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# ON MARKOVIAN PERTURBATIONS OF THE GROUP OF SHIFTS ON THE REAL LINE

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**Abstract.** We study perturbations $(\tilde{\tau}_t)_{t\in\mathbb{R}}$ of the group of shifts $(\tau_t)_{t\in\mathbb{R}}$ acting on $L^2(\mathbb{R})$ by $(\tau_t f)(x) = f(x-t)$, such that $\tilde{\tau}_t - \tau_t$ belongs to a certain Schatten–von Neumann class $\mathfrak{S}_p$ with $p \geq 1$, and the perturbations have the Markovian properties in the following sense: if $f \equiv 0$ on $\mathbb{R}_+$, then, for $t < 0$, $\tau_t f = \tilde{\tau}_t f$. The problem is to describe possible spectral types of the unitary component of $\tilde{\tau}_t$. Previously, we studied this problem for the isometric semigroup $(\tilde{\tau}_t|_{L^2(\mathbb{R}_+)})_{t\geq 0}$ using a special model in the Hardy space. It was shown that, for $\mathfrak{S}_1$-perturbations of the isometric semigroups, any singular spectral type may be obtained, whereas for $\mathfrak{S}_p$ perturbations the perturbed semigroup $(\tilde{\tau}_t|_{L^2(\mathbb{R}_+)})_{t\geq 0}$ may have arbitary spectral type of the unitary component. For $p > 1$ the statement remains true for the unitary dilations $(\tilde{\tau}_t)_{t\in\mathbb{R}}$; however, in terms of our model the difference $\tilde{\tau}_t - \tau_t$ cannot be in $\mathfrak{S}_1$.

**Keywords.** Group of shifts, Markovian perturbation, Schatten–von Neumann ideals, Hardy space, inner function.

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## 1. Introduction

Consider the unitary group $(\tau_t)_{t\in\mathbb{R}}$ of shifts on the space $L^2(\mathbb{R})$,

$$(\tau_t f)(x) = f(x-t), \quad f \in L^2(\mathbb{R}),$$

which will be thought of as the unperturbed group. In this paper we are concerned with perturbations $(\tilde\tau_t)$ of the group $(\tau_t)$ satisfying the following properties:

– $(\tilde\tau_t)$ is a strongly continuous group of unitary operators on $L^2(\mathbb{R})$;

– the difference $\tilde\tau_t - \tau_t$ belongs to a certain Schatten–von Neumann class $\mathfrak{S}_p$ for every $t \in \mathbb{R}$;

– the perturbation satisfies the so-called Markovian property, which means that the perturbed operators coincide with the unperturbed ones on the left semi-axis $\mathbb{R}_-$ as $t < 0$, i.e.,

$$\tau_t f = \tilde\tau_t f \quad \text{if} \quad f = 0 \ \text{ on } \ \mathbb{R}_+, \qquad t < 0. \tag{1}$$

As usual, the Markovian property may be interpreted in the sense that "the past does not depend on the future". Perturbations with this property (with $p = 2$) were investigated by the first author [3, 4, 5] in connection with the study of cocycle perturbations of the flow of Powers shifts [17, 10, 11].

We identify the spaces $L^2(\mathbb{R}_+)$ and $L^2(\mathbb{R}_-)$ with the subspaces of functions from $L^2(\mathbb{R})$ identically equal to zero on the semiaxes $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. As follows from the Markovian property, $L^2(\mathbb{R}_+)$ is invariant under the operators $\tilde\tau_t$ with $t > 0$. Therefore, one may consider the semigroup of isometric operators that are restrictions of $(\tilde\tau_t)$ to $L^2(\mathbb{R}_+)$, $t > 0$, which is also of our interest. The unitary group $(\tilde\tau_t)$ is the *dilation* of this isometric semigroup.

The central problem considered in this paper is to describe all possible spectral types of the perturbed unitary groups and isometric semigroups, which are determined by the spectral type of their generators or cogenerators. For $p = 2$ this problem was solved in [7, 8] where is was shown that any spectral type is possible. Moreover, for the case of singular spectral measures a model for such perturbations was provided. We will show that the same is true for any $p > 1$. However, for $p = 1$, it follows from the stability of the absolutely continuous spectrum that, for the cogenerators of the groups $(\tau_t)$ and $(\tilde\tau_t)$, their absolutely continuous parts are unitarily equivalent. Since the spectral measure of the cogenerator of the group $(\tau_t)$ is the Lebesgue measure of multiplicity 1 on the unit circle, for $p = 1$, our aim thus reduces to a description of all possible singular parts. We show that for the case of the semigroup $(\tilde\tau_t|_{L^2(\mathbb{R}_+)})_{t\ge 0}$ any singular spectral type can be realized.

The present paper is a continuation of [9], where the same problem was completely solved for the isometric semigroup $(\tilde\tau_t|_{L^2(\mathbb{R}_+)})_{t\ge 0}$. To state the main result of [9], we introduce a model on the unit circle unitarily equivalent to the original one (see [9, Section 3]). The symbol $L^2$ (without specifying the measure) stands for the $L^2$-space with respect to the Lebesgue measure $m$ on $\mathbb{T}$ normalized so that $m(\mathbb{T}) = 1$. The cogenerator of the unperturbed unitary group is the operator $U$ of multiplication by $z$ in $L^2$. As is well known, $U$ is the shift operator with respect to the orthonormal basis $(z^n)_{n\in\mathbb{Z}}$, i.e., $Uz^n = z^{n+1}$. The Markovian property (1) for a perturbation $\tilde U$ of $U$, translated into the terms of the model on the unit circle, is equivalent to

$$\tilde U f = U f, \quad f \in \bar z H^2_-.$$

Hence $\tilde{U}$ acts as a shift along an orthonormal family $(s_n)$ such that $s_n = z^n$ for $n < 0$. Therefore, the orthogonal complement $K$ of all functions $(s_n)$ is a subspace of $H^2$ reducing $\tilde{U}$. Since the restriction of $\tilde{U}$ to the closed linear span of all $s_n$ is unitarily equivalent to $U$, the stability theorem mentioned above implies that in the case $p = 1$ the restriction of $\tilde{U}$ to $K$ is a singular unitary operator. Our problem is thus reduced to the study of the spectral structure of the restriction of $\tilde{U}$ to $K$.

The restriction of $\tilde{U}$ to $H^2$ will be considered as a perturbation of $S$ and denoted by $\tilde{S}$; $\tilde{U}$ may be regarded as a dilation of $\tilde{S}$.

Now we introduce a new parameter of our construction, namely, an inner function $\theta$ in the unit disk (a function $\theta \in H^2$ is said to be inner if $|\theta| = 1$ almost everywhere on $\mathbb{T}$). The space $K_\theta$ is defined by

$$K_\theta = H^2 \ominus \theta H^2,$$

it will be taken as $K$. Then $L^2$ splits into the direct sum:

$$L^2 = H^2_- \oplus K_\theta \oplus \theta H^2.$$

Our model is described as follows. The operators $\tilde{S}$, $\tilde{U}$ (which are thought of as perturbations of $S$, $U$) act on $H^2$ and $L^2$, respectively, and, by our assumptions, they have the following properties:

(i) $\tilde{S}$ is a diagonal operator with respect to the decomposition $H^2 = K_\theta \oplus \theta H^2$;
(ii) $\tilde{S}$ acts on $\theta H^2$ as multiplication by $z$;
(iii) the restriction of $\tilde{S}$ to $K_\theta$ is a unitary operator;
(iv) $H^2$ is an invariant subspace of $\tilde{U}$, and $\tilde{S}$ is the restriction of $\tilde{U}$ to $H^2$;
(v) $H^2_-$ is invariant for $\tilde{U}^*$ (which is a consequence of (iv)), and on $H^2_-$ the operator $\tilde{U}^*$ coincides with $U^*$, i.e., with multiplication by $\bar{z}$; equivalently, $\tilde{U}$ acts on $\bar{z}H^2_-$ as multiplication by $z$;
(vi) $\tilde{U}\bar{z} = \alpha\theta$, where $\alpha$ is a constant with $|\alpha| = 1$ (one can always make $\alpha = 1$ by taking the inner function $\alpha\theta$ in place of $\theta$).

From conditions (i) – (iii) it follows that $\tilde{S}$ is an isometry; properties (iv) – (vi) imply that $\tilde{U}$ is a unitary operator. The subspace $K_\theta$ is reducing for $\tilde{U}$. Properties of the restriction of $\tilde{U}$ to $K_\theta$ will be specified later; now we only mention that 1 is not an eigenvalue of $\tilde{U}|_{K_\theta}$.

On the orthogonal complement of $K_\theta$, $\tilde{U}$ acts as a shift along the orthonormal family $(s_n)$, where $s_n = z^n$ if $n < 0$, and $s_n = \theta z^n$ if $n \geq 0$. Hence 1 does not belong to the point spectrum of $\tilde{U}$, and one may consider the group of unitary operators for which $\tilde{U}$ is the cogenerator.

Elements of the group have the form $\varphi_t(\tilde{U})$, where the function $\varphi_t$ is defined by

$$\varphi_t(z) = \exp\left(t\frac{z+1}{z-1}\right), \quad t \in \mathbb{R}. \tag{2}$$

Note that $\varphi_t \in H^\infty$ for $t \geq 0$. The elements of the unperturbed groups are of the form $\varphi_t(U)$.

Suppose that $\tilde{U}$ is a trace class perturbation of $U$, or, equivalently, $\tilde{S}$ is a trace class perturbation of $S$. From theorem about stability of the absolutely continuous spectrum it follows that the absolutely continuous parts of $U$ and $\tilde{U}$ are unitarily equivalent, hence the spectral measure of the unitary operator $\tilde{S}|_{K_\theta}$ must be singular relative to the Lebesgue measure.

Now we formulate the main results of [9] in terms of the model on the circle, which shows that any singular summand $V$ may be obtained for trace class perturbations of the isometric semigroup of shifts.

**Theorem 1.1.** *Let $V$ be a singular unitary operator of multiplicity $n \le \infty$ such that 1 is not an eigenvalue of $V$, $\varepsilon > 0$. Then there exists an operator $\tilde{S}$ with the properties* (i)–(iii) *such that*

$a_1$) *the restriction of $\tilde{S}$ to $K_\theta$ is unitarily equivalent to $V$;*
$b_1$) *rank* $(\tilde{S} - S) \le n$;
$c_1$) $||\tilde{S} - S||_{\mathfrak{S}_1} \le \varepsilon$;
$d_1$) $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_1$ *for all* $t > 0$.

One of the first results of the present paper is that the direct analog of this theorem for the unitary dilations, where the operators $S$, $\tilde{S}$ are replaced by $U$, $\tilde{U}$, respectively, is not true.

**Theorem 1.2.** *If $\tilde{U}$ satisfies* (i) – (vi)*, then $\varphi_t(\tilde{U}) - \varphi_t(U)$ cannot be in $\mathfrak{S}_1$ for any $t$ unless $\theta$ is identically equal to a unimodular constant.*

However, the statement of Theorem 1.1 remains true for all classes $\mathfrak{S}_p$ with $p > 1$.

**Theorem 1.3.** *Assume that $V$ is a singular unitary operator of multiplicity $n \le \infty$ such that 1 is not an eigenvalue of $V$. Fix $\varepsilon > 0$.*

*There exist operators $\tilde{S}$, $\tilde{U}$ satisfying* (i) – (vi) *and such that*

$a_2$) *the restriction of $\tilde{U}$ to $K_\theta$ is unitarily equivalent to $V$;*
$b_2$) *rank* $(\tilde{U} - U) \le n + 1$;
$c_2$) $||\tilde{U} - U||_{\mathfrak{S}_1} \le \varepsilon$;
$d_2$) $\varphi_t(\tilde{U}) - \varphi_t(U) \in \mathfrak{S}_p$ *for all* $p > 1$.

As an immediate consequence of Theorems 1.1, 1.3, we obtain the following result for perturbations of the group of shifts on the real line.

**Theorem 1.4.** *Let $(\tau_t)$, $t \in \mathbb{R}$, be the group of shifts on $L^2(\mathbb{R})$, $(\tau_t f)(x) = f(x - t)$, and let $V$ an arbitrary singular unitary operator with the only restriction that 1 does not belong to the point spectrum of $V$. Then there exists a unitary group $(\tilde{\tau}_t)$, $t \in \mathbb{R}$, such that*

– *$(\tilde{\tau}_t)$ is a Markovian perturbation of the group $(\tau_t)$;*
– *the cogenerator of the group $(\tilde{\tau}_t)$ is unitarily equivalent to $U \oplus V$;*
– *$\tilde{\tau}_t - \tau_t \in \mathfrak{S}_p$ for any $p > 1$, $t \in \mathbb{R}$;*
– *for $t > 0$, the restrictions to $L^2(\mathbb{R}_+)$ of the operators $\tilde{\tau}_t - \tau_t$ belong to the trace class $\mathfrak{S}_1$.*

A certain analog of Theorem 1.3 holds without the assumption that the unitary operator $V$ is singular.

**Theorem 1.5.** *Let $V$ be a unitary operator such that 1 is not an eigenvalue of $V$. Then there exists a construction satisfying* (i) – (vi)*, for which the restriction of $\tilde{U}$ to $K_\theta$ is unitarily equivalent to $V$ and $\varphi_t(\tilde{U}) - \varphi_t(U) \in \mathfrak{S}_p$ for all $p > 1$.*

For the class $\mathfrak{S}_2$, analogs of Theorems 1.1, 1.3 were obtained in [7, 8]. The new results in this paper are obtained for more narrow classes, i.e., $\mathfrak{S}_p$ with $p < 2$, and especially with $p = 1$. In [6, 7] the construction of the operator $V$ was connected with the triangulation studied by Ahern and Clark [2]. Instead, we use Clark's construction [12], which allows us to isometrically identify the space $K_\theta$ with an $L^2$-space on the unit circle.

The paper is organized as follows. In Section 2 we recall the main properties of the construction introduced in [9]. Also we use an equivalent model on the real line instead of the unit circle described in Section 3. In Section 4 we discuss the structure of the unitary dilation $\tilde{U}$ of the isometry $\tilde{S}$ and show that in the matrix representation of operators for the decomposition $L^2 = H^2_- \oplus K_\theta \oplus \theta H^2$ the crucial role is played by the entry $H^2_- \to \theta H^2$. In Section 5 Parfenov's results [15] about embeddings of the Paley–Wiener space of entire functions to weighted $L^2$-spaces on the real line are used to prove Theorems 1.2, 1.3 and 1.5.

## 2. The spaces $K_\theta$ and the model construction

In this section we recall the details of the construction of the model satisfying (i) – (vi) starting from a singular measure $\mu$ on the unit circle $\mathbb{T}$ and an inner function $\theta$ associated with $\mu$ (see [9, Sections 4-5]).

Let $\mu$ be a measure on $\mathbb{T}$ singular relative to the Lebesgue measure $m$. Define the function $\theta$ by

$$\frac{1+\theta(z)}{1-\theta(z)} = \int_{\mathbb{T}} \frac{1+\bar{\xi}z}{1-\bar{\xi}z}\, d\mu(\xi), \qquad z \in \mathbb{D}. \tag{3}$$

The measure $\mu$ is concentrated on the set where angular boundary limits of $\theta$ exist and are equal to 1. For $u \in L^2(\mu)$ set

$$(\Omega u)(z) = (1-\theta(z)) \int_{\mathbb{T}} \frac{u(\xi)d\mu(\xi)}{1-\bar{\xi}z}. \tag{4}$$

It is well known [12] that $\Omega$ is a unitary operator from $L^2(\mu)$ to $K_\theta$. Moreover, angular boundary values of the function $\Omega u$ exist and coincide with $u$ $\mu$-almost everywhere [16]. Analogs of these results for the spaces of vector-valued functions may be found in [14].

In this section $V$ is the operator of multiplication by the independent variable $\xi$ on $L^2(\mu)$. Then, for $h = \Omega u$, $u \in L^2(\mu)$, we have

$$(\Omega V \Omega^* h)(z) - zh(z) = (1-\theta(z)) \int_{\mathbb{T}} \xi u(\xi) d\mu(\xi).$$

Since $\int_{\mathbb{T}} \xi u(\xi) d\mu(\xi) = (u, \bar{\xi})_{L^2(\mu)} = (h, \Omega\bar{\xi})_{K_\theta}$, we obtain

$$\Omega V \Omega^* h = zh + (h, g)(1-\theta), \quad h \in K_\theta,$$

where $g = \Omega\bar{\xi}$. Recall that $S$ denotes the shift operator on $H^2$, and define the operator $\tilde{S}$ on $H^2$ by

$$\tilde{S} = S + (\cdot, g)(1-\theta). \tag{5}$$

The operator $\tilde{S}$ differs from the multiplication by $z$ by a rank-one operator $\tilde{S} - S = (\cdot, g)(1-\theta)$, and $\|\tilde{S} - S\| \leq 2\sqrt{\mu(\mathbb{T})}$. Thus, the operator $\tilde{S}$ has the properties (i)

– (iii) and the restriction of $\tilde{S}$ to $K_\theta$ is unitarily equivalent to the operator $V$ of multiplication by the independent variable on the space $L^2(\mu)$. Thus, $\tilde{S}$ is unitarily equivalent to $S \oplus V$.

In what follows we work with measures $\mu$ satisfying

$$\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^2} < \infty, \tag{6}$$

which, in particular, means that 1 is not an eigenvalue of $V$. Then $\theta(1)$ is well defined in the snce of nontangential boundary values and $\frac{1-\overline{\theta(1)}\theta}{1-z} \in K_\theta$; namely

$$\left\| \frac{1-\overline{\theta(1)}\theta}{1-z} \right\|_2^2 \le 4 \int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^2}. \tag{7}$$

Define the operator $\tilde{U}$ on $L^2$,

$$\tilde{U} = U + (\cdot, g)(1-\theta) - (\cdot, \bar{z})(1-\overline{\theta(1)}\theta). \tag{8}$$

Then $\tilde{U}$ has the properties (iv) – (vi) and is unitarily equivalent to $U \oplus V$.

Now we express the difference $\varphi_t(\tilde{S}) - \varphi_t(S)$ in terms of $\mu$ and $\theta$. On $\theta H^2$ the operator $\tilde{S}$ coincides with $S$, hence $\varphi(\tilde{S})$ coincides with $\varphi(S)$. Thus, we need to study the restriction of $\varphi(\tilde{S}) - \varphi(S)$ to $K_\theta$, which makes natural a consideration of the operator

$$X : L^2(\mu) \to H^2, \quad X = (\varphi(\tilde{S}) - \varphi(S))\Omega,$$

with $\Omega$ defined by (4). Take $u \in L^2(\mu)$. Then

$$(Xu)(z) = (1-\theta(z)) \int_{\mathbb{T}} \frac{\varphi_t(\xi) - \varphi_t(z)}{\xi - z} \xi u(\xi) d\mu(\xi).$$

Consider the integral operator $K : L^2(\mu) \to H^2$,

$$(Ku)(z) = \int_{\mathbb{T}} \frac{\varphi_t(\xi) - \varphi_t(z)}{\xi - z} u(\xi) d\mu(\xi).$$

If $K \in \mathfrak{S}_p$, then $X \in \mathfrak{S}_p$ as well, and we obviously have $\|\varphi_t(\tilde{S}) - \varphi_T(S)\|_{\mathfrak{S}_p} = \|X\|_{\mathfrak{S}_p} \le 2 \cdot \|K\|_{\mathfrak{S}_p}$.

It is almost immediate that $K \in \mathfrak{S}_2$ if and only if $\mu$ satisfies (6) (see [9] for details). In [9] an analogous condition sufficient for the inclusion $K \in \mathfrak{S}_1$ was found: if, for some $q > 8$,

$$\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^{\,q}} < \infty, \tag{9}$$

then $K \in \mathfrak{S}_1$. Since $\mu(\{1\}) = 0$, we can always multiply $\mu$ by a weight to achieve (9), and this proves Theorem 1.1 for the case of multiplicity 1. It should be mentioned that recently we were able to show that (9) implies $K \in \mathfrak{S}_1$ already for any $q > 3$. However, the proof in [9] is simple and elementary, and also leads to the required result.

Consider the case of multiplicity $N$, $1 < N \le \infty$, that is, the case when $V$ is the direct sum of operators of multiplication by $z$ in spaces $L^2(\mu_n)$, $1 \le n \le N$. Let $\theta_n$

be defined by (3) with $\mu = \mu_n$. Multiplying, if necessary, by positive weights, we may assume that

$$\sum_n \left( \int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1-\xi|^{\,2}} \right)^{1/2} < \infty;$$

in this case the product $\prod_n \theta_n$ converges (may be, up to some constant factors) to an inner function $\theta$. Define the operator $\tilde{S}$ on $H^2$ by

$$\tilde{S} = S + \sum_n (\cdot, \hat{\theta}_n g_n)\, \hat{\theta}_n \cdot (1-\theta_n), \tag{10}$$

Then the operator $\tilde{S}$ is diagonal with respect to the decomposition $H^2 = \sum \oplus \hat{\theta}_n K_{\theta_n} \oplus \theta H^2$. Indeed, on each subspace $\hat{\theta}_n K_{\theta_n}$, $\tilde{S}$ is a unitary transplantation of multiplication by $z$ on $L^2(\mu_n)$: take $f = \Omega_n u \in K_{\theta_n}$; then, by a direct computation, $V(\hat{\theta}_n f) = \hat{\theta}_n \Omega_n(zu) \in \hat{\theta}_n K_{\theta_n}$; on $\theta H^2$, $\tilde{S}$ coincides with $S$.

A condition sufficient for the inclusion $\varphi_t(\tilde{S}) - \varphi_t(S) \in \mathfrak{S}_1$ is that

$$\sum_n \left( \int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1-\xi|^{\,q}} \right)^{1/2} \tag{11}$$

for some $q > 8$, and for the trace class norm of $\varphi_t(\tilde{S}) - \varphi_t(S)$ we have

$$\|\varphi_t(\tilde{S}) - \varphi_t(S)\|_{\mathfrak{S}_1} \le M_q \cdot t \cdot \sum_n \left( \int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1-\xi|^{\,q}} \right)^{1/2}. \tag{12}$$

## 3. Some formulas for the model on the real line

When we study unitary groups, sometimes it is more convenient to work on the real line instead of the unit circle. We will then work with an analog of the construction presented in Section 2, where we considered the space $L^2 = L^2(\mathbb{T}, m)$. The space $K_\theta$ is the orthogonal difference of two spaces, $H^2$ and $\theta H^2$, that are invariant under the shift operator $S$. Now we describe the function model on the real line and its connection with the model on the unit circle. For the model on the real line, the model space is $L^2(\mathbb{R})$ with the usual Lebesgue measure.

The unitary equivalence between the two models is established by the unitary operator sending a function $u \in L^2(\mathbb{R})$ to the function $f \in L^2$, $f(z) = \frac{2i\sqrt{\pi}}{1-z} u(i\frac{1+z}{1-z})$. The inverse operator acts from $L^2$ to $L^2(\mathbb{R})$ and takes a function $f \in L^2$ to the function on the real line, whose value at a point $x \in \mathbb{R}$ is $\frac{1}{\sqrt{\pi}(x+i)} f(\frac{x-i}{x+i})$. This operator sends $H^2$ to the Hardy space $H^2(\mathbb{C}_+)$ in the upper half-plane, $H^2_-$ to $H^2(\mathbb{C}_-)$, and $\theta H^2$ to $\Theta H^2(\mathbb{C}_+)$, where $\Theta$ is the transplantation of the inner function $\theta$ to the upper half-plane, $\Theta(s) = \theta(\frac{s-i}{s+i})$.

With a measure $\mu$ on $\mathbb{T}$ we associate a measure $\nu$ on $\mathbb{R}$ by

$$d\mu(\xi) = \frac{d\nu(x)}{\pi(1+x^2)}, \tag{13}$$

where $\xi = \frac{x-i}{x+i}$. The Lebesgue measure on $\mathbb{R}$ then corresponds to the measure $m$ on $\mathbb{T}$. If an inner function $\theta$ in the unit disk is determined by the relation

$$\frac{1+\theta(z)}{1-\theta(z)} = \int_{\mathbb{T}} \frac{1+\bar\xi z}{1-\bar\xi z}\, d\mu(\xi),$$

then for $\nu$ and $\Theta$ after the substitution $z = \frac{s-i}{s+i}$ we obtain

$$\frac{1-\Theta(s)}{1+\Theta(s)} = \frac{2}{\pi i}\int_{\mathbb{R}} \left(\frac{1}{x-s} - \frac{x}{x^2+1}\right) d\nu(x). \tag{14}$$

For measures $\mu$ and $\nu$ related by (13), condition (6) is equivalent to

$$\nu(\mathbb{R}) < \infty;$$

in this case the limit $\Theta(\infty) = \lim_{y\to+\infty}\Theta(iy) \in \mathbb{T}$ is well defined, $1-\overline{\Theta(\infty)}\Theta \in L^2(\mathbb{R})$ and, similarly to (7),

$$\|1-\overline{\Theta(\infty)}\Theta\|_{L^2(\mathbb{R})} = |1-\Theta(\infty)|\sqrt{\nu(\mathbb{R})} \le 2\sqrt{\nu(\mathbb{R})}.$$

Starting with a family of measures $(\nu_n)$ on the real line, construct inner functions $(\Theta_n)$ according to (14), where $n$ runs over either the set $\mathbb{N}$ or its finite segment $1,\dots,N$. Like on the model on the circle, for a natural number $n$ we define

$$\hat\Theta_n = \prod_{k=1}^{n-1} \overline{\Theta_k(\infty)}\Theta_k.$$

If the measures $\nu_n$ satisfy

$$\sum_n \big(\nu_n(\mathbb{R})\big)^{1/2} < \infty,$$

then the series

$$\sum_n \hat\Theta_n(1-\overline{\Theta_n(\infty)}\Theta_n)$$

converges in the norm of the space $L^2(\mathbb{R})$. The partial sums have the form $1-\hat\Theta_n$. Hence the limit function can be written as $1-\Theta$, where $\Theta$ is an inner function with $\Theta(\infty)=1$, and $\Theta$ coincides with the product of the functions $\Theta_n$ up to a multiplicative unimodular constant.

## 4. The structure of unitary dilations

Now we return back to the unitary dilation $\tilde U$ of the operator $\tilde S$ defined by (10) in the general case of singular unitary operator $V$. For the particular case of multiplicity 1, $\tilde U$ was defined by formula (8). Here we assume that the measures $\mu_n$ satisfy condition (9) for some $q > 8$, as in Section 2. The function $\theta$ was constructed there; it coincides with the product of all $\theta_n$ multiplied by unimodular constants, so that we have $\frac{1-\theta}{1-z} \in L^2$.

Define the operator $\tilde U$ on $L^2$ by

$$\tilde U = U + \sum_n (\cdot, \hat\theta_n g_n)\hat\theta_n \cdot (1-\theta_n) - (\cdot, \bar z)(1-\theta).$$

It is easy to check properties (iv) – (vi). The expression $\sum_n \left(\int_{\mathbb{T}} \frac{d\mu_n(\xi)}{|1-\xi|^2}\right)^{1/2}$ can be made small, and then properties $a_2), b_2), c_2)$ required in Theorem 1.3 follow from the construction. Thus, to prove the theorem, it remains to find a condition on the measures $\mu_n$ analogous to (9) and sufficient for the property $d_2$).

To prove Theorems 1.2 and 1.3, we need to investigate whether $\varphi_t(\tilde{U}) - \varphi_t(U)$ may belong to the classes $\mathfrak{S}_p$. Considering the matrix of $\varphi_t(\tilde{U}) - \varphi_t(U)$ with respect to the decomposition $L^2 = H^2_- \oplus K_\theta \oplus \theta H^2$ one can show that all entries but one may belong to $\mathfrak{S}_1$. Indeed, for the matrix block $K_\theta \oplus \theta H^2 \to K_\theta \oplus \theta H^2$ the statement follows from Theorem 1.1, namely, estimate (12) provides sufficient conditions for the restriction to $H^2$ of the difference $\varphi_t(\tilde{U}) - \varphi_t(U)$ to belong to the trace class. If we consider the adjoint operators, we see that the same conditions of Theorem 1.1 imply that the block $H^2_- \oplus K_\theta \to H^2_- \oplus K_\theta$ is of trace class. By construction, the entry $H^2 \to H^2_-$ is zero. Thus, the remaining entry is that corresponding to the operator $H^2_- \to \theta H^2$. Futhermore, on the space $\bar{\varphi}_t H^2_-$ both operators $\varphi_t(\tilde{U}), \varphi_t(U)$ act as multiplication by $\varphi_t$, hence $\varphi_t(\tilde{U}) - \varphi_t(U) = 0$ on $\bar{\varphi}_t H^2_-$. It remains to consider the action of $\varphi_t(\tilde{U}) - \varphi_t(U)$ on the subspace $\bar{\varphi}_t H^2 \ominus H^2 = \bar{\varphi}_t K_{\varphi_t}$.

Let $Q : \bar{\varphi}_t K_{\varphi_t} \to H^2$ be the restriction of $\varphi_t(\tilde{U}) - \varphi_t(U)$ to $\bar{\varphi}_t K_{\varphi_t}$.

**Lemma 4.1.** *For $v \in K_{\varphi_t}$ we have $Q(\bar{\varphi}_t v) = -(1-\theta)v$.*

**Proof.** For $u \in H^2_-$, we have

$$P_+\varphi_t(\tilde{U})u = \theta \cdot P_+(\varphi_t u), \quad P_-\varphi_t(\tilde{U})u = P_-(\varphi_t u). \tag{15}$$

which is actually true for any function $\varphi \in H^\infty$ in place of $\varphi_t$. Indeed, it is easy to check this property for functions $\varphi(z) = z^n$, $n > 0$, if the elements of the basis $z^m$, $m < 0$, are taken as $u$; therefore, by linearity and $*$-weak continuity, it holds for all $\varphi \in H^\infty$ as well.

Since $\varphi_t(U)u = \varphi_t u$, it follows from (15) that

$$\big(\varphi_t(\tilde{U}) - \varphi_t(U)\big)u = (\theta - 1) \cdot P_+(\varphi_t u), \quad u \in H^2_-,$$

and it remains to plug $u = \bar{\varphi}_t v$. □

Here it is convenient to pass to the model on the real line, see Section 3. From the subspace $K_{\varphi_t}$ on the unit circle we obtain the space $H^2(\mathbb{C}_+) \ominus e^{itz}H^2(\mathbb{C}_+)$ on the real line, which after multiplication by the function $e^{itz/2}$ becomes the *Paley–Wiener space* $\mathcal{PW}_{t/2}$. Recall that the space $\mathcal{PW}_a$, $a > 0$, consists of all entire functions of exponential type at most $a$, whose restrictions to $L^2(\mathbb{R})$ are square summable.

By Lemma 4.1, to decide whether the operator $Q$ may belong to a class $\mathfrak{S}_p$, we need to answer the following question: *for which inner functions $\Theta$ does the operator $f \mapsto (1-\Theta)f$ on $H^2(\mathbb{C}_+)\ominus e^{itz}H^2(\mathbb{C}_+)$ belong to the class $\mathfrak{S}_p$?* Clearly, this is equivalent to the fact that the embedding of the Paley–Wiener space $\mathcal{PW}_{t/2}$ into the $L^2$-space on $\mathbb{R}$ with the weight $|1-\Theta|^2$ belongs to $\mathfrak{S}_p$. Now it is natural to use the following known result.

**Theorem 4.2.** (O.G. Parfenov, [15]) *The operator of embedding of the Paley–Wiener space $\mathcal{PW}_a$ (for any fixed $a > 0$) into the $L^2$-space with a weight $w$ on the real line belongs to the class $\mathfrak{S}_p$, $p > 0$, if and only if*

$$\sum_k \left( \int_k^{k+1} w(x)dx \right)^{p/2} < \infty. \tag{16}$$

## 5. Applications of the Parfenov's criterion

Our aim now is to find conditions on the measure $\nu$ such that for the inner function $\Theta$ related to $\nu$ by (14) the weight $w = |1-\Theta|^2$ satisfies (16). Obviously, for $p = 2$ this is equivalent to $1 - \Theta \in H^2(\mathbb{C}_+)$, just like in [6, 7]. We show now that for $p = 1$ this is impossible, and this will complete the proof of Theorem 1.2.

**Proof of Theorem 1.2.** It suffices to prove that relation (16) cannot be fulfilled for the weight $w = |1-\Theta|^2$ with $p = 1$. From (16) and the Cauchy inequality we obtain

$$\int_{\mathbb{R}} |1-\Theta(t)|\,dt \le \sum_{k\in\mathbb{Z}} \left( \int_k^{k+1} |1-\Theta(t)|^2\,dt \right)^{1/2} < \infty.$$

We conclude that $1-\Theta$ is in the Hardy space $H^1$, which yields $\int_{\mathbb{R}} \big(1-\Theta(t)\big)\,dt = 0$. However, the last relation is impossible since $\mathrm{Re}\,(1-\Theta) > 0$ almost everywhere on $\mathbb{R}$. $\square$

Now let $\mu$ be a singular measure on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} \frac{d\mu(\xi)}{|1-\xi|^q} < \infty$$

for some $q \in \mathbb{N}$, $q > 2$, which in terms of the measure $\nu$ on $\mathbb{R}$ associated with $\mu$ (13) looks like

$$\int_{\mathbb{R}} (1+|t|)^{q-2} d\nu(t) < \infty. \tag{17}$$

Construct $\theta$ by (3) and set $\Theta(s) = \theta\left(\frac{s-i}{s+i}\right)$; or, equivalently, use relation (14) to construct $\Theta$ by $\nu$.

**Lemma 5.1.** *If $\nu$ satisfies (17) and $\Theta$ is connected with $\nu$ by (14), then $\Theta$ satisfies Parfenov's condition*

$$\sum_{k\in\mathbb{Z}} \left( \int_k^{k+1} |1-\overline{\Theta(\infty)}\Theta(t)|^2 dt \right)^{p/2} < \infty \tag{18}$$

*with $p > q' = \frac{q}{q-1}$.*

**Proof.** Recall that for $u \in L^2(\mu)$, the operator $\Omega$,

$$(\Omega u)(z) = (1-\theta(z)) \int_{\mathbb{T}} \frac{u(\xi)d\mu(\xi)}{1-\bar{\xi}z},$$

is unitary from $L^2(\mu)$ onto $K_\theta$. It follows from the inclusion $(1-\xi)^{-q} \in L^1(\mu)$ that for each function $f \in K_\theta$ its derivative of order $q-1$, $f^{(q-1)}$, has a finite radial limit at 1 (for related results see [13]). It is easy to show that the functional $f^{(q-1)}(1)$ on $K_\theta$ is essentially of the form

$$f^{(q-1)}(1) = \int_{\mathbb{T}} \left( \frac{1-\overline{\theta(1)}\theta(\xi)}{1-\xi} \right)^q f(\xi)\,dm(\xi)$$

and we conclude that

$$\int_{\mathbb{T}} \left| \frac{1-\overline{\theta(1)}\theta(\xi)}{1-\xi} \right|^{2q} dm(\xi) < \infty.$$

(Strictly speaking, the function $\left(\frac{1-\overline{\theta(1)}\theta(\xi)}{1-\xi}\right)^q$ is not in $K_\theta$, but the norm of its projection onto $\theta H^2$ is smaller, than for the projection to $K_\theta$.) Changing the variables to the real line we get

$$C := \int_{\mathbb{R}} |1 - \overline{\Theta(\infty)}\Theta(t)|^{2q}(|t|+1)^{2q-2}dt < \infty.$$

By the Hölder inequality,

$$\begin{aligned}\int_k^{k+1} |1 - \overline{\Theta(\infty)}\Theta(t)|^2 dt \leq \\ &\leq \left(\int_k^{k+1} |1 - \overline{\Theta(\infty)}\Theta(t)|^{2q}(|t|+1)^{2q-2}dt\right)^{1/q}\left(\int_k^{k+1}\frac{dt}{(|t|+1)^2}\right)^{1/q'} \\ &\leq \frac{C^{1/q}}{(|k|+1)^{2/q'}}.\end{aligned}$$

Let $p > q'$. We obtain

$$\sum_{k\in\mathbb{Z}}\left(\int_k^{k+1} |1 - \overline{\Theta(\infty)}\Theta(t)|^2 dt\right)^{p/2} \leq C^{p/2q}\sum_{k\in\mathbb{Z}}\frac{1}{(|k|+1)^{p/q'}} < \infty,$$

and (18) follows. □

**Proof of Theorem 1.3.** As in the proof of Theorem 1.1, take measures $\mu_n$ on the unit circle such that $V$ is unitarily equivalent to the direct sum of the operators of multiplication by $z$ in the spaces $L^2(\mu_n)$. Multiplying the measures by appropriate weights, we get measures satisfying

$$\sum_n \left(\int_{\mathbb{T}}\frac{d\mu_n(\xi)}{|1-\xi|^q}\right)^{1/q} < \infty \tag{19}$$

for all $q > 2$. Then (12) is fulfilled, and, similarly to the proof of Theorem 1.1, the question is reduced to whether the operator $Q$ may belong to the classes $\mathfrak{S}_p$. This is conveniently reformulated in terms of the Paley–Wiener spaces $\mathcal{PW}_t$. To complete the proof, we will show that the embedding operator of $\mathcal{PW}_t$ into the $L^2$-space on $\mathbb{R}$ with the weight $|1 - \overline{\Theta(\infty)}\Theta|^2$ belongs to $\mathfrak{S}_p$, $p > 1$, for any $t > 0$.

In terms of measures $\nu_n$, for a fixed $q$, relation (19) may be rewritten as

$$\sum_n \left(\int_{\mathbb{R}}(1+|t|)^{q-2}\, d\nu_n(t)\right)^{1/q} < \infty. \tag{20}$$

Consider the class $\mathcal{H}_{2,p}$ of functions from $L^2(\mathbb{R})$, for which the norm

$$\|f\|_{2,p} = \left(\sum_k\left(\int_k^{k+1}|f(x)|^2dx\right)^{p/2}\right)^{1/p}$$

is finite, where $p > q' = \frac{q}{q-1}$. It follows from (18) and (20) that the series $\sum_n \tilde{\Theta}_n(1 - \overline{\Theta_n(\infty)}\Theta_n)$ converges in $\mathcal{H}_{2,p}$-norm. Hence the limit function $1 - \Theta$ belongs to $\mathcal{H}_{2,p}$ .

Since this is true for all $q > 2$, we obtain

$$\sum_k \left( \int_k^{k+1} |1 - \overline{\Theta(\infty)}\Theta(x)|^2 dx \right)^{p/2} < \infty$$

whenever $1 < p < 2$, as required. $\square$

**Proof of Theorem 1.4.** The statement immediately follows from the proof of Theorem 1.3 after passing to the model of the groups of shifts on the real line. Indeed, for the construction presented in the proof of Theorem 1.3, the differences of the elements of the semigroups on the space $H_+$ are of trace class, as required. $\square$

**Proof of Theorem 1.5.** We give a brief sketch of the proof. Take a unitary operator $V$. Since 1 is not its eigenvalue, $V$ may be represented as a direct sum of unitary operators $V_n$, whose spectra are separated from the point 1. Since the functions $\varphi_t$ are smooth outside of any neighborhood of the point 1, the operators $V_n$ may be slightly perturbed so that the resulting operators $V_n'$ will be unitary with singular spectral measures, and the differences $\varphi_t(V_n) - \varphi_t(V_n')$ will be in $\mathfrak{S}_p$ for all $p > 1$ with arbitrarily small norms for $1 \le t \le 2$. It remains to apply the procedure from the proof of theorem 1.3 to the direct sum of the operators $V_n'$. $\square$

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