})} + \|\nabla q_0\|_{\widetilde{W}_2^{l,l/2}(Q_{T+1})} + \|q_0\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_{T+1})}$$

$$+c\Big(\|\boldsymbol{v}\|_{W_2^{2+l,1+l/2}(Q_{T,T+1})} + \|\nabla p\|_{W_2^{l,l/2}(Q_{T,T+1})} + \|p\|_{W_2^{l+1/2,l/2+1/4}(G_{T,T+1})}\Big)$$

$$+cT\Big(\|\boldsymbol{v}\|_{W_2^{1+l,1/2+l/2}(Q_{T,T+1})} + \|\nabla p\|_{W_2^{l-1,l/2-1/2}(Q_{T,T+1})} + \|p\|_{W_2^{l-1/2,l/2-1/4}(G_{T,T+1})}\Big)$$

$$\le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \|q\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}\Big) \le c\widetilde{Y}(T).$$

Together with the function $r(\xi,t) = r(\xi,T) + \int_T^t \boldsymbol{N}(\bar{X}) \cdot \boldsymbol{u}(\xi,\tau)d\tau = R(X)$, $\boldsymbol{u}$ and q satisfy (1.12). It is easily seen that

$$\sup_{t\in(T,T+1)} \Big(\|r(\cdot,t)\|_{W_2^{l+1}(\Gamma_0)} + T\|r(\cdot,t)\|_{W_2^{l}(\Gamma_0)}\Big) \le \|r(\cdot,T)\|_{W_2^{l+1}(\Gamma_0)}$$

$$+T\|r(\cdot,T)\|_{W_2^{l}(\Gamma_0)} + c\int_T^{T+1} \Big(\|\boldsymbol{u}(\cdot,\tau)\|_{W_2^{l+1}(\Gamma_0)} + T\|\boldsymbol{u}(\cdot,\tau)\|_{W_2^{l}(\Gamma_0)}\Big)d\tau$$

$$\le c\widetilde{Y}(T).$$

It follows that we have constructed the extension of $\boldsymbol{u}, q, r$ into the time interval $(0, T+1)$ such that $\widetilde{Y}(T+1) \le c_2\widetilde{Y}(T)$ with a certain c_2 independent of T. Hence $\widetilde{Y}(T+1) \le c_1c_2N_0$, and if we impose on ϵ one more last restriction $c_1c_2\epsilon \le \delta$, then, by Theorem 3.6, $\widetilde{Y}(T+1) \le c_1N_0$. So we can repeat the extension step by step into the infinite time interval $t > 0$ and obtain estimate (3.6). This completes the proof of Theorem 3.1.

4 Proof of Theorems 3.2 and 3.3

The problems (3.7) and (3.8) in the domain bounded only by the surface $\mathcal{G}$ are treated in the paper [4] for other (greater) values of l. The proofs are somewhat incomplete, and we repeat here the main ideas of the proofs.

Proof of theorem 3.3. We follow the arguments in [4] and reduce the problem (3.8) to a similar problem with $f = 0$, $\boldsymbol{v}_0 = 0$ and with $\boldsymbol{f}$ and $\boldsymbol{d}$ that can be extended by zero into $\mathcal{F}\times(-\infty, 0)$ and $\mathcal{G} \times (-\infty, 0)$ with preservation of class. We introduce $\boldsymbol{v}_1 = \nabla\Phi$ where Φ is a solution of the Dirichlet-Neumann problem

$$\nabla^2\Phi(x, t) = f(x, t), \quad x \in \mathcal{F},$$
$$\Phi(x, t) = 0, \quad x \in \mathcal{G}, \quad \frac{\partial\Phi(x, t)}{\partial n} = 0, \quad x \in S. \tag{4.1}$$

By a well-known coercive estimate for this problem,

$$\|\boldsymbol{v}_1\|_{W_2^{2+l,0}(\mathfrak{Q}_T)} \leq c\|f\|_{W_2^{l+1,0}(\mathfrak{Q}_T)}.$$

Moreover, since Φ_t is a solution of

$$\nabla^2\Phi_t = \nabla\cdot\boldsymbol{F}_t, \quad x \in \mathcal{F}, \quad \Phi_t\Big|_{x\in\mathcal{G}} = 0, \quad \frac{\partial\Phi_t}{\partial n}\Big|_{x\in S} = 0,$$

and $\boldsymbol{F}\cdot\boldsymbol{n}|_{x\in S} = 0$, we have

$$\|\nabla\Phi_t\|_{L_2(\mathfrak{Q}_T)} = \|\boldsymbol{v}_{1t}\|_{L_2(\mathfrak{Q}_T)} \leq c\|\boldsymbol{F}_t\|_{L_2(\mathfrak{Q}_T)}.$$

Applying this estimate also to finite differences of Φ_t with respect to time, we obtain

$$\|\boldsymbol{v}_{1t}\|_{W_2^{0,l/2}(\mathfrak{Q}_T)} \leq c\|\boldsymbol{F}_t\|_{W_2^{0,l/2}(\mathfrak{Q}_T)},$$

hence

$$\|\boldsymbol{v}_1\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} \leq c\Big(\|f\|_{W_2^{l+1,0}(\mathfrak{Q}_T)} + \|\boldsymbol{F}_t\|_{W_2^{0,l/2}(\mathfrak{Q}_T)}\Big). \tag{4.2}$$

The difference $\boldsymbol{w}_1 = \boldsymbol{v} - \boldsymbol{v}_1$ is a solution of the problem

$$\boldsymbol{w}_{1t} - \nu\nabla^2\boldsymbol{w}_1 + \nabla p = \boldsymbol{f}_1(x, t), \quad \nabla\cdot\boldsymbol{w}_1 = 0, \quad x \in \mathcal{F},$$
$$T(\boldsymbol{w}_1, p)\boldsymbol{N} = \boldsymbol{d}_1(x, t), \quad x \in \mathcal{G}, \quad \boldsymbol{w}_1(x, t) = \boldsymbol{a}_1(x, t), \quad x \in S,$$
$$\boldsymbol{w}_1(x, 0) = \boldsymbol{w}_{10}(x), \qquad x \in \mathcal{F}$$

where $\boldsymbol{a}_1 = \boldsymbol{a} - \boldsymbol{v}_1$,

$$\boldsymbol{f}_1 = \boldsymbol{f} - \boldsymbol{v}_{1t} + \nu\nabla^2\boldsymbol{v}_1, \quad \boldsymbol{d}_1 = \boldsymbol{d} - \nu S(\boldsymbol{v}_1)\boldsymbol{N}, \quad \boldsymbol{w}_{10} = \boldsymbol{v}_0(x) - \boldsymbol{v}_1(x, 0) \tag{4.3}$$

are vector fields satisfying the inequalities

$$\|\boldsymbol{f}_1\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} \leq c\Big(\|\boldsymbol{f}\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|\boldsymbol{v}_1\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}\Big),$$

$$\|\boldsymbol{d}_1\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \le c\Big(\|\boldsymbol{d}\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|\boldsymbol{v}_1\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}\Big).$$

$$\|\boldsymbol{w}_{10}\|_{W_2^{l+1}(\mathcal{F})} \le \|\boldsymbol{v}_0\|_{W_2^{l+1}(\mathcal{F})} + c\|\boldsymbol{v}_1\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)},$$

$$\|\boldsymbol{a}_1\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)} \le \|\boldsymbol{a}\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)} + c\|\boldsymbol{v}_1\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}.$$

Clearly, $\boldsymbol{a}_1 \cdot \boldsymbol{n} = 0$. Next, we decompose $\boldsymbol{f}_1$ into an orthogonal sum of a solenoidal and potential vector fields:

$$\boldsymbol{f}_1 = \boldsymbol{f}_1' + \nabla\varphi_1, \tag{4.4}$$

where φ_1 is a solution to the problem

$$\nabla^2\varphi_1 = \nabla\cdot\boldsymbol{f}_1, \quad x \in \mathcal{F}, \quad \varphi_1|_{x\in\mathcal{G}} = 0, \quad \frac{\partial\varphi_1}{\partial n}\Big|_{x\in S} = \boldsymbol{f}_1\cdot\boldsymbol{n}.$$

We have

$$\|\nabla\varphi_1\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|\boldsymbol{f}_1'\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} \le c\|\boldsymbol{f}_1\|_{W_2^{l,l/2}(\mathfrak{Q}_T)}. \tag{4.5}$$

The functions $\boldsymbol{w}_1$ and $q_1 = p - \varphi_1$ satisfy the relations

$$\boldsymbol{w}_{1t} - \nu\nabla^2\boldsymbol{w}_1 + \nabla q_1 = \boldsymbol{f}_1'(x,t), \quad \nabla\cdot\boldsymbol{w}_1 = 0, \quad x \in \mathcal{F},$$

$$T(\boldsymbol{w}_1, q_1)\boldsymbol{N} = \boldsymbol{d}_1(x,t), \quad x \in \mathcal{G}, \quad \boldsymbol{w}_1 = \boldsymbol{a}_1, \quad x \in S,$$

$$\boldsymbol{w}_1(x,0) = \boldsymbol{w}_{10}(x), \quad x \in \mathcal{F}.$$

The next step is the construction of $\boldsymbol{v}_2$ and p_2 satisfying appropriate initial and boundary conditions. We notice that $\boldsymbol{w}_1\cdot\boldsymbol{n}|_S = 0$ and q_1 can be regarded as a solution of the problem

$$\nabla^2 q_1(x,t) = 0, \quad x \in \mathcal{F},$$

$$q_1(x,t) = \nu\boldsymbol{N}\cdot S(\boldsymbol{w}_1)\boldsymbol{N} - \boldsymbol{d}_1\cdot\boldsymbol{N}, \quad x \in \mathcal{G}, \quad \frac{\partial q_1}{\partial n} = \nu\nabla^2\boldsymbol{w}_1(x,t)\cdot\boldsymbol{n}(x), \quad x \in S.$$

In the limit as $t \to 0$ we obtain

$$\begin{gathered} \nabla^2 q_1(x,0) = 0, \quad x \in \mathcal{F}, \\ q_1(x,0) = \nu\boldsymbol{N}\cdot S(\boldsymbol{w}_{10}(x))\boldsymbol{N} - \boldsymbol{d}_1(x,0)\cdot\boldsymbol{N}, \quad x \in \mathcal{G}, \\ \frac{\partial q_1}{\partial n} = \nu\nabla^2\boldsymbol{w}_{10}(x)\cdot\boldsymbol{n}(x), \quad x \in S. \end{gathered} \tag{4.6}$$

Since the function

$$\nu\boldsymbol{N}\cdot S(\boldsymbol{w}_{10}(x))\boldsymbol{N} - \boldsymbol{d}_1(x,0)\cdot\boldsymbol{N} \equiv p_0(x)$$

belongs to $W_2^{l-1/2}(\mathcal{G})$, we can construct its extension $p_0^* \in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_\infty)$ such that $p_2(x,0) = p_0(x)$ and

$$\begin{aligned} \|p_0^*\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_\infty)} &\le c\|p_0\|_{W_2^{l-1/2}(\mathcal{G})} \\ &\le c\Big(\|\boldsymbol{w}_{10}\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{d}_1\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}\Big). \end{aligned} \tag{4.7}$$

We may assume that $p_0^*(x,t)=0$ for $t>1$ (this can be achieved by multiplication of the extended function by an appropriate cut-off function of t). We also construct a divergence free vector field $\boldsymbol{w}^*\in W_2^{l+2,l/2+1}(\mathfrak{Q}_\infty)$ such that $\boldsymbol{w}^*(x,0)=\boldsymbol{w}_{10}(x)$ and

$$\|\boldsymbol{w}^*\|_{W_2^{l+2,l/2+1}(\mathfrak{Q}_\infty)}\le c\|\boldsymbol{w}_{10}\|_{W_2^{l+1}(\mathcal{F})}.$$

First we extend $\boldsymbol{w}_{10}$ from $\mathcal{F}$ in $\mathbb{R}^3$ with the preservation of class and solenoidality (see [5]), and then we set

$$\boldsymbol{w}^*(x,t)=\zeta(t)\int_{\mathbb{R}^3}\Gamma(x-y,t)\boldsymbol{w}_{10}(y)dy, \tag{4.8}$$

where $\zeta(t)$ is a smooth monotone function equal to one for small t and to zero for $t>1$, and

$$\Gamma(x,t)=\frac{1}{(4\pi t)^{3/2}}\exp\Big(-\frac{|x|^2}{4t}\Big)$$

is a fundamental solution of the heat equation. Well-known estimates of the heat potential imply

$$\|\boldsymbol{w}^*\|_{W_2^{l+2,l/2+1}(\mathfrak{Q}_\infty)}\le c\|\boldsymbol{w}_{10}\|_{W_2^{l+1}(\mathbb{R}^3)}\le c\|\boldsymbol{w}_{10}\|_{W_2^{l+1}(\mathcal{F})}. \tag{4.9}$$

Now, we define $p_2(x,t)$ as a solution of the problem

$$\nabla^2 p_2(x,t)=0,\quad x\in\mathcal{F},$$

$$p_2(x,t)=p_0^*(x,t),\quad x\in\mathcal{G},\quad \frac{\partial p_2(x,t)}{\partial n}=\nu\nabla^2\boldsymbol{w}^*(x,t)\cdot\boldsymbol{n},\quad x\in S.$$

We have

$$\|\nabla p_2\|_{W_2^{l,0}(\mathfrak{Q}_\infty)}\le c\Big(\|p_0^*\|_{W_2^{l+1/2,0}(\mathfrak{G}_\infty)}+\|\nabla^2\boldsymbol{w}^*\cdot\boldsymbol{n}\|_{W_2^{l-1/2,0}(\Sigma_\infty)}\Big), \tag{4.10}$$

$$\|\nabla p_2\|_{L_2(\mathcal{F})}\le c\Big(\|p_0^*\|_{W_2^{1/2}(\mathcal{G})}+\|\nabla^2\boldsymbol{w}^*\cdot\boldsymbol{n}\|_{W_2^{-1/2}(S)}\Big)$$
$$\le c\Big(\|p_0^*\|_{W_2^{1/2}(\mathcal{G})}+\|\nabla^2\boldsymbol{w}^*\|_{L_2(\mathcal{F})}\Big)$$

(for $\nabla^2\boldsymbol{w}^*$ is a solenoidal vector field), and, as a consequence of the last estimate,

$$\|\nabla p_2\|_{W_2^{0,l/2}(\mathfrak{Q}_\infty)}\le c\Big(\|p_0^*\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_\infty)}+\|\nabla^2\boldsymbol{w}^*\|_{W_2^{0,l/2}(\mathfrak{Q})}\Big). \tag{4.11}$$

Inequalities (4.10), (4.11) imply

$$\|\nabla p_2\|_{W_2^{l,l/2}(\mathfrak{Q}_\infty)}\le c\Big(\|p_0\|_{W_2^{l-1/2}(\mathcal{G})}+\|\nabla^2\boldsymbol{w}^*\|_{W_2^{l,l/2}(\mathfrak{Q}_\infty)}\Big)$$
$$\le c\Big(\|\boldsymbol{w}_{10}\|_{W_2^{l+1}(\mathcal{F})}+\|\boldsymbol{d}_1\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}\Big). \tag{4.12}$$

The function $\nabla^2\boldsymbol{w}^*\cdot\boldsymbol{n}\in L_2(0,T,W_2^{l-1/2}(\mathcal{G}))\cap W_2^{l/2}(0,T,W_2^{-1/2}(\mathcal{G}))$ has a limit as $t\to 0$ that is equal to $\nabla^2\boldsymbol{w}_{10}\cdot\boldsymbol{n}$, hence $p_2(x,0)$ is a solution of (4.6).

Let

$$\boldsymbol{w}_{11}(x) = \boldsymbol{f}'_1(x,0) - \nabla p_2(x,0) + \nu\nabla^2\boldsymbol{w}_{10}(x) \in W_2^{l-1}(\mathcal{F}).$$

We construct a solenoidal vector field $\boldsymbol{v}_2 \in W_2^{2+l,1+l/2}(\mathfrak{Q}_\infty)$ such that

$$\boldsymbol{v}_2(x,0) = \boldsymbol{w}_{10}(x), \quad \boldsymbol{v}_{2t}(x,0) = \boldsymbol{w}_{11}(x) \tag{4.13}$$

and

$$\|\boldsymbol{v}_2\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_\infty)} \le c\Big(\|\boldsymbol{w}_{10}\|_{W_2^{l+1}(\mathcal{F})} + \|\boldsymbol{w}_{11}\|_{W_2^{l-1}(\mathcal{F})}\Big). \tag{4.14}$$

To this end, we extend $\boldsymbol{w}_{10}$ and $\boldsymbol{w}_{11}$ from $\mathcal{F}$ to $\mathbb{R}^3$ with the preservation of their classes and solenoidality [5] and define a solenoidal $\boldsymbol{h} \in W_2^{l,l/2}(\mathfrak{Q}_\infty)$ such that $\boldsymbol{h}(x,t) = 0$ for $t > 1$, $\boldsymbol{h}(x,0) = \boldsymbol{w}_{11}(x) - \nabla^2\boldsymbol{w}_{10}(x)$ and

$$\|\boldsymbol{h}\|_{W_2^{l,l/2}(\mathfrak{Q}_\infty)} \le c\|\boldsymbol{w}_{11} - \nabla^2\boldsymbol{w}_{10}\|_{W_2^{l-1}(\mathcal{F})}.$$

This can be done by the formula similar to (4.8):

$$\boldsymbol{h}(x,t) = \zeta(t)\int_{\mathbb{R}^3}\Gamma(x-y,t)(\boldsymbol{w}_{11}(y) - \nabla^2\boldsymbol{w}_{10}(y))dy.$$

Now we set

$$\boldsymbol{v}_2(x,t) = \zeta(t)\int_{\mathbb{R}^3}\Gamma(x-y,t)\boldsymbol{w}_{10}(y)dy + \zeta(t)\int_0^t d\tau\int_{\mathbb{R}^3}\Gamma(x-y,t-\tau)\boldsymbol{h}(y,\tau)dy.$$

This vector field satisfies (4.13). Inequality (4.14) follows from the estimates of the heat potentials.

The functions $\boldsymbol{w}_2 = \boldsymbol{w}_1 - \boldsymbol{v}_2$ and $q_2 = q_1 - p_2$ satisfy the equations

$$\boldsymbol{w}_{2t} - \nu\nabla^2\boldsymbol{w}_2 + \nabla q_2 = \boldsymbol{f}'_1(x,t) - \boldsymbol{v}_{2t} + \nu\nabla^2\boldsymbol{v}_2 - \nabla p_2 \equiv \boldsymbol{f}_2,$$

$$\nabla\cdot\boldsymbol{w}_2 = 0, \quad x \in \mathcal{F},$$

$$T(\boldsymbol{w}_2, q_2)\boldsymbol{N} = \boldsymbol{d}_1 - T(\boldsymbol{v}_2, p_2)\boldsymbol{N} \equiv \boldsymbol{d}_2(x,t), \quad x \in \mathcal{G},$$

$$\boldsymbol{w}_2 = \boldsymbol{a}_1 - \boldsymbol{v}_2 \equiv \boldsymbol{a}_2(x,t), \quad x \in S,$$

$$\boldsymbol{w}_2(x,0) = 0, \qquad x \in \mathcal{F}.$$

It is clear that

$$\|\boldsymbol{f}_2\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} \le c\Big(\|\boldsymbol{f}'_1\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|\boldsymbol{v}_2\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + \|\nabla p_2\|_{W_2^{l,l/2}(\mathfrak{Q}_T)}\Big),$$

$$\|\boldsymbol{d}_2\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \le c\Big(\|\boldsymbol{d}_1\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|\boldsymbol{v}_2\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + \|p_2\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}\Big),$$

$$\|\boldsymbol{a}_2\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)} \le \|\boldsymbol{a}_1\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)} + c\|\boldsymbol{v}_2\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}.$$

By the definition of $\boldsymbol{v}_2$, the vector field $\boldsymbol{f}_2$ vanishes for $t = 0$ and its zero extension into $\mathcal{F}\times(-\infty, 0)$ is an extension with the preservation of class. Moreover, in view of the compatibility condition and of the definition of p_2, we have $\boldsymbol{d}_2(x, 0) = 0$, hence $\boldsymbol{d}_2$ can be also extended by zero into $\mathcal{G}\times(-\infty, 0)$ with the preservation of class $W_2^{l+1/2,l/2+1/4}$. The same is true for $\boldsymbol{a}_2$. Let

$$\boldsymbol{v}_3(x, t) = \nabla\Psi(x, t),$$

where $\Psi(x, t)$ is a solution to the problem

$$\nabla^2\Psi(x, t) = 0, \quad x \in \mathcal{F},$$

$$\Psi(x, t) = 0, \quad x \in \mathcal{G}, \quad \frac{\partial\Psi}{\partial n} = \boldsymbol{a}_2 \cdot \boldsymbol{n} = -\boldsymbol{v}_2(x, t) \cdot \boldsymbol{n}, \quad x \in S.$$

We have

$$\|\boldsymbol{v}_3\|_{W_2^{2+l,0}(\mathfrak{Q}_T)} \le c\|\boldsymbol{v}_2 \cdot \boldsymbol{n}\|_{W_2^{l+3/2,0}(\Sigma_T)}, \tag{4.16}$$

and from the energy relation

$$\int_{\mathcal{F}} |\nabla\Psi_t|^2 dx = \int_{\mathcal{G}} \boldsymbol{v}_{2t} \cdot \boldsymbol{n}\Psi_t dS$$

we conclude that

$$\|\boldsymbol{v}_{3t}\|_{L_2(\mathcal{F})} \le c\|\boldsymbol{v}_{2t} \cdot \boldsymbol{n}\|_{W_2^{-1/2}(\mathcal{G})} \le c\|\boldsymbol{v}_{2t}\|_{L_2(\mathcal{F})},$$

since $\nabla\cdot\boldsymbol{v}_2 = 0$. Applying this inequality to the finite differences of Ψ with respect to t and taking account of (4.16) we obtain

$$\|\boldsymbol{v}_3\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} \le c\|\boldsymbol{v}_2\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}. \tag{4.17}$$

We also have $\boldsymbol{v}_3(x, 0) = 0$ and $\boldsymbol{v}_{3t}(x, 0) = 0$, for, by the compatibility conditions,

$$\boldsymbol{v}_2(x, 0) \cdot \boldsymbol{n} = \boldsymbol{w}_{10}(x) \cdot \boldsymbol{n} = 0, \quad x \in S;$$

moreover,

$$\boldsymbol{v}_{2t}(x, 0)|_S \cdot \boldsymbol{n} = \boldsymbol{w}_{11}(x) \cdot \boldsymbol{n} = \Big(\nu\nabla^2\boldsymbol{w}_{10} \cdot \boldsymbol{n} - \frac{\partial p_2(x, 0)}{\partial n}\Big)\Big|_S = 0.$$

The functions $\boldsymbol{w}_3 = \boldsymbol{w}_2 - \boldsymbol{v}_3$ and q_2 satisfy

$$\boldsymbol{w}_{3t} - \nu\nabla^2\boldsymbol{w}_3 + \nabla q_2 = \boldsymbol{f}_2(x, t) - \boldsymbol{v}_{3t} + \nu\nabla^2\boldsymbol{v}_3 \equiv \boldsymbol{f}_3,$$

$$\nabla \cdot \boldsymbol{w}_3 = 0, \quad x \in \mathcal{F},$$

$$T(\boldsymbol{w}_3, q_2)\boldsymbol{N} = \boldsymbol{d}_2 - \nu S(\boldsymbol{v}_3)\boldsymbol{N} \equiv \boldsymbol{d}_3(x, t), \quad x \in \mathcal{G},$$

$$\boldsymbol{w}_3 = \boldsymbol{a}_2 - \boldsymbol{v}_3 \equiv \boldsymbol{a}_3, \quad x \in S,$$

$$\boldsymbol{w}_3(x, 0) = 0.$$

Our last step in this chain of transformations is the decomposition of $\boldsymbol{f}_3$, as in (4.4):

$$\boldsymbol{f}_3 = \boldsymbol{f}_3' + \nabla \varphi_3,$$

where φ_3 is a solution of

$$\nabla^2 \varphi_3 = \nabla \cdot \boldsymbol{f}_3, \quad x \in \mathcal{F}, \quad \varphi_3|_{x\in\mathcal{G}} = 0, \quad \frac{\partial \varphi_3}{\partial n}\Big|_{x\in S} = \boldsymbol{f}_3 \cdot \boldsymbol{n}.$$

The following inequality analogous to (4.5) holds:

$$\|\nabla \varphi_3\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|\boldsymbol{f}_3'\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} \le c\|\boldsymbol{f}_3\|_{W_2^{l,l/2}(\mathfrak{Q}_T)}. \tag{4.18}$$

The functions $\boldsymbol{w}_3$ and $q_3 = q_2 - \varphi_3$ satisfy the relations

$$\begin{gathered} \boldsymbol{w}_{3t} - \nu \nabla^2 \boldsymbol{w}_3 + \nabla q_3 = \boldsymbol{f}_3'(x,t), \quad \nabla \cdot \boldsymbol{w}_3 = 0, \quad x \in \mathcal{F}, \\ T(\boldsymbol{w}_3, q_3)\boldsymbol{N} = \boldsymbol{d}_3(x,t), \quad x \in \mathcal{G}, \quad \boldsymbol{w}_3 = \boldsymbol{a}_3, \quad x \in S, \\ \boldsymbol{w}_3(x,0) = 0, \quad x \in \mathcal{F}. \end{gathered} \tag{4.19}$$

The vector fields $\boldsymbol{f}_3'$, $\boldsymbol{d}_3$, $\boldsymbol{a}_3$ vanish for $t = 0$ and admit the zero extension in the domain $t < 0$ with the preservation of class. In addition, $\boldsymbol{a}_3 \cdot \boldsymbol{n} = 0$.

The estimate (3.12) for the solution of problem (4.19) can be obtained by using Schauder's localization method that reduces the proof of (3.12) to the proof of similar estimates for the model problems in the half-space $\mathbb{R}^3_+$, for whose solution explicit formulas are available (see [6-8]). The proof can be carried out following the arguments in [7], Theorems 3.2, 3.3, 4.1 (with $\sigma = 0$, $\gamma = 0$), which can be somewhat simplified. The details are omitted. The final result of the application of the Schauder procedure is the estimate of the type (4.32) in [7], i.e.,

$$\begin{aligned} &\|\boldsymbol{w}_3\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + \|\nabla q_3\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|q_3\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} \\ &\le c\Big(\|\boldsymbol{f}_3'\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|\boldsymbol{d}_3\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|\boldsymbol{a}_3\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)} \\ &\quad + \|\boldsymbol{w}_3\|_{L_2(\mathfrak{Q}_T)} + \|q_3\|_{W_2^{0,l/2}(\mathfrak{Q}_T)} + \|q_3\|_{L_2(\mathfrak{Q}_T)}\Big). \end{aligned} \tag{4.20}$$

To estimate the norms of q_3 in (4.20), we consider q_3 as a solution of the problem

$$\nabla^2 q_3(x,t) = 0, \quad x \in \mathcal{F},$$

$$q_3|_{x\in\mathcal{G}} = \nu \boldsymbol{N} \cdot S(\boldsymbol{w}_3)\boldsymbol{N} - \boldsymbol{d}_3 \cdot \boldsymbol{N}, \quad \frac{\partial q_3}{\partial n}\Big|_{x\in S} = \nu \nabla^2 \boldsymbol{w}_3 \cdot \boldsymbol{n},$$

and we define $\psi(x,t)$ by

$$\nabla^2 \psi(x,t) = q_3(x,t), \quad x \in \mathcal{F},$$

$$\psi|_{x\in\mathcal{G}} = 0, \quad \frac{\partial \psi(x,t)}{\partial n}\Big|_{x\in S} = 0.$$

It satisfies the well-known coercive estimate

$$\|\psi\|_{W_2^2(\mathcal{F})} \leq c\|q_3\|_{L_2(\mathcal{F})}.$$

From this estimate and from the Green formula

$$\int_{\mathcal{F}} q_3^2(x,t)dx = \int_{\mathcal{G}} \frac{\partial \psi}{\partial n} q_3(x,t)dS - \int_S \psi(x,t)\frac{\partial q_3}{\partial N}dS$$

$$= \int_{\mathcal{G}} \frac{\partial \psi}{\partial n}(\nu \boldsymbol{N}\cdot S(\boldsymbol{w}_3)\boldsymbol{N} - \boldsymbol{d}_3\cdot \boldsymbol{N})dS - \nu\int_S \boldsymbol{N}\cdot(\mathrm{rot}\boldsymbol{w}_3 \times \nabla\psi)dS$$

it follows that

$$\|q_3\|_{L_2(\mathcal{F})} \leq c\Big(\|\nabla \boldsymbol{w}_3\|_{L_2(\mathcal{G}\cup S)} + \|\boldsymbol{d}_3\|_{L_2(\mathcal{G})}\Big).$$

Applying this inequality to the finite differences of q_3 with respect to t, we obtain

$$\|q_3\|_{W_2^{0,l/2}(\mathfrak{Q}_T))} \leq c\Big(\|\nabla \boldsymbol{w}_3\|_{W_2^{0,l/2}(\mathfrak{G}_T\cup\Sigma_T)} + \|\boldsymbol{d}_3\|_{W_2^{0,l/2}(\mathfrak{G}_T)}\Big). \tag{4.21}$$

The norms of $\boldsymbol{w}_3$ on the right-hand side can be estimated by the interpolation inequality with a small $\epsilon > 0$

$$\|\nabla \boldsymbol{w}_3\|_{W_2^{0,l/2}(\mathfrak{G}_T\cup\Sigma_T)} \leq \epsilon\|\boldsymbol{w}_3\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + c(\epsilon)\|\boldsymbol{w}_3\|_{L_2(\mathfrak{Q}_T)}, \tag{4.22}$$

and from (4.20)-(4.22) we conclude that

$$\|\boldsymbol{w}_3\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)} + \|\nabla q_3\|_{W_2^{l,l/2}(\mathfrak{Q}_T)} + \|q_3\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}$$

$$\leq c\Big(\|\boldsymbol{f}'_3\|_{W_2^{l,l/2}(\mathfrak{Q}_T))} + \|\boldsymbol{d}_3\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)} + \|\boldsymbol{a}_3\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)} + \|\boldsymbol{w}_3\|_{L_2(\mathfrak{Q}_T)}\Big). \tag{4.23}$$

Finally, we estimate $\|\boldsymbol{w}_3\|_{L_2(\mathfrak{Q}_T)}$ by the energy inequality. We multiply the first equation in (4.19) by $\boldsymbol{w}_3$ and integrate over $\mathcal{F}$. This leads to

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_3\|^2_{L_2(\mathcal{F})} + \frac{\nu}{2}\|S(\boldsymbol{w}_3)\|^2_{L_2(\mathcal{F})}$$

$$= \int_{\mathcal{F}} \boldsymbol{f}'_3\cdot \boldsymbol{w}_3 dx + \int_{\mathcal{G}} \boldsymbol{d}_3\cdot\boldsymbol{w}_3 dS + \int_S T(\boldsymbol{w}_3, q_3)\boldsymbol{n}\cdot\boldsymbol{a}_3 dS.$$

Making use of the Korn inequality

$$\|\boldsymbol{w}_3\|_{W_2^1(\mathcal{F})} \leq c\|S(\boldsymbol{w}_3)\|_{L_2(\mathcal{F})},$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}_3\|^2_{L_2(\mathcal{F})} + \beta\|\boldsymbol{w}_3\|^2_{L_2(\mathcal{F})} \leq c\Big(\|\boldsymbol{f}'_3\|^2_{L_2(\mathcal{F})} + \|\boldsymbol{d}_3\|^2_{L_2(\mathcal{G})}\Big) + \|T(\boldsymbol{w}_3,q_3)\boldsymbol{n}\|_{L_2(S)}\|\boldsymbol{a}_3\|_{L_2(S)},$$

where $\beta = const > 0$. This implies

$$\frac{d}{dt}e^{2\beta t}\|\boldsymbol{w}_3\|^2_{L_2(\mathcal{F})} \leq 2ce^{2\beta t}\Big(\|\boldsymbol{f}'_3\|^2_{L_2(\mathcal{F})} + \|\boldsymbol{d}_3\|^2_{L_2(\mathcal{G})}\Big) + 2e^{2\beta t}\|T(\boldsymbol{w}_3,q_3)\boldsymbol{n}\|_{L_2(S)}\|\boldsymbol{a}_3\|_{L_2(S)},$$

$$\|\boldsymbol{w}_3\|^2_{L_2(\mathcal{F})} \leq 2c\int_0^t e^{-2\beta(t-\tau)}\Big(\|\boldsymbol{f}'_3\|^2_{L_2(\mathcal{F})}+\|\boldsymbol{d}_3\|^2_{L_2(\mathcal{G})}\Big)d\tau+2\int_0^t e^{-2\beta(t-\tau)}\|T(\boldsymbol{w}_3,q_3)\boldsymbol{n}\|_{L_2(S)}\|\boldsymbol{a}_3\|_{L_2(S)}d\tau,$$

$$\int_0^t \|\boldsymbol{w}_3\|^2_{L_2(\mathcal{F})}d\tau \leq c\int_0^t \Big(\|\boldsymbol{f}'_3\|^2_{L_2(\mathcal{F})}+\|\boldsymbol{d}_3\|^2_{L_2(\mathcal{G})}+\|T(\boldsymbol{w}_3,q_3)\boldsymbol{n}\|_{L_2(S)}\|\boldsymbol{a}_3\|_{L_2(S)}\Big)d\tau.$$

Together with (4.23), the last inequality yields estimate (3.12) for the solution of the problem (4.19):

$$\|\boldsymbol{w}_3\|_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}+\|\nabla q_3\|_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|q_3\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}$$
$$\leq c\Big(\|\boldsymbol{f}'_3\|_{W_2^{l,l/2}(\mathfrak{Q}_T))}+\|\boldsymbol{d}_3\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}+\|\boldsymbol{a}_3\|_{W_2^{l+3/2,l/2+3/4}(\Sigma_T)}\Big), \tag{4.24}$$

and (4.24), (4.2), (4.14), (4.16) imply (3.12).

We omit the proof of the solvability of problem (4.19) in the Sobolev spaces that can be carried out in the same way as in [4], Theorem 1.

In fact, Theorem 3.3 holds for arbitrary $l \in [0,3/2)$; in the case $l \geq 3/2$ additional compatibility conditions are required. In the proof of (3.11) we use the estimate (3.12) with $l \in (0,1/2)$. In this case, the compatibility condition $\nu\Pi_0 S(\boldsymbol{v}_0)\boldsymbol{N} = \Pi_0\boldsymbol{d}(x,0)$ makes no sense and is not required; moreover, the zero extension in the domain $t < 0$ with the preservation of class is possible for arbitrary $\boldsymbol{f} \in W_2^{l,l/2}(\mathfrak{Q}_T)$, $\boldsymbol{d} \in W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)$, since $l/2+1/4 < 1/2$. This enables us to repeat the above arguments without constructing p_2 and with $\boldsymbol{v}_2 = \boldsymbol{w}^*$. So we arrive at (4.19) with $\boldsymbol{w}_3 = \boldsymbol{v}-\boldsymbol{v}_1-\boldsymbol{w}^*$, $q_3 = p-\varphi_1-\varphi_3$. It is easily seen that $\boldsymbol{a}_3$ vanishes for $t=0$, so at the end we obtain (3.12) with $l \in (0,1/2)$.

We note in conclusion that inserting a weak linear term $2\omega(\boldsymbol{e}_3\times\boldsymbol{v})$ into the first equation (4.19) does not change the result: Theorem 3.3 remains valid.

Proof of theorem 3.2 The solvability of the problem (3.7) can be deduced from Theorem 3.3, because the function ρ can be excluded with the help of the formula $\rho(x,t) = \rho_0(x)+\int_0^t(\boldsymbol{v}\cdot\boldsymbol{N}+g)d\tau$. So we only need to obtain the estimates (3.10), (3.11). Without loss of generality we can assume $\omega = 0$. As above, we introduce the function $\boldsymbol{v}_1 = \nabla\Phi$ where Φ is a solution of (4.1), and we reduce the problem (3.7) to a similar problem with zero divergence:

$$\begin{gathered}\boldsymbol{w}_{1t}-\nu\nabla^2\boldsymbol{w}_1+\nabla p = \boldsymbol{f}_1(x,t), \quad \nabla\cdot\boldsymbol{w}_1 = 0, \quad x\in\mathcal{F},\\ T(\boldsymbol{w}_1,p)\boldsymbol{N}+\boldsymbol{N}b(x)\rho = \boldsymbol{d}_1(x,t), \quad \rho_t-\boldsymbol{N}(x)\cdot\boldsymbol{w}_1 = g_1(x,t), \quad x\in\mathcal{G},\\ \boldsymbol{w}_1(x,t) = \boldsymbol{a}-\boldsymbol{v}_1 \equiv \boldsymbol{a}_1(x,t), \quad x\in S,\\ \boldsymbol{w}_1(x,0) = \boldsymbol{w}_{10}(x), \quad x\in\mathcal{F}, \quad \rho(x,0) = \rho_0(x), \quad x\in\mathcal{G},\end{gathered} \tag{4.25}$$

where $\boldsymbol{w}_1 = \boldsymbol{v}-\boldsymbol{v}_1$, $g_1 = g+\boldsymbol{N}\cdot\boldsymbol{v}_1$, and $\boldsymbol{f}_1$, $\boldsymbol{d}_1$, $\boldsymbol{w}_{10}$ are defined in (4.3). The function $\boldsymbol{v}_1$ satisfies (4.2).

We obtain (3.10) following arguments in [4], Sec.3. We consider a model problem in the half-space $\mathbb{R}^3_+ = \{x_3 > 0\}$:

$$\boldsymbol{v}_t-\nu\nabla^2\boldsymbol{v}+\nabla p = \boldsymbol{f}(x,t), \quad \nabla\cdot\boldsymbol{v} = f(x,t) \qquad x\in\mathbb{R}^3_+,$$

$$
\begin{gathered}
T_{i3}(\boldsymbol{v},p)+\delta_{i3}\beta\rho=-d_i(x,t), \quad \beta=const>0, \quad i=1,2,3,\\
\rho_t+v_3=g(x,t), \quad \rho(x,0)=\rho_0(x), \quad x\in\mathbb{R}^2,\\
\boldsymbol{v}(x,0)=\boldsymbol{v}_0(x), \qquad x\in\mathbb{R}^3_+,
\end{gathered}
\tag{4.26}
$$

and we use Lemma 1 in [4]:

Proposition 4.1 *Let* $\mathbb{R}_T=\mathbb{R}^3_+\times(0,T)$, $\mathbb{R}'_T=\mathbb{R}^2\times(0,T)$ *and let* $\boldsymbol{v}\in W_2^{2+l,1+l/2}(\mathbb{R}_T)$, $\nabla p\in W_2^{l,l/2}(\mathbb{R}_T)$, $\rho\in W_2^{l+1/2,l/2+1/4}(\mathbb{R}'_T)$ *be a solution of the model problem (4.26) having for all* $t\le T$ *a compact support contained in* $C_\lambda=B_\lambda\times(0,\lambda)$, *where* B_λ *is a disc* $|x'|\le\lambda$ *in* $\mathbb{R}^2$ *and* $\lambda\in(0,1)$ *is a small positive number. The solution satisfies the inequality*

$$
\begin{gathered}
\sup_{t<T}\Big(\langle\langle\boldsymbol{v}(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}+\|\rho(\cdot,t)\|^2_{W_2^{l+1}(\mathbb{R}^2)}\Big)\\
+\int_0^T\Big(\langle\langle\nabla\boldsymbol{v}(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}+\|\rho(\cdot,t)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}\Big)dt\\
\le c\Big(\langle\langle\boldsymbol{v}_0\rangle\rangle^2_{l+1,\mathbb{R}^3_+}+\|\rho_0\|^2_{W_2^{l+1}(\mathbb{R}^2)}\Big)\\
+c\int_0^T\Big(\langle\langle\boldsymbol{f}(\cdot,t)\rangle\rangle^2_{l,\mathbb{R}^3_+}+\langle\langle f(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}+\|\boldsymbol{d}(\cdot,t)\|^2_{W_2^{l+1/2}(\mathbb{R}^2)}\\
+\|g(\cdot,t)\|^2_{W_2^{l+3/2}(\mathbb{R}^2)}\Big)dt\\
+c\Big(\int_0^T\langle\langle f(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}dt\Big)^{1/2}\Big(\int_0^T\langle\langle p(\cdot,t)\rangle\rangle^2_{l+1,\mathbb{R}^3_+}dt\Big)^{1/2}
\end{gathered}
\tag{4.27}
$$

where

$$
\langle\langle u\rangle\rangle_{l,R^3_+}=\Big(\int_0^\infty\|u(\cdot,x_3\|^2_{W_2^l(\mathbb{R}^2)}dx_3\Big)^{1/2}
$$

is the W_2^l*-norm of* u *with respect to the tangential variables* x_1,x_2 *and* c *is a constant independent of* T.

Then, as described in [4], we estimate the norm

$$
\mathcal{R}^2(T)\equiv\|\rho(\cdot,t)\|^2_{W_2^{l+1}(\mathcal{G})}+\int_0^t\|\rho(\cdot,\tau)\|^2_{W_2^{l+1/2}(\mathcal{G})}d\tau
$$

of the solution of (4.25) using the localization method. Let $x_0\in\mathcal{G}$ and let $\chi(x)$ be a smooth cut-off function equal to one for $|x-x_0|\le\lambda/2$ and to zero for $|x-x_0|\ge\lambda$ where λ is a small positive parameter. The functions

$$
\boldsymbol{u}=\boldsymbol{w}_1\chi, \quad q=p\chi, \quad r=\rho\chi
$$

satisfy

$$
\begin{gathered}
\boldsymbol{u}_t-\nu\nabla^2\boldsymbol{u}+\nabla q=\boldsymbol{f}_1\chi+\boldsymbol{f}'_1, \quad \nabla\cdot\boldsymbol{u}=f'_1,\\
T(\boldsymbol{u},q)\boldsymbol{N}+\boldsymbol{N}br=\boldsymbol{d}_1\chi+\boldsymbol{d}'_1, \quad r_t=\boldsymbol{u}\cdot\boldsymbol{N}+g_1\chi,
\end{gathered}
\tag{4.28}
$$

$$r(x,0)=\rho_0(x)\chi\equiv r_0(x),\quad \boldsymbol{u}(x,0)=\boldsymbol{w}_{10}(x)\chi\equiv\boldsymbol{u}_0$$

in a neighborhood of x_0 with

$$\boldsymbol{f}_1'=-\nu(\nabla^2(\boldsymbol{w}_1\chi)-\chi\nabla^2\boldsymbol{w}_1)+p\nabla\chi,\quad f_1'=\boldsymbol{w}_1\cdot\nabla\chi,\quad \boldsymbol{d}_1'=\nu(S(\chi\boldsymbol{w}_1)-\chi S(\boldsymbol{w}_1))\boldsymbol{N}.$$

We pass to the local Cartesian coordinates $\{y_1,y_2,y_3\}$ with the origin at x_0 and with the y_1 and y_2-axes located on the tangential plane to $\mathcal{G}$ at x_0, and we write (4.28) in the form (4.26) setting $\beta=b(x_0)$ and leaving only principal linear terms on the left-hand side of all the equations. We use Proposition 4.1. Then we cover $\mathcal{G}$ by a finite number of the subsets $\mathcal{G}_k=\{x\in\mathcal{G}:|x-x_0|\le\lambda\}$, write estimates (4.27) obtained in the neighborhoods of all x_k and add them. We fix λ sufficiently small and arrive at the inequality analogous to (3.21) in [4], namely,

$$\mathcal{R}^2(T)\le c\Big(\mathcal{F}_1^2(T)+\lambda\mathcal{V}_1^2(T)+\lambda\mathcal{R}^2(T)+\|\boldsymbol{w}_1\|^2_{W_2^{1+l,1/2+l/2}(\mathfrak{Q}_T)}+\|p\|^2_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|\rho\|^2_{W_2^{l,0}(\mathfrak{G}_T)}\Big),$$

where

$$\mathcal{F}_1^2(T)=\|\boldsymbol{f}_1\|^2_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|\boldsymbol{d}_1\|^2_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}+\|g_1\|^2_{W_2^{l+3/2,l/2+3/4}(\mathfrak{G}_T)}+\|\boldsymbol{w}_{10}\|^2_{W_2^{l+1}(\mathcal{F})}+\|\rho_0\|^2_{W_2^{l+1}(\mathcal{G})},$$

$$\mathcal{V}_1^2(T)=\|\boldsymbol{w}_1\|^2_{W_2^{2+l,1+l/2}(\mathfrak{Q}_T)}+\|\nabla p\|^2_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|p\|^2_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}.$$

Next, we consider $\boldsymbol{w}_1,p$ as the solution of the problem (3.8) with $\boldsymbol{f}=\boldsymbol{f}_1$, $f=0$, $\boldsymbol{d}=\boldsymbol{d}_1-\boldsymbol{N}b\rho$, $\boldsymbol{a}=\boldsymbol{a}_1$, $\boldsymbol{v}_0=\boldsymbol{w}_{10}$ and apply the inequality (3.12). This leads to

$$\mathcal{V}_1^2(T)\le c\Big(\|\boldsymbol{f}_1\|^2_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|\boldsymbol{d}_1\|^2_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}+\|\boldsymbol{w}_{10}\|^2_{W_2^{l+1}(\mathcal{F})}+\mathcal{R}^2(T)+\|\rho\|^2_{W_2^{0,l/2+1/4}(\mathfrak{G}_T)}\Big),$$

hence,

$$\mathcal{V}_1^2(T)+\mathcal{R}^2(T)\le c\Big(\mathcal{F}_1^2(T)+\|\boldsymbol{w}_1\|^2_{W_2^{1+l,1/2+l/2}(\mathfrak{Q}_T)}+\|p\|^2_{W_2^{l,l/2}(\mathfrak{Q}_T)}+\|\rho\|^2_{W_2^{l,0}(\mathfrak{G}_T)}+\|\rho\|_{W_2^{0,l/2+1/4}(\mathfrak{G}_T)}\Big),$$

if λ is sufficiently small.

Now, we estimate the norms of the solution on the right-hand side. We use the interpolation inequality

$$\|p\|^2_{W_2^{l,0}(\mathfrak{Q}_T)}\le\epsilon_1\|\nabla p\|^2_{W_2^{l,0}(\mathfrak{Q}_T)}+c(\epsilon_1)\|p\|^2_{L_2(\mathfrak{Q}_T)}$$

with arbitrarily small $\epsilon_1>0$. To estimate the L_2-norm of p, we regard p as a sum $p=p_1+p_2$, where the p_i are solutions to the problems

$$\nabla^2p_1=\nabla\cdot\boldsymbol{f}_1,\quad x\in\mathcal{F},\quad p_1|_{\mathcal{G}}=0,\quad \frac{\partial p_1}{\partial n}\Big|_S=\boldsymbol{f}_1\cdot\boldsymbol{n},$$

$$\nabla^2p_2=0,\quad x\in\mathcal{F},$$

$$p_2|_{\mathcal{G}}=\nu\boldsymbol{N}\cdot S(\boldsymbol{w}_1)\boldsymbol{N}+b\rho-\boldsymbol{d}_1\cdot\boldsymbol{N},\quad \frac{\partial p_1}{\partial n}\Big|_S=\nu\nabla^2\boldsymbol{w}_1\cdot\boldsymbol{n}.$$

Since $\|\nabla p_1\|_{L_2(\mathcal{F})} \leq \|\boldsymbol{f}_1\|_{L_2(\mathcal{F})}$, we have

$$\|p_1\|_{L_2(\mathcal{F})} \leq c\|\nabla p_1\|_{L_2(\mathcal{F})} \leq c\|\boldsymbol{f}_1\|_{L_2(\mathcal{F})}.$$

The function p_2 can be estimated precisely in the same way as q_3 in the preceding theorem:

$$\|p_2\|_{L_2(\mathcal{F})} \leq c\Big(\|\boldsymbol{d}_1\|_{L_2(\mathcal{G})} + \|\rho\|_{L_2(\mathcal{G})} + \|\nabla \boldsymbol{w}_1\|_{L_2(\mathcal{G}\cup S)}\Big).$$

Similar inequalities hold for the finite differences of p_i with respect to t, hence

$$\|p\|_{W_2^{0,l/2}(\mathfrak{Q}_T)} \leq c\Big(\|\boldsymbol{f}_1\|_{W_2^{0,l/2}(\mathfrak{Q}_T)}$$

$$+\|\boldsymbol{d}_1\|_{W_2^{0,l/2}(\mathfrak{G}_T)} + \|\rho\|_{W_2^{0,l/2}(\mathfrak{G}_T)} + \|\nabla \boldsymbol{w}_1\|_{W_2^{0,l/2}(\mathfrak{G}_T\cup\Sigma_T)}\Big).$$

We estimate the $W_2^{0,l/2}$-norm of ρ, using the equation $\rho_t = \boldsymbol{w}_1 \cdot \boldsymbol{N} + g_1$:

$$\|\rho\|_{W_2^{0,l/2}(\mathfrak{G}_T)} \leq c\Big(\|\rho\|_{L_2(\mathfrak{G}_T)} + \|\rho_t\|_{L_2(\mathfrak{G}_T)}\Big)$$

$$\leq c\Big(\|\rho\|_{L_2(\mathfrak{G}_T)} + \|\boldsymbol{w}_1\|_{L_2(\mathfrak{G}_T)} + \|g_1\|_{L_2(\mathfrak{G}_T)}\Big),$$

and the $W_2^{l,0}$-norm, using the interpolation inequality

$$\|\rho\|_{W_2^{l,0}(\mathfrak{G}_T)} \leq \epsilon_2\|\rho\|_{W_2^{l+1/2,0}(\mathfrak{G}_T)} + c(\epsilon_2)\Big(\int_0^T \|\rho\|^2_{W_2^{-1/2}(\mathcal{G})}\Big)^{1/2}.$$

Finally, we have

$$\|\nabla \boldsymbol{w}_1\|_{W_2^{0,l/2}(\mathfrak{G}_T\cup\Sigma_T)} + \|\boldsymbol{w}_1\|_{W_2^{l+1,l/2+1/2}(\mathfrak{Q}_T)} \leq \epsilon_3\|\boldsymbol{w}_1\|_{W_2^{l+2,l/2+1}(\mathfrak{Q}_T)} + c(\epsilon_3)\|\boldsymbol{w}_1\|_{L_2(\mathfrak{Q}_T)}.$$

Choosing ϵ_i sufficiently small, we easily obtain (3.10) for the solution of problem (4.25) which implies (3.10) for $\boldsymbol{v}, p, \rho$.

The above proof of the inequality (3.10) is valid also for $l \in [0, 1/2)$.

In order to prove (3.11), we should obtain additionally estimates of the norms of $t\boldsymbol{v}$, tp, $t\rho$. They can be deduced from (3.10). Let us consider the problem (4.25) and estimate $t\boldsymbol{w}_1$, tp, $t\rho$. Indeed, these functions can be regarded as a solution of the problem

$$(t\boldsymbol{w})_{1t} + 2\omega(\boldsymbol{e}_3 \times t\boldsymbol{w}_1) - \nu\nabla^2 t\boldsymbol{w}_1 + \nabla tp = t\boldsymbol{f}(x,t) + \boldsymbol{w}_1,$$

$$\nabla \cdot t\boldsymbol{w}_1(x,t) = 0, \quad x \in \mathcal{F}, \quad t > 0,$$

$$T(t\boldsymbol{w}_1, tp)\boldsymbol{N} + \boldsymbol{N}b(x)t\rho = t\boldsymbol{d}(x,t),$$

$$(t\rho)_t - \boldsymbol{N}(x) \cdot t\boldsymbol{w}_1(x,t) = tg(x,t) + \rho, \quad x \in \mathcal{G},$$

$$t\boldsymbol{w}_1(x,t) = t\boldsymbol{a}(x,t), \quad x \in S,$$

$$(t\boldsymbol{w}_{10})(x,0)=0, \quad x\in\mathcal{F}, \qquad (t\rho)(x,0)=0, \quad x\in\mathcal{G}.$$

By the inequality (3.10) with $l-1$ instead of l, we have

$$\|t\boldsymbol{w}_1\|_{W_2^{1+l,1/2+l/2}(\mathfrak{Q}_T)}+\|t\nabla p\|_{W_2^{l-1,l/2-1/2}(\mathfrak{Q}_T)}+\|tp\|_{W_2^{l-1/2,l/2-1/4}(\mathfrak{G}_T)}$$
$$+\|t\rho\|_{W_2^{l-1/2,0}(\mathfrak{G}_T)}+\sup_{t<T} t\|\rho(\cdot,t)\|_{W_2^{l+}(\Gamma_0)}$$
$$\leq c\Big(\|t\boldsymbol{f}\|_{W_2^{l-1,l/2-1/2}(\mathfrak{Q}_T)}+\|tf\|_{W_2^{l,0}(\mathfrak{Q}_T)}+\|t\boldsymbol{F}\|_{W_2^{0,1/2+l/2}(\mathfrak{Q}_T)}$$
$$+\|t\boldsymbol{d}_1\|_{W_2^{l-1/2,l/2-1/4}(\mathfrak{G}_T)}+\|tg\|_{W_2^{l+1/2,l/2+1/4}(\mathfrak{G}_T)}+\|t\boldsymbol{a}\|_{W_2^{l+1/2,l/2+1/4}(\Sigma_T)}\Big)$$
$$+c\Big(\int_0^T t^2(\|\boldsymbol{w}_1\|^2_{L_2(\mathcal{F})}+\|\rho\|^2_{W_2^{-1/2}(\mathcal{G})}\Big)^{1/2}$$
$$+c\Big(\|\boldsymbol{w}_1\|_{W_2^{l,l/2}(Q_T)}+\|\rho\|_{W_2^{l+1/2,l/2+1}(\mathfrak{G}_T)}\Big) \tag{4.29}$$

The last two terms can be estimated by the inequality (3.10) (written for the solution of (4.25)). When we add the resulting estimate to (3.10), we obtain (3.11) for $\boldsymbol{w}_1, p, \rho$ and consequently for $\boldsymbol{v}, p, \rho$. This completes the proof of Theorem 3.2.

5 Proof of Proposition 3.1.

.

In this section we estimate the nonlinear terms in (1.12). For this we need some auxiliary propositions.

Proposition 5.1. *Arbitrary functions $u(x), v(x)$ given in a domain $\Omega\subset\mathbb{R}^n$ satisfy the inequality*

$$\|uv\|_{W_2^l(\Omega)}\leq c\Big(\sup_\Omega|v(x)|\|u\|_{W_2^l(\Omega)}+\|u\|_{L_p(\Omega)}\|v\|_{W_2^{l+n/p}(\Omega)}\Big), \tag{5.1}$$

where $2<p\leq\infty$. In particular,

$$\|uv\|_{W_2^l(\Omega)}\leq c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^s(\Omega)}, \quad s>n/2, \tag{5.2}$$

if $l\leq n/2$, and

$$\|uv\|_{W_2^l(\Omega)}\leq c\|u\|_{W_2^l(\Omega)}\Big(\sup_\Omega|v(x)|+\|v\|_{W_2^{n/2}(\Omega)}\Big), \tag{5.3}$$

if $l<n/2$.

Inequalities (5.1)-(5.3) hold also for functions given on smooth manifolds. The proof of Proposition 5.1 can be found, for instance, in [3].

In addition, we have

$$\|uv\|_{L_2(\Omega)}\leq\|u\|_{L_p(\Omega)}\|v\|_{L_q(\Omega)}\leq c\|u\|_{W_2^l(\Omega)}\|v\|_{W_2^{n/2-l}(\Omega)}, \tag{5.4}$$

where $1/p+1/q=1/2$, $l=n/2-n/p=n/q$.

Proposition 5.2 [3] *Let $b_1(x),...,b_M(x)$ be functions of class $W_2^r(\Omega)\cap W_2^{3/2}(\Omega)$ defined in the domain $\Omega\subset\mathbb{R}^3$, and let $f(b)$, $b=(b_1,...b_M)$, be a smooth function uniformly bounded together with its derivatives with respect to b_k when $b=b(x)$, $x\in\Omega$. Then*

$$\|f(b(\cdot)\|_{W_2^r(\Omega)}\le\|f\|_{L_2(\Omega)}+c\|b\|_{W_2^r(\Omega)}, \tag{5.5}$$

if $r<1$, and

$$\|\nabla_x f(b)\|_{W_2^{r-1}(\Omega)}\le c\|\nabla b\|_{W_2^{r-1}(\Omega)}, \tag{5.6}$$

if $r\ge1$. The constant in (5.6) depends on $\|b\|_{W_2^{r-1}(\Omega)}$ and $\|b\|_{W_2^{3/2}(\Omega)}$.

The inequalities (5.5), (5.6) hold also in the two-dimensional case under the assumption $b\in W_2^r(\Omega)\cap W_2^1(\Omega)$.

Proposition 5.3 *1. An arbitrary function $u\in W_2^\mu(0,T)$, $\mu\in(0,1)$, satisfies the inequality*

$$\|u\|^2_{L_2(0,T)}+\int_0^T\int_0^T\frac{|u(t)-u(t')|^2}{|t-t'|^{1+2\mu}}dtdt'$$
$$\le c\Big(\|u\|^2_{L_2(0,T)}+\int_0^{\min(T,1)}\frac{dh}{h^{1+2\mu}}\int_h^T|u(t-h)-u(t)|^2dt\Big). \tag{5.7}$$

2. If $u\in W_2^1(0,T)$ and $\mu\in(0,1)$, then

$$\|u\|_{W_2^\mu(0,T)}\le c\|u\|_{W_2^1(0,T)}, \tag{5.8}$$

The constants in (5.7),(5.8) are independent of T.

We omit an elementary proof of this proposition. Applying (5.7) to the function $tu(t)$, we easily obtain

$$\|tu\|^2_{L_2(0,T)}+\int_0^T\int_0^T\frac{|tu(t)-t'u(t')|^2}{|t-t'|^{1+2\mu}}dtdt'$$
$$\le c\Big(\|(1+t)u\|^2_{L_2(0,T)}+\int_0^{\min(T,1)}\frac{dh}{h^{1+2\mu}}\int_h^T t^2|u(t-h)-u(t)|^2dt\Big)$$
$$\le c\Big(\|(1+t)u\|^2_{L_2(0,T)}+\int_0^{\min(T,1)}\frac{dh}{h^{1+2\mu}}\int_h^T|(t-h)u(t-h)-tu(t)|^2dt\Big) \tag{5.9}$$

Let $\Delta_t(-h)u(t)=u(t-h)-u(t)$. In what follows we often use the relation

$$\Delta_t(-h)(u(t)v(t))=(\Delta_t(-h)u(t))v(t-h)+u(t)\Delta_t(-h)v(t). \tag{5.10}$$

If u and v depend also on $x\in\Omega\subset\mathbb{R}^n$, then

$$\|\Delta_t(-h)(u(\cdot,t)v(\cdot,t))\|_{L_2(\Omega)}\le\sup_\Omega|v(x,t-h)|\|\Delta_t(-h)u(\cdot,t)\|_{L_2(\Omega)}$$
$$+\int_0^h\|v_t(\cdot,t-\tau)\|_{L_q(\Omega)}d\tau\|u(\cdot,t)\|_{L_p(\Omega)}, \tag{5.11}$$

where $1/p+1/q=1/2$. If $n/2-n/p=l$, then, by (5.4),

$$\|\Delta_t(-h)(u(\cdot,t)v(\cdot,t))\|_{L_2(\Omega)} \leq \sup_{\Omega}|v(x,t-h)|\|\Delta_t(-h)u(\cdot,t)\|_{L_2(\Omega)}$$

$$+c\int_0^h \|v_t(\cdot,t-\tau)\|_{W_2^{n/2-l}(\Omega)}d\tau\|u(\cdot,t)\|_{W_2^l(\Omega)}. \tag{5.12}$$

The estimates of nonlinear terms in (1.12) are based on the analysis of the elements A_{ij} of the matrix A. They are second degree polynomials of $D_{km}(\xi,t)=\int_0^t \frac{\partial u_k(\xi,\tau)}{\partial \xi_m}d\tau$.

Proposition 5.4 *Assume that* $\boldsymbol{u}(\xi,t)$*, defined for* $t\in[0,T]$*, satisfies the inequality*

$$\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)} \leq \delta \leq 1. \tag{5.13}$$

Then for arbitrary $l_1\in[0,l+1/2)$

$$\|D_{km}(\cdot,t)\|_{W_2^{l_1}(\Omega_0)} \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}. \tag{5.14}$$

Moreover,

$$\sup_{\Omega_0}|D_{km}(\xi,t)| \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{5.15}$$

$$\|D_{km}(\cdot,t)\|_{W_2^{l+1}(\Omega_0)} \leq c\sqrt{t}\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{5.16}$$

$$\|I-A\|_{W_2^{l_1}(\Omega_0)} \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{5.17}$$

$$\sup_{\Omega_0}|I-A(\xi,t)| \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{5.18}$$

$$\|A\|_{W_2^{l_1}(\Omega_0)} \leq c, \quad \|A\|_{W_2^{l+1}(\Omega_0)} \leq c(1+\sqrt{t}), \tag{5.19}$$

$$\|I-A\|_{W_2^{l+1}(\Omega_0)} \leq c\sqrt{t}\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{5.20}$$

$$\|A_t(\cdot,t)\|_{W_2^l(\Omega_0)} \leq c\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}, \tag{5.21}$$

$$\|A_{tt}(\cdot,t)\|_{L_2(Q_T)} \leq c\|\nabla \boldsymbol{u}\|_{W_2^{l+1,l/2+1/2}(Q_T)}, \tag{5.22}$$

$$\|\nabla A_t(\cdot,t)\|_{W_2^l(\Omega_0)} \leq c\Big(\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{l+1}(\Omega_0)}+\sqrt{t}(\sup_{\Omega_0}|\nabla \boldsymbol{u}(\xi,t)|+\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{3/2}(\Omega_0)})\Big), \tag{5.23}$$

and, as a consequence,

$$\|\nabla A_t\|_{W_2^{l,0}(Q_T)} \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}, \tag{5.24}$$

where $\|A\|=\max_{k,m}\|A_{km}\|$*. The constants in all these inequalities are independent of* T*.*

Proof. By the Hölder inequality,

$$\|D_{km}(\cdot,t)\|_{W_2^{l_1}(\Omega_0)} \leq \int_0^t \|\frac{\partial u_k(\cdot,\tau)}{\partial \xi_m}\|_{W_2^{l_1}(\Omega_0)}d\tau$$

$$\leq \Big(\int_0^t (1+\tau)^{2\beta}\|\nabla \boldsymbol{u}(\cdot,\tau)\|_{W_2^{l_1}(\Omega_0)}d\tau\Big)^{1/2}\Big(\int_0^t (1+\tau)^{-2\beta}d\tau\Big)^{1/2} \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)},$$

where $\beta = 1 + l - l_1 > 1/2$; moreover,

$$\|D_{km}(\cdot,t)\|_{W_2^{l+1}(\Omega_0)} \leq \int_0^t \|\frac{\partial u_k(\cdot,\tau)}{\partial \xi_m}\|_{W_2^{l+1}(\Omega_0)} d\tau \leq c\sqrt{t}\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}.$$

Estimate (5.15) is a consequence of (5.14) and of the imbedding of $W_2^s(\Omega_0)$, $s > 3/2$, in $C(\Omega_0)$.

Since $\delta_{km} - A_{km}$ is a linear combination of D_{ij} and $D_{ij}D_{qs}$, (5.17)-(5.20) follow easily from (5.14)-(5.16) and Proposition 5.1. The time derivatives A_{ijt} are linear combinations of $D_{kmt} = \frac{\partial u_k}{\partial \xi_m}$ and $\frac{\partial u_k}{\partial \xi_m} D_{qi}$, and A_{ijtt} are linear combinations of $\frac{\partial}{\partial t}\frac{\partial u_k}{\partial \xi_m}$, $\frac{\partial}{\partial t}\frac{\partial u_k}{\partial \xi_m} D_{qs}$ and $\frac{\partial u_k}{\partial \xi_m}\frac{\partial u_i}{\partial \xi_j}$. Hence

$$\|A_t(\cdot,t)\|_{W_2^l(\Omega_0)} \leq c\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big(1 + \sup_{\Omega_0}|D(\xi,t)| + \|D(\cdot,t)\|_{W_2^{3/2}(\Omega_0)}\Big)$$

$$\leq c(1+\delta)\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)},$$

$$\|A_{tt}\|_{L_2(Q_T)} \leq c\Big(\|\nabla \boldsymbol{u}_t\|_{L_2(Q_T)} + \sup_{t<T}\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\|\nabla \boldsymbol{u}\|_{W_2^{l,0}(Q_T)}\Big),$$

which implies (5.22).

Finally, (5.23) is a consequence of the fact that $\frac{\partial}{\partial \xi_s}A_{ijt}(\xi,t)$ is a linear combination of $\frac{\partial^2 u_k}{\partial \xi_m \partial \xi_j}$, $\frac{\partial^2 u_k}{\partial \xi_m \partial \xi_j} D_{qi}$, $\frac{\partial u_k}{\partial \xi_m}\frac{\partial D_{qi}}{\partial \xi_j}$:

$$\|\nabla A_t(\cdot,t)\|_{W_2^l(\Omega_0)} \leq c\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{l+1}(\Omega_0)}\Big(1 + \sup_{\Omega_0}|D(\xi,t)| + \|D(\cdot,t)\|_{W_2^{3/2}(\Omega_0)}\Big)$$

$$+c\Big(\sup_{\Omega_0}|\nabla \boldsymbol{u}(\xi,t)| + \|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{3/2}(\Omega_0)}\Big)\int_0^t \|\nabla \boldsymbol{u}(\cdot,\tau)\|_{W_2^{l+1}(\Omega_0)} d\tau$$

$$\leq c\Big(\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{l+1}(\Omega_0)} + \sqrt{t}\sup_{\Omega_0}|\nabla \boldsymbol{u}(\xi,t)| + \sqrt{t}\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{3/2}(\Omega_0)}\Big)$$

$$\leq c\Big(\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{l+1}(\Omega_0)} + \sqrt{t}\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{l+1/2}(\Omega_0)}\Big).$$

This implies (5.24), for

$$\int_0^T (1+t)\|\nabla \boldsymbol{u}(\cdot,t)\|^2_{W_2^{l+1/2}(\Omega_0)} dt$$

$$\leq c\int_0^T \Big(\|\nabla \boldsymbol{u}(\cdot,t)\|^2_{W_2^{l+1}(\Omega_0)} + (1+t)^2\|\nabla \boldsymbol{u}(\cdot,t)\|^2_{W_2^l(\Omega_0)}\Big)dt \leq c\|\nabla \boldsymbol{u}(\cdot,t)\|^2_{\widetilde{W}_2^{l+1}(Q_T)}.$$

The proposition is proved.

Now, we proceed to the estimates of $\boldsymbol{l}_1(\boldsymbol{u},q), l_2(\boldsymbol{u}), \boldsymbol{L}(\boldsymbol{u})$.

Proposition 5.5. *If (5.13) holds, then the expressions $\boldsymbol{l}_1(\boldsymbol{u},q)$, $l_2(\boldsymbol{u})$, $\boldsymbol{L}(\boldsymbol{u})$ satisfy the inequalities*

$$\|\boldsymbol{l}_1(\boldsymbol{u},q)\|_{\widetilde{W}_2^{l,l/2}(Q_T)}$$

$$\leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}\Big(\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)}\Big), \tag{5.25}$$

$$\|l_2(\boldsymbol{u})\|_{\widetilde{W}_2^{l+1,0}(Q_T)} + \|\boldsymbol{L}(\boldsymbol{u})\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)} + \|\boldsymbol{u}_t\|_{\widetilde{W}_2^{l,l/2}(Q_T)} + \sup_{Q_T}|\boldsymbol{u}(\xi,t)|$$

$$+\|(1+t)\boldsymbol{u}\|_{W_2^{l+1/2,0}(Q_T)}\Big) \leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_T)} \tag{5.26}$$

with constants independent of T.

Proof. We start with the estimate of $\|(I-A)\nabla q\|_{W_2^{l,0}(Q_T)}$. It follows from (5.3), (5.17), (5.18) that

$$\|(I-A)\nabla q\|_{W_2^l(\Omega_0)} \leq c\Big(\sup_{\Omega_0}|I-A(\xi,t)| + \|I-A\|_{W_2^{3/2}(\Omega_0)}\Big)\|\nabla q\|_{W_2^l(\Omega_0)}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla q(\cdot,t)\|_{W_2^l(\Omega_0)}, \tag{5.27}$$

$$\|t(I-A)\nabla q\|_{W_2^{l-1}(\Omega_0)} \leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|t\nabla q(\cdot,t)\|_{W_2^{l-1}(\Omega_0)},$$

hence,

$$\|(I-A)\nabla q\|_{\widetilde{W}_2^{l,0}(Q_T)} \leq \|(I-A)\nabla q\|_{W_2^{l,0}(Q_T)}$$

$$+\|t(I-A)\nabla q\|_{W_2^{l-1,0}(Q_T)} \leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\|\nabla q(\cdot,t)\|_{\widetilde{W}_2^{l,0}(Q_T)}. \tag{5.28}$$

The expression $l_2(\boldsymbol{u}) = (I-A)\nabla\cdot\boldsymbol{u}$ satisfies

$$\|l_2(\boldsymbol{u})\|_{W_2^{l+1}(\Omega_0)} = \|(I-A)\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)}$$

$$\leq c\Big(\sup_{\Omega_0}|(I-A)|\|\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)} + \|I-A\|_{W_2^{l+1}(\Omega_0)}\sup_{\Omega_0}|\nabla\boldsymbol{u}|\Big)$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)} + \sqrt{t}\sup_{\Omega_0}|\nabla\boldsymbol{u}|\Big), \tag{5.29}$$

$$\|t(I-A)\nabla\boldsymbol{u}\|_{W_2^l(\Omega_0)} \leq ct\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla\boldsymbol{u}\|_{W_2^l(\Omega_0)},$$

which implies

$$\|l_2(\boldsymbol{u})\|_{\widetilde{W}_2^{l+1,0}(Q_T)} \leq c\|\nabla\boldsymbol{u}\|^2_{\widetilde{W}_2^{l+1,0}(Q_T)}. \tag{5.30}$$

Now, we consider

$$(A\nabla\cdot A\nabla)\boldsymbol{u} - \nabla^2\boldsymbol{u} = A\nabla\cdot(A-I)\nabla)\boldsymbol{u} + ((A-I)\nabla\cdot\nabla)\boldsymbol{u}.$$

The inequality

$$\|(A-I)\nabla\cdot\nabla\boldsymbol{u}\|_{W_2^l(\Omega_0)} \leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)}$$

is proved in the same way as (5.27). Moreover, by (5.19) and (5.29),

$$\|A\nabla\cdot(A-I)\nabla\boldsymbol{u}\|_{W_2^l(\Omega_0)} \leq c\|(A-I)\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)} + \sqrt{t}\sup_{\Omega_0}|\nabla\boldsymbol{u}(\xi,t)|\Big),$$

hence

$$\|(A\nabla\cdot A\nabla)\boldsymbol{u}-\nabla^2\boldsymbol{u}\|_{W_2^l(\Omega_0)}\le c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)}+\sqrt{t}\sup_{\Omega_0}|\nabla\boldsymbol{u}|\Big).$$

Replacing l by $l-1$ we obtain

$$\|(A\nabla\cdot A\nabla)\boldsymbol{u}-\nabla^2\boldsymbol{u}\|_{W_2^{l-1}(\Omega_0)}\le c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla\boldsymbol{u}\|_{W_2^l(\Omega_0)}.$$

The last two inequalities imply

$$\|(A\nabla\cdot A\nabla)\boldsymbol{u}-\nabla^2\boldsymbol{u}\|_{\widetilde{W}_2^{l,0}(Q_T)}\le c\|\nabla\boldsymbol{u}\|^2_{\widetilde{W}_2^{l+1,0}(Q_t)}.$$

This estimate completes the proof of

$$\|\boldsymbol{l}_1(\boldsymbol{u},q)\|_{\widetilde{W}_2^{l,0}(Q_T)}\le c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}+\|\nabla q\|_{\widetilde{W}_2^{l,0}(Q_T)}\Big). \tag{5.31}$$

Next, we estimate the $\widetilde{W}_2^{0,l/2}(Q_T)$-norm of $(A-I)\nabla q$. By (5.18),

$$\|(1+t)(A-1)\nabla q\|_{L_2(Q_T)}\le c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|(1+t)\nabla q\|_{L_2(Q_T)}. \tag{5.32}$$

Moreover, by virtue of (5.12),

$$\begin{aligned}\|\Delta_t(-h)((I-A)\nabla q)\|_{L_2(\Omega_0)}&\le\sup_{\Omega_0}|(1-A(\xi,t-h)|\|\Delta_t(-h)\nabla q\|_{L_2(\Omega_0)}\\&+c\int_0^h\|A_t(\cdot,t-\tau)\|_{W_2^{3/2-l}(\Omega_0)}d\tau\|\nabla q(\cdot,t)\|_{W_2^l(\Omega_0)}\\&\le c\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\Delta_t(-h)\nabla q\|_{L_2(\Omega_0)}\\&+h\sup_{\tau\in(t-h,t)}\|\nabla\boldsymbol{u}(\cdot,\tau)\|_{W_2^l(\Omega_0)}\|\nabla q(\cdot,t)\|_{W_2^l(\Omega_0)}\Big),\end{aligned} \tag{5.33}$$

$$\begin{aligned}\|\Delta_t(-h)((I-A)\nabla q)\|_{L_2(\Omega_0)}&\le\sup_{\Omega_0}|(1-A(\xi,t-h)|\|\Delta_t(-h)\nabla q\|_{L_2(\Omega_0)}\\&+c\int_0^h\|\nabla\boldsymbol{u}(\cdot,t-\tau)\|_{W_2^{5/2-l}(\Omega_0)}d\tau\|\nabla q(\cdot,t)\|_{W_2^{l-1}(\Omega_0)}\\&\le c\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\Delta_t(-h)\nabla q\|_{L_2(\Omega_0)}+\sqrt{h}\|\nabla\boldsymbol{u}\|_{W_2^{l+1,0}(Q_t)}\|\nabla q(\cdot,t)\|_{W_2^{l-1}(\Omega_0)}\Big),\end{aligned} \tag{5.34}$$

which implies

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+l}}\int_h^T\|\Delta_t(-h)((I-A)\nabla q(\cdot,t))\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$
$$\le c\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}+\sup_{t<T}\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big)\|\nabla q\|_{W_2^{l,l/2}(Q_T)},$$

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^l}\int_h^T t^2\|\Delta_t(-h)((I-A)\nabla q(\cdot,t))\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$
$$\le c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\Big(\Big(\int_0^{min(T,1)}\frac{dh}{h^l}\int_h^T t^2\|\Delta_t(-h)\nabla q\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$
$$+\|t\nabla q\|_{W_2^{l-1}(Q_T)}\Big).$$

By Proposition 5.3, these inequalities, together with (5.32), yield

$$\|(A-1)\nabla q\|_{\widetilde{W}_2^{0,l/2}(Q_T)}\le c\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}+\sup_{t<T}\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big)\|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)}. \tag{5.35}$$

Next, we estimate $\|\boldsymbol{L}(\boldsymbol{u})\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)}$. It is easily seen that

$$\|(1+t)\boldsymbol{L}(\boldsymbol{u})\|_{L_2(Q_T)}\le c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\|(1+t)\boldsymbol{u}\|_{L_2(Q_T)}.$$

Let us consider $\boldsymbol{L}_t(\boldsymbol{u})=(I-A^T)\boldsymbol{u}_t-A_t^T\boldsymbol{u}$. The inequality

$$\|(I-A^T)\boldsymbol{u}_t\|_{\widetilde{W}_2^{0,l/2}(Q_T)}\le c\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}+\sup_{t<T}\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big)\|\boldsymbol{u}_t\|_{\widetilde{W}_2^{l,l/2}(Q_T)}$$

is obtained in the same way as (5.35), so it remains to estimate $\|A_t^T\boldsymbol{u}\|_{\widetilde{W}_2^{0,l/2}(Q_T)}$. First, we have

$$\|(1+t)A_t^T\boldsymbol{u}\|_{L_2(Q_T)}\le c\sup_{t<T}\|\nabla\boldsymbol{u}(\xi,t)\|_{W_2^l(\Omega_0)}\|(1+t)\boldsymbol{u}\|_{W_2^{l,0}(Q_T)}. \tag{5.36}$$

Now we estimate the norms containing the finite difference

$$\Delta_t(-h)A_t^T\boldsymbol{u}=A_t^T(\xi,t-h)\Delta_t(-h)\boldsymbol{u}(\xi,t)+(\Delta_t(-h)A_t(\xi,t))\boldsymbol{u}(\xi,t).$$

Using (5.4) we obtain

$$\|\Delta_t(-h)A_t^T\boldsymbol{u}\|_{L_2(\Omega_0)}\le c\|A_t(\cdot,t-h)\|_{W_2^{3/2-l}(\Omega_0)}\int_0^h\|\boldsymbol{u}_t(\cdot,t-\tau)\|_{W_2^l(\Omega_0)}d\tau$$
$$+\sup_{\Omega_0}|\boldsymbol{u}(\xi,t)|\int_0^h\|A_{tt}(\cdot,t-\tau)\|_{L_2(\Omega_0)}d\tau.$$

Since $3/2-l<l$, we have

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+l}}\int_h^T\|\Delta_t(-h)A_t\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$
$$\le c\Big(\sup_{t<T}\|A_t(\cdot,t)\|_{W_2^l(\Omega_0)}\|\boldsymbol{u}_t\|_{W_2^{l,0}(Q_T)}+\sup_{Q_T}|\boldsymbol{u}(\xi,t)|\|A_{tt}\|_{L_2(Q_T)}\Big)$$
$$\le c\|\nabla\boldsymbol{u}\|_{W_2^{l+1,l/2+1/2}(Q_T)}\Big(\|\boldsymbol{u}_t\|_{W_2^{l,0}(Q_T)}+\sup_{Q_T}|\boldsymbol{u}(\xi,t)|\Big).$$

We can estimate $\Delta_t(-h)A_t^T\boldsymbol{u}$ in a different way, namely,

$$\|\Delta_t(-h)A_t^T\boldsymbol{u}\|_{L_2(\Omega_0)}$$

$$\leq c\sqrt{h}\Big(\|A_t(\cdot,t-h)\|_{W_2^l(\Omega_0)}\|\boldsymbol{u}_t\|_{W_2^{l,0}(Q_t)}+\sup_{\Omega_0}|\boldsymbol{u}(\xi,t)|\|A_{tt}\|_{L_2(Q_t)}\Big),$$

and obtain

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^l}\int_h^T t^2\|\Delta_t(-h)A_t^T(\boldsymbol{u})\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}\Big(\|\boldsymbol{u}_t\|_{W_2^{l,0}(Q_T)}+\|t\boldsymbol{u}\|_{W_2^{l+1/2,0}(Q_T)}\Big).$$

This completes the proof of

$$\|\boldsymbol{L}(\boldsymbol{u})\|_{\widetilde{W}_2^{0,1+l/2}(Q_T)}\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}\Big(\sup_{Q_T}|\boldsymbol{u}(\xi,t)|$$

$$+\|\boldsymbol{u}_t\|_{\widetilde{W}_2^{l,l/2}(Q_T)}+\|(1+t)\boldsymbol{u}\|_{W_2^{l+1/2,0}(Q_T)}\Big)$$

and of inequality (5.26).

In order to conclude the proof of (5.25), we need to estimate the $\widetilde{W}_2^{0,l/2}(Q_T)$- norm of

$$(A\nabla\cdot A\nabla-\nabla^2)\boldsymbol{u}=((A-I)\nabla\cdot A\nabla+\nabla\cdot(A-I)\nabla))\boldsymbol{u}\equiv\boldsymbol{\Lambda}.$$

We have $\Delta_t(-h)\boldsymbol{\Lambda}=\boldsymbol{\Lambda}_1+\boldsymbol{\Lambda}_2$,

$$\boldsymbol{\Lambda}_1=(A(\xi,t-h)-I)\nabla\cdot A(\xi,t-h)\nabla-\nabla\cdot(A(\xi,t-h)-I)\nabla)\boldsymbol{v},\quad \boldsymbol{v}=\Delta_t(-h)\boldsymbol{u},$$

$$\boldsymbol{\Lambda}_2=(A(\xi,t-h)-I)(\nabla\cdot(\Delta_t(-h)A)\nabla)\boldsymbol{u}+(\Delta_t(-h)A)\nabla\cdot A\nabla\boldsymbol{u}+\nabla\cdot(\Delta_t(-h)A\nabla)\boldsymbol{u}.$$

By (5.4),

$$\|\boldsymbol{\Lambda}_1\|_{L_2(\Omega_0)}\leq c\Big(\sup_{Q_t}|A(\xi,t)-I|\|D^2\boldsymbol{v}\|_{L_2(\Omega_0)}+\|\nabla A\|_{W_2^{1/2}(\Omega_0)}\|\nabla\boldsymbol{v}\|_{W_2^1(\Omega_0)}\Big)$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{v}\|_{L_2(\Omega_0)}+\|D^2\boldsymbol{v}\|_{L_2(\Omega_0)}\Big),$$

where $D^2\boldsymbol{v}=\Big(\frac{\partial^2 v_i}{\partial\xi_j\partial\xi_k}\Big)_{i,j,k=1.2.3}$. This implies

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+l}}\int_h^T\|\boldsymbol{\Lambda}_1\|^2_{L_2(\Omega_0)}\Big)^{1/2}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\Big(\|\nabla\boldsymbol{u}\|_{W_2^{0,l/2}(Q_T)}+\|D^2\boldsymbol{u}\|_{W_2^{0,l/2}(Q_T)}\Big),$$

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^l}\int_h^T\|t\boldsymbol{\Lambda}_1\|^2_{L_2(\Omega_0)}\Big)^{1/2}$$

$$\leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\Big(\|t\nabla \boldsymbol{u}\|_{W_2^{0,(l-1)/2}(Q_T)}+\|tD^2\boldsymbol{u}\|_{W_2^{0,(l-1)/2}(Q_T)}+\|\nabla \boldsymbol{u}\|_{W_2^{1,0}(Q_T)}\Big).$$

The expression $\boldsymbol{\Lambda}_2$ satisfies the inequality

$$\|\boldsymbol{\Lambda}_2\|_{L_2(\Omega_0)}\leq c\int_0^h\|A_t(\cdot,t-\tau)\|_{W_2^l(\Omega_0)}d\tau\Big(\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{1+l}(\Omega_0)}+\|\nabla\cdot A\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big)$$

$$+c\int_0^h\|\nabla A_t(\cdot,t-\tau)\|_{W_2^l(\Omega_0)}d\tau\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}. \tag{5.37}$$

By (5.1), (5.19),

$$\|\nabla\cdot A\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\leq c\|A\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{l+1}(\Omega_0)}\leq c\Big(\|\nabla \boldsymbol{u}\|_{W_2^{1+l}(\Omega_0)}+\sqrt{t}\sup_{\Omega_0}|\nabla \boldsymbol{u}(\xi,t)|\Big),$$

whence

$$\|\boldsymbol{\Lambda}_2\|_{L_2(\Omega_0)}\leq c\Big(\int_0^h\|\nabla A_t(\cdot,t-\tau)\|_{W_2^l(\Omega_0)}d\tau\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}$$

$$+h\sup_{\tau<t}\|A_t(\cdot,\tau)\|_{W_2^l(\Omega_0)}\Big(\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{1+l}(\Omega_0)}+\sqrt{t}\sup_{\Omega_0}|\nabla \boldsymbol{u}(\xi,t)|\Big),$$

and, by virtue of (5.24),

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+l}}\int_h^T\|\boldsymbol{\Lambda}_2\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$

$$\leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\Big(\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}+\sup_{t<T}\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big),$$

Along with (5.37), we have

$$\|\boldsymbol{\Lambda}_2\|_{L_2(\Omega_0)}\leq c\int_0^h\|A_t(\cdot,t-\tau)\|_{W_2^{l+1}(\Omega)}d\tau\Big(\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}+\|\nabla\cdot A\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^{l-1}(\Omega_0)}\Big)$$

$$+c\int_0^h\|\nabla A_t(\cdot,t-\tau)\|_{W_2^l(\Omega_0)}d\tau\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}$$

$$\leq c\sqrt{h}\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\|\nabla \boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)},$$

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^l}\int_h^T\|t\boldsymbol{\Lambda}_2\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}\leq c\|\nabla \boldsymbol{u}\|^2_{\widetilde{W}_2^{l+1,0}(Q_T)}.$$

This shows that

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+l}}\int_h^T\|\Delta_t(-h)\boldsymbol{\Lambda}\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$

$$+\Big(\int_0^{min(T,1)}\frac{dh}{h^l}\int_h^T t^2\|\Delta_t(-h)\boldsymbol{\Lambda}\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}\leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1.0}(Q_T)}\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1.l/2+1/2}(Q_T)}$$

and completes the proof of (5.25) and of the proposition.

Let us pass to the estimates of $\boldsymbol{l}_3(\boldsymbol{u})$ and $l_4(\boldsymbol{u})$.

Proposition 5.6. *Under the assumption (5.13), the expressions* $\boldsymbol{l}_3(\boldsymbol{u})$ *and* $l_4(\boldsymbol{u})$ *satisfy*

$$\|\boldsymbol{l}_3(\boldsymbol{u})\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} + \|l_4(\boldsymbol{u})\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}. \tag{5.38}$$

Proof. We start with the estimate of $S_u(\boldsymbol{u}) - S(\boldsymbol{u})$. By virtue of (5.29),

$$\|S_u(\boldsymbol{u}) - S(\boldsymbol{u})\|_{W_2^{l+1/2}(\Gamma_0)} \leq c\|(A-I)\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)} + \sqrt{t}\sup_{\Omega_0}|\nabla\boldsymbol{u}(\xi,t)|\Big),$$

$$\|S_u(\boldsymbol{u}) - S(\boldsymbol{u})\|_{W_2^{l-1/2}(\Gamma_0)} \leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla\boldsymbol{u}\|_{W_2^{l}(\Omega_0)},$$

which implies

$$\|S_u(\boldsymbol{u}) - S(\boldsymbol{u})\|_{\widetilde{W}_2^{l+1/2,0}(\Gamma_T)} \leq c\|\nabla\boldsymbol{u}\|^2_{\widetilde{W}_2^{l+1,0}(Q_T)}.$$

By (5.12), further we have

$$\|\Delta_t(-h)(S_u(\boldsymbol{u}) - S(\boldsymbol{u}))\|_{L_2(\Gamma_0)} \leq c\Big(\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\Delta_t(-h)\nabla\boldsymbol{u}\|_{L_2(\Gamma_0)}$$

$$+\int_0^h \|\nabla\boldsymbol{u}(\cdot,t-\tau)\|_{W_2^{l-1/2}(\Gamma_0)}d\tau\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^{l-1/2}(\Gamma_0)}\Big),$$

whence

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+2\lambda_2}}\int_h^T\|\Delta_t(-h)(S_u(\boldsymbol{u}) - S(\boldsymbol{u})\|^2_{L_2(\Gamma_0)}dt\Big)^{1/2}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\Big(\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+2\lambda_2}}\int_h^T\|\Delta_t(-h)\nabla\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Gamma_0)}dt\Big)^{1/2}$$

$$+\sup_{t<T}\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big) \leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\|\nabla\boldsymbol{u}\|_{W_2^{l+1,l/2+1/2}(Q_T)},$$

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+2\lambda_3}}\int_h^T\|t\Delta_t(-h)(S_u(\boldsymbol{u}) - S(\boldsymbol{u})\|^2_{L_2(\Gamma_0)}dt\Big)^{1/2}$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+2\lambda_3}}\int_h^T\|t\Delta_t(-h)\nabla\boldsymbol{u}(\cdot,t)\|^2_{L_2(\Gamma_0)}dt\Big)^{1/2}$$

$$+c\|(1+t)\nabla\boldsymbol{u}\|_{W_2^{l,0}(Q_T)}\sup_{t<T}\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}\Big),$$

where $\lambda_2 = l/2+1/4$, $\lambda_3 = l/2-1/4$. The last two inequalities conclude the proof of

$$\|S_u(\boldsymbol{u}) - S(\boldsymbol{u})\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)} \leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}. \tag{5.39}$$

Now, we estimate $\|S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}$. Suppose we have shown that

$$\|\boldsymbol{n}(X)\|_{W_2^{l-1/2}(\Gamma_0)}\leq c,\quad \|\boldsymbol{n}(X)\|_{W_2^{l+1/2}(\Gamma_0)}\leq c(1+\sqrt{t}), \tag{5.40}$$

$$|\boldsymbol{n}_0(\xi)-\boldsymbol{n}(X)|\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{5.41}$$

$$\|\boldsymbol{n}_0-\boldsymbol{n}\|_{W_2^{l_2}(\Gamma_0)}\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)},\quad l_2<l, \tag{5.42}$$

$$\|\boldsymbol{n}_0-\boldsymbol{n}\|_{W_2^{l+1/2}(\Gamma_0)}\leq c(1+\sqrt{t})\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}. \tag{5.43}$$

Then, making use of the relation

$$S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0=(S_u(\boldsymbol{u})-S(\boldsymbol{u}))\boldsymbol{n}+S(\boldsymbol{u})(\boldsymbol{n}-\boldsymbol{n}_0), \tag{5.44}$$

we obtain

$$\|S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0\|_{W_2^{l+1/2}(\Gamma_0)}\leq c\Big(\|S_u(\boldsymbol{u})-S(\boldsymbol{u})\|_{W_2^{l+1/2}(\Gamma_0)}$$

$$+\|\boldsymbol{n}\|_{W_2^{l+1/2}(\Gamma_0)}\sup_{\Gamma}|S_u(\boldsymbol{u})-S(\boldsymbol{u})|+\sup_{\Gamma}|S(\boldsymbol{u})|\|\boldsymbol{n}-\boldsymbol{n}_0\|_{W_2^{l+1/2}(\Gamma_0)}$$

$$+\sup_{\Gamma_0}|\boldsymbol{n}-\boldsymbol{n}_0|\|S(\boldsymbol{u})\|_{W_2^{l+1/2}(\Gamma_0)}\Big)\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{u}\|_{W_2^{l+1}(\Omega_0)}+\sqrt{t}\sup_{\Omega_0}|\nabla\boldsymbol{u}(\xi,t)|\Big),$$

$$\|S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0\|_{W_2^{l-1/2}(\Gamma_0)}\leq c\Big(\|S_u(\boldsymbol{u})-S(\boldsymbol{u})\|_{W_2^{l-1/2}(\Gamma_0)}(\sup_{\Gamma_0}|\boldsymbol{n}|+\|\boldsymbol{n}\|_{W_2^1(\Gamma_0)})$$

$$+\|S_u(\boldsymbol{u})\|_{W_2^{l-1/2}(\Gamma_0)}(\sup_{\Gamma_0}|\boldsymbol{n}-\boldsymbol{n}_0|+\|\boldsymbol{n}-\boldsymbol{n}_0\|_{W_2^1(\Gamma_0)})\Big)$$

$$\leq c\|\nabla\boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^l(\Omega_0)}$$

and, as a consequence,

$$\|S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0\|_{\widetilde{W}_2^{l+1/2,0}(G_T)}\leq c\|\nabla\boldsymbol{u}\|^2_{\widetilde{W}_2^{l+1,0}(Q_T)}.$$

To estimate $\|S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0\|_{\widetilde{W}_2^{0,l/2+1/4}(G_T)}$, we apply the operation $\Delta_t(-h)$ to (5.44) which gives

$$\Delta_t(-h)(S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0)=\Delta_t(-h)(S_u(\boldsymbol{u})-S(\boldsymbol{u}))\boldsymbol{n}(X(\xi,t-h))$$

$$+\Delta_t(-h)(S(\boldsymbol{u}))(\boldsymbol{n}(X(\xi,t-h))-\boldsymbol{n}_0(\xi))+S_u(\boldsymbol{u})\Delta_t(-h)\boldsymbol{n}(X)$$

and

$$\|\Delta_t(-h)(S_u(\boldsymbol{u})\boldsymbol{n}-S(\boldsymbol{u})\boldsymbol{n}_0)\|_{L_2(\Gamma_0)}\leq c\Big(\|\Delta_t(-h)(S_u(\boldsymbol{u})-S(\boldsymbol{u}))\|_{L_2(\Gamma_0)}$$

$$+\|\Delta_t(-h)S(\boldsymbol{u})\|_{L_2(\Gamma_0)}\sup_{\Gamma_0}|\boldsymbol{n}(X(\xi,t-h))-\boldsymbol{n}_0(\xi)|+\|S_u(\boldsymbol{u})\Delta_t(-h)\boldsymbol{n}\|_{L_2(\Gamma_0)}\Big).$$

Since $|\boldsymbol{n}_t|\leq c|\nabla\boldsymbol{u}|$, the last term does not exceed

$$\|\nabla\boldsymbol{u}(\cdot,t)\|_{W_2^{l-1/2}(\Gamma_0)}\int_0^h\|\nabla\boldsymbol{u}(\cdot,t-\tau)\|_{W_2^{l-1/2}(\Gamma_0)}d\tau.$$

Taking (5.39) into account, we show that

$$\Big(\int_0^{min(T,1)} \frac{dh}{h^{1+2\lambda_2}} \int_h^T \|\Delta_t(-h)(S_u(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{n}_0)\|^2_{L_2(\Gamma_0)} dt\Big)^{1/2}$$

$$+\Big(\int_0^{min(T,1)} \frac{dh}{h^{1+2\lambda_3}} \int_h^T \|t\Delta_t(-h)(S_u(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{n}_0)\|^2_{L_2(\Gamma_0)} dt\Big)^{1/2}$$

$$\leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)} \|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}$$

and conclude the proof of

$$\|S_u(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{n}_0\|_{\widetilde{W}_2^{0,l/2+1/4}(G_T)} \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)} \|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,l/2+1/2}(Q_T)}.$$

So it remains to verify (5.40)-(5.43). We have

$$\boldsymbol{n}(X) - \boldsymbol{n}_0(\xi) = (A - I)\boldsymbol{n}_0 + A\boldsymbol{n}_0(|A\boldsymbol{n}_0|^{-1} - 1),$$

$$\frac{1}{|A\boldsymbol{n}_0|} - 1 = \frac{(\boldsymbol{n}_0 - A\boldsymbol{n}_0)\cdot \boldsymbol{n}_0 + A\boldsymbol{n}_0(\boldsymbol{n}_0 - A\boldsymbol{n}_0)}{|A\boldsymbol{n}_0|(1 + |A\boldsymbol{n}_0|)}.$$

Under the assumption (5.13), the function $|A\boldsymbol{n}_0|$ is bounded from below. From this fact and from

$$|(A - I)\boldsymbol{n}_0| \leq c \sup_{\Gamma_0} |A(\xi, t) - I)| \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)},$$

we easily deduce (5.41). We also have

$$\|(A - I)\boldsymbol{n}_0\|_{W^{l_2}(\Gamma_0)} \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)},$$

$$\|(A - I)\boldsymbol{n}_0\|_{W^{l+1/2}(\Gamma_0)} \leq c(1 + \sqrt{t})\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)},$$

By Proposition 5.2, for an arbitrary regular function $f(A\boldsymbol{n}_0)$ bounded together with its derivatives for $\xi \in \Gamma_0$, the following inequality holds:

$$|f(A\boldsymbol{n}_0)| + \|f(A\boldsymbol{n}_0)\|_{W_2^{l_2}(\Gamma_0)} \leq c,$$

$$\|f(A\boldsymbol{n}_0)\|_{W_2^{l+1/2}(\Gamma_0)} \leq c(1 + \sqrt{t}).$$

It follows that

$$\||A\boldsymbol{n}_0|^{-1} - 1\|_{W_2^{l_2}(\Gamma_0)} \leq c\|(A - I)\boldsymbol{n}_0\|_{W_2^{l_2}(\Gamma_0)} \leq c\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_t)},$$

$$\||A\boldsymbol{n}_0|^{-1} - 1\|_{W_2^{l+1/2}(\Gamma_0)} \leq c(1 + \sqrt{t})\|\nabla \boldsymbol{u}\|_{\widetilde{W}_2^{l+1,0}(Q_T)}.$$

Hence, the difference $\boldsymbol{n} - \boldsymbol{n}_0$ satisfies (5.42), (5.43), and from those inequalities it is easy to deduce (5.40). So we have justified (5.40)-(5.43) and estimated $S_u(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{n}_0$. Other terms in $\boldsymbol{l}_3(\boldsymbol{u})$

and $l_4(\boldsymbol{u})$ are treated in a similar way and satisfy the same inequalities as $S_u(\boldsymbol{u})\boldsymbol{n} - S(\boldsymbol{u})\boldsymbol{n}_0$. The proposition is proved.

For the estimates of $\boldsymbol{l}_5(\boldsymbol{u}, r)$ and $l_6(\boldsymbol{u})$ we need the following auxiliary proposition.

Proposition 5.7. *Assume that Γ_t, $t \in (0,T)$, is defined by equation (1.10) with sufficiently small $\rho(z,t)$ (i.e., Γ_t is located in a certain small neighborhood of $\mathcal{G}$), $\boldsymbol{u}$ satisfies the inequality*

$$\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} \leq \delta, \tag{5.45}$$

and let $f(z)$ be a smooth function given on $\mathcal{G}$. Then

$$\|f(\bar{X}) - f(\bar{\xi})\|_{W_2^{l+1/2}(\Gamma_0)} \leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)}, \tag{5.46}$$

$$\|f(\bar{X}) - f(\bar{\xi})\|_{W_2^{l+3/2}(\Gamma_0)} \leq c(1+\sqrt{t})\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)}. \tag{5.47}$$

Proof. We recall that the point $\bar{x}$ is connected with x by

$$\bar{x} = x - R(x)\nabla R(x) \equiv \mathfrak{R}(x).$$

The function $\mathfrak{R}(x)$ is smooth in a certain neighborhood of $\mathcal{G}$. Let us extend f in this neighborhood by setting, for instance, $f(x) = f(\bar{x})$. Then $f(\bar{x}) = f(\mathfrak{R}(x)) \equiv \mathfrak{f}(x)$ is also a smooth function, and

$$f(\bar{X}) - f(\bar{\xi}) = \int_0^1 \frac{\partial}{\partial s}\mathfrak{f}(X_s)ds = \int_0^1 \nabla\mathfrak{f}(X_s)ds \cdot \int_0^t \boldsymbol{u}(\xi,\tau)d\tau, \tag{5.48}$$

where $X_s(\xi,t) = \xi + s\int_0^t \boldsymbol{u}(\xi,\tau)d\tau$. It is clear that

$$\|X_s\|_{W_2^{l-1/2}(\Gamma_0)} \leq c(1 + \|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}) \leq c,$$

$$\|X_s\|_{W_2^{l+1/2}(\Gamma_0)} \leq c(1 + \sqrt{t}\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)})$$

with constants independent of T. Therefore inequalities (5.46), (5.47) follow from (5.48) and from Proposition 5.1. For instance,

$$\begin{aligned}\|f(\bar{X}) - f(\bar{\xi})\|_{W_2^{l+3/2}(\Gamma_0)} &\leq c\int_0^1 \Big(\sup_{\Gamma_0}|\nabla\mathfrak{f}(X_s)| \int_0^t \|\boldsymbol{u}\|_{W_2^{l+3/2}(\Gamma_0)}d\tau \\ &\quad + \|\nabla\mathfrak{f}(X_s)\|_{W_2^{l+3/2}(\Gamma_0)} \int_0^t \sup_{\Gamma_0}|\boldsymbol{u}(\xi,\tau)|d\tau\Big)ds \\ &\leq c\Big(\sqrt{t}\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)} + (1+\sqrt{t}\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)})\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)}\Big) \\ &\leq c(1+\sqrt{t})\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_T)},\end{aligned}$$

and (5.46) is established in the same way. The proposition is proved.

Proposition 5.8. *If $\boldsymbol{u}$ satisfies (5.45), then*

$$\|l_6(\boldsymbol{u})\|_{\widetilde{W}_2^{l+3/2,l/2+3/4}(G_T)} \le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)} + \sup_{Q_T}|\boldsymbol{u}(\xi,t)|\Big)\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)}. \tag{5.49}$$

If, in addition, $\|\rho_0\|_{W_2^{l+1/2}(\mathcal{G})} \ll 1$, then

$$\|l_5(\boldsymbol{u},r)\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}$$

$$\le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)} + \sup_{t<T}\|r(\cdot,t)\|_{W_2^{l+1/2}(\mathcal{G})}\Big)\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)} + \|r\|_{\widetilde{W}_2^{l+1/2,0}(G_T)}\Big). \tag{5.50}$$

Proof. We start with the proof of (5.50). By (5.1), (5.3) and (5.46),

$$\|(b(\bar{X}) - b(\bar{\xi}))r\|_{W_2^{l+1/2}(\Gamma_0)} \le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\|r(\cdot,t)\|_{W_2^{l+1/2}(\Gamma_0)},$$

$$\|(b(\bar{X}) - b(\bar{\xi}))r\|_{W_2^{l-1/2}(\Gamma_0)} \le c\|r(\cdot,t)\|_{W_2^{l-1/2}(\Gamma_0)}\Big(\sup_{\Gamma_0}\Big(|(b(\bar{X}) - b(\bar{\xi})|$$

$$+\|b(\bar{X}) - b(\bar{\xi})\|_{W_2^1(\Gamma_0)}\Big) \le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\|r(\cdot,t)\|_{W_2^{l-1/2}(\Gamma_0)}.$$

Moreover, if

$$\sup_{t<T}\|r(\cdot,t)\|_{W_2^{l+1/2}(\Gamma_0)} \ll 1, \tag{5.51}$$

then the norm $\|b_1(\boldsymbol{u},r)\|_{W_2^{l+1/2}(\Gamma_0)}$ (with b_1 defined in (1.11)) is bounded, and

$$\|b_1(\boldsymbol{u},r)r^2\|_{W_2^{l+1/2}(\Gamma_0)} \le c\|r^2\|_{W_2^{l+1/2}(\Gamma_0)} \le c\|r\|^2_{W_2^{l+1/2}(\Gamma_0)},$$

$$\|b_1(\boldsymbol{u},r)r^2\|_{W_2^{l-1/2}(\Gamma_0)} \le c\Big(\sup_{\Gamma_0}|b_1(\xi,r)| + \|b_1(\cdot,r)\|_{W_2^1(\Gamma_0)}\Big)\|r^2\|_{W_2^{l-1/2}(\Gamma_0)}$$

$$\le c\|r\|_{W_2^{l+1/2}(\Gamma_0)}\|r\|_{W_2^{l-1/2}(\Gamma_0)}.$$

Combining the above estimates, we obtain

$$\|l_5(\boldsymbol{u},r)\|_{\widetilde{W}_2^{l+1/2,0}(G_T)}$$

$$\le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)} + \sup_{t<T}\|r(\cdot,t)\|_{W_2^{l+1/2}(\Gamma_0)}\Big)\|r\|_{\widetilde{W}_2^{l+1/2,0}(G_T)}. \tag{5.52}$$

It is shown in [3] that

$$\|r\|_{W_2^{l+1/2}(\Gamma_0)} \le c\Big(\|\rho_0\|_{W_2^{l+1/2}(\mathcal{G})} + \|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\Big),$$

so the condition (5.51) holds if ρ_0 and $\boldsymbol{u}$ are small.

Now we pass to the estimate of $\|l_5(\boldsymbol{u},r)\|_{\widetilde{W}_2^{0,l/2+1/4}(G_T)}$. Since $l/2+1/4<1$, we can use Proposition 5.3:

$$\|l_5(\boldsymbol{u},r)\|_{\widetilde{W}_2^{0,l/2+1/4}(G_T)} \le c\Big(\|(1+t)l_5(\boldsymbol{u},r)\|_{L_2(G_T)} + \|(1+t)l_{5t}(\boldsymbol{u},r)\|_{L_2(G_T)}\Big). \tag{5.53}$$

The first term on the right -hand side has already been estimated in (5.52), so we need to consider the time derivative of l_5. We have

$$\frac{\partial}{\partial t}(b(\bar{X})-b(\bar{\xi}))r(\xi,t) = \nabla_X \mathfrak{b}(X)\cdot\boldsymbol{u}(\xi,t)r(\xi,t) + (b(\bar{X})-b(\bar{\xi}))r_t(\xi,t)$$

where $\mathfrak{b}(X)=b(\bar{X})$, as well as $\nabla_X\mathfrak{b}(X)$, is a bounded function. We use the equation $r_t=\boldsymbol{N}(\bar{X})\cdot\boldsymbol{u}$ and obtain

$$\|\frac{\partial}{\partial t}(b(\bar{X})-b(\bar{\xi}))r\|_{L_2(\Gamma_0)} \le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,0}(Q_t)}\|\boldsymbol{u}\|_{L_2(\Gamma_0)}$$

$$+c\||\boldsymbol{u}||r|\|_{L_2(\Gamma_0)} \le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)} + \sup_{\Gamma_0}|r(\xi,t)|\Big)\|\boldsymbol{u}\|_{L_2(\Gamma_0)}.$$

The time derivative of $b_1(\boldsymbol{u},r)$ is also bounded, whence

$$\|\frac{\partial}{\partial t}b_1(\boldsymbol{u},r)r^2\|_{L_2(\Gamma_0)} \le c\Big(\||\boldsymbol{u}||r|\|_{L_2(\Gamma_0)} + \|r^2\|_{L_2(\Gamma_0)}\Big)$$

$$\le c\sup_{\Gamma_0}|r(\xi,t)|\Big(\|\boldsymbol{u}\|_{L_2(\Gamma_0)} + \|r\|_{L_2(\Gamma_0)}\Big).$$

The last two inequalities imply

$$\|(1+t)l_{5t}(\boldsymbol{u},r)\|_{L_2(G_T)}$$

$$\le c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)} + \sup_{G_T}|r(\xi,t)|\Big)(\|(1+t)\boldsymbol{u}\|_{L_2(G_T)} + \|(1+t)r\|_{L_2(G_T)}\Big),$$

which completes the proof of (5.50).

We turn to the inequality (5.49). By (5.1), (5.47) and (5.46),

$$\|l_6(\boldsymbol{u})\|_{W_2^{l+3/2}(\Gamma_0)}$$

$$\le c\Big(\sup_{\Gamma_0}|\boldsymbol{N}(\bar{X})-\boldsymbol{N}(\bar{\xi})|\|\boldsymbol{u}(\cdot,t)\|_{W_2^{l+3/2}(\Gamma_0)} + \|\boldsymbol{N}(\bar{X})-\boldsymbol{N}(\bar{\xi})\|_{W_2^{l+3/2}(\Gamma_0)}\sup_{\Gamma_0}|\boldsymbol{u}(\xi,t)|\Big)$$

$$\le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\Big(\|\boldsymbol{u}(\cdot,t)\|_{W_2^{1+3/2}(\Gamma_0)} + \sqrt{t}\sup_{\Gamma_0}|\boldsymbol{u}(\xi,t)|\Big),$$

$$\|l_6(\boldsymbol{u})\|_{W_2^{l+1/2}(\Gamma_0)}$$

$$\le c\Big(\sup_{\Gamma_0}|\boldsymbol{N}(\bar{X})-\boldsymbol{N}(\bar{\xi})|\|\boldsymbol{u}(\cdot,t)\|_{W_2^{l+1/2}(\Gamma_0)} + \|\boldsymbol{N}(\bar{X})-\boldsymbol{N}(\bar{\xi})\|_{W_2^{l+1/2}(\Gamma_0)}\sup_{\Gamma_0}|\boldsymbol{u}(\xi,t)|\Big)$$

$$\le c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\|\boldsymbol{u}(\cdot,t)\|_{W_2^{l+1/2}(\Gamma_0)},$$

which implies

$$\|l_6(\boldsymbol{u})\|_{\widetilde{W}_2^{l+3/2,0}(G_T)} \leq c\|\boldsymbol{u}\|^2_{\widetilde{W}_2^{2+l,0}(Q_T)}. \tag{5.54}$$

Now we estimate $\|l_6(\boldsymbol{u})\|_{\widetilde{W}_2^{0,l/2+3/4}(G_T)}$. We notice that $l/2+3/4 \in (5/4, 3/2)$ and $\mu = l/2+1/4 \in (3/4, 1)$. We consider the finite difference

$$\Delta_t(-h)l_6(\boldsymbol{u}) = (\Delta_t(-h)\boldsymbol{N}(\bar{X}))\cdot \boldsymbol{u}(\xi,t) + (\boldsymbol{N}(\bar{X}(\xi,t-h)) - \boldsymbol{N}(\bar{\xi}))\cdot \Delta_t(-h)\boldsymbol{u}(\xi,t)$$

$$= (\boldsymbol{N}(\bar{X}(\xi,t-h)) - \boldsymbol{N}(\bar{\xi}))\cdot \Delta_t(-h)\boldsymbol{u}(\xi,t) + \int_0^h \frac{\partial}{\partial t}\boldsymbol{N}(\bar{X}(\xi,t-\tau))d\tau \cdot \boldsymbol{u}(\xi,t).$$

Since $|\frac{\partial}{\partial t}\boldsymbol{N}(\bar{X})| \leq c|\boldsymbol{u}(\xi,t)|$, we have

$$\|\Delta_t(-h)l_6(\boldsymbol{u})\|_{L_2(\Gamma_0)} \leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\|\Delta_t(-h)\boldsymbol{u}\|_{L_2(\Gamma_0)}$$

$$+ch\sup_{Q_t}|\boldsymbol{u}(\xi,\tau)|\|\boldsymbol{u}\|_{L_2(\Gamma_0)},$$

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+2\mu}}\int_h^T t^2\|\Delta_t(-h)l_6(\boldsymbol{u})\|^2_{L_2(\Gamma_0)}dt\Big)^{1/2}$$

$$\leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)}\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+2\mu}}\int_h^T t^2\|\Delta_t(-h)\boldsymbol{u}\|^2_{L_2(\Gamma_0)}dt\Big)^{1/2}$$

$$+c\sup_{Q_T}|\boldsymbol{u}(\xi,t)|\|(1+t)\boldsymbol{u}\|_{L_2(G_T)}. \tag{5.55}$$

We should also consider the time derivative of $l_6(\boldsymbol{u})$,

$$\frac{\partial}{\partial t}l_6(\boldsymbol{u}) = (\boldsymbol{N}(\bar{X}) - \boldsymbol{N}(\bar{\xi}))\cdot \boldsymbol{u}_t + \frac{\partial \boldsymbol{N}(\bar{X})}{\partial t}\cdot \boldsymbol{u}$$

By Proposition 5.7, we have

$$\|\frac{\partial}{\partial t}l_6(\boldsymbol{u})\|_{L_2(\Gamma_0)} \leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\|\boldsymbol{u}_t\|_{L_2(\Gamma_0)} + c\sup_{\Gamma_0}|\boldsymbol{u}(\xi,t)|\|\boldsymbol{u}\|_{L_2(\Gamma_0)}\Big)$$

and

$$\|l_{6t}\|_{L_2(G_T)} \leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)}\|\boldsymbol{u}_t\|_{L_2(G_T)} + \sup_{G_T}|\boldsymbol{u}|\|\boldsymbol{u}\|_{L_2(G_T)}\Big). \tag{5.56}$$

Finally, we should estimate the norm

$$\Big(\int_0^{min(T,1)}\frac{dh}{h^{1+2\mu_1}}\int_h^T \|\Delta_t(-h)l_{6t}\|^2_{L_2(\Gamma_0)}dt\Big)^{1/2} \tag{5.57}$$

where $\mu_1 = l/2 - 1/4 \in (1/4, 1/2)$. We have

$$\Delta_t(-h)l_{6t} = (\boldsymbol{N}(\bar{X}(\xi,t-h)) - \boldsymbol{N}(\bar{\xi}))\cdot \Delta_t(-h)\boldsymbol{u}_t + \boldsymbol{u}_t\cdot \Delta_t(-h)\boldsymbol{N}(\bar{X}) + \Delta_t(-h)\Big(\frac{\partial \boldsymbol{N}(\bar{X})}{\partial t}\cdot \boldsymbol{u}\Big).$$

Since $\boldsymbol{N}^*(\Re(x))$ is a smooth function in a neighborhood of $\mathcal{G}$, we obtain

$$\|\Delta_t(-h)l_{6t}\|_{L_2(\Gamma_0)} \leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_t)}\|\Delta_t(-h)\boldsymbol{u}_t(\cdot,t)\|_{L_2(\Gamma_0)} + h\sup_{Q_t}|\boldsymbol{u}(\xi,t)|\|\boldsymbol{u}_t\|_{L_2(\Gamma_0)}$$

$$+\int_0^h \||\boldsymbol{u}_t(\xi,t-\tau)||\boldsymbol{u}(\xi,t-\tau)|\|_{L_2(\Gamma_0)}d\tau\Big),$$

from which it follows that (5.57) does not exceed

$$c\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)}\|\boldsymbol{u}_t\|_{W_2^{0,l/2-1/4}(G_t)} + c\sup_{G_T}|\boldsymbol{u}(\xi,t)|\Big(\|\boldsymbol{u}_t\|_{L_2(G_T)} + \|\boldsymbol{u}\|_{L_2(G_T)}\Big)$$

Thus,

$$\|l_6(\boldsymbol{u})\|_{\widetilde{W}_2^{0,l/2+3/4}(G_T)} \leq c\Big(\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,0}(Q_T)} + \sup_{Q_T}|\boldsymbol{u}(\xi,t)|\Big)\|\boldsymbol{u}\|_{\widetilde{W}_2^{0,l/2+3/4}(G_T)}.$$

This completes the proof of both (5.49) and the proposition.

It is clear that the estimates (3.22) are consequences of (5.25), (5.26), (5.38), (5.49), (5.50).

6 Proof of Proposition 3.2.

In this section, we estimate the norms $\|\boldsymbol{u}\|_{L_2(\Omega_0)}$ and $\|r\|_{W_2^{-1/2}(\Gamma_0)}$. We start with the following auxiliary proposition.

Proposition 6.1. *For an arbitrary $f_0 \in W_2^{1/2}(\mathcal{G})$ such that $\int_{\mathcal{G}} f_0(z)dS = 0$ one can construct a divergence free vector field $\boldsymbol{W}_0(z)$, $z \in \mathcal{F}$, satisfying the conditions*

$$\boldsymbol{W}_0|_S = 0, \quad \boldsymbol{W}_0|_{\mathcal{G}} = \boldsymbol{N}f_0 \tag{6.1}$$

and the inequalities

$$\|\boldsymbol{W}_0\|_{W_2^1(\mathcal{F})} \leq c\|f_0\|_{W_2^{1/2}(\mathcal{G})},$$
$$\|\boldsymbol{W}_0\|_{L_2(\mathcal{F})} \leq c\|f_0\|_{L_2(\mathcal{G})}. \tag{6.2}$$

The relation between $\boldsymbol{W}_0$ and $\boldsymbol{f}_0$ is linear.

Proof. We define $\boldsymbol{W}_0(x)$ as a solution of the stationary Stokes problem

$$-\nabla^2\boldsymbol{W}_0(x) + \nabla Q_0(x) = 0, \quad \nabla\cdot\boldsymbol{W}_0(x) = 0, \quad x \in \mathcal{F},$$

$$\boldsymbol{W}_0|_S = 0, \quad \boldsymbol{W}_0|_{\mathcal{G}} = \boldsymbol{N}(x)f_0(x).$$

For an arbitrary $f_0 \in W_2^{1/2}$ this problem has a unique generalized solution $\boldsymbol{W}_0 \in W_2^1(\Omega_0)$, $Q_0 \in L_2(\mathcal{F})$, satisfying the normalization condition $\int_{\mathcal{F}} Q_0(x)dx = 0$. By the trace theorem for the Sobolev spaces, we can construct the vector field $\boldsymbol{W}_1 \in W_2^1(\mathcal{F})$ (not necessarily divergence free) satisfying (6.1),(6.2). The difference $\boldsymbol{W}_0 - \boldsymbol{W}_1 = \boldsymbol{U}$ is a generalized solution of the problem

$$-\nabla^2\boldsymbol{U}(x) + \nabla Q_0(x) = \nabla^2\boldsymbol{W}_1(x), \quad \nabla\cdot\boldsymbol{U}(x) = -\nabla\cdot\boldsymbol{W}_1(x), \quad x \in \mathcal{F},$$

$$\boldsymbol{U}|_S = 0, \quad \boldsymbol{U}|_{\mathcal{G}} = 0, \quad \int_{\mathcal{F}} Q_0(x)dx = 0.$$

We multiply the first equation by $\boldsymbol{U}$ and integrate over $\mathcal{F}$. Then we integrate by parts which leads to

$$\int_{\mathcal{F}} |\nabla \boldsymbol{U}|^2 dx = -\int_{\mathcal{F}} Q_0(x) \nabla \cdot \boldsymbol{W}_1(x) dx - \int_{\mathcal{F}} \nabla \boldsymbol{W}_1 : \nabla \boldsymbol{U} dx$$
$$\leq c \|\nabla \boldsymbol{W}_1\|_{L_2(\mathcal{F})} \Big(\|Q_0\|_{L_2(\mathcal{F})} + \|\nabla \boldsymbol{U}\|_{L_2(\mathcal{F})} \Big).$$

Since the pressure Q_0 satisfies the inequality

$$\|Q_0\|_{L_2(\mathcal{F})} \leq c \Big(\|\nabla \boldsymbol{U}\|_{L_2(\mathcal{F})} + \|\nabla \boldsymbol{W}_1\|_{L_2(\mathcal{F})} \Big),$$

we obtain

$$\|\nabla \boldsymbol{U}\|_{L_2(\mathcal{F})} \leq c \|\nabla \boldsymbol{W}_1\|_{L_2(\mathcal{F})} \leq c \|f_0\|_{W_2^{1/2}(\mathcal{G})},$$

which proves (6.2_1).

To prove (6.2_2), we define $\boldsymbol{\varphi}(x)$ and $\psi(x)$ as a solution of the problem

$$-\nabla^2 \boldsymbol{\varphi}(x) + \nabla \psi(x) = \boldsymbol{W}_0(x), \quad \nabla \cdot \boldsymbol{\varphi}(x) = 0, \quad x \in \mathcal{F},$$

$$\boldsymbol{\varphi}|_S = 0, \quad \boldsymbol{\varphi}|_{\mathcal{G}} = 0, \quad \int_{\mathcal{F}} \psi(x) dx = 0,$$

and we recall that $\boldsymbol{\varphi}(x)$, $\psi(x)$ satisfy the well-known coercive estimate

$$\|\boldsymbol{\varphi}\|_{W_2^2(\mathcal{F})} + \|\psi\|_{W_2^1(\mathcal{F})} \leq c \|\boldsymbol{W}_0\|_{L_2(\mathcal{F})}.$$

By the Green identity, we have

$$\int_{\mathcal{F}} |\boldsymbol{W}_0(x)|^2 dx = -\int_{\mathcal{G}} f_0(x) \boldsymbol{N}(x) \cdot T(\boldsymbol{\varphi}, \psi) \boldsymbol{N}(x) dS.$$

Inequality (6.2_2) follows from this relation and the coercive estimate. The proposition is proved.

In our applications the function f_0 depends also on $t \in (0,T)$. Since the relation between $\boldsymbol{W}_0$ and $\boldsymbol{f}_0$ is linear, (6.2_2) implies

$$\|\boldsymbol{W}_{0t}\|_{L_2(\mathcal{F})} \leq c \|f_{0t}\|_{L_2(\mathcal{G})}. \tag{6.3}$$

We assume that Γ_t is given by the equation (1.10), and we map $\mathcal{F}$ on Ω_t by the transformation

$$x = z + \boldsymbol{N}^*(z) \rho^*(z,t) \equiv e_\rho(z), \quad z \in \mathcal{F},$$

with ρ^* subject to

$$\|\rho^*(\cdot,t)\|_{W_2^{l_1+1/2}(\mathcal{F})} \leq c \|\rho(\cdot,t)\|_{W_2^{l_1}(\mathcal{G})} \ll 1,$$

$\forall l_1 \in [0, l+1-\epsilon]$, $\epsilon \in (0, l-1)$. Let $\boldsymbol{\mathcal{L}}$ be the Jacobi matrix of this transformation, $L = \det \boldsymbol{\mathcal{L}}$, $\widehat{\boldsymbol{\mathcal{L}}} = L \boldsymbol{\mathcal{L}}^{-1}$. We define the vector field $\boldsymbol{W}(x,t)$, $x \in \Omega_t$, by

$$\boldsymbol{W}(e_\rho(z), t) = \frac{\mathcal{L}(z,\rho)}{L(z,\rho)} \boldsymbol{W}_0(z,t) \equiv \widetilde{\boldsymbol{W}}(z,t). \tag{6.4}$$

We have

$$\boldsymbol{W}_0(z,t)=\widehat{\mathcal{L}}(z,t)\widetilde{\boldsymbol{W}}(z,t),$$

which implies

$$0=\nabla_z\cdot\boldsymbol{W}_0=\sum_{k,m=1}^3\widehat{L}_{km}\frac{\partial\widetilde{W}_m}{\partial z_k}=L(z,\rho)\nabla_x\cdot\boldsymbol{W}(x,t)|_{x=e_\rho(z)}.$$

Hence, $\boldsymbol{W}(x,t)$ is also divergence free and

$$\boldsymbol{W}\cdot\boldsymbol{n}|_{\Gamma_t}=\frac{\widetilde{\boldsymbol{W}}\cdot\widehat{\mathcal{L}}^T\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|}=\frac{\boldsymbol{W}_0\cdot\boldsymbol{N}}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|}=\frac{f_0(z,t)}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|}\equiv f(x,t),\quad x=e_\rho(z).$$

Now we estimate the norms $\|\boldsymbol{W}\|_{W_2^1(\Omega_t)}$ and $\|\boldsymbol{W}_t\|_{L_2(\Omega_t)}$. Since the norm $\|\rho\|_{W_2^{l+1-\epsilon}(\mathcal{G})}$ is small, we have $L\ge c>0$ and

$$\|\boldsymbol{W}\|_{L_2(\Omega_t)}\le c\|\widetilde{\boldsymbol{W}}\|_{L_2(\mathcal{F})}\le c\|\boldsymbol{W}_0\|_{L_2(\mathcal{F})}\le c\|f_0\|_{L_2(\mathcal{G})}. \tag{6.5}$$

Moreover, from the relation

$$\frac{\partial}{\partial x_k}\boldsymbol{W}(x,t)=\sum_{m=1}^3\ell^{mk}\Big(\Big(\frac{\partial}{\partial y_m}\frac{\mathcal{L}}{L}\Big)\boldsymbol{W}_0+\frac{\mathcal{L}}{L}\frac{\partial\boldsymbol{W}_0}{\partial z_m}\Big),\quad x=e_\rho(z),$$

where ℓ^{mk} are elements of $\mathcal{L}^{-1}$, we obtain

$$\|\nabla\boldsymbol{W}\|_{L_2(\Omega_t)}\le c\Big(\sum_{m=1}^3\|\Big(\frac{\partial}{\partial y_m}\frac{\mathcal{L}}{L}\Big)\|_{L_3(\mathcal{F})}\|\boldsymbol{W}_0\|_{L_6(\mathcal{F})}$$

$$+\sup_{\Gamma_0}\Big|\frac{\mathcal{L}}{L}\Big|\|\nabla_z\boldsymbol{W}_0\|_{L_2(\mathcal{F})}\Big)\le c\|\boldsymbol{W}_0\|_{W_2^1(\mathcal{F})}\le c\|f_0\|_{W_2^{1/2}(\mathcal{F})}, \tag{6.6}$$

because

$$\|\ell^{mk}\Big(\frac{\partial}{\partial y_m}\frac{\mathcal{L}}{L}\Big)\|_{L_3(\mathcal{F})}\le c\sum_{|j|\le 2}\|D^j\rho^*\|_{L_3(\mathcal{G})}\le c\|\rho^*\|_{W_2^{5/2}(\mathcal{F})}\le c\|\rho\|_{W_2^2(\mathcal{F})}\le c.$$

We pass to the estimate of $\boldsymbol{W}_t$. First we consider

$$\widetilde{\boldsymbol{W}}_t=\frac{\mathcal{L}}{L}\frac{\partial\boldsymbol{W}_0}{\partial t}+\Big(\frac{\partial}{\partial t}\frac{\mathcal{L}}{L}\Big)\boldsymbol{W}_0.$$

By (6.2), (6.3), we have

$$\|\widetilde{\boldsymbol{W}}_t\|_{L_2(\mathcal{F})}\le\sup_{\mathcal{F}}|\frac{\mathcal{L}}{L}|\|\boldsymbol{W}_{0t}\|_{L_2(\mathcal{F})}$$

$$+c\|\boldsymbol{W}_0\|_{L_6(\Omega_t)}(\|\nabla\rho_t^*(\cdot,t)\|_{L_3(\mathcal{F})}+\|\rho_t^*\|_{L_3(\mathcal{F})}\Big)\le c\Big(\|f_{0t}\|_{L_2(\mathcal{G})}+\|f_0\|_{W_2^{1/2}(\mathcal{G})}\Big).$$

Finally, from

$$\widetilde{\boldsymbol{W}}_t(z,t) = \boldsymbol{W}_t(x,t) + \sum_{k=1}^{3} \boldsymbol{W}_{x_k} N_k^*(z)\rho_t^*(z,t)\Big|_{x=e_\rho(z)}$$

we obtain

$$\|\boldsymbol{W}_t\|_{L_2(\Omega_t)} \leq \|\widetilde{\boldsymbol{W}}_t\|_{L_2(\mathcal{F})} + c\|\nabla_x \boldsymbol{W}\|_{L_2(\Omega_t)}$$
$$\leq c\Big(\|f_{0t}\|_{L_2(\mathcal{G})} + \|f_0\|_{W_2^{1/2}(\mathcal{G})}\Big). \tag{6.7}$$

Proof of proposition 3.2. We follow the arguments in [9]. We multiply the first equation in (1.4) by $\boldsymbol{w}(x,t)$ and integrate over Ω_t. After simple calculations we arrive at

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t} |\boldsymbol{w}(x,t)|^2 dx + \frac{\nu}{2}\int_{\Omega_t} |S(\boldsymbol{w})|^2 dx - \int_{\Gamma_t} (m(x) - a^2)\boldsymbol{w}\cdot\boldsymbol{n}dS = 0.$$

The surface integral is equal to

$$\frac{d}{dt}\int_{\Omega_t} (m(x) - a^2)dx = \frac{d}{dt}\Big(\int_{\Omega_t} (m(x) - a^2)dx - \int_{\mathcal{F}} (m(z) - a^2)dz\Big).$$

As shown in [10] (see (2.16)-(2.18)),

$$\int_{\Omega_t} (m(x) - a^2)dx - \int_{\mathcal{F}} (m(z) - a^2)dz = \int_{\mathcal{G}} (m(z) - a^2)\rho(z,t)dS$$
$$+ \int_{\mathcal{G}} \Big(\frac{\partial m(z)}{\partial N} - (m(z) - a^2)\mathcal{H}(z)\Big)\rho^2(z,t) + q(\rho)$$
$$= \int_{\mathcal{G}} \frac{\partial m(z)}{\partial N}\rho^2(z,t)dS + q(\rho)$$

where $\mathcal{H}$ is the doubled mean curvature of $\mathcal{G}$ and $q(\rho)$ is a remainder term satisfying the inequality

$$|q(\rho)| \leq c\int_{\mathcal{G}} |\rho(z,t)|^3 dS.$$

Hence,

$$\frac{d}{dt}\Big(\frac{1}{2}\int_{\Omega_t} |\boldsymbol{w}(x,t)|^2 dx + \int_{\mathcal{G}} b(z)\rho^2(z,t)dS - q(\rho)\Big) + \frac{\nu}{2}\int_{\Omega_t} |S(\boldsymbol{w})|^2 dx = 0, \tag{6.8}$$

where $b(z) = -\frac{\partial m(z)}{\partial N} \geq b_0 > 0$.

We obtain one more relation, multiplying the first equation in (1.4) by the vector field $\boldsymbol{W}(x,t)$ defined in (6.4) and integrating over Ω_t (the function f_0 will be chosen later). This leads to

$$\frac{d}{dt}\int_{\Omega_t} \boldsymbol{w}\cdot\boldsymbol{W}dx - \int_{\Omega_t} \boldsymbol{w}\cdot(\boldsymbol{W}_t + (\boldsymbol{w}\cdot\nabla)\boldsymbol{W})dx + 2\omega\int_{\Omega_t} (\boldsymbol{e}_3\times\boldsymbol{w})\cdot\boldsymbol{W}dx$$

$$+\frac{\nu}{2}\int_{\Omega_t} S(\boldsymbol{w}):S(\boldsymbol{W})dx-\int_{\Gamma_t}(m(x)-a^2)f dS=0. \tag{6.9}$$

Now, we multiply (6.9) by a small $\gamma>0$ and add to (6.8). This gives (3.24) with

$$E(t)=\frac{1}{2}\int_{\Omega_t}|\boldsymbol{w}(x,t)|^2dx+\int_{\mathcal{G}} b(z)\rho^2(z,t)dS-q(\rho)+\gamma\int_{\Omega_t}\boldsymbol{w}(x,t)\cdot\boldsymbol{W}(x,t)dx,$$

$$E_1(t)=\frac{\nu}{2}\int_{\Omega_t}|S(\boldsymbol{w})|^2dx-\gamma\int_{\Omega_t}\boldsymbol{w}\cdot(\boldsymbol{W}_t+(\boldsymbol{w}\cdot\nabla)\boldsymbol{W})dx$$

$$+2\omega\gamma\int_{\Omega_t}(\boldsymbol{e}_3\times\boldsymbol{w})\cdot\boldsymbol{W}dx+\frac{\nu\gamma}{2}\int_{\Omega_t}S(\boldsymbol{w}):S(\boldsymbol{W})dx-\gamma\int_{\Gamma_t}(m(x)-a^2)f dS. \tag{6.10}$$

It is clear that (3.25) holds if γ is sufficiently small. Let us consider the surface integral in (6.10). We introduce a projection of ρ on the subspace $\widehat{L}_2(\mathcal{G})$ of functions orthogonal to constants in $L_2(\mathcal{G})$:

$$P\rho=\rho(z,t)-\frac{1}{|\mathcal{G}|}\int_{\mathcal{G}}\rho(y,t)dS,$$

and set

$$f=\frac{f_0}{|\widehat{\mathcal{L}}^T\boldsymbol{N}|}$$

where $f_0\in\widehat{L}_2(\mathcal{G})$ is a solution of the equation

$$Pbf_0=PbPf_0=(-\Delta_{\mathcal{G}})^{-1/2}P\rho$$

and $\Delta_{\mathcal{G}}$ is the Laplace-Beltrami operator on $\mathcal{G}$. This equation is uniquely solvable because the operator PbP is positive definite in the subspace $\widehat{L}_2(\mathcal{G})$, and the Laplace-Beltrami operator acts in this subspace. It is clear that

$$\|f_0\|_{W_2^{1/2}(\mathcal{G})}\le c\|(-\Delta_{\mathcal{G}})^{-1/2}P\rho\|_{W_2^{1/2}(\mathcal{G})}\le c\|P\rho\|_{W_2^{-1/2}(\mathcal{G})}. \tag{6.11}$$

Since the function $\varphi(z,\rho)$ (1.15) satisfies (1.14), we have

$$\Big|\int_{\mathcal{G}}\rho(z,t)dz\Big|=\Big|\int_{\mathcal{G}}(\rho-\varphi)dS\Big|=\Big|\int_{\mathcal{G}}(\frac{\rho^2}{2}\mathcal{H}-\frac{\rho^3}{3}\mathcal{K})dS\Big|$$

$$\le\|\rho\|_{W_2^{-1/2}(\mathcal{G})}\|\frac{\rho}{2}\mathcal{H}-\frac{\rho^2}{3}\mathcal{K}\|_{W_2^{1/2}(\mathcal{G})}\le\delta\|\rho\|_{W_2^{-1/2}(\mathcal{G})}.$$

This shows that

$$\|\rho-P\rho\|_{W_2^{-1/2}(\mathcal{G})}\le c\delta\|\rho\|_{W_2^{-1/2}(\mathcal{G})}$$

and for small δ

$$c^{-1}\|\rho\|_{W_2^{-1/2}(\mathcal{G})}\le\|P\rho\|_{W_2^{-1/2}(\mathcal{G})}\le c\|\rho\|_{W_2^{-1/2}(\mathcal{G})}.$$

The difference $m(x) - a^2$ can be written in the form

$$m(x) - a^2 = m(e_\rho(z)) - m(z) = -b(z)\rho + \int_0^1 (1-s)\frac{\partial^2 m(e_{s\rho})}{\partial s^2} ds$$
$$\equiv -b(z)\rho + m'(z,\rho)\rho^2 \tag{6.12}$$

where

$$m'(z,\rho) = \int_0^1 (1-s) \sum_{j,k=1}^3 m_{jk}(z + s\boldsymbol{N}(z)\rho) ds N_j(z) N_k(z).$$

Hence

$$-\int_{\Gamma_t} (m(x) - a^2) f dS = \int_{\mathcal{G}} (b(z)\rho(z,t) - \rho^2 m'(z,\rho)) f_0 dS = \int_{\mathcal{G}} P\rho(-\Delta_{\mathcal{G}}^{-1/2}) P\rho dS$$
$$+ \int_{\mathcal{G}} b(z)(\rho - P\rho) f_0 dS - \int_{\mathcal{G}} \rho^2 m'(z,\rho) f_0 dS. \tag{6.13}$$

The integral $\int_{\mathcal{G}} P\rho(-\Delta_{\mathcal{G}}^{-1/2})P\rho dS$ is equivalent to $\|P\rho\|^2_{W_2^{-1/2}(\mathcal{G})}$, and the last two terms in (6.13) do not exceed

$$\|\rho - P\rho\|_{W_2^{-1/2}(\Omega_0)} \|bf_0\|_{W_2^{1/2}(\Omega_0)} + \|\rho\|_{W_2^{-1/2}(\Omega_0)} \|\rho m'(\cdot,\rho) f_0\|_{W_2^{1/2}(\Omega_0)}$$
$$\leq c\delta \|\rho\|^2_{W_2^{-1/2}(\mathcal{G})}.$$

This shows that for small δ

$$-\int_{\Gamma_t} (m(x) - a^2) f dS \geq c\|\rho\|^2_{W_2^{-1/2}(\mathcal{G})}.$$

By (6.5)-(6.7), other terms in (6.10) are estimated in the following way:

$$\Big| 2\omega\gamma \int_{\Omega_t} (\boldsymbol{e}_3 \times \boldsymbol{w}) \cdot \boldsymbol{W} dx + \frac{\nu\gamma}{2} \int_{\Omega_t} S(\boldsymbol{w}) : S(\boldsymbol{W}) dx \Big|$$
$$\leq c\gamma \|\boldsymbol{w}\|_{W_2^1(\Omega_t)} \|\boldsymbol{W}\|_{W_2^1(\Omega_t)} \leq c\gamma \|\boldsymbol{w}\|_{W_2^1(\Omega_t)} \|\rho\|_{W_2^{-1/2}(\mathcal{G})},$$
$$\gamma \Big| \int_{\Omega_t} \boldsymbol{w} \cdot (\boldsymbol{w} \cdot \nabla) \boldsymbol{W} dx \Big| \leq c\gamma \|\boldsymbol{w}\|^2_{L_4(\Omega_t)} \|\rho\|_{W_2^{-1/2}(\mathcal{G})},$$
$$\gamma \Big| \int_{\Omega_t} \boldsymbol{w} \cdot \boldsymbol{W}_t dx \Big| \leq c\gamma \|\boldsymbol{w}\|_{L_2(\Omega_t)} \Big(\|\rho_t\|_{L_2(\mathcal{G})} + \|\rho\|_{W_2^{-1/2}(\mathcal{G})} \Big).$$

Since the kinematic boundary conditions in (1.4), $V'_n = \boldsymbol{w} \cdot \boldsymbol{n}'$, can be written in an equivalent form

$$\rho_t(z,t) = \frac{\boldsymbol{w}(e_\rho(z),t) \cdot \widehat{\boldsymbol{\mathcal{L}}}^T(z,t)\boldsymbol{N}(z)}{\boldsymbol{N}(z) \cdot \widehat{\boldsymbol{\mathcal{L}}}^T(z,\rho)\boldsymbol{N}(z)},$$

we have $\|\rho_t\|_{L_2(\mathcal{G})} \leq c\|\boldsymbol{w}\|_{L_2(\Gamma_t)}$. Finally, we use the Korn inequality

$$\|\boldsymbol{w}\|_{W_2^1(\Omega_t)} \leq c\|S(\boldsymbol{w})\|_{L_2(\Omega_t)}$$

and show that (3.26) holds in the case of small δ and γ. Proposition 3.2 is proved.

7 On the solvability of the problems (3.29) and (3.31).

Let us go back to Theorems 3.7 and 3.8 on the solvability of the problems (3.29) and (3.31). We consider the second slightly more complicated problem. It is studied in the spaces $\widehat{W}_2^{l,l/2}(Q_{T,T+1}),\ \ l > 1$, with the norm

$$\|u\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})} = \|u\|_{W_2^{l,l/2}(Q_{T,T+1})} + T\|u\|_{W_2^{l-1,l/2-1/2}(Q_{T,T+1})};$$

we also set

$$\|u\|_{\widehat{W}_2^{l,0}(Q_{T,T+1})} = \|u\|_{W_2^{l,0}(Q_{T,T+1})} + T\|u\|_{W_2^{l-1,0}(Q_{T,T+1})},$$

$$\|u\|_{\widehat{W}_2^{0,l/2}(Q_{T,T+1})} = \|u\|_{W_2^{0,l/2}(Q_{T,T+1})} + T\|u\|_{W_2^{0,l/2-1/2}(Q_{T,T+1})}.$$

The spaces $\widehat{W}_2^{l,l/2}(G_{T,T+1})$ of functions defined on $G_{T,T+1}$ are introduced in a similar way.

For the analysis of the problem (3.31) we need some additional estimates of the expressions $\boldsymbol{l}_1(\boldsymbol{u}, q), l_2(\boldsymbol{u}), \boldsymbol{l}_3(\boldsymbol{u}), l_4(\boldsymbol{u})$.

Proposition 7.1. *Assume that $\boldsymbol{u}, q$ are defined in Q_T, extended in the time interval $(T, T+1)$ according to the rule (3.30), and that*

$$\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)} + \|\nabla q\|_{\widetilde{W}_2^{l,l/2}(Q_T)} \leq \delta;$$

moreover, let $\boldsymbol{v}, \boldsymbol{v}', p, p'$ satisfy the conditions $\boldsymbol{v}(\xi,t) = \boldsymbol{v}'(\xi,t) = 0$, $p(\xi,t) = p'(\xi,t) = 0$ for $t \leq T$, $\boldsymbol{v}, \boldsymbol{v}' \in \widehat{W}_2^{l+2,l/2+1}(Q_{T,T+1})$, $\nabla p, \nabla p' \in \widehat{W}_2^{l,l/2}(Q_{T,T+1})$ and

$$\|\boldsymbol{v}\|_{\widehat{W}_2^{l+2,l/2+1/2}(Q_{T,T+1})} + \|\nabla p\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})} \leq \delta,$$

$$\|\boldsymbol{v}'\|_{\widehat{W}_2^{l+2,l/2+1/2}(Q_{T,T+1})} + \|\nabla p'\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})} \leq \delta$$

with sufficiently small $\delta > 0$. Then

$$\|\boldsymbol{l}_1(\boldsymbol{u}_0 + \boldsymbol{v}, q_0 + p) - \boldsymbol{l}_1(\boldsymbol{u}_0 + \boldsymbol{v}', q_0 + p')\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}$$

$$\leq c\delta\Big(\|\boldsymbol{v} - \boldsymbol{v}'\|_{\widehat{W}_2^{l+2,l/2+1}(Q_{T,T+1})} + \|\nabla(p - p')\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}\Big),$$

$$\|l_2(\boldsymbol{u}_0 + \boldsymbol{v}) - l_2(\boldsymbol{u}_0 + \boldsymbol{v}')\|_{\widehat{W}_2^{l+1,0}(Q_{T,T+1})} + \|\boldsymbol{L}(\boldsymbol{u}_0 + \boldsymbol{v}) - \boldsymbol{L}(\boldsymbol{u}_0 + \boldsymbol{v}')\|_{\widehat{W}_2^{0,1+l/2}(Q_{T,T+1})}$$

$$+\|\boldsymbol{l}_3(\boldsymbol{u}_0 + \boldsymbol{v}) - \boldsymbol{l}_3(\boldsymbol{u}_0 + \boldsymbol{v}')\|_{\widehat{W}_2^{l+1/2,l/1+1/4}(G_{T,T+1})}$$

$$+\|l_4(\boldsymbol{u}_0 + \boldsymbol{v}) - l_4(\boldsymbol{u}_0 + \boldsymbol{v}')\|_{\widehat{W}_2^{l+1/2,l/1+1/4}(G_{T,T+1})} \leq c\delta\|\boldsymbol{v} - \boldsymbol{v}'\|_{\widehat{W}_2^{2+l,1+l/2}(Q_{T,T+1})}, \tag{7.1}$$

where $\boldsymbol{u}_0, q_0$ are defined in (3.30).

In addition,

$$\|\ell_4(\boldsymbol{v}) - \ell_4(\boldsymbol{v}')\|_{\widehat{W}_2^{l+1,l/2+1/4}(G_{T,T+1})} \leq c\delta\|\boldsymbol{v} - \boldsymbol{v}'\|_{\widehat{W}_2^{2+l,1+l/2}(Q_{T,T+1})}. \tag{7.2}$$

Inequality (7.2) is obvious. The proof of (7.1) is based on the estimates of the differences $A_{ij}-A'_{ij}$, where A_{ij} and A'_{ij} are co-factors corresponding to the transformations $x=\xi+\int_0^t \boldsymbol{u}(\xi,\tau)d\tau$ and $x=\xi+\int_0^t \boldsymbol{u}'(\xi,\tau)d\tau$, respectively.

Proposition 7.2. *If $\boldsymbol{u}$, $\boldsymbol{u}'$ satisfy (5.13), then*

$$\|A(\cdot,t)-A'(\cdot,t)\|_{W_2^{l_1}(\Omega_0)} \le c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{7.3}$$

$$\sup_{\Omega_0}|A(\xi,t)-A'(\xi,t)| \le c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{7.4}$$

$$\|A(\cdot,t)-A'(\cdot,t)\|_{W_2^{l+1}(\Omega_0)} \le c(1+\sqrt{t})\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)}, \tag{7.5}$$

$$\|A_t(\cdot,t)-A'_t(\cdot,t)\|_{W_2^{l}(\Omega_0)}$$
$$\le c\Big(\|\nabla(\boldsymbol{u}(\cdot,t)-\boldsymbol{u}'(\cdot,t))\|_{W_2^l(\Omega_0)}+\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla\boldsymbol{u}'\|_{W_2^l(\Omega_0)}\Big), \tag{7.6}$$

$$\|A_{tt}-A'_{tt}\|_{L_2(Q_t)} \le c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{W_2^{l+1,l/2+1/2}(Q_t)}, \tag{7.7}$$

$$\|\nabla A_t-\nabla A'_t\|_{W_2^{l,0}(Q_t)} \le c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{W_2^{l,0}(Q_t)}, \tag{7.8}$$

where $l_1<l+1/2$.

Proof. Inequalities (7.3)-(7.5) follow easily from the obvious estimates

$$\|D(\cdot,t)-D'(\cdot,t)\|_{W_2^{l_1}(\Omega_0)} \le c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)},$$

$$\sup_{\Omega_0}|D(\xi,t)-D'(\xi,t)| \le c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)},$$

$$\|D(\cdot,t)-D'(\cdot,t)\|_{W_2^{l+1}(\Omega_0)} \le c(1+\sqrt{t})\|\nabla(\boldsymbol{u}-\boldsymbol{u}'\|_{\widetilde{W}_2^{l+1,0}(Q_t)}.$$

Other estimates follow from the fact that $A_{ijt}-A'_{ijt}$ are linear combinations of $\frac{\partial(u_k-u'_k)}{\partial\xi_m}$, $\frac{\partial(u_k-u'_k)}{\partial\xi_m}D_{qi}$, $\frac{\partial u'_k}{\partial\xi_m}(D_{qi}-D'_{qi})$, $A_{ijtt}-A'_{ijtt}$ are linear combinations of $\frac{\partial^2(u_k-u'_k)}{\partial\xi_m\partial t}$ $\frac{\partial^2(u_k-u'_k)}{\partial\xi_m\partial t}D_{qi}$, $\frac{\partial^2 u'_k}{\partial\xi_m\partial t}(D_{qi}-D'_{qi})$, $\frac{\partial(u_k-u'_k)}{\partial\xi_m}\frac{\partial u_q}{\partial\xi_i}$, $\frac{\partial}{\partial\xi_s}(A_{ijt}-A'_{ijt})$ are linear combinations of $\frac{\partial^2(u_k-u'_k)}{\partial\xi_m\partial\xi_s}$, $\frac{\partial^2(u_k-u'_k)}{\partial\xi_m\partial\xi_s}D_{qi}$, $\frac{\partial^2 u'_k}{\partial\xi_m\partial\xi_s}(D_{qi}-D'_{qi})$, $\frac{\partial(u_k-u'_k)}{\partial\xi_m}\frac{\partial D_{qi}}{\partial\xi_s}$, $\frac{\partial u'_k}{\partial\xi_m}\frac{\partial(D_{qi}-D'_{qi})}{\partial\xi_s}$. We have

$$\|A_t-A'_t\|_{W_2^l(\Omega_0)} \le c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{W_2^l(\Omega_0)}\Big(1+\sup_{\Omega_0}|D(\xi,t)|+\|D\|_{W_2^{3/2}(\Omega_0)}\Big)$$
$$+c\|\nabla\boldsymbol{u}'\|_{W_2^l(\Omega_0)}\Big(\sup_{\Omega_0}|D(\xi,t)-D'(\xi,t)|+\|D-D'\|_{W_2^{3/2}(\Omega_0)}\Big)$$
$$\le c\Big(\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{W_2^l(\Omega_0)}+\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\|\nabla\boldsymbol{u}'\|_{W_2^l(\Omega_0)}\Big),$$

$$\|A_{tt}-A'_{tt}\|_{L_2(Q_t)} \le c\Big(\|\nabla(\boldsymbol{u}_t-\boldsymbol{u}'_t)\|_{L_2(Q_t)}+\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)}$$
$$+\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{W_2^{l,0}(Q_t)}\sup_{t'<t}\|\nabla\boldsymbol{u}(\cdot,t')\|_{W_2^l(\Omega_0)}\Big),$$

which implies (7.7). Finally,

$$\|\nabla(A_t - A'_t)\|_{W_2^l(\Omega_0)} \le c\Big(\|D^2(\boldsymbol{u} - \boldsymbol{u}')\|_{W_2^l(\Omega_0)} + \sqrt{t}\sup_{\Omega_0}|\nabla(\boldsymbol{u}(\xi,t) - \boldsymbol{u}'(\xi,t))|$$

$$+\sqrt{t}\|\nabla(\boldsymbol{u}(\cdot,t) - \boldsymbol{u}'(\cdot,t))\|_{W_2^{3/2}(\Omega_0)}\Big) + c\|\nabla(\boldsymbol{u} - \boldsymbol{u}')\|_{\widetilde{W}_2^{l+1,0}(Q_t)}\Big(\|\nabla\boldsymbol{u}'\|_{W_2^{l+1}(\Omega_0)}$$

$$+\sqrt{t}\sup_{\Omega_0}|\nabla\boldsymbol{u}'(\xi,t)| + \sqrt{t}\|\nabla\boldsymbol{u}'(\cdot,t)\|_{W_2^{3/2}(\Omega_0)}\Big),$$

which implies (7.8). The proposition is proved.

The proof of (7.1) is analogous to that of Propositions 5.5 and 5.6. We illustrate the method of obtaining these inequalities for the simple case

$$\mathcal{P} = (I - A)\nabla q - (I - A')\nabla q' = \mathcal{P}_1 + \mathcal{P}_2,$$

where $q = q_0 + p$, $q' = q_0 + p'$, $\mathcal{P}_1 = (I - A')\nabla(q - q')$, $\mathcal{P}_2 = (A' - A)\nabla q$. We also set $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{v}$, $\boldsymbol{u}' = \boldsymbol{u}_0 + \boldsymbol{v}'$. The norm $\|\mathcal{P}_1\|_{\widehat{W}_2^{l,0}(Q_{T,T+1})}$ can be estimated precisely as in Proposition 5.5:

$$\|\mathcal{P}_1\|_{\widehat{W}_2^{l,0}(Q_{T,T+1})} \le c\|\nabla\boldsymbol{u}'\|_{\widetilde{W}_2^{l+1,0}(Q_{T+1})}\|\nabla(q - q')\|_{\widehat{W}_2^{l,0}(Q_{T+1})}$$

$$\le c\delta\|\nabla(p - p')\|_{\widehat{W}_2^{l,0}(Q_{T,T+1})}.$$

Moreover, we have

$$\|\Delta_t(-h)\mathcal{P}_1\|_{L_2(\Omega_0)} \le \sup_{\Omega_0}|I - A'(\xi, t-h)|\|\Delta_t(-h)\nabla(q - q')\|_{L_2(\Omega_0)}$$

$$+ch\sup_{t<T}\|A'_t(\cdot,t)\|_{W_2^l(\Omega_0)}\|\nabla(q - q')\|_{W_2^l(\Omega_0)},$$

$$\|\Delta_t(-h)\mathcal{P}_1\|_{L_2(\Omega_0)} \le \sup_{\Omega_0}|I - A'(\xi, t-h)|\|\Delta_t(-h)\nabla(q - q')\|_{L_2(\Omega_0)}$$

$$+c\sqrt{h}\|A'_t\|_{W_2^{l+1,0}(Q_t)}\|\nabla(q - q')\|_{W_2^{l-1}(\Omega_0)},$$

and, as a consequence,

$$\Big(\int_0^1 \frac{dh}{h^{1+l}}\int_{T+h}^{T+1}\|\Delta_t(-h)\mathcal{P}_1\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$

$$\le c\|\boldsymbol{u}'\|_{\widetilde{W}_2^{l+1,0}(Q_{T+1}}\Big(\int_0^1 \frac{dh}{h^{1+l}}\int_{T+h}^{T+1}\|\Delta_t(-h)\nabla(q - q')\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$

$$+c\sup_{\tau<h}\|\boldsymbol{u}'(\cdot, t-\tau)\|_{W_2^l(\Omega)}\|\nabla(q - q')\|_{W_2^{l,0}(Q_{T,T+1})},$$

$$\Big(\int_0^1 \frac{dh}{h^{l}}\int_{T+h}^{T+1}t^2\|\Delta_t(-h)\mathcal{P}_1\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$

$$\le c\|\nabla\boldsymbol{u}'\|_{\widetilde{W}_2^{l+1,0}(Q_{T+1}}\Big(\int_0^1 \frac{dh}{h^{l}}\int_{T+h}^{T+1}t^2\|\Delta_t(-h)\nabla(q - q')\|^2_{L_2(\Omega_0)}dt\Big)^{1/2}$$

$$+c\|\nabla \boldsymbol{u}'\|_{W_2^{l,0}(Q_{T,T+1})}\|t\nabla(q-q')\|_{W_2^{l-1,0}(Q_{T,T+1})},$$

which implies

$$\|\mathcal{P}_1\|_{\widehat{W}_2^{0,l/2}(Q_{T,T+1})} \leq c\delta\|\nabla(p-p')\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}.$$

Hence,

$$\|\mathcal{P}_1\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})} \leq c\delta\|\nabla(p-p')\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}.$$

The function $\mathcal{P}_2$ is estimated in a similar way. We have

$$\|\mathcal{P}_2\|_{\widehat{W}_2^{l,0}(Q_{T,T+1})} \leq c\|\nabla(\boldsymbol{u}-\boldsymbol{u}')\|_{\widehat{W}_2^{l+1,0}(Q_{T+1})}\|\nabla q\|_{\widehat{W}_2^{l,0}(Q_{T+1})}$$
$$\leq c\delta\|\nabla(\boldsymbol{v}-\boldsymbol{v}')\|_{\widehat{W}_2^{l,0}(Q_{T,T+1})}$$

(we have used the fact that $\boldsymbol{u}-\boldsymbol{u}'=\boldsymbol{v}-\boldsymbol{v}'=0$ for $t\leq T$); moreover, from

$$\|\Delta_t(-h)\mathcal{P}_2\|_{L_2(\Omega_0)} \leq \sup_{\Omega_0}|A(\xi,t-h)-A'(\xi,t-h)|\|\Delta_t(-h)\nabla q\|_{L_2(\Omega_0)}$$
$$+ch\sup_{t<T}\|A_t(\cdot,t)-A_t'(\cdot,t)\|_{W_2^l(\Omega_0)}\|\nabla q\|_{W_2^l(\Omega_0)},$$
$$\|\Delta_t(-h)\mathcal{P}_2\|_{L_2(\Omega_0)} \leq \sup_{\Omega_0}|A(\xi,t-h)-A'(\xi,t-h)|\|\Delta_t(-h)\nabla q\|_{L_2(\Omega_0)}$$
$$+c\sqrt{h}\|A_t-A_t'\|_{W_2^{l+1}(Q_t)}\|\nabla q\|_{W_2^{l-1}(\Omega_0)}$$

we obtain

$$\|\mathcal{P}_2\|_{\widehat{W}_2^{0,l/2}(Q_{T,T+1})} \leq c\delta\|\nabla(\boldsymbol{v}-\boldsymbol{v}')\|_{\widehat{W}_2^{l+1,0}(Q_{T,T+1})}.$$

Hence

$$\|\mathcal{P}\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})} \leq c\delta\Big(\|\nabla(\boldsymbol{v}-\boldsymbol{v}')\|_{\widehat{W}_2^{l+1,0}(Q_{T,T+1})}+\|\nabla(p-p')\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}\Big).$$

Other inequalities in Proposition 7.1 are also obtained by the arguments presented in the proof of Propositions 5.5 and 5.6, on the basis of (7.3)-(7.8). The details are omitted.

As in [3], we establish the solvability of the problem (3.31) by the method of successive approximations:

$$\begin{gathered}
\boldsymbol{v}_{m+1t}+2\omega(\boldsymbol{e}_3\times\boldsymbol{v}_{m+1})-\nu\nabla^2\boldsymbol{v}_{m+1}+\nabla p_{m+1} \\
=\boldsymbol{l}_1(\boldsymbol{u}_0+\boldsymbol{v}_m,q_0+p_m)-\boldsymbol{l}_1(\boldsymbol{u}_0,q_0)+\boldsymbol{f}(\xi,t), \\
\nabla\cdot\boldsymbol{v}_{m+1}=l_2(\boldsymbol{u}_0+\boldsymbol{v}_m)-l_2(\boldsymbol{u}_0)+f(\xi,t), \quad \xi\in\Omega_0,\ \ t>T, \\
\Pi_0 S(\boldsymbol{v}_{m+1})\boldsymbol{n}_0=\boldsymbol{l}_3(\boldsymbol{u}_0+\boldsymbol{v}_m)-\boldsymbol{l}_3(\boldsymbol{u}_0)+\boldsymbol{d}(\xi,t), \\
-p_{m+1}+\nu\boldsymbol{n}_0\cdot S(\boldsymbol{v}_{m+1})\boldsymbol{n}_0-\ell_3(\boldsymbol{v}_{m+1})=l_4(\boldsymbol{u}_0+\boldsymbol{v}_m)-l_4(\boldsymbol{u}_0) \\
+\ell_4(\boldsymbol{v}_m)+d(\xi,t), \quad \xi\in\Gamma_0, \\
\boldsymbol{v}_{m+1}(\xi,t)=0, \quad \xi\in S, \\
\boldsymbol{v}_{m+1}(\xi,T)=0, \quad \xi\in\Omega_0,
\end{gathered} \tag{7.9}$$

$m=0,1,\ldots$. As the zero approximation, we take $\boldsymbol{v}_0=0$, $p_0=0$, so $\boldsymbol{v}_1,p_1$ are found as a solution of a linear problem

$$\begin{gathered}
\boldsymbol{v}_{1t}+2\omega(\boldsymbol{e}_3\times\boldsymbol{v}_1)-\nu\nabla^2\boldsymbol{v}_1+\nabla p_1=\boldsymbol{f}(\xi,t),\\
\nabla\cdot\boldsymbol{v}_1=f(\xi,t),\quad \xi\in\Omega_0,\quad t>T,\\
\Pi_0 S(\boldsymbol{v}_1)\boldsymbol{n}_0=\boldsymbol{d}(\xi,t),\\
-p_1+\nu\boldsymbol{n}_0\cdot S(\boldsymbol{v}_1)\boldsymbol{n}_0-\ell_3(\boldsymbol{v}_1)=d(\xi,t),\quad \xi\in\Gamma_0,\\
\boldsymbol{v}_1(\xi,t)=0,\quad \xi\in S,\\
\boldsymbol{v}_1(\xi,T)=0,\quad \xi\in\Omega_0.
\end{gathered}\tag{7.10}$$

The functions $\boldsymbol{f}$, $f=\nabla\cdot\boldsymbol{F}$, $\boldsymbol{d}$, d are defined in (3.32). They satisfy the inequality

$$\begin{gathered}
\|\boldsymbol{f}\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}+\|f\|_{\widehat{W}_2^{l+1,0}(Q_{T,T+1})}+\|\boldsymbol{F}\|_{\widehat{W}_2^{0,1+l/2}(Q_{T,T+1})}\\
+\|\boldsymbol{d}\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_{T,T+1})}+\|d\|_{\widehat{W}_2^{l+1/2,l/2+1/4}(G_{T,T+1})}\leq cU(T),
\end{gathered}$$

where

$$U(T)=\|\boldsymbol{u}\|_{\widetilde{W}_2^{2+l,1+l/2}(Q_T)}+\|\nabla q\|_{\widetilde{W}_2^{l,l/2}(G_T)}.$$

This inequality follows from the estimates of nonlinear terms obtained in Sec. 5. The difference $m(X[\boldsymbol{u}_0])-m^{(0)}(X[\boldsymbol{u}_0])$ needs a special treatment. It can be written in the form

$$\begin{gathered}
m(X[\boldsymbol{u}_0])-m^{(0)}(X[\boldsymbol{u}_0])=-3\Big(m(X_0(\xi,t))-m((X_0(\xi,2T-t))\Big)\\
+4\Big(m(X_0(\xi,t))-m((X_0(\xi,3T/2-t/2))\Big),
\end{gathered}$$

where $X_0(\xi,t)=X[\boldsymbol{u}_0](\xi,t)$, $t\in(T,T+1)$. Since

$$X_0(\xi,t)=X_0(\xi,2T-t)+\int_{2T-t}^t\boldsymbol{u}_0(\xi,\tau)d\tau,$$

we have

$$\begin{gathered}
m(X_0(\xi,t))-m((X_0(\xi,2T-t))\\
=\int_0^1\nabla m\Big(X_0(\xi,2T-t)+s\int_{2T-t}^t\boldsymbol{u}_0(\xi,\tau)d\tau\Big)ds\cdot\int_{2T-t}^t\boldsymbol{u}_0(\xi,\tau)d\tau,\\
m(X_0(\xi,t))-m((X_0(\xi,3T/2-t/2))\\
=\int_0^1\nabla m\Big(X_0(\xi,3T/2-t/2)+s\int_{3T/2-t/2}^t\boldsymbol{u}_0(\xi,\tau)d\tau\Big)ds\cdot\int_{3T/2-t/2}^t\boldsymbol{u}_0(\xi,\tau)d\tau.
\end{gathered}$$

These expressions are estimated in the same way as the difference (5.48). The norm $\|m(X[\boldsymbol{u}_0])-m^{(0)}(X[\boldsymbol{u}_0])\|_{\widetilde{W}_2^{0,l/2+1/4}(G_T)}$ is estimated with the help of (5.8). As result, we obtain

$$\|m(X[\boldsymbol{u}_0])-m^{(0)}(X[\boldsymbol{u}_0])\|_{\widetilde{W}_2^{l+1/2,l/2+1/4}(G_T)}\leq c\|\boldsymbol{u}\|_{\widetilde{W}_2^{l+2,l/2+1}(Q_T)}.$$

The problem (7.10) differs from (3.8) by the presence of additional linear terms $2\omega(\boldsymbol{e}_3 \times \boldsymbol{v}_1)$ and $\ell_3(\boldsymbol{v}_1)$. But they are of a lower order, so estimate (3.12) is still applicable to (7.10). The compatibility conditions are satisfied, hence, this problem is uniquely solvable and the solution satisfies the inequality

$$X_1(T) \equiv \|\boldsymbol{v}_1\|_{\widehat{W}_2^{l+2,l/2+1}(Q_{T,T+1})} + \|\nabla p_1\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})} \leq cU(T).$$

By (3.28), $U(T) \leq c\epsilon$. It is easily seen that $\boldsymbol{v}_{1t}|_{t=T} = 0$, $p_1|_{t=T} = 0$.

If $\boldsymbol{v}_m$ and p_m are found and the functions $\boldsymbol{u}$,q, $\boldsymbol{v}_m, p_m$, $\boldsymbol{v}' = 0, p' = 0$ satisfy the assumptions of Proposition 7.1, then we can find $\boldsymbol{v}_{m+1}, p_{m+1}$ as a solution of the problem (7.9). By virtue of (3.12). (7.1), (7.2),

$$X_{m+1}(T) = \|\boldsymbol{v}_{m+1}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_{T,T+1})} + \|\nabla p_{m+1}\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})} \leq c_1\delta X_m(T) + c_2U(T). \tag{7.11}$$

Let ϵ in (3.4) be so small that $c_2(1-c_1\delta)^{-1}U(T) \leq \delta$. If

$$X_m(T) \leq c_2(1-c_1\delta)^{-1}U(T), \tag{7.12}$$

then it follows from (7.11) that X_{m+1} satisfies the same inequality and $\boldsymbol{u}$,q, $\boldsymbol{v}_{m+1}, p_{m+1}$, $\boldsymbol{v}' = 0, p' = 0$ satisfy the assumptions of Proposition 7.1. This allows us to find $\boldsymbol{v}_{m+2}, p_{m+2}$ and so forth. In this way we find all the approximations $\boldsymbol{v}_m, p_m$ and obtain a uniform estimate (7.12) for them.

Since the functions (3.32) vanish for $t = T$, we have $\boldsymbol{v}_t|_{t=T} = 0$, $p|_{t=T} = 0$.

To prove the convergence of the sequence $\boldsymbol{v}_m, p_m$, we estimate the differences $\boldsymbol{w}_{m+1} = \boldsymbol{v}_{m+1} - \boldsymbol{v}_m$, $s_{m+1} = p_{m+1} - p_m$, $m = 1, 2, ...$ They satisfy the relations

$$\boldsymbol{w}_{m+1t} + 2\omega(\boldsymbol{e}_3 \times \boldsymbol{w}_{m+1}) - \nu\nabla^2\boldsymbol{w}_{m+1} + \nabla s_{m+1}$$

$$= \boldsymbol{l}_1(\boldsymbol{u}_0 + \boldsymbol{v}_m, q_0 + p_m) - \boldsymbol{l}_1(\boldsymbol{u}_0 + \boldsymbol{v}_{m-1}, q_0 + p_{m-1}),$$

$$\nabla \cdot \boldsymbol{w}_{m+1} = l_2(\boldsymbol{u}_0 + \boldsymbol{v}_m) - l_2(\boldsymbol{u}_0 + \boldsymbol{v}_{m-1}), \quad \xi \in \Omega_0, \ \ t > T,$$

$$\Pi_0 S(\boldsymbol{w}_{m+1})\boldsymbol{n}_0 = \boldsymbol{l}_3(\boldsymbol{u}_0 + \boldsymbol{v}_m) - \boldsymbol{l}_3(\boldsymbol{u}_0 + \boldsymbol{v}_{m-1}),$$

$$-s_{m+1} + \nu\boldsymbol{n}_0 \cdot S(\boldsymbol{w}_{m+1})\boldsymbol{n}_0 - \ell_3(\boldsymbol{w}_{m+1}) = l_4(\boldsymbol{u}_0 + \boldsymbol{v}_m) - l_4(\boldsymbol{u}_0 + \boldsymbol{v}_{m-1})$$

$$+\ell_4(\boldsymbol{v}_m) - \ell_4(\boldsymbol{v}_{m-1}), \quad \xi \in \Gamma_0,$$

$$\boldsymbol{w}_{m+1}(\xi, t) = 0, \quad \xi \in S,$$

$$\boldsymbol{w}_{m+1}(\xi, T) = 0, \quad \xi \in \Omega_0.$$

By inequalities (3.12), (7.12), and Proposition 7.1, we have

$$Z_{m+1}(T) \leq c\delta Z_m(T),$$

where

$$Z_m(T) = \|\boldsymbol{w}_m\|_{\widehat{W}_2^{l+2,l/2+1}(Q_{T,T+1})} + \|\nabla s_m\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}.$$

It follows that $\sum_{m=1}^{M} Z_{m+1}(T) \le c\delta \sum_{m=1}^{M} Z_m(T)$, and if $c\delta < 1/2$, which can be guaranteed by the choice of a small ϵ, then $\sum_{m=1}^{M} Z_{m+1}(T) \le 2Z_1(T) = 2X_1(T)$, which means the convergence of $\boldsymbol{v}_m, p_m$ to a solution of the problem (3.31). It is unique, since for the difference $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}'$, $s = p - p'$ of the two solutions we have

$$\|\boldsymbol{w}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_{T,T+1})} + \|\nabla s\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}$$
$$\le \frac{1}{2}\Big(\|\boldsymbol{w}\|_{\widehat{W}_2^{l+2,l/2+1}(Q_{T,T+1})} + \|\nabla s\|_{\widehat{W}_2^{l,l/2}(Q_{T,T+1})}\Big),$$

so these solutions coincide. Theorem 3.8 is proved.

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