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Schemes of a finite projective plane and their extensions

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Abstract

There are several schemes (coherent configurations) related to a finite projective plane $\mathcal{P}$. In this paper we construct a new scheme which contains in some sense all of them. It turns out that this scheme is nothing else than the 2-extension of the non-homogeneous scheme of $\mathcal{P}$. This fact has interesting applications in the theory of multi-dimensional extensions of schemes and similarities.

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1 Introduction

A *projective plane* is a triple consisting of a set of points, a set of lines, and an incidence relation between their elements, the defining property of which is the following: any two distinct points (resp. lines) are incident to a unique line (resp. point). To avoid degenerate cases usually a certain non-degeneracy condition is also added. An isomorphism of projective planes is a bijection taking the points (resp. lines) of one plane to the points (resp. lines) of the other and preserving the incidence relation.

In this paper we deal with finite projective planes. For such a plane there exists a positive integer q such that each point (resp. line) is incident to exactly $q+1$ lines (resp. points), and the number of points equals the number of lines and equals q^2+q+1 (see [14]). The number q is called the *order* of the projective plane. For any prime power q, the projective plane the points and the lines of which are the lines and the planes of a 3-dimensional linear space over a Galois field of order q and the incidence is defined by inclusion, is called a *Galois plane* of order q.[1] The Galois planes are precisely the finite projective planes for which the Desargues theorem is true. Though there is a lot of non-Desarguesian finite projective planes, up to now it is not known if there exists such a plane of prime order.

There are several schemes (coherent configurations) related to a finite projective plane: scheme of type $\left[\begin{smallmatrix} 2 & 2 \\ & 2 \end{smallmatrix}\right]$ (non-homogeneous scheme), scheme of a distance-regular graph of diameter 3 (homogeneous scheme) and schemes of flags and antiflags [13, 1, 12, 8]. In this paper we construct a new scheme $\mathcal{C}$ which contains in some sense all of them. The underlying set of it equals V^2 where V is the set of elements (the points and the lines) of the plane $\mathcal{P}$ in question. (In the Desarguesian case $\mathcal{C}$ is some fusion of the scheme corresponding to the componentwise action of $\mathrm{Aut}(\mathcal{P})$ on V^2.) It turns out that $\mathcal{C}$ is nothing else than the 2-extension of the non-homogeneous scheme of $\mathcal{P}$ in sense of [4]. This fact has interesting applications in the theory of multi-dimensional extensions of schemes and similarities.

The basis relations of the new scheme mentioned above will be constructed by means of special complete colored graphs Γ with vertex sets Ω such that $\Xi \subset \Omega \subset \Xi \cup \Xi^{(2)}$ where Ξ is the set consisting of four distinct commuting variables $\xi_1, \xi_2, \xi_3, \xi_4$, and $\Xi^{(2)}$ is the symmetric square of Ξ the elements of which are written as products of two distinct variables. The colors are called types and come from the types of elements and relations between them in a projective plane. Each vertex of Γ has type $\mathfrak{p}$ (point) or $\mathfrak{l}$ (line), whereas each edge has type $\mathfrak{e}$ (equality), $\mathfrak{e}'$ (non-equality), $\mathfrak{i}$ (incidence) or $\mathfrak{i}'$ (non-incidence). We also assume some natural compatibility conditions: (a) the edge between vertices of the same (resp. distinct) types must have type $\mathfrak{e}$ or $\mathfrak{e}'$ (resp. $\mathfrak{i}$ or $\mathfrak{i}'$), (b) if an edge between variables ξ and η is of type $\mathfrak{e}'$, then $\xi\eta \in \Omega$ and the type of the edge between ξ and $\xi\eta$) is of type $\mathfrak{i}$. Any such graph Γ is called a Ξ-*configuration*.

Let $x = (x_1, \ldots, x_4)$ where $x_i \in V$ for all i. We associate to x a Ξ-configuration as follows. Set Ω to be the union of Ξ and the set of all $\xi_i\xi_j \in \Xi^{(2)}$ such that x_i and x_j are distinct points or distinct lines in $\mathcal{P}$. The type of a vertex $\xi \in \Omega$ (resp. an edge between ξ and η) is defined to be the

[1]The "plane of order 1" (triangle) is also treated here as a Galois plane.

type of the element $f(\xi)$ (resp. the relation between $f(\xi)$ and $f(\eta)$) in the plane $\mathcal{P}$ where $f(\xi) = x_i$ for $\xi = \xi_i$ and $f(\xi)$ equals the unique element of $\mathcal{P}$ incident to both x_i and x_j for $\xi = \xi_i\xi_j$. The graph Γ constructed in such a way is denoted by $\mathrm{Der}(x) = \mathrm{Der}(x, \mathcal{P})$. For a Ξ-configuration Γ set

$$R(\Gamma, \mathcal{P}) = \{x \in V^4 : \ \mathrm{Der}(x) = \Gamma\}.$$

Clearly, when Γ runs over all Ξ-configurations, the nonempty sets of this kind form a partition Π of V^4 called the *derivative partition* associated with $\mathcal{P}$. Given a set $R \in \Pi$ the graph $\mathrm{Der}(x)$ obviously does not depend on the choice of $x \in R$. We denote it by $\mathrm{Der}(R)$ and set

$$\mathrm{Der}(\mathcal{P}) = \{\mathrm{Der}(R) : \ R \in \Pi\}.$$

Below classes of the partition $\mathcal{P}$ are identified with binary relations on V^2 by means of the natural bijection from V^4 to $V^2 \times V^2$ that takes (x_1, x_2, x_3, x_4) to $((x_1, x_2), (x_3, x_4))$.

Theorem 1.1 *Let $\mathcal{P}$ be a finite projective plane of order q and Π the derivative partition associated with $\mathcal{P}$. Then*

(1) *the pair (V^2, Π) is a scheme,*

(2) *the set $\mathrm{Der}(\mathcal{P})$ does not depend on $\mathcal{P}$ for a fixed q, and even does not depend on q for sufficiently large q,*

(3) *given $R, S, T \in \Pi$ the intersection number $c_{R,S}^T$ of the scheme (V^2, Π) equals $f(q)$ where $f(x)$ is a polynomial with integral coefficients depending only on $\mathrm{Der}(R)$, $\mathrm{Der}(S)$ and $\mathrm{Der}(T)$ (but not depending on $\mathcal{P}$).*

The Ξ-configurations belonging to $\mathrm{Der}(\mathcal{P})$ are described in the table of Appendix. It follows that $\mathrm{Der}(\mathcal{P})$ does not depend on q for $q \geq 3$, and the cardinality of Π equals 200, 400 and 416 for $q = 1$, $q = 2$ and $q \geq 3$ respectively. The reflexive relations of Π are given by the eight Ξ-configurations uniquely determined by their restrictions to the set $\{\xi_1, \xi_2\}$:

$$\mathfrak{p} \overset{\mathfrak{e}}{\text{——}} \mathfrak{p} \qquad \mathfrak{p} \overset{\mathfrak{e}'}{\text{——}} \mathfrak{p} \qquad \mathfrak{p} \overset{\mathfrak{i}}{\text{——}} \mathfrak{l} \qquad \mathfrak{p} \overset{\mathfrak{i}'}{\text{——}} \mathfrak{l}$$

$$\mathfrak{l} \overset{\mathfrak{e}}{\text{——}} \mathfrak{l} \qquad \mathfrak{l} \overset{\mathfrak{e}'}{\text{——}} \mathfrak{l} \qquad \mathfrak{l} \overset{\mathfrak{i}}{\text{——}} \mathfrak{p} \qquad \mathfrak{l} \overset{\mathfrak{i}'}{\text{——}} \mathfrak{p}$$

They give rise to eight homogeneity sets of the scheme (V^2, Π):

$$E_{\mathfrak{t}} = \{(x, y) \in V_{\mathfrak{t}} \times V_{\mathfrak{t}} : \ x \text{ is equal to } y\}, \quad E'_{\mathfrak{t}} = (V_{\mathfrak{t}} \times V_{\mathfrak{t}}) \setminus E_{\mathfrak{t}}, \tag{1}$$

$$I_{\mathfrak{t}} = \{(x, y) \in V_{\mathfrak{t}} \times V_{\mathfrak{t}'} : \ x \text{ is incident to } y\}, \quad I'_{\mathfrak{t}} = (V_{\mathfrak{t}} \times V_{\mathfrak{t}'}) \setminus I_{\mathfrak{t}} \tag{2}$$

where $\mathfrak{t} \in \{\mathfrak{p}, \mathfrak{l}\}$, $\mathfrak{t}'$ is defined from condition $\{\mathfrak{t}, \mathfrak{t}'\} = \{\mathfrak{p}, \mathfrak{l}\}$ and $V_{\mathfrak{p}}$ and $V_{\mathfrak{l}}$ are the points and the lines of $\mathcal{P}$. These eight sets considered as binary relations on V form a partition $\mathcal{R}$ of V^2 and the pair $(V, \mathcal{R})$ is just the scheme introduced by D. Higman (see [13]). We call $(V, \mathcal{R})$ and (V^2, Π) the (*non-homogeneous*) *scheme* and the *derivative scheme* of the projective plane $\mathcal{P}$ respectively. The mapping $x \mapsto (x,x)$ takes the basis relations of the former scheme to the relations $R(\Gamma,\mathcal{P})$ where Γ runs over Ξ-configurations obtained from the above eight Ξ-configurations by interchange ξ_2 and ξ_3. Thus the non-homogeneous scheme is isomorphic to the restriction of the derivative scheme to $E_{\mathfrak{p}} \cup E_{\mathfrak{l}}$. Analogously, one can see that the restrictions of the derivative scheme to $I_{\mathfrak{p}}$ and $I'_{\mathfrak{p}}$ are the scheme of flags (see [12]) and an extension of the scheme of antiflags respectively (see [8]).

From Theorem 1.1 it follows that for a sufficiently large q being the order of a finite projective plane $\mathcal{P}$, there exists a uniquely determined non-commutative algebra over $\mathbb{R}$ of dimension 416, the adjacency algebra of the derivative scheme of $\mathcal{P}$. The elements of the standard linear base of it are parametrized by Ξ-configurations from the set $\mathrm{Der}(\mathcal{P})$ not depending on q by statement (2) of the theorem; the structure constants are given by tensor $T(q)$ the entries of which are defined from statement (3). Since any prime power is the order of some projective plane, for every real q there exists the algebra with structure constant tensor $T(q)$. Moreover, this algebra has a structure of a generalized C-algebra in the sense of [2].

Theorem 1.1 has important applications to the theory of m-extensions of schemes and similarities developed in [7, 3, 4, 5]. The m-extension of a scheme $\mathcal{C}$ on V is by definition a special scheme on the V^m whereas the m-extension of a similarity between $\mathcal{C}$ and $\mathcal{C}'$ is a special similarity between their m-extensions (for details see Section 3). It was proved that

$$\mathcal{C} = \overline{\mathcal{C}}^{(1)} \le \ldots \le \overline{\mathcal{C}}^{(n)} = \ldots = \overline{\mathcal{C}}^{(\infty)}, \tag{3}$$

where $\overline{\mathcal{C}}^{(m)}$ is the m-closure of $\mathcal{C}$ (i.e. the restriction of the m-extension to the diagonal of V^m identified with V), $n = |V|$ and $\overline{\mathcal{C}}^{(\infty)}$ is the scheme of the group $\mathrm{Aut}(\mathcal{C})$, and

$$\mathrm{Sim}(\mathcal{C},\mathcal{C}') = \mathrm{Sim}_1(\mathcal{C},\mathcal{C}') \supset \ldots \supset \mathrm{Sim}_n(\mathcal{C},\mathcal{C}') = \ldots = \mathrm{Sim}_\infty(\mathcal{C},\mathcal{C}'), \tag{4}$$

where $\mathrm{Sim}_m(\mathcal{C},\mathcal{C}')$ is the set of all m-similarities from $\mathcal{C}$ to $\mathcal{C}'$ (i.e. similarities having m-extension) and $\mathrm{Sim}_\infty(\mathcal{C},\mathcal{C}')$ is the set of all similarities from $\mathcal{C}$ to $\mathcal{C}'$ induced by isomorphisms of these schemes. In fact, the starting point to this paper was to determine the 2-extension of the scheme of a projective plane and to find conditions for the latter scheme to be 2-closed. Generally, the problem of finding the m-extension of a scheme seems to be very hard. Apart from trivial cases it was completely solved only for cyclotomic schemes over a finite field [5]. Surprisingly, in our case the answer is given by the following statement.

Theorem 1.2 *Let $\mathcal{C}$ be a scheme of a finite projective plane $\mathcal{P}$ and $\widehat{\mathcal{C}}$ its 2-extension. Then*

(1) *$\widehat{\mathcal{C}}$ equals the derivative scheme of $\mathcal{P}$,*

(2) *the scheme $\mathcal{C}$ is 2-closed.*■

Let $\mathcal{C}$ and $\mathcal{C}'$ be schemes of finite projective planes $\mathcal{P}$ and $\mathcal{P}'$ of the same order with the sets of basis relations $\mathcal{R}$ and $\mathcal{R}'$ respectively. Then the bijection from $\mathcal{R}$ to $\mathcal{R}'$ preserving the types of basis relations defines a similarity $\varphi : \mathcal{C} \to \mathcal{C}'$. It is called the *canonical similarity* from $\mathcal{C}$ to $\mathcal{C}'$. By statement (2) of Theorem 1.1 we have $\mathrm{Der}(\mathcal{P}) = \mathrm{Der}(\mathcal{P}')$. So the mapping $\widehat{\varphi}$ from Π to Π' such that

$$R(\Gamma, \mathcal{P})^{\widehat{\varphi}} = R(\Gamma, \mathcal{P}'), \quad \Gamma \in \mathrm{Der}(\mathcal{P}),$$

where Π and Π' are the derivative partitions associated with $\mathcal{P}$ and $\mathcal{P}'$ respectively, is a bijection. Thus from statement (3) of Theorem 1.1 and statement (1) of Theorem 1.2 it follows that $\widehat{\varphi}$ defines a similarity from $\widehat{\mathcal{C}}$ to $\widehat{\mathcal{C}'}$ which is obviously the 2-extension of φ. This proves the following result.

Theorem 1.3 *The canonical similarity between the schemes of two finite projective planes of the same order, is in fact a* 2*-similarity.*$\blacksquare$

There are natural analogs of Theorems 1.1 and 1.3 for the *homogeneous scheme* of a projective plane $\mathcal{P}$ defined as follows. Let $\mathcal{C}$ be the scheme of the plane $\mathcal{P}$. Then it is easily seen that the group $\Phi = \mathrm{Sim}(\mathcal{C}, \mathcal{C})$ consists of two elements. Denote by φ the nontrivial one. Clearly, φ coincides with the canonical similarity from $\mathcal{C}$ to the scheme of the projective plane dual to $\mathcal{P}$. (So by Theorem 1.3 the similarity φ has the 2-extension $\widehat{\varphi}$.) The scheme $\mathcal{D} = \mathcal{C}^{\Phi}$ obtained from $\mathcal{C}$ by merging of Φ-orbits is called the *homogeneous scheme* of $\mathcal{P}$. One can see that $\mathcal{D}$ is the scheme of the incidence graph of $\mathcal{P}$ which is a bipartite distance-regular graph of diameter 3. If $\mathcal{C}'$ is the scheme of a projective plane $\mathcal{P}'$ of the same order as $\mathcal{P}$, then the canonical similarity from $\mathcal{C}$ to $\mathcal{C}'$ induces by restriction a similarity $\psi : \mathcal{D} \to \mathcal{D}'$ where $\mathcal{D}'$ is the homogeneous scheme of $\mathcal{P}'$. The similarity ψ being obviously the only element of $\mathrm{Sim}(\mathcal{D}, \mathcal{D}')$ is called the *canonical similarity* from $\mathcal{D}$ to $\mathcal{D}'$.

Theorem 1.4 *In the above notations we have:*

(1) $\widehat{\mathcal{D}} = \widehat{\mathcal{C}}^{\widehat{\Phi}}$ *where* $\widehat{\Phi} = \{1_{\widehat{\mathcal{C}}}, \widehat{\varphi}\}$,

(2) *the scheme* $\mathcal{D}$ *is* 2*-closed,*

(3) *the canonical similarity from* $\mathcal{D}$ *to the homogeneous scheme of another projective plane is in fact a* 2*-similarity.*

In studying permutation groups of rank 3 D. Higman [11] introduced for strongly regular graphs the notion of t-condition, which was generalized to arbitrary colored graphs in [9] (see also [4]). Below, by the colored graph of a scheme $(V, \mathcal{R})$ we mean a graph the vertex set and the set of colored classes of which are V and $\mathcal{R}$ respectively. By [4, Theorem 6.4] the colored graph of any 2-closed scheme satisfies the 6-condition. So from statement (2) of Theorem 1.4 we obtain the following result which in particular implies that there are infinitely many distance-regular graphs of diameter 3 that satisfy the 6-condition and are not distance-transitive.

Corollary 1.5 *The colored graph of the scheme (homogeneous or not) of any finite projective plane satisfies the* 6*-condition.*$\blacksquare$

In the framework of the theory of m-extensions the important invariants of a scheme $\mathcal{C}$ are the schurity number $t(\mathcal{C})$ and the separability number $s(\mathcal{C})$ introduced in [4] by means of inclusions (3) and (4) respectively (see Section 3). The rest of results is about these numbers in the case when $\mathcal{C}$ is the (homogeneous or not) scheme of a projective plane $\mathcal{P}$ of order q. We observe that by the Ostrom-Wagner theorem [14, Theorem 14.13] the plane $\mathcal{P}$ is a Galois plane iff its group of collineations acts 2-transitively on the points of $\mathcal{P}$. Thus $t(\mathcal{C}) = 1$ iff $\mathcal{P}$ is a Galois plane (cf. [4, Theorem 7.9]). Besides, one can see that $s(\mathcal{C}) = 1$ iff any projective plane of order q is isomorphic either to $\mathcal{P}$ or to its dual plane in the homogeneous case, and to $\mathcal{P}$ in the non-homogeneous case.

Next, from the definitions it follows that if $t(\mathcal{C}) = 2$ (resp. $s(\mathcal{C}) = 2$), then $\mathcal{C} < \overline{\mathcal{C}}^{(\infty)} = \overline{\mathcal{C}}^{(2)}$ (resp. $\mathrm{Sim}(\mathcal{C},\mathcal{C}') \supsetneqq \mathrm{Sim}_\infty(\mathcal{C},\mathcal{C}') = \mathrm{Sim}_2(\mathcal{C},\mathcal{C}')$ for some scheme $\mathcal{C}'$). However, if $\mathcal{C}$ is the non-homogeneous scheme of a projective plane, the latter contradicts statement (2) of Theorem 1.2 (resp. Theorem 1.3), whereas if $\mathcal{C}$ is the homogeneous one, it contradicts statement (2) of Theorem 1.4 (resp. statement (3) of Theorem 1.4). Thus we come to the following result.

Corollary 1.6 *The schurity and separability numbers of the scheme (homogeneous or not) of a projective plane cannot equal 2.*■

In [4, Theorem 7.9] it was proved that $s(\mathcal{C}) \le 6$ whenever $\mathcal{C}$ is a homogeneous scheme of a Galois plane of order q. The following theorem generalizes and strengthens this result. From Corollary 1.6 it follows that the new upper bound is attained whenever there exists a non-Galois plane of order q (there are infinitely many of such q, see [14]).

Theorem 1.7 *Let $\mathcal{C}$ be the scheme (homogeneous or not) of a Galois plane. Then $s(\mathcal{C}) \le 3$.*

In fact, proof of Theorem 1.1 occupies Sections 4–8. In the first two of them we introduce the concept of a configuration (generalizing that of Ξ-configuration) and study general properties of configurations as well as their embeddings to projective planes.[2] The proof proper is in Section 6. The key point is to show that the numbers of embeddings of special configurations in a projective plane of order q depend only on q; those numbers are in fact the intersection numbers of the derivative scheme. These configurations turn to be admissible in sense of Section 7 (Theorem 7.3). In Section 8 we prove that the embedding number of any admissible configuration can be expressed via those of admissible configurations of smaller size (Theorem 7.4). The proofs of Theorems 1.2, 1.4 and 1.7 are contained in Sections 9, 10 and 11 respectively. In Appendix we present for any projective plane the list of Ξ-configurations parameterizing the basis relations of the derivative scheme. To make the paper as far as possible self-contained we cite the background on schemes and their multi-dimensional extensions in Sections 2 and 3 (the details can be found in [2]).

A part of results related to homogeneous schemes of projective planes was announced in [6]. Concerning theory of finite projective planes we refer to [14].

Notation. As usual we denote by $\mathbb{Z}$ and $\mathbb{R}$ the ring of integers and the field of reals respectively.

[2]The idea to use configurations was inspired by paper [10].

The cardinality of a finite set V is denoted by $|V|$ and the diagonal of V^2 by $\Delta(V)$. For a (binary) relation R on V and $v \in V$ we set $R^T = \{(u,v) \in V^2 : (v,u) \in R\}$,

$$R_{in}(v) = \{u \in V : (u,v) \in R\}, \quad R_{out}(v) = \{u \in V : (v,u) \in R\}.$$

For relations $R, S \subset V^2$ we set

$$R \cdot S = \{(u,w) \in V^2 : (u,v) \in R, \ (v,w) \in S \text{ for some } v \in V\},$$

$$R \otimes S = \{((u_1,u_2),(v_1,v_2)) \in (V^2)^2 : (u_1,v_1) \in R, \ (u_2,v_2) \in S\}.$$

For an equivalence relation E on V we denote by V/E the set of its classes. If R is a relation on V, then we set

$$R_{V/E} = \{(X,Y) \in (V/E)^2 : R_{X,Y} \neq \emptyset\}$$

where $R_{X,Y} = R \cap (X \times Y)$. For $X \subset V$ we set $R_X = R_{X,X}$ and treat it as a relation on X.

The group of all permutations of V is denoted by $\mathrm{Sym}(V)$.

Each bijection $f : V \to V'$ $(v \mapsto v^f)$ naturally defines a bijection $R \mapsto R^f$ from the relations on V onto the relations on V' and a group isomorphism $g \mapsto g^f$ from $\mathrm{Sym}(V)$ onto $\mathrm{Sym}(V')$. For an equivalence relation E on V the bijection f induces a bijection $f_{V/E} : V/E \to V'/E'$ where $E' = E^f$.

For a positive integer m the ith coordinate of $x \in V^m$ is denoted by x_i.

For a projective plane $\mathcal{P}$ the sets of points and lines are denoted by $V_\mathfrak{p}$ and $V_\mathfrak{l}$ respectively where $\mathfrak{p}$ and $\mathfrak{l}$ are distinct symbols.

To determine the *type* of an element of $\mathcal{P}$ we introduce a function $t_{1,\mathcal{P}} : V \to \mathfrak{T}_1$ where $V = V_\mathfrak{p} \cup V_\mathfrak{l}$ and $\mathfrak{T}_1 = \{\mathfrak{p}, \mathfrak{l}\}$. By definition $t_{1,\mathcal{P}}(x) = \mathfrak{t}$ if and only if $x \in V_\mathfrak{t}$. The elements $\mathfrak{p}$ and $\mathfrak{l}$ are called *dual* to each other.

To determine the *type* of the relation between two elements of $\mathcal{P}$ we introduce the set $\mathfrak{T}_2 = \{\mathfrak{e}, \mathfrak{e}', \mathfrak{i}, \mathfrak{i}'\}$ of four distinct symbols and a function $t_{2,\mathcal{P}} : V^2 \to \mathfrak{T}_2$. By definition $t_{2,\mathcal{P}}(x,y)$ equals $\mathfrak{e}$ or $\mathfrak{e}'$ (resp. $\mathfrak{i}$ or $\mathfrak{i}'$) if and only if x, y are equal or non-equal elements of the same type (resp. incident or non-incident elements of distinct types).

When it does not lead to misunderstanding we will omit the subscript $\mathcal{P}$ in $t_{1,\mathcal{P}}$ and $t_{2,\mathcal{P}}$.

Given distinct elements x, y of $\mathcal{P}$ of the same type the unique element incident to both of them is denoted by xy.

2 Schemes and similarities

Let V be a finite set and $\mathcal{R}$ a partition of V^2 closed with respect to transposition. Denote by $\mathcal{R}^*$ the set of all unions of the elements of $\mathcal{R}$. A pair $\mathcal{C} = (V, \mathcal{R})$ is called a *coherent configuration* or a *scheme* on V if the diagonal $\Delta(V)$ of V^2 belongs to $\mathcal{R}^*$ and given $R, S, T \in \mathcal{R}$, the number

$$c_{R,S}(u,w) = |\{v \in V : (u,v) \in R, \ (v,w) \in S\}| \tag{5}$$

does not depend on the choice of $(u,w) \in T$. The elements of V, $\mathcal{R} = \mathcal{R}(\mathcal{C})$, $\mathcal{R}^* = \mathcal{R}^*(\mathcal{C})$ and the numbers (5) are called the *points*, the *basis relations*, the *relations* and the *intersection numbers* of $\mathcal{C}$, respectively; the latter are denoted by $c_{R,S}^T$. From the definition it follows $R \cdot S \in \mathcal{R}^*$ for all $R, S \in \mathcal{R}^*$. The number $\mathrm{rk}(\mathcal{C}) = |\mathcal{R}|$ is called the *rank* of $\mathcal{C}$.

The set V is the disjoint union of *homogeneity* sets of $\mathcal{C}$, i.e. those $X \subset V$ for which $\Delta(X) \in \mathcal{R}$. Given $R \in \mathcal{R}$ there exist uniquely determined homogeneity sets X and Y such that $R \subset X \times Y$. Set $d_{out}(R) = c_{R,R^T}^{\Delta(X)}$ and $d_{in}(R) = c_{R^T,R}^{\Delta(Y)}$. Then

$$d_{out}(R) = |R_{out}(x)|, \quad d_{in}(R) = |R_{in}(y)|, \qquad x \in X,\ y \in Y.$$

The scheme $\mathcal{C}$ is called *homogeneous* if V is a homogeneity set of it. By a *set of* $\mathcal{C}$ we mean any union of homogeneity sets. It is easily seen that X and Y are sets of $\mathcal{C}$ if and only if $X \times Y \in \mathcal{R}^*$.

Two schemes are called *isomorphic* if there exists a bijection between their point sets preserving the basis relations. Any such bijection is called an *isomorphism* of these schemes. The group of all isomorphisms of a scheme $\mathcal{C}$ contains a normal subgroup

$$\mathrm{Aut}(\mathcal{C}) = \{f \in \mathrm{Sym}(V) :\ R^f = R,\ R \in \mathcal{R}\}$$

called the *automorphism group* of $\mathcal{C}$. On the other hand, let $G \leq \mathrm{Sym}(V)$ be a permutation group and $\mathcal{R}$ the set of orbits of the componentwise action of G on V^2. Then $(V, \mathcal{R})$ is a scheme and we call it the *scheme of* G.

Given a set $X \subset V$ denote by $\mathcal{R}_X$ the set of all nonempty relations R_X, $R \in \mathcal{R}$. If X is a set of $\mathcal{C}$, then $\mathcal{C}_X = (X, \mathcal{R}_X)$ is a scheme. Given an equivalence relation E on V denote by $\mathcal{R}_{V/E}$ the set of all nonempty relations $R_{V/E}$, $R \in \mathcal{R}$. If $E \in \mathcal{R}^*$, then $\mathcal{C}_{V/E} = (V/E, \mathcal{R}_{V/E})$ is a scheme. The set of all equivalence relations $E \in \mathcal{R}^*$ is denoted by $\mathcal{E}$.

Given schemes $\mathcal{C}_1 = (V_1, \mathcal{R}_1)$ and $\mathcal{C}_2 = (V_2, \mathcal{R}_2)$ denote by $\mathcal{R}_1 \otimes \mathcal{R}_2$ the set of all relations $R_1 \otimes R_2$, $R_1 \in \mathcal{R}_1$, $R_2 \in \mathcal{R}_2$. Then the pair $\mathcal{C}_1 \otimes \mathcal{C}_2 = (V_1 \times V_2, \mathcal{R}_1 \otimes \mathcal{R}_2)$ is a scheme. It is called the *tensor product* of $\mathcal{C}_1$ and $\mathcal{C}_2$.

The set of all schemes on V is partially ordered by inclusion: namely, $\mathcal{C} \leq \mathcal{C}'$ iff $\mathcal{R}^* \subset (\mathcal{R}')^*$; in this case we call $\mathcal{C}'$ an *extension* of $\mathcal{C}$ and $\mathcal{C}$ a *fusion* of $\mathcal{C}'$. For sets $\mathcal{R}_1, \ldots, \mathcal{R}_s$ of binary relations on V we denote by $[\mathcal{R}_1, \ldots, \mathcal{R}_s]$ the smallest scheme $\mathcal{C}$ on V such that $\mathcal{R}_i \subset \mathcal{R}^*$ for all i; we omit the braces if $\mathcal{R}_i = \{R_i\}$ and write $\mathcal{C}_i$ instead of $\mathcal{R}_i$ if the latter is the set of basis relations of $\mathcal{C}_i$. For an equivalence relation E on V we set $\mathcal{C}_E = [\mathcal{C}, \{\Delta(X) :\ X \in V/E\}]$.

Schemes $\mathcal{C}$ and $\mathcal{C}'$ are called *similar* if

$$c_{R,S}^T = c_{R^\varphi,S^\varphi}^{T^\varphi}, \qquad R, S, T \in \mathcal{R}, \tag{6}$$

for some bijection $\varphi : \mathcal{R} \to \mathcal{R}'$, $R \mapsto R^\varphi$ called a *similarity* from $\mathcal{C}$ to $\mathcal{C}'$. The set of all similarities is denoted by $\mathrm{Sim}(\mathcal{C}, \mathcal{C}')$. Each isomorphism from $\mathcal{C}$ to $\mathcal{C}'$ induces in a natural way a similarity between these schemes. The set of all isomorphisms from $\mathcal{C}$ to $\mathcal{C}'$ inducing a similarity φ is denoted by $\mathrm{Iso}(\mathcal{C}, \mathcal{C}', \varphi)$.

The similarity φ induces bijections between the sets as well as relations of $\mathcal{C}$ and $\mathcal{C}'$ respectively for the notation of which we use the same letter. One can see that $\mathcal{E}^\varphi = \mathcal{E}'$, $V^\varphi = V'$ and

$$(R^T)^\varphi = (R^\varphi)^T, \quad d_{out}(R) = d_{out}(R^\varphi), \quad d_{in}(R) = d_{in}(R^\varphi), \qquad R \in \mathcal{R}. \tag{7}$$

Given $E \in \mathcal{E}$ the similarity φ induces a similarity $\varphi_{V/E} : \mathcal{C}_{V/E} \to \mathcal{C}'_{V'/E'}$ where $E' = E^\varphi$. One can see that given a group $\Phi \le \mathrm{Sim}(\mathcal{C}, \mathcal{C})$ the pair $\mathcal{C}^\Phi = (V, \mathcal{R}^\Phi)$ where $\mathcal{R}^\Phi = \{R^\Phi : \ R \in \mathcal{R}\}$ with $R^\Phi = \bigcup_{\varphi \in \Phi} R^\varphi$, is a scheme.

A scheme $\mathcal{C} = (V, \mathcal{R})$ is called *semiregular* if $d_{out}(R) \le 1$ and $d_{in}(R) \le 1$ for all $R \in \mathcal{R}$. For homogeneous schemes the first three statements of the following theorem were proved in [15].

Theorem 2.1 *Let $\mathcal{C} = (V, \mathcal{R})$ be a scheme and $E \in \mathcal{E}$. Suppose that the scheme $\mathcal{C}_{V/E}$ is semiregular. Then*

(1) $\mathcal{R}(\mathcal{C}_E) = \{R_{X,Y} : \ R \in \mathcal{R}, \ X, Y \in V/E, \ R_{X,Y} \ne \emptyset\}$,

(2) *given $g \in \mathrm{Aut}(\mathcal{C}_{V/E})$ the mapping $\psi_g : R_{X,Y} \mapsto R_{X^g,Y^g}$ is a similarity of $\mathcal{C}_E$,*

(3) $\mathcal{C} = (\mathcal{C}_E)^\Psi$ *where* $\Psi = \{\psi_g : \ g \in \mathrm{Aut}(\mathcal{C}_{V/E})\}$,

(4) *for any similarity $\varphi : \mathcal{C} \to \mathcal{C}'$ there exists a similarity $\varphi_E : \mathcal{C}_E \to \mathcal{C}'_{E'}$ extending φ where $E' = E^\varphi$. Moreover, given $f \in \mathrm{Iso}(\mathcal{C}_{V/E}, \mathcal{C}'_{V'/E'}, \varphi_{V/E})$ the similarity φ_E can be chosen so that $(\varphi_E)_{V/E}$ be induced by f.*

Proof. First we prove that given $R, S, T \in \mathcal{R}$ and $X, X', X'', Y, Y', Y'' \in V/E$ with $R_{X,Y} \ne \emptyset$ and $S_{X',Y'} \ne \emptyset$ we have

$$c_{R_{X,Y}, S_{X',Y'}}(u, v) = \delta_{X'',X} \delta_{Y,X'} \delta_{Y',Y''} c_{R,S}^T, \quad (u, v) \in T_{X'',Y''}. \tag{8}$$

Without loss of generality we assume that $X'' = X$, $Y = X'$ and $Y' = Y''$. Since $R_{X,Y} \ne \emptyset$ and $S_{X',Y'} \ne \emptyset$, the semiregularity of $\mathcal{C}_{V/E}$ implies that $(R_{V/E})_{out}(X) = \{Y\}$ and $(S_{V/E})_{in}(Y') = \{X'\}$ for some $X', Y \in V/E$. So

$$c_{R,S}(u, v) = c_{R_{X,Y}, S_{X',Y'}}(u, v), \quad (u, v) \in X \times Y'.$$

If $(u, v) \in T$, then the left-hand side equals $c_{R,S}^T$ and (8) follows.

Denote by $\mathcal{R}'$ the right-hand side of the equality in statement (1) of the theorem. Clearly, $\mathcal{R}'$ is a partition of V^2 closed with respect to transposition and $\Delta(V) \in (\mathcal{R}')^*$. So (8) implies that $\mathcal{C}' = (V, \mathcal{R}')$ is a scheme the intersection numbers of which are as follows:

$$c_{R_{X,Y}, S_{X',Y'}}^{T_{X'',Y''}} = \delta_{X'',X} \delta_{Y,X'} \delta_{Y',Y''} c_{R,S}^T. \tag{9}$$

Since obviously $\mathcal{C}' \ge \mathcal{C}$ and $\mathcal{C}_E \ge \mathcal{C}'$, we conclude that $\mathcal{C}' = \mathcal{C}_E$. Statement (1) is proved. Statement (2) immediately follows from (9) and the definition of ψ_g. Finally, by [4, Theorem 4.4] the automorphism group of a semiregular scheme acts transitively on any of its basis relations. So given $R \in \mathcal{R}$ we have

$$R = \bigcup_{(X,Y) \in R_{V/E}} R_{X,Y} = \bigcup_{g \in \mathrm{Aut}(\mathcal{C}_{V/E})} R_{X_0^g, Y_0^g}$$

where $X_0, Y_0 \in V/E$ with $R_{X_0,Y_0} \neq \emptyset$. This proves statement (3).

To prove statement (4) let $\varphi : \mathcal{C} \to \mathcal{C}'$ be a similarity. Then $\mathcal{C}'_{V'/E'}$ coincides with $\varphi_{V/E}$-image of $\mathcal{C}_{V/E}$ and hence is semiregular (see (7)). So $\mathrm{Iso}(\mathcal{C}_{V/E}, \mathcal{C}'_{V'/E'}, \varphi_{V/E}) \neq \emptyset$ by [4, Theorem 4.4]. Take $f \in \mathrm{Iso}(\mathcal{C}_{V/E}, \mathcal{C}'_{V'/E'}, \varphi_{V/E})$. Due to (9) the mapping $R_{X,Y} \mapsto R'_{X^f,Y^f}$, $R_{X,Y} \in \mathcal{R}(\mathcal{C}_E)$ induces a similarity from $\mathcal{C}_E$ to $\mathcal{C}'_{E'}$. Since the restriction of it to $\mathcal{C}$ coincides with φ, we are done.■

Let $\mathcal{P}$ be a projective plane of order q, $V = V_{\mathfrak{p}} \cup V_{\mathfrak{l}}$ the set of its elements and $\mathcal{R}$ the set of eight relations (cf. (1) and (2))

$$E_{\mathfrak{t}} = t_{2,\mathcal{P}}^{-1}(\mathfrak{e})_{V_{\mathfrak{t}},V_{\mathfrak{t}}}, \quad E'_{\mathfrak{t}} = t_{2,\mathcal{P}}^{-1}(\mathfrak{e}')_{V_{\mathfrak{t}},V_{\mathfrak{t}}}, \quad I_{\mathfrak{t}} = t_{2,\mathcal{P}}^{-1}(\mathfrak{i})_{V_{\mathfrak{t}},V_{\mathfrak{t}'}}, \quad I'_{\mathfrak{t}} = t_{2,\mathcal{P}}^{-1}(\mathfrak{i}')_{V_{\mathfrak{t}},V_{\mathfrak{t}'}}, \qquad \mathfrak{t} \in \mathfrak{T}_1$$

where $\mathfrak{t}'$ is the dual to $\mathfrak{t}$ (see Notation). Then it is easily seen that the pair $\mathcal{C} = (V, \mathcal{R})$ is a scheme on $2(q^2 + q + 1)$ points; it is called the *(non-homogeneous) scheme of* $\mathcal{P}$. This scheme has two homogeneity sets $V_{\mathfrak{p}}$ and $V_{\mathfrak{l}}$ and satisfies the following conditions with $X = V_{\mathfrak{p}}$, $Y = V_{\mathfrak{l}}$ and $R = I_{\mathfrak{p}}$, or $X = V_{\mathfrak{l}}$, $Y = V_{\mathfrak{p}}$ and $R = I_{\mathfrak{l}}$:

(P1) $\mathrm{rk}(\mathcal{C}) = 8$ and $\mathrm{rk}(\mathcal{C}_X) = \mathrm{rk}(\mathcal{C}_Y) = 2$,

(P2) $c_{R,R^T}^S = 1$ for some basis relation $R \subset X \times Y$ and $S = X^2 \setminus \Delta(X)$.

In fact, any scheme $\mathcal{C}$ with two homogeneity sets X and Y satisfying (P1) and (P2) is the scheme of some projective plane. We note that the degrees (d_{in} and d_{out}) of the relations $E_{\mathfrak{t}}$, $E'_{\mathfrak{t}}$, $I_{\mathfrak{t}}$ and $I'_{\mathfrak{t}}$ equal 1, $q^2 + q$, $q + 1$ and q^2 respectively where $\mathfrak{t} \in \{\mathfrak{p}, \mathfrak{l}\}$.

Let $\varphi : \mathcal{C} \to \mathcal{C}'$ be a similarity. Then from above and the properties of similarities it follows that $\mathcal{C}'$ is also the scheme of a projective plane of order q. Moreover, φ takes the basis relations of $\mathcal{C}$ either to the corresponding basis relations of $\mathcal{C}'$, or to the dual ones. Conversely, any of these two mappings induces a similarity from $\mathcal{C}$ to $\mathcal{C}'$ (the first of them is called a *canonical* one). In particular, $\Phi = \mathrm{Sim}(\mathcal{C}, \mathcal{C})$ is a group of order 2. The scheme $\mathcal{D} = \mathcal{C}^\Phi$ is called the *homogeneous scheme of* $\mathcal{P}$. As above any scheme similar to $\mathcal{D}$ is the homogeneous scheme of a projective plane of the same order. Moreover, given two homogeneous schemes of projective planes of the same order there exists a unique similarity between them called *canonical*.

3 Extensions of schemes and similarities

Let $\mathcal{C}$ be a scheme on V and m a positive integer. The *m-extension* of $\mathcal{C}$ is by definition the following scheme on V^m:

$$\widehat{\mathcal{C}}^{(m)} = [\mathcal{C}^m, \Delta_m] \tag{10}$$

where $\mathcal{C}^m$ is the m-fold tensor product of $\mathcal{C}$ and Δ_m is the diagonal of V^m. We observe that with the exception for trivial cases the m-extension is a non-homogeneous scheme for all $m \geq 2$. From the definition it follows that $\widehat{\mathcal{C}}^{(1)} = \mathcal{C}$ and $\widehat{\mathcal{C}}^{(m)}$ is a fusion of the scheme corresponding to the componentwise action of the group $\mathrm{Aut}(\mathcal{C})$ on V^m. Moreover,

$$\mathrm{Aut}(\mathcal{C}) \cong \mathrm{Aut}(\widehat{\mathcal{C}}^{(m)})_{\Delta_m} \cong \mathrm{Aut}(\widehat{\mathcal{C}}^{(m)}).$$

The intersection numbers of the m-extension of a scheme give rise to new isomorphism invariants of it. Namely, a similarity $\psi : \widehat{\mathcal{C}}^{(m)} \to \widehat{\mathcal{C}'}^{(m)}$ is called an *m-extension* of a similarity $\varphi : \mathcal{C} \to \mathcal{C}'$ if

(1) $(\Delta_m)^\psi = \Delta'_m$,

(2) $R^\psi = R^{\varphi^m}$ for all $R \in \mathcal{R}(\mathcal{C}^m)$

where $\varphi^m : \mathcal{C}^m \to (\mathcal{C}')^m$ is the m-fold tensor product of φ. From the definition it follows that each similarity has 1-extension coinciding with it. Moreover, for any m the existence of an m-extension implies its uniqueness. We note that not every similarity has m-extension. However, if φ is induced by some isomorphism, then φ has m-extension for all m.

Given a scheme $\mathcal{C}$ on V set

$$\overline{\mathcal{C}} = \overline{\mathcal{C}}^{(m)} = ((\widehat{\mathcal{C}}^{(m)})_\Delta)^{\delta^{-1}}, \tag{11}$$

where $\delta : v \mapsto (v, \ldots, v)$ is the diagonal mapping from V to V^m. The scheme $\overline{\mathcal{C}}$ is called the *m-closure* of $\mathcal{C}$; the scheme $\mathcal{C}$ is called *m-closed*, if $\mathcal{C} = \overline{\mathcal{C}}^{(m)}$. A similarity between two schemes is called an *m-similarity* if it has m-extension. The set of all m-similarities from $\mathcal{C}$ to $\mathcal{C}'$ is denoted by $\mathrm{Sim}_m(\mathcal{C}, \mathcal{C}')$. The following theorem was proved in [3].

Theorem 3.1 *For an arbitrary scheme $\mathcal{C}$ on n points the following statements hold:*

$$\mathcal{C} = \overline{\mathcal{C}}^{(1)} \le \ldots \le \overline{\mathcal{C}}^{(n)} = \ldots = \overline{\mathcal{C}}^{(\infty)}, \tag{12}$$

$$\mathrm{Sim}(\mathcal{C}, \mathcal{C}') = \mathrm{Sim}_1(\mathcal{C}, \mathcal{C}') \supset \ldots \supset \mathrm{Sim}_n(\mathcal{C}, \mathcal{C}') \ldots = \mathrm{Sim}_\infty(\mathcal{C}, \mathcal{C}'). \tag{13}$$

where $\overline{\mathcal{C}}^{(\infty)}$ is the scheme of the group $\mathrm{Aut}(\mathcal{C})$ and $\mathrm{Sim}_\infty(\mathcal{C}, \mathcal{C}')$ is the set of all similarities from $\mathcal{C}$ to $\mathcal{C}'$ induced by isomorphisms of these schemes.$\blacksquare$

A scheme $\mathcal{C}$ is called *m-schurian* if $\overline{\mathcal{C}}^{(m)} = \overline{\mathcal{C}}^{(\infty)}$; it is called *$m$-separable* if $\mathrm{Sim}_m(\mathcal{C}, \mathcal{C}') = \mathrm{Sim}_\infty(\mathcal{C}, \mathcal{C}')$ for all $\mathcal{C}'$. The integers

$$t(\mathcal{C}) = \min\{m : \ \mathcal{C} \text{ is } m\text{-schurian}\}, \quad s(\mathcal{C}) = \min\{m : \ \mathcal{C} \text{ is } m\text{-separable}\},$$

are called the *schurity number* and the *separability number* of $\mathcal{C}$ respectively. One can prove that $t(\mathcal{C}) = s(\mathcal{C}) = 1$ whenever the scheme $\mathcal{C}$ is semiregular [4, Theorem 4.4].

The following theorem to be used throughout the paper is straightforward from [4, Lemma 6.2]. Below given $R \subset V^2$ and $i, j \in \{1, \ldots, 2m\}$ we set $\mathrm{Cyl}_R(i, j) = \{x \in V^{2m} : \ (x_i, x_j) \in R\}$.

Theorem 3.2 *Let $\mathcal{C}$ be a scheme on V, $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}^{(m)}$ and $\overline{\mathcal{C}} = \overline{\mathcal{C}}^{(m)}$ where m is a positive integer. Then given a relation R of the scheme $\overline{\mathcal{C}}$ and $i, j \in \{1, \ldots, 2m\}$ the following statements hold:*

(1) *the set $\mathrm{Cyl}_R(i, j)$ is a relation of the scheme $\widehat{\mathcal{C}}$,*

(2) $\mathrm{Cyl}_R(i, j)^{\widehat{\varphi}} = \mathrm{Cyl}_{R^{\overline{\varphi}}}(i, j)$ *for any m-similarity $\varphi : \mathcal{C} \to \mathcal{C}'$ where $\widehat{\varphi} = \widehat{\varphi}^{(m)}$ and $\overline{\varphi} = \overline{\varphi}^{(m)}$.*$\blacksquare$

4 Configurations

Let Ω be a finite set. We will consider colored undirected graphs Γ on Ω with loops the vertices and the edges of which are labeled with elements of the sets $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively (see Notation). The edge set of Γ is identified with a symmetric binary relation $\Delta(\Omega) \subset E \subset \Omega^2$. The coloring is given by functions

$$t_1 : \Omega \to \mathfrak{T}_1, \quad t_2 : E \to \mathfrak{T}_2.$$

Thus Γ is represented by the triple (Ω, E, t) where $t = (t_1, t_2)$.

Definition 4.1 *The colored graph $\Gamma = (\Omega, E, t)$ is called a configuration on Ω if $\Delta(\Omega) \subset E_{\mathfrak{e}}$ and*

$$E_{\mathfrak{e}} \cup E_{\mathfrak{e}'} \subset (\Omega_{\mathfrak{p}} \times \Omega_{\mathfrak{p}}) \cup (\Omega_{\mathfrak{l}} \times \Omega_{\mathfrak{l}}), \qquad E_{\mathfrak{i}} \cup E_{\mathfrak{i}'} \subset (\Omega_{\mathfrak{p}} \times \Omega_{\mathfrak{l}}) \cup (\Omega_{\mathfrak{l}} \times \Omega_{\mathfrak{p}})$$

where $\Omega_{\mathfrak{t}} = t_1^{-1}(\mathfrak{t})$ for $\mathfrak{t} \in \mathfrak{T}_1$ and $E_{\mathfrak{t}} = t_2^{-1}(\mathfrak{t})$ for $\mathfrak{t} \in \mathfrak{T}_2$.

The configuration Γ is called *complete* if $E = \Omega^2$. Given a set $\Omega' \subset \Omega$ the configuration obtained by the restriction of Γ to Ω' is denoted by $\Gamma_{\Omega'}$.

Let $\mathcal{P}$ be a finite projective plane. Then there is a natural complete configuration on the set V of its elements where $t_1 = t_{1,\mathcal{P}}$ and $t_2 = t_{2,\mathcal{P}}$ (see Notation). Let $i_0 : \Omega_0 \to V$ be a mapping where $\Omega_0 \subset \Omega$.

Definition 4.2 *A mapping $i : \Omega \to V$ is called a strict Γ-extension of i_0 if $i|_{\Omega_0} = i_0$ and*

$$t_1(\xi) = t_{1,\mathcal{P}}(i(\xi)), \quad \xi \in \Omega, \quad \text{and} \qquad t_2(\xi, \eta) = t_{2,\mathcal{P}}(i(\xi), i(\eta)), \quad (\xi, \eta) \in E.$$

If the latter equality holds only if $t_2(\xi, \eta) \in \{\mathfrak{e}, \mathfrak{i}\}$, then the mapping i is called a non-strict Γ-extension.

The set and the number of all strict Γ-extensions (resp. non-strict Γ-extensions) of the mapping i_0 is denoted by $\mathrm{Emb}_{str}(\Gamma, i_0, \mathcal{P})$ and $\mathrm{e}_{str}(\Gamma, i_0, \mathcal{P})$ (resp. $\mathrm{Emb}(\Gamma, i_0, \mathcal{P})$ and $\mathrm{e}(\Gamma, i_0, \mathcal{P})$). If $\Omega_0 = \emptyset$, we omit i_0 and call the elements of the sets $\mathrm{Emb}_{str}(\Gamma, \mathcal{P})$ and $\mathrm{Emb}(\Gamma, \mathcal{P})$ strict and non-strict *embeddings* of the configuration Γ into the plane $\mathcal{P}$ respectively. Clearly, $\mathrm{Emb}_{str}(\Gamma, i_0, \mathcal{P}) \subset \mathrm{Emb}(\Gamma, i_0, \mathcal{P})$ and

$$\mathrm{Emb}_{str}(\Gamma, i_0, \mathcal{P}) \subset \mathrm{Emb}_{str}(\Gamma, \mathcal{P}), \qquad \mathrm{Emb}(\Gamma, i_0, \mathcal{P}) \subset \mathrm{Emb}(\Gamma, \mathcal{P})$$

for all i_0. Besides, if $\mathrm{Emb}_{str}(\Gamma, i_0, \mathcal{P}) \neq \emptyset$ (resp. $\mathrm{Emb}(\Gamma, i_0, \mathcal{P}) \neq \emptyset$), then i_0 is a strict (resp. non-strict) embedding of Γ_{Ω_0} into $\mathcal{P}$. We observe that the set $\mathrm{Emb}(\Gamma, i_0, \mathcal{P})$ does not change if we remove some edges of Γ belonging to the set $E' = (E_{\mathfrak{e}'} \cup E_{\mathfrak{i}'}) \setminus \Omega_0^2$. Below, the elements $\mathfrak{e}$ and $\mathfrak{e}'$, as well as $\mathfrak{i}$ and $\mathfrak{i}'$ of the set $\mathfrak{T}_2$ (see Notation) will be called *opposite* to each other

Theorem 4.3 *In the above notation, if $i_0 \in \mathrm{Emb}_{str}(\Gamma_{\Omega_0}, \mathcal{P})$, then*

$$\mathrm{e}_{str}(\Gamma, i_0, \mathcal{P}) = \sum_{S \subset E'} (-1)^{|S|} \, \mathrm{e}(\Gamma_S, i_0, \mathcal{P})$$

where Γ_S is obtained from Γ by replacing the type of any edge from S with its opposite.

Proof. From the definitions of strict and non-strict extensions it follows that

$$\mathrm{Emb}(\Gamma, i_0, \mathcal{P}) \setminus \mathrm{Emb}_{str}(\Gamma, i_0, \mathcal{P}) = \bigcup_{s \in E'} \mathrm{Emb}(\Gamma_{\{s\}}, i_0, \mathcal{P}).$$

Since $\mathrm{Emb}(\Gamma_S, i_0, \mathcal{P}) = \bigcap_{s \in S} \mathrm{Emb}(\Gamma_{\{s\}}, i_0, \mathcal{P})$ for all nonempty $S \subset E'$ and $\Gamma_\emptyset = \Gamma$, the required statement follows by the inclusion-exclusion principle.■

Let Γ be a configuration on Ω. The connectivity components of the graph $(\Omega, E_\mathfrak{e})$ induce a partition of Ω. Denote by $\overline{\Omega}$ the set of classes of this partition. Clearly, all elements of a class $A \in \overline{\Omega}$ have the same type; we denote it by $t_1(A)$. For a subset $\Omega_0 \subset \Omega$ we set

$$\overline{\Omega}_0 = \overline{\Omega}_{0,\Gamma} = \{A \in \overline{\Omega} : \ A \cap \Omega_0 \neq \emptyset\}.$$

The class containing an element (vertex) $\xi \in \Omega$ is denoted by $\overline{\xi}$.

Two elements $\xi, \eta \in \Omega$ (resp. classes $A, B \in \overline{\Omega}$) are called *incident* if $(\xi, \eta) \in E_\mathfrak{i}$ (resp. $(A \times B) \cap E_\mathfrak{i} \neq \emptyset$). For $\xi \in \Omega$ and $\Omega_0 \subset \Omega$ (resp. for $A \in \overline{\Omega}$ and $\overline{\Omega}_0 \subset \overline{\Omega}$) we set $N(\xi, \Omega_0)$ (resp. $N(A, \overline{\Omega}_0)$) to be the set of all elements from Ω_0 (resp. classes from $\overline{\Omega}_0$) incident to ξ (resp. to A); the cardinality of this set is denoted by $d(\xi, \Omega_0)$ (resp. $d(A, \overline{\Omega}_0)$). If $\Omega_0 = \Omega$ (resp. $\overline{\Omega}_0 = \overline{\Omega}$) we omit Ω_0 (resp. $\overline{\Omega}_0$).

Let Γ be a configuration on Ω, $\Omega_0 \subset \Omega$ and $\Gamma_0 = \Gamma_{\Omega_0}$. Then the natural injection from Ω_0 to Ω induces a mapping $f : \overline{\Omega}_{0,\Gamma_0} \to \overline{\Omega}_\Gamma$ that preserves the incidence of classes. We say that Ω_0 is a Γ-*subset* of Ω if f is injective and classes A_0, B_0 of Γ_0 are incident whenever the classes $f(A_0)$, $f(B_0)$ of Γ are incident. The configuration Γ is called *final* with respect to Ω_0 if $\overline{\Omega}_\Gamma = \overline{\Omega}_{0,\Gamma}$ and the configuration Γ_0 is complete.

Lemma 4.4 *Let $\Gamma = (\Omega, E, t)$ be a final configuration with respect to a set $\Omega_0 \subset \Omega$, and $i_0 \in \mathrm{Emb}_{str}(\Gamma_0, \mathcal{P})$ where $\Gamma_0 = \Gamma_{\Omega_0}$ and $\mathcal{P}$ is a projective plane of order q. Then* $\mathrm{e}(\Gamma, i_0, \mathcal{P}) = 0$ *or* 1, *and the latter holds if and only if Ω_0 is an Γ-subset of Ω.*

Proof. Clearly, any non-strict Γ-extension of i_0 maps all elements of any class of Γ to the same element of $\mathcal{P}$. Thus the first statement follows from the finality of the configuration Γ. To prove the second statement first suppose that i is a non-strict Γ-extension of i_0. Let A_0, B_0 be classes of Γ_0 and $A = f(A_0)$, $B = f(B_0)$. If A and B are equal (resp. incident), then so are the elements $i(A) = i_0(A_0)$ and $i(B) = i_0(B_0)$ of $\mathcal{P}$. Since the configuration Γ_0 is complete, this implies that the classes A_0 and B_0 are equal (resp. incident). Conversely, let Ω_0 be a Γ-subset of Ω. Since Γ is final, the set $A_0 = A \cap \Omega_0$ is nonempty for all $A \in \overline{\Omega}_\Gamma$. From the injectivity of f it follows that A_0 is a class of Γ_0. Define a mapping $i : \Omega \to V$ where V is the set of elements of $\mathcal{P}$, by setting $i(\alpha) = i_0(A_0)$ where A_0 is as above with A being the class of Γ containing α. Clearly, $i|_{\Omega_0} = i_0$. Next, if $(\alpha, \beta) \in E_\mathfrak{e}$ (resp. $(\alpha, \beta) \in E_\mathfrak{i}$), then the classes A and B are equal (resp. incident) where A and B are the classes of Γ containing α and β respectively. Since Ω_0 is a Γ-subset of Ω, this implies that the classes A_0 and B_0 are equal (resp. incident). So the elements $i(\alpha) = i_0(A_0)$ and $i(\beta) = i_0(B_0)$ of $\mathcal{P}$ are equal (resp. incident). Thus $i \in \mathrm{Emb}(\Gamma, i_0, \mathcal{P})$.■

An important example of a configuration is given by any Ξ-configuration and, in particular, the configuration $\Gamma = \mathrm{Der}(x, \mathcal{P})$ where $x \in V^4$ is a quadruple of elements of a projective plane $\mathcal{P}$ (see Section 1). We observe that Γ is a complete configuration on the set

$$\Omega_x = \Xi \cup \{\xi_k\xi_l \in \Xi^{(2)} : \ x_k \neq x_l, \ t_{1,\mathcal{P}}(x_k) = t_{1,\mathcal{P}}(x_l)\},$$

whence it follows that the mapping

$$i : \Omega_x \to V, \quad \xi_j \mapsto x_j, \ \xi_k\xi_l \mapsto x_kx_l$$

is well defined where x_kx_l is the unique element of $\mathcal{P}$ incident to both x_k and x_l. Set $t_x = (i \circ t_{1,\mathcal{P}}, (i,i) \circ t_{2,\mathcal{P}})$. Then $\Gamma = (\Omega_x, \Omega_x^2, t_x)$ and $i \in \mathrm{Emb}_{str}(\Gamma, \mathcal{P})$. We call i the *embedding of* Γ *associated with* x.

5 Counting embedding numbers

Let $\Gamma = (\Omega, E, t)$ be a configuration. Given $A, B \in \overline{\Omega}$ with $t_1(A) = t_1(B)$ we set

$$\Gamma_{A=B} = (\Omega, E', t'), \quad E' = E \cup (A \times B) \cup (B \times A), \quad t' = (t_1, t_2'),$$

where $t_2'(\xi, \eta) = t_2'(\eta, \xi) = \mathfrak{e}$ for all $\xi \in A$, $\eta \in B$, and $t_2'(\xi, \eta) = t_2(\xi, \eta)$ for the other pairs $(\xi, \eta) \in \Omega^2$. Let A, B, C, D be pairwise distinct elements of $\overline{\Omega}$. Below we say that $Q = A\,B\,C\,D$ is a *quadrangle* in Γ, if each element of the set $\{A, C\}$ is incident to each element of the set $\{B, D\}$.

Lemma 5.1 *Let Γ be a configuration on Ω, $\Omega_0 \subset \Omega$ and $i_0 \in \mathrm{Emb}_{str}(\Gamma_0, \mathcal{P})$ where $\Gamma_0 = \Gamma_{\Omega_0}$ and $\mathcal{P}$ a projective plane of order q. Suppose that the configuration Γ_0 is complete. Then*

(1) *given $A \notin \overline{\Omega} \setminus \overline{\Omega}_0$ we have*

$$\mathrm{e}(\Gamma, i_0, \mathcal{P}) = \begin{cases} (q^2+q+1)\,\mathrm{e}(\Gamma_1, i_0, \mathcal{P}), & \text{if } d(A) = 0, \\ (q+1)\,\mathrm{e}(\Gamma_1, i_0, \mathcal{P}), & \text{if } d(A) = 1, \\ \mathrm{e}(\Gamma_1, i_0, \mathcal{P}), & \text{if } d(A) = 2 \text{ and } N(A, \overline{\Omega}_0) \subset \overline{\Omega}_0, \\ \mathrm{e}(\Gamma_1, i_0, \mathcal{P}) + q\,\mathrm{e}(\Gamma_2, i_0, \mathcal{P}), & \text{otherwise} \end{cases}$$

where $\Gamma_1 = \Gamma_{\Omega \setminus A}$, $\Gamma_2 = (\Gamma_1)_{B=C}$ with $N(A, \overline{\Omega}_0) = \{B, C\}$.

(2) *given a quadrangle $Q = A\,B\,C\,D$ in Γ not all vertices of which are in $\overline{\Omega}_0$ we have*

$$\mathrm{e}(\Gamma, i_0, \mathcal{P}) = \begin{cases} \mathrm{e}(\Gamma_1, i_0, \mathcal{P}), & \text{if } \{B, D\} \subset \overline{\Omega}_0, \\ \mathrm{e}(\Gamma_2, i_0, \mathcal{P}), & \text{if } \{A, C\} \subset \overline{\Omega}_0, \\ \mathrm{e}(\Gamma_1, i_0, \mathcal{P}) + \mathrm{e}(\Gamma_2, i_0, \mathcal{P}) - \mathrm{e}(\Gamma_3, i_0, \mathcal{P}), & \text{otherwise,} \end{cases}$$

where $\Gamma_1 = \Gamma_{A=C}$, $\Gamma_2 = \Gamma_{B=D}$, $\Gamma_3 = (\Gamma_1)_{B=D}$.

Proof. We observe that if p_1, p_2 (resp. l_1, l_2) are points (resp. lines) of the projective plane $\mathcal{P}$ such that p_i incident to l_j for all i, j, then either $p_1 = p_2$ or $l_1 = l_2$. So from the definition of non-strict embedding it follows that

$$\mathrm{Emb}(\Gamma, i_0, \mathcal{P}) = \mathrm{Emb}(\Gamma_1, i_0, \mathcal{P}) \cup \mathrm{Emb}(\Gamma_2, i_0, \mathcal{P}),$$

$$\mathrm{Emb}(\Gamma_3, i_0, \mathcal{P}) = \mathrm{Emb}(\Gamma_1, i_0, \mathcal{P}) \cap \mathrm{Emb}(\Gamma_2, i_0, \mathcal{P}).$$

Thus the third equality of statement (2) is obtained by the inclusion-exclusion principle. If $B, D \in \overline{\Omega}_0$, then the completeness of Γ_0 implies that $i(B) \neq i(D)$ for all $i \in \mathrm{Emb}(\Gamma, i_0, \mathcal{P})$. So $i(A) = i(C)$ and hence $\mathrm{e}(\Gamma, i_0, \mathcal{P}) = \mathrm{e}(\Gamma_1, i_0, \mathcal{P})$, which proves the first equality of statement (2). The second equality of statement (2) is proved similarly.

To prove statement (1) we observe that if $d(A) = 0$ (resp. $d(A) = 1$), then each $i_1 \in \mathrm{Emb}(\Gamma_1, i_0, \mathcal{P})$ can be extended to $i \in \mathrm{Emb}(\Gamma, i_0, \mathcal{P})$ in $q^2 + q + 1$ (resp. $q + 1$) ways by sending all elements of A to an arbitrary element of $\mathcal{P}$ of type $t_1(A)$ (resp. to an arbitrary element of $\mathcal{P}$ incident to the i_1-image of the neighbor of A). Thus $\mathrm{e}(\Gamma, i_0, \mathcal{P}) = (q^2 + q + 1)\,\mathrm{e}(\Gamma_1, i_0, \mathcal{P})$ if $d(A) = 0$, and $\mathrm{e}(\Gamma, i_0, \mathcal{P}) = (q + 1)\,\mathrm{e}(\Gamma_1, i_0, \mathcal{P})$ if $d(A) = 1$. This proves the first two equalities. The third equality follows from the completeness of the configuration Γ_0.

Let $d(A) = 2$ and $\{B, C\} \not\subset \overline{\Omega}_0$. For $j = 1, 2$ denote by $\mathrm{Emb}_j(\Gamma, i_0, \mathcal{P})$ (resp. $\mathrm{Emb}_j(\Gamma_1, i_0, \mathcal{P})$) the set of all $i \in \mathrm{Emb}(\Gamma, i_0, \mathcal{P})$ (resp. $i \in \mathrm{Emb}(\Gamma_1, i_0, \mathcal{P})$) such that $i(B) \neq i(C)$ for $j = 1$, and $i(B) = i(C)$ for $j = 2$. Then obviously

$$\mathrm{Emb}(\Gamma, i_0, \mathcal{P}) = \mathrm{Emb}_1(\Gamma, i_0, \mathcal{P}) \cup \mathrm{Emb}_2(\Gamma, i_0, \mathcal{P}),$$

$$\mathrm{Emb}(\Gamma_1, i_0, \mathcal{P}) = \mathrm{Emb}_1(\Gamma_1, i_0, \mathcal{P}) \cup \mathrm{Emb}_2(\Gamma_1, i_0, \mathcal{P}),$$

and the unions are disjoint. Set $\mathrm{e}_j(\Gamma, i_0, \mathcal{P}) = |\,\mathrm{Emb}_j(\Gamma, i_0, \mathcal{P})|$ and $\mathrm{e}_j(\Gamma_1, i_0, \mathcal{P}) = |\,\mathrm{Emb}_j(\Gamma_1, i_0, \mathcal{P})|$. Since two distinct elements of the same type in a projective plane are incident to a unique element of the other type, each $i_1 \in \mathrm{Emb}_1(\Gamma_1, i_0, \mathcal{P})$ can be uniquely extended to $i \in \mathrm{Emb}_1(\Gamma, i_0, \mathcal{P})$. So

$$\mathrm{e}_1(\Gamma, i_0, \mathcal{P}) = \mathrm{e}_1(\Gamma_1, i_0, \mathcal{P}). \tag{14}$$

On the other hand, each $i_1 \in \mathrm{Emb}_2(\Gamma_1, i_0, \mathcal{P})$ can be extended to $i \in \mathrm{Emb}_2(\Gamma, i_0, \mathcal{P})$ in $q + 1$ ways by sending all elements of A to an arbitrary element of $\mathcal{P}$ incident to $i_1(B) = i_1(C)$. Since obviously $\mathrm{e}_2(\Gamma_1, i_0, \mathcal{P}) = \mathrm{e}(\Gamma_2, i_0, \mathcal{P})$, this implies that

$$\mathrm{e}_2(\Gamma, i_0, \mathcal{P}) = (q + 1)\,\mathrm{e}_2(\Gamma_1, i_0, \mathcal{P}) = \mathrm{e}_2(\Gamma_1, i_0, \mathcal{P}) + q\ \mathrm{e}(\Gamma_2, i_0, \mathcal{P}). \tag{15}$$

From (14) and (15) it follows that $\mathrm{e}(\Gamma, i_0, \mathcal{P}) = \mathrm{e}(\Gamma_1, i_0, \mathcal{P}) + q\ \mathrm{e}(\Gamma_2, i_0, \mathcal{P})$, which proves the third equality of statement (1).∎

Throughout the rest of the section we fix a set Ω_0 and denote by $\boldsymbol{\Gamma}$ the class of all configurations on sets containing Ω_0. Let $\Gamma \in \boldsymbol{\Gamma}$. Given a class A of Γ not in $\overline{\Omega}_0$ with $d(A) \leq 2$ set

$$N(\Gamma, A) = \begin{cases} \{\Gamma_1\}, & \text{if } d(A) \leq 1 \text{ or } N(A) \subset \overline{\Omega}_0, \\ \{\Gamma_1, \Gamma_2\}, & \text{otherwise,} \end{cases}$$

where $\Gamma_1 = \Gamma_{\Omega\setminus A}$ and $\Gamma_2 = \Gamma_{B=C}$ with B and C being the neighbors of A (cf. statement (1) of Lemma 5.1). Similarly, given a quadrangle $Q = A\,B\,C\,D$ in Γ not all vertices of which are in $\overline{\Omega}_0$ set

$$N(\Gamma, Q) = \begin{cases} \{\Gamma_1\}, & \text{if } \{B, D\} \subset \overline{\Omega}_0, \\ \{\Gamma_2\}, & \text{if } \{A, C\} \subset \overline{\Omega}_0, \\ \{\Gamma_1, \Gamma_2, \Gamma_3\}, & \text{otherwise,} \end{cases}$$

where $\Gamma_1 = \Gamma_{A=C}$, $\Gamma_2 = \Gamma_{B=D}$ and $\Gamma_3 = (\Gamma_1)_{B=D}$ (cf. statement (2) of Lemma 5.1). Clearly, $N(\Gamma, A), N(\Gamma, Q)$ are subsets of $\mathbf{\Gamma}$.

Definition 5.2 *By a d-deduction in $\mathbf{\Gamma}$ we mean a rooted tree of depth d with vertices in $\mathbf{\Gamma}$ such that the set of sons of any non-leaf Γ' equals either $N(\Gamma', A)$ for some class A of Γ' not in $\overline{\Omega}_{0,\Gamma'}$ with $d(A) \leq 2$, or $N(\Gamma', Q)$ for some quadrangle Q in Γ' not all vertices of which are in $\overline{\Omega}_{0,\Gamma'}$. If Γ is the root of this tree and a set $\mathbf{\Gamma}' \subset \mathbf{\Gamma}$ contains all of its leaves, we say that it is a d-deduction of Γ from $\mathbf{\Gamma}'$. In this case for any $d' \geq d$ we also say that Γ is d'-deducible from $\mathbf{\Gamma}'$.*

From the definition it follows that Γ is 0-deducible from $\mathbf{\Gamma}'$ if and only if $\Gamma \in \mathbf{\Gamma}'$. Also if Γ is d_1-deducible from the set $\mathbf{\Gamma}_1$ each element of which is d_2-deducible from $\mathbf{\Gamma}_2$, then Γ is $(d_1 + d_2)$-deducible from $\mathbf{\Gamma}_2$. From Lemmas 5.1 and 4.4 and Theorem 4.3 we immediately obtain the following statement.

Theorem 5.3 *Let $\Gamma \in \mathbf{\Gamma}$ and $i_0 \in \mathrm{Emb}_{str}(\Gamma_{\Omega_0}, \mathcal{P})$ where $\mathcal{P}$ is a projective plane of order q. Suppose that Γ is d-deducible from the set of all final configurations belonging to $\mathbf{\Gamma}$ for some $d \geq 0$. Then*

$$\mathrm{e}_{str}(\Gamma, i_0, \mathcal{P}) = f_\Gamma(q), \quad \mathrm{e}(\Gamma, i_0, \mathcal{P}) = g_\Gamma(q)$$

where $f_\Gamma(x)$ and $g_\Gamma(x)$ are polynomials with integral coefficients not depending on $\mathcal{P}$ and i_0.■

6 Proof of Theorem 1.1

Let $\Gamma = (\Omega, E, t)$ where $t = (t_1, t_2)$, be a Ξ-configuration (see Section 4). Then it easy to see that $R(\Gamma, \mathcal{P})^T = R(\Gamma^T, \mathcal{P})$ where $\Gamma^T = (\Omega, E, t')$ with $t' = (f_1 \circ t_1, f_2 \circ t_2)$ and $f_1 \in \mathrm{Sym}(\Omega)$ and $f_2 \in \mathrm{Sym}(E)$ the permutations induced by the involution $(\xi_1, \xi_3)(\xi_2, \xi_4) \in \mathrm{Sym}(\Xi)$. Besides,

$$\Delta(V^2) = \bigcup_{\Gamma} R(\Gamma, \mathcal{P})$$

where Γ runs over all Ξ-configurations such that $t_2(\xi_1, \xi_3) = t_2(\xi_2, \xi_4) = \mathfrak{e}$. Thus the partition Π is closed with respect to transposition and the set $\Delta(V^2)$ is the union of classes of Π. To complete the proof of statement (1) it only suffices to verify that given $R, S, T \in \Pi$ the number $c_{R,S}(u, w)$ defined by (5) does not depend on the choice of $(u, w) \in T$. We need some notation.

Let Σ be the set of six distinct commuting variables $\xi_1, \xi_2, \xi_3, \xi_4$ and σ, τ. Thus, $\Sigma = \Xi \cup \{\sigma, \tau\}$ where Ξ was defined in Section 1. Set $\mathbf{\Omega} = \Sigma \cup \Sigma^{(2)}$ where $\Sigma^{(2)}$ is the symmetric square of Σ.

Obviously, $|\mathbf{\Omega}| = 21$ and $\mathbf{\Omega}$ contains the set $\mathbf{\Omega}_0 = \Xi \cup \Xi^{(2)}$. For $k = 0, 1, 2$ let us define an injection $f_k : \mathbf{\Omega}_0 \to \mathbf{\Omega}$ such that

$$(f_k(\xi_1), f_k(\xi_2), f_k(\xi_3), f_k(\xi_4)) = \begin{cases} (\xi_1, \xi_2, \xi_3, \xi_4), & \text{if } k = 0, \\ (\xi_1, \xi_2, \sigma, \tau), & \text{if } k = 1, \\ (\sigma, \tau, \xi_3, \xi_4), & \text{if } k = 2, \end{cases}$$

and $f_k(\xi_i \xi_j) = f_k(\xi_i) f_l(\xi_j)$ for all $i \neq j$. Set $\mathbf{\Omega}_k = f_k(\mathbf{\Omega}_0)$ $(k = 1, 2)$.

Let Γ_k be a Ξ-configuration, $k = 0, 1, 2$. Denote by Γ'_k the configuration on obtained from Γ_k by translation of structure along f_k. Obviously, $\Gamma'_0 = \Gamma_0$. Suppose that the configurations $\Gamma_0, \Gamma_1, \Gamma_2$ are *consistent*, i.e.

$$(\Gamma'_k)_{\Omega_k \cap \Omega_l} = (\Gamma'_l)_{\Omega_k \cap \Omega_l}, \qquad k, l = 0, 1, 2,$$

where Ω_k is the underlying set of Γ'_k. Then there exists a uniquely determined configuration Γ on the set $\Omega = \bigcup_{k=0}^{2} \Omega_k$ such that $\Gamma_{\Omega_k} = \Gamma'_k$ for all k. We denote it by $[\Gamma_0, \Gamma_1, \Gamma_2]$ and call the *initial* configuration associated with $\Gamma_0, \Gamma_1, \Gamma_2$. The following statement is straightforward from definitions.

Lemma 6.1 *Let $\mathcal{P}$ be a projective plane, V the set of its elements and let $\Gamma_0, \Gamma_1, \Gamma_2$ be Ξ-configurations with $\Gamma_0 = \mathrm{Der}(x, \mathcal{P})$ where $x \in V^4$. Suppose that the configurations $\Gamma_0, \Gamma_1, \Gamma_2$ are consistent and $\Gamma = [\Gamma_0, \Gamma_1, \Gamma_2]$. Then for $R = R(\Gamma_1, \mathcal{P})$, $S = R(\Gamma_2, \mathcal{P})$ and $u = (x_1, x_2)$, $w = (x_3, x_4)$ we have*

$$c_{R,S}(u, w) = \mathrm{e}_{str}(\Gamma, i_0, \mathcal{P})$$

where $i_0 \in \mathrm{Emb}_{str}(\Gamma_0, \mathcal{P})$ is the embedding associated with x.■

Let us complete the proof of statement (1). Take $R, S, T \in \Pi$ and set $\Gamma_0 = \mathrm{Der}(T)$, $\Gamma_1 = \mathrm{Der}(R)$, $\Gamma_2 = \mathrm{Der}(S)$. If Γ_0, Γ_1 and Γ_2 are not consistent, then obviously $c_{R,S}(u, w) = 0$ for all $(u, w) \in T$. Suppose that they are consistent and take $(u, w) \in T$. Then $\Gamma_0 = \mathrm{Der}(x, \mathcal{P})$ where $x = (u, w)$. So by Lemma 6.1 we have $c_{R,S}(u, w) = \mathrm{e}_{str}(\Gamma, i_0, \mathcal{P})$ where Γ and i_0 as in the lemma. By the theorem below the latter number does not depend on the choice of (u, w), which proves statement (1).

Theorem 6.2 *Let Γ be an initial configuration and $i_0 \in \mathrm{Emb}_{str}(\Gamma_0, \mathcal{P})$ where $\mathcal{P}$ is a projective plane of order q. Then*

$$\mathrm{e}_{str}(\Gamma, i_0, \mathcal{P}) = f_\Gamma(q)$$

where $f_\Gamma(x)$ is a polynomial with integral coefficients not depending on $\mathcal{P}$ and i_0.

Proof. Follows from Theorems 7.3, 7.4 and 5.3.■

Statement (3) is obtained as a byproduct of Theorem 6.2. To prove statement (2) let $\Gamma' \in \mathrm{Der}(\mathcal{P})$. Then there exists $x \in V^4$ such that $\Gamma' = \mathrm{Der}(x, \mathcal{P})$. Set $\Gamma_0 = \mathrm{Der}(y, \mathcal{P})$ where $y =$

(x_1, x_2, x_1, x_2), $\Gamma_1 = \Gamma'$ and $\Gamma_2 = (\Gamma')^T$. Since these configurations are obviously consistent, one can form the initial configuration $\Gamma = [\Gamma_0, \Gamma_1, \Gamma_2]$. Then

$$R(\Gamma_0, \mathcal{P}) = \{(z_1, z_2, z_1, z_2) \in V^4 : \ t_1(z_i) = t_1(x_i) \ (i = 1, 2), \ t_2(z_1, z_2) = t_2(x_1, x_2)\}. \tag{16}$$

Take another projective plane $\mathcal{P}'$ of order q' and elements $x'_1, x'_2 \in V'$ such that $t_1(x'_1) = t_1(x_1)$, $t_1(x'_2) = t_1(x_2)$, $t_2(x'_1, x'_2) = t_2(x_1, x_2)$. From (16) it follows that $\mathrm{Der}(y, \mathcal{P}) = \mathrm{Der}(y', \mathcal{P}') = \Gamma_0$ where $y' = (x'_1, x'_2, x'_1, x'_2)$. Denote by $i_0 \in \mathrm{Emb}_{str}(\Gamma_0, \mathcal{P})$ and $i'_0 \in \mathrm{Emb}_{str}(\Gamma_0, \mathcal{P}')$ the embeddings associated with the quadruples y and y' respectively. Then by Theorem 6.2 we have

$$\mathrm{e}_{str}(\Gamma, i_0, \mathcal{P}) = f_\Gamma(q), \quad \mathrm{e}_{str}(\Gamma, i'_0, \mathcal{P}') = f_\Gamma(q').$$

Since obviously $\mathrm{e}_{str}(\Gamma, i_0, \mathcal{P}) \neq 0$, the polynomial f_Γ is a nonzero one. So $\mathrm{e}_{str}(\Gamma_1, \mathcal{P}') \neq 0$ whenever $q' = q$ or q' is larger than the maximum absolute value of a root of f_Γ. In both cases we have $R(\Gamma', \mathcal{P}') = R(\Gamma_1, \mathcal{P}') \neq \emptyset$ whence it follows that $\Gamma' \in \mathrm{Der}(\mathcal{P}')$. This proves the first part of statement (2) and also the second part because the number of all Ξ-configurations is obviously finite.■

7 Admissible configurations

7.1 Let Γ be an initial configuration on Ω (see Section 6). Then from the definition it follows that $\Omega \subset \mathbf{\Omega}$ and

$$\Gamma_{\Omega_0} \text{ is a } \Xi\text{-configuration.} \tag{17}$$

Besides, Γ satisfies the following condition with $\xi, \eta, \zeta \in \Sigma$:

$$\xi, \xi\eta \in \Omega \ \Rightarrow \ t_2(\xi, \xi\eta) = \mathfrak{i}, \quad \text{and} \quad \xi\eta, \xi\zeta \in \Omega \ \Rightarrow \ t_1(\xi\eta) = t_1(\xi\zeta) \tag{18}$$

(it is an easy consequence of the fact that the same is true for any Ξ-configuration on Ω and $\xi, \eta, \zeta \in \Xi$). Moreover, conditions (17) and (18) are preserved when passing from a configuration Γ to the configuration $\Gamma_{\Omega \setminus A}$ with $A \in \overline{\Omega} \setminus \overline{\Omega}_0$ as well as to the configuration $\Gamma_{A=B}$ with $A, B \in \overline{\Omega}$ such that $\{A, B\} \not\subset \overline{\Omega}_0$. From now on under a configuration we mean any configuration Γ on $\Omega \subset \mathbf{\Omega}$ satisfying conditions (17) and (18) (we will refer to the latter one as the *Type Argument*). Below we set

$$\Omega^*_1 = \Omega \cap \{\sigma\xi_1, \sigma\xi_2, \tau\xi_1, \tau\xi_2\}, \quad \Omega^*_2 = \Omega \cap \{\sigma\xi_3, \sigma\xi_4, \tau\xi_3, \tau\xi_4\}. \tag{19}$$

Clearly, $\Omega^*_i = \Omega_i \setminus (\Omega_0 \cap \Omega_{3-i})$, $i = 1, 2$.

Definition 7.1 *We say that a configuration Γ on Ω is admissible at a class $A \in \overline{\Omega}$ if either $A \not\subset \Omega^*_1 \cup \Omega^*_2$ or*

$$A \cap \Omega^*_i \neq \emptyset \ \Rightarrow \ N(A) \subset \overline{\Omega}_i, \qquad i = 1, 2. \tag{20}$$

The configuration Γ is called admissible if it is admissible at any of its classes.

The following lemma is straightforward from the definition.

Lemma 7.2 *If a configuration Γ on Ω is admissible at a class $A \in \overline{\Omega}$, then so are the configurations $\Gamma_{\Omega \setminus B}$ and $\Gamma_{C=D}$ where B, C, D are classes of Γ other than A such that B and at least one of C, D belong to $\overline{\Omega} \setminus \overline{\Omega}_0$.*[3]$\blacksquare$

Theorem 7.3 *Any initial configuration is admissible.*

Proof. Let Γ be an initial configuration associated with Ξ-configurations Γ_0, Γ_1 and Γ_2. It suffices to verify that Γ is admissible at each class $A \in \overline{\Omega}$ such that $A \subset \Omega_1^* \cup \Omega_2^*$. Suppose that $A \cap \Omega_i^* \neq \emptyset$ for some $i = 1, 2$. If $A \cap \Omega_{3-i}^* \neq \emptyset$, then one can find $\alpha_0, \ldots, \alpha_k \in A$ such that $t_2(\alpha_j, \alpha_{j+1}) = \mathfrak{e}$ for all $j = 0, \ldots, k-1$ where $\alpha_0 \in \Omega_i^*$ and $\alpha_k \in \Omega_{3-i}^*$. Set $l = \max_{\alpha_j \in \Omega_i^*} j$. Then $l < k$ and $\alpha_{l+1} \in A \setminus \Omega_i^* \subset \Omega_{3-i}^*$. On the other hand, $\alpha_l \in \Omega_i^*$ and hence the definition of initial configuration implies that $\alpha_{l+1} \in \Omega_i$. Contradiction. Thus $A \subset \Omega_i^*$ whence it follows that $N(\alpha) \subset \Omega_i$ for all $\alpha \in A$, and so $N(A) \subset \overline{\Omega}_i$.$\blacksquare$

An admissible configuration Γ on Ω is called *d-reducible* for some $d > 0$ if there exists a d'-deduction in $\mathbf{\Gamma}$ with $0 < d' \leq d$ from the admissible configurations (see Definition 5.2) where $\mathbf{\Gamma}$ is the class of all configurations the underlying sets of which contain Ω_0. In Section 8 we will prove the following theorem.

Theorem 7.4 *Each admissible configuration other than a final one is* 2*-reducible. In particular, any admissible configuration is d-deducible from final ones for some d.*

7.2 In this subsection we find sufficient conditions for an admissible configuration to be 1-reducible or 2-reducible (Theorem 7.7 and Lemma 7.8). For this purpose the following definition will be useful.

Definition 7.5 *Let Γ be an admissible configuration on Ω. Two classes $A_1, A_2 \in \overline{\Omega}$ of the same type are called comparable (resp. quasi-comparable) in Γ if either $A_1, A_2 \in \overline{\Omega}_0$ or the configuration $\Gamma_{A_1=A_2}$ is admissible (resp.* 1*-deducible from admissible ones).*

Below we frequently use the following sufficient conditions for comparability and quasi-comparability.

Lemma 7.6 *Let Γ be an admissible configuration on Ω. Suppose that $A_1, A_2 \in \overline{\Omega}$ are classes of the same type one of which does not belong to $\overline{\Omega}_0$. Then*

(1) *A_1 and A_2 are comparable whenever either $A_i \not\subset \Omega_1^* \cup \Omega_2^*$ for some i, or $N(A_1) = N(A_2)$,*

(2) *A_1 and A_2 are quasi-comparable whenever $A_1, A_2 \notin \overline{\Omega}_0$ and Γ contains a path $P = A_1\, B\, C\, D\, A_2$ with $C \in \overline{\Omega}_0$ such that either $B, D \in \overline{\Omega}_0$, or $N(A_1) \setminus \{B\} = N(A_2) \setminus \{D\}$ and $N(B) \setminus \{A_1\} = N(D) \setminus \{A_2\}$.*

[3]Obviously, A is a class of any of the configurations $\Gamma_{\Omega \setminus B}$ and $\Gamma_{C=D}$.

Proof. From Lemma 7.2 it follows that the configuration $\Gamma' = \Gamma_{A_1=A_2}$ is admissible at each class other than $A = A_1 \cup A_2$. Suppose that the hypothesis of statement (1) holds. If $A_i \not\subset \Omega_1^* \cup \Omega_2^*$ for some i, then Γ' is obviously admissible at A (Definition 7.1) and hence the classes A_1 and A_2 are comparable. Otherwise, the neighborhood of A in Γ' coincides with $N(A_1) = N(A_2)$. Since Γ is admissible at both A_1 and A_2, this implies that Γ' is admissible at A (see (20)). Statement (1) is proved.

Suppose that the hypothesis of statement (2) holds. Then $Q = A\,B\,C\,D$ is a quadrangle in Γ'. Besides, since $C \in \overline{\Omega}_0$, the configuration $\Gamma'_1 = \Gamma'_{A=C}$ is admissible at class $A \cup C$, and hence is admissible by Lemma 7.2. If $B, D \in \overline{\Omega}_0$, then Γ' is 1-deducible from Γ'_1 and we are done. Otherwise, Γ' is 1-deducible from Γ'_1, $\Gamma'_2 = \Gamma'_{B=D}$ and $\Gamma'_3 = (\Gamma'_2)_{A=C} = (\Gamma'_1)_{B=D}$. Thus by Lemma 7.2 it suffices to verify that the configuration Γ'_2 is admissible. However, by the same lemma this configuration is admissible at each class other than A and $B \cup D$. From the hypothesis it follows that the neighborhoods of A_1 and A_2 in $\Gamma'' = \Gamma_{B=D}$ coincide. Since Γ'' is admissible at both A_1 and A_2, this implies that the configuration $\Gamma'_2 = \Gamma''_{A_1=A_2}$ is admissible at the class A (see the above paragraph). Similarly, the neighborhoods of B and D in Γ' coincide and Γ' is admissible at both B and D. Thus the configuration Γ'_2 is admissible at the class $B \cup D$.$\blacksquare$

Theorem 7.7 *Let Γ be an admissible configuration on Ω. Then the following two statements hold:*

(1) *Let $A \in \overline{\Omega} \setminus \overline{\Omega}_0$ and $d(A) \leq 2$. Suppose that the neighbors of A are comparable (resp. quasi-comparable) in $\Gamma_{\Omega\setminus A}$ whenever $d(A) = 2$. Then Γ is 1-reducible (resp. 2-reducible).*

(2) *Let $Q = A\,B\,C\,D$ be a quadrangle in Γ not all vertices of which belong to $\overline{\Omega}_0$. Suppose that A and C are comparable in Γ (resp. quasi-comparable in Γ and also in $\Gamma_{B=D}$ if B or D does not belong to $\overline{\Omega}_0$). Then Γ is 1-reducible (resp. 2-reducible) whenever B and D are comparable.*

Proof. Let us prove statement (1). Since $d(A) \leq 2$, the configuration Γ is 1-deducible from the set $\{\Gamma_1\}$ if $d(A) < 2$, and from the set $\{\Gamma_1, \Gamma_2\}$ if $d(A) = 2$ where $\Gamma_1 = \Gamma_{\Omega\setminus A}$, and $\Gamma_2 = (\Gamma_1)_{B=C}$ with B, C being the neighbors of A. By Lemma 7.2 the configuration Γ_1 is admissible. Suppose that Γ_2 enters the deduction. Then at least one of B, C does not belong to $\overline{\Omega}_{0,\Gamma_1} = \overline{\Omega}_0 \setminus \{A\}$. So if the classes B and C are comparable (resp. quasi-comparable) in Γ_1, then by Definition 7.1 the configuration Γ_2 is admissible (resp. 1-deducible from admissible ones), and hence the configuration Γ is 1-reducible (resp. 2-reducible).

Let us prove statement (2). Since not all vertices of Q belong to $\overline{\Omega}_0$, the configuration Γ is 1-deducible from the set $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ where $\Gamma_1 = \Gamma_{A=C}$, $\Gamma_2 = \Gamma_{B=D}$ and $\Gamma_3 = (\Gamma_1)_{B=D} = (\Gamma_2)_{A=C}$. If Γ_1 enters the deduction, then A or C does not belong to $\overline{\Omega}_0$. So by Definition 7.5 the configuration Γ_1 is admissible (resp. 1-deducible from admissible ones) whenever the classes A and C are comparable (resp. quasi-comparable) in Γ. Suppose that the classes B and D are comparable in Γ. Then if Γ_2 enters the deduction, it is admissible by the above argument. Suppose Γ_3 enters the deduction. Then so do Γ_1 and Γ_2. Therefore if A and C are comparable in Γ, then these configurations are admissible by above, and hence by Lemma 7.2 the configuration Γ_3 is admissible. Finally, let A and C be quasi-comparable in Γ. Since B or D does not belong to $\overline{\Omega}_0$, the hypothesis implies that A and C are quasi-comparable in Γ_2, and hence the configuration

Γ_3 is 1-deducible from the admissible configurations. Thus in the comparability (resp. quasi-comparability) case the configuration Γ is 1-deducible (resp. 2-deducible) from the admissible configurations, i.e. 1-reducible (resp. 2-reducible).■

We will refer to the sufficient condition of 1-reducibility (resp. 2-reducibility) from statements (1) and (2) of Theorem 7.7 as the Degree Argument and the Quadrangle Argument (resp. the Generalized Degree Argument and the Generalized Quadrangle Argument) respectively. From statement (1) of Lemma 7.6 it follows that two classes of a configuration Γ are comparable whenever one of them belongs to the set

$$X^* = X \setminus \overline{\Omega_1 \cap \Omega_2}, \quad X = \overline{\Omega} \setminus \overline{\Omega}_0. \tag{21}$$

In these cases we will use the Degree Argument and the Quadrangle Argument without any reference to the above statement (and even without any mention that the classes are comparable).

Below we will use the following property of an admissible but not 1-reducible configuration Γ on Ω: given $A \in X$ and $\xi, \eta \in \Omega_0$ we have

$$\overline{\xi}, \overline{\eta} \in N(A, \overline{\Omega}_0),\ \overline{\xi} \neq \overline{\eta} \ \Rightarrow\ \{\xi, \eta\} = \{\xi_i\xi_j, \xi_k\} \text{ or } \{\xi, \eta\} = \{\xi_i\xi_j, \xi_k\xi_l\} \tag{22}$$

with distinct $i, j, k, l \in \{1, 2, 3, 4\}$. Indeed, if $\xi = \xi_i$, $\eta = \xi_j$, (resp. $\xi = \xi_i\xi_j$, $\eta = \xi_i\xi_k$) then by the Quadrangle Argument with $Q = A\,\overline{\xi_i}\,\overline{\xi_i\xi_j}\,\overline{\xi_j}$ (resp. $Q = A\,\overline{\xi_i\xi_j}\,\overline{\xi_i}\,\overline{\xi_i\xi_k}$) the configuration Γ is 1-reducible. We observe that $t_1(\xi) = t_1(\eta)$. So by the Type Argument the set $\{\xi, \eta\}$ cannot contain both a variable and the product of this variable by another one. Thus claim (22) is proved.

Lemma 7.8 *Let Γ be an admissible configuration on Ω. Suppose that at least one of the following conditions is satisfied:*

(1) *$d(A, \overline{\Omega}_0) \geq 3$ for some $A \in X$,*

(2) *$d(A, \overline{\Omega}_0) + d(B, \overline{\Omega}_0) \geq 4$ for some incident classes $A, B \in X$,*

(3) *$d(B, \overline{\Omega}_0) + d(C, \overline{\Omega}_0) \geq 4$ and $d(A, \overline{\Omega}_0) \geq 1$ for some distinct comparable classes $B, C \in X$ such that A is incident to both B and C.*

Then the configuration Γ is 1-reducible.

Proof. Let $\overline{\xi}, \overline{\eta}, \overline{\zeta}$ be pairwise distinct elements of $N(A, \overline{\Omega}_0)$ where $\xi, \eta, \zeta \in \Omega_0$. Since all 2-subsets of the set $\{\xi, \eta, \zeta\}$ can not simultaneously be of the form as in (22), statement (1) follows. Let us prove statement (2). By statement (1) we can assume that $d(A, \overline{\Omega}_0) = d(B, \overline{\Omega}_0) = 2$. From claim (22) (applied to A and B) it follows that $\overline{\xi_i\xi_j} \in N(A, \overline{\Omega}_0)$ and $\overline{\xi_k\xi_l} \in N(B, \overline{\Omega}_0)$ for some i, j, k, l. Then the Type Argument shows that $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and the neighbor of A in $\overline{\Omega}_0$ other than $\overline{\xi_i\xi_j}$, is either $\overline{\xi_k}$ or $\overline{\xi_l}$, say $\overline{\xi_k}$. However, by the Quadrangle Argument with $Q = A\,B\,\overline{\xi_k\xi_l}\,\overline{\xi_k}$ the configuration Γ is 1-reducible.

Let us prove statement (3). By statement (1) we can assume that $d(B, \overline{\Omega}_0) = d(C, \overline{\Omega}_0) = 2$. If $D \in N(B, \overline{\Omega}_0) \cap N(C, \overline{\Omega}_0)$, then by the Quadrangle Argument with $Q = D\,B\,A\,C$ the configuration

Γ is 1-reducible. Contradiction. Thus from (22) (for $A = B, C$) and the Type Argument it follows that at least one of the sets $N(B, \overline{\Omega}_0)$, $N(C, \overline{\Omega}_0)$, say the first one, is of the form $N(B, \overline{\Omega}_0) = \{\overline{\xi_i\xi_j}, \overline{\xi_k\xi_l}\}$ where i, j, k, l are pairwise distinct. From the hypothesis of the statement it follows that there exists $\xi \in \Omega_0$ such that $\overline{\xi} \in N(A, \overline{\Omega}_0)$. The Type Argument shows that $\xi \in \Xi$. Denote by D the class $\overline{\xi_i\xi_j}$ if $\xi \in \{\xi_i, \xi_j\}$, and the class $\overline{\xi_k\xi_l}$ if $\xi \in \{\xi_k, \xi_l\}$. Then by the Quadrangle Argument with $Q = A\,\overline{\xi}\,D\,B$ the configuration Γ is 1-reducible.■

8 Proof of Theorem 7.4

Given $\xi \in \Sigma$ we set $\xi' = \xi^f$ where $f = (\xi_1, \xi_2)(\xi_3, \xi_4)(\sigma, \tau) \in \mathrm{Sym}(\Sigma)$; if $\xi = \eta\zeta$ with $\eta, \zeta \in \Sigma$, then by definition $\xi' = \eta'\zeta'$. The element ξ' is called the *twin* of ξ. Below by convention given a configuration Γ on Ω we set $\overline{\xi} = \emptyset$ whenever $\xi \in \mathbf{\Omega} \setminus \Omega$.

Lemma 8.1 *Suppose that a non-final admissible configuration Γ is not 1-reducible. Then $X^* \neq \emptyset$ and $d(A) = 3$ for each $A \in X^*$. Moreover, exactly one of the two following statements hold:*

(I) *for all $\upsilon \in \{\sigma, \tau\}$ and $\xi \in \Xi$ such that $A = \overline{\upsilon\xi} \in X^*$, we have $N(A) = \{\overline{\xi}, \overline{\sigma}, \overline{\tau}\}$,*

(II) *for all $\upsilon \in \{\sigma, \tau\}$ and $\xi \in \Xi$ such that $A = \overline{\upsilon\xi} \in X^*$, we have $N(A) = \{\overline{\xi}, \overline{\upsilon}, A'\}$,*

where A' is the class containing the twin of $\upsilon\xi$.

Proof. Suppose that $X^* = \emptyset$. Then all elements of $\overline{\Omega}$ are pairwise comparable. Since Γ is not final, the set X is not empty. So by the Degree Argument without loss of generality we can assume that $d(A, \overline{\Omega}) \geq 3$ for all $A \in X$. In particular, if $d(A, X) = 0$ for some $A \in X$, then the configuration Γ is 1-reducible by statement (1) of Lemma 7.8. Contradiction. Otherwise $|X| \geq 2$ and $d(A, X) \geq 1$ for all $A \in X$. If $|X| = 2$, then Γ is 1-reducible by statement (2) of Lemma 7.8. Contradiction. Thus $|X| = 3$ and $X = \{\overline{\sigma}, \overline{\tau}, \overline{\sigma\tau}\}$. Then $N(\overline{\sigma\tau}) = \{\overline{\sigma}, \overline{\tau}\} \cup N(\overline{\sigma\tau}, \overline{\Omega}_0)$ by the Type Argument. So the configuration Γ is 1-reducible by the Degree Argument with $A = \overline{\sigma\tau}$ if $d(\overline{\sigma\tau}, \overline{\Omega}_0) = 0$, and by statement (3) of Lemma 7.8 with $A = \overline{\sigma\tau}$, $B = \overline{\sigma}$ and $C = \overline{\tau}$ if $d(\overline{\sigma\tau}, \overline{\Omega}_0) > 0$. Contradiction. Thus $X^* \neq \emptyset$.

Let $A = \overline{\upsilon\xi} \in X^*$ where $\upsilon \in \{\sigma, \tau\}$ and $\xi \in \Xi$. Then $A \subset \Omega_1^* \cup \Omega_2^*$. Without loss of generality we can assume that $\upsilon = \sigma$ and $\xi = \xi_1$. Since the configuration Γ is admissible, we have $N(A) \subset \overline{\Omega}_1$. So by the Type Argument we conclude that

$$N(A) \subset \{A', \overline{\xi}_1, \overline{\xi}_2, \overline{\sigma}, \overline{\tau}\}.$$

If $\overline{\xi}_1 \neq \overline{\xi}_2$ and $\overline{\xi}_2 \in N(A)$, then the configuration Γ is 1-reducible by the Quadrangle Argument with $Q = \overline{\xi}_1 A \overline{\xi}_2 \overline{\xi_1\xi_2}$. Contradiction. Thus $N(A) \subset \{A', \overline{\xi}_1, \overline{\sigma}, \overline{\tau}\}$. By the Type Argument we conclude that either $N(A) \subset \{\overline{\xi_1}, \overline{\sigma}, \overline{\tau}\}$ or $N(A) \subset \{\overline{\xi_1}, \overline{\sigma}, A'\}$. In any case the elements of $N(A)$ are pairwise comparable. Since Γ is not 1-reducible, the Degree Argument implies that $d(A) = 3$. In particular, the neighborhood of each class from X^* is of type I or of type II. If there are no neighborhoods of type I, we are done. Otherwise $\sigma, \tau \in \Omega$ and

$$t_1(\sigma) = t_1(\tau). \tag{23}$$

In this case if there are no neighborhoods of type II, we are done. Otherwise there exists a class $A = \overline{\upsilon\xi} \in X^*$ with the neighborhood of type II. Since the class A' contains the element $\upsilon'\xi'$, we have $t_1(\upsilon) = t_1(\upsilon'\xi') \neq t_1(\upsilon')$ (the Type Argument). As $\{\upsilon, \upsilon'\} = \{\sigma, \tau\}$ this implies that $t_1(\sigma) \neq t_1(\tau)$, which contradicts (23). Thus exactly one of statements (I), (II) holds.■

Now let us turn to the proof of Theorem 7.4. Let Γ be a non-final admissible configuration. Without loss of generality we can assume that Γ is not 1-reducible. Then from Lemma 8.1 it follows that $X^* \neq \emptyset$ and $d(A) = 3$ for all $A \in X^*$. Let us consider two cases depending on which statement (I) or (II) of this lemma holds.

Case I. In this case $\sigma, \tau \in \Omega$ and $t_1(\sigma) = t_1(\tau)$. Let us consider two cases.

Let $|X^*| = 1$. Then $X^* = \{A\}$ where $A = \overline{\upsilon_0\xi}$ with $\upsilon_0 \in \{\sigma, \tau\}$, $\xi \in \Xi$, and $N(A) = \{\overline{\xi}, \overline{\sigma}, \overline{\tau}\}$. So $\sigma\tau \notin \Omega$, for otherwise by the Quadrangle Argument with $Q = \overline{\sigma}\, A\, \overline{\tau}\, \overline{\sigma\tau}$ the configuration Γ is 1-reducible ($\sigma\tau$ is incident to both σ and τ by the Type Argument). Contradiction. So $N(\overline{\upsilon}) \subset \overline{\Omega}_0 \cup \{A\}$ where $\upsilon \in \{\sigma, \tau\}$. If $\overline{\upsilon} \in X$, then $d(\overline{\upsilon}, \overline{\Omega}_0) \geq 2$ by the Degree Argument. Thus we get contradiction with statement (1) of Lemma 7.8 if $\overline{\sigma}, \overline{\tau} \in \overline{\Omega}_0$, with statement (2) if only one of these elements, say $\overline{\sigma}$, belongs to $\overline{\Omega}_0$ ($B = \overline{\sigma}$), and with statement (3) if $\overline{\sigma}, \overline{\tau} \notin \overline{\Omega}_0$ ($B = \overline{\sigma}$, $C = \overline{\tau}$).

Let $|X^*| \geq 2$. Take in X^* two distinct classes $A_1 = \overline{\upsilon_1\xi}$ and $A_2 = \overline{\upsilon_2\eta}$ where $\upsilon_1, \upsilon_2 \in \{\sigma, \tau\}$ and $\xi, \eta \in \Xi$. Then $N(A_1) = \{\overline{\xi}, \overline{\sigma}, \overline{\tau}\}$ and $N(A_2) = \{\overline{\eta}, \overline{\sigma}, \overline{\tau}\}$. If $\overline{\xi} = \overline{\eta}$, then $N(A_1) = N(A_2)$. By statement (1) of Lemma 7.6 this shows that A_1 and A_2 are comparable. So by the Quadrangle Argument with $Q = \overline{\sigma}\, A_1\, \overline{\tau}\, A_2$ the configuration Γ is 1-reducible. Contradiction. Thus $\overline{\xi} \neq \overline{\eta}$. Then by statement (2) of Lemma 7.6 with $P = A_1\, \overline{\xi}\, \overline{\xi\eta}\, \overline{\eta}\, A_2$ the classes A_1 and A_2 are quasi-comparable. So by the Generalized Quadrangle Argument with $Q = \overline{\sigma}\, A_1\, \overline{\tau}\, A_2$ the configuration Γ is 2-reducible and we are done.

Case II. In this case from the Type Argument it follows that $\tau\sigma \notin \Omega$ and if $\sigma, \tau \in \Omega$, then $t_1(\sigma) \neq t_1(\tau)$. Furthermore, we have the following statement.

Lemma 8.2 *Let $\overline{\upsilon\xi} \in X^*$ where $\upsilon \in \{\sigma, \tau\}$ and $\xi \in \Xi$. Then*

(1) $\overline{\upsilon} \in X$,

(2) $d(\overline{\upsilon}) = d(\overline{\upsilon}, X^*) = 2$, *and* $N(\overline{\upsilon}) = \{\overline{\upsilon\xi}, \overline{\upsilon\eta}\}$ *for some* $\eta \in \Xi$,

(3) $t_1(\xi) = t_1(\eta) = t_1(\upsilon)$, $t_1(\xi') = t_1(\eta') \neq t_1(\upsilon)$.

Proof. Without loss of generality we assume that $\upsilon = \sigma$ and $\xi = \xi_1$. Set $A = \overline{\upsilon\xi}$. From Lemma 8.1 it follows that $d(A) = 3$ and $N(A) = \{\overline{\xi}_1, \overline{\sigma}, A'\}$ with $A' = \overline{\tau\xi_2}$. In particular, $t_1(\xi_1) \neq t_1(\xi_2)$. Moreover, if $\tau \in \Omega$, then by the Quadrangle Argument with $Q = \overline{\sigma}\, \overline{\tau}\, A'\, A$ the classes σ and τ are not incident.

Suppose that $\overline{\sigma} \in \overline{\Omega}_0$. Then by (22) we have $\overline{\sigma} = \overline{\xi_j\xi_k}$ for distinct $j, k \in \{2, 3, 4\}$. Moreover $j, k \neq 2$ by the Quadrangle Argument with $Q = A\, \overline{\sigma}\, \overline{\xi}_2\, A'$. Thus $\overline{\sigma} = \overline{\xi_3\xi_4}$. Besides, by statement (1) of Lemma 7.8 we have $A' \notin \overline{\Omega}_0$. So $A' \in X^*$, for otherwise $A' = \overline{\tau}$ and $t_1(A') = t_1(\sigma) \neq t_1(\tau)$

in contrast to the Type Argument. Therefore by Lemma 8.1 we have $\tau \in \Omega$ and $\overline{\tau} \neq \overline{\xi}_2$. By statement (2) of Lemma 7.8 with $B = A'$, we see that $\overline{\tau} \notin \overline{\Omega}_0$. Since $t_1(\xi_1) \neq t_1(\xi_2) = t_1(\xi_3) = t_1(\xi_4)$ it follows that

$$N(\overline{\tau}, \overline{\Omega}_0) \subset \{\overline{\xi}_1, \overline{\xi_2\xi_3}, \overline{\xi_2\xi_4}, \overline{\xi_3\xi_4}\}.$$

So the Quadrangle Argument with $Q = A\,A'\,\overline{\tau}\,\overline{\xi}_1$, $Q = \overline{\tau}\,A'\,\overline{\xi}_2\,\overline{\xi_2\xi_3}$, $Q = \overline{\tau}\,A'\,\overline{\xi}_2\,\overline{\xi_2\xi_4}$, $Q = A\,A'\,\overline{\tau}\,\overline{\sigma}$, shows that $N(\overline{\tau}, \overline{\Omega}_0) = \emptyset$. Since the classes $\overline{\sigma}$ and $\overline{\tau}$ are not incident, this implies that $N(\overline{\tau}) \subset X^*$ and hence

$$N(\overline{\tau}) \subset X^* \cap \{\overline{\tau\xi_2}, \overline{\tau\xi_3}, \overline{\tau\xi_4}\}$$

(Type Argument). From the Degree Argument it follows that $d(\overline{\tau}) > 1$ and so $\overline{\tau\xi_i} \in N(\overline{\tau})$ for some $i \in \{3, 4\}$, say $i = 3$. Since $\overline{\tau\xi_3} \in X^*$, we conclude by Lemma 8.1 that $\overline{\sigma\xi_4} \in N(\overline{\tau\xi_3})$ which contradicts to the fact that $t_1(\xi_3) = t_1(\xi_4)$. This proves statement (1).

Suppose that $d(\overline{\sigma}, X^*) < 1$. Then since $\overline{\tau} \notin N(\overline{\sigma})$, the Degree Argument with $A = \overline{\sigma}$ implies that $d(\overline{\sigma}, \overline{\Omega}_0) \geq 2$. Since $t_1(\xi_1) \neq t_1(\xi_2)$, formula (22) implies that $N(\overline{\sigma}, \overline{\Omega}_0) \supset \{\overline{\xi}_i, \overline{\xi_j\xi_k}\}$ where i, j, k are pairwise distinct elements of $\{1, 2, 3, 4\}$, $i \neq 1$, $j, k \neq 2$ and $\overline{\xi}_i \neq \overline{\xi_j\xi_k}$. However, $i \neq 2$ (resp. $j \neq 1$, $k \neq 1$) by the Quadrangle Argument with $Q = A\,A'\,\overline{\xi}_2\,\overline{\sigma}$ (resp. $Q = A\,\overline{\sigma}\,\overline{\xi_1\xi_k}\,\overline{\xi}_1$, $Q = A\,\overline{\sigma}\,\overline{\xi_1\xi_j}\,\overline{\xi}_1$). So $i, j, k \in \{3, 4\}$. Contradiction. Thus $d(\overline{\sigma}, X^*) \geq 2$. So there exists $B \in N(\overline{\sigma}, X^*)$ other than A. Then by the Type Argument we have $B \in \{\overline{\sigma\xi_3}, \overline{\sigma\xi_4}\}$, say $B = \overline{\sigma\xi_3}$. In this case Lemma 8.1 implies that $N(B) = \{B', \overline{\sigma}, \overline{\xi}_3\}$ where $B' = \overline{\tau\xi_4}$. So

$$t_1(\xi_1) = t_1(\xi_3) = t_1(\sigma), \quad t_1(\xi_2) = t_1(\xi_4) \neq t_1(\sigma),$$

which proves statement (3). It also shows that $N(\overline{\sigma}, X^*) = \{A, B\}$ and $N(\overline{\sigma}, \overline{\Omega}_0) \subset \{\overline{\xi_1\xi_3}, \overline{\xi}_2, \overline{\xi}_4\}$ (Type Argument). However, from the Quadrangle Argument with $Q = A\,\overline{\sigma}\,\overline{\xi_1\xi_3}\,\overline{\xi}_1$ (resp. $Q = A\,A'\,\overline{\xi}_2\,\overline{\sigma}$, $Q = B\,B'\,\overline{\xi}_4\,\overline{\sigma}$) it follows that $\overline{\xi_1\xi_3}$ (resp. $\overline{\xi}_2$, $\overline{\xi}_4$) does not belong to $N(\overline{\sigma}, \overline{\Omega}_0)$. Thus $N(\overline{\sigma}) = \{A, B\}$.■

Since $X^* \neq \emptyset$, from statement (2) of Lemma 8.2 it follows that X^* contains distinct classes $A_1 = \overline{\upsilon\xi}$ and $A_2 = \overline{\upsilon\eta}$ where $\upsilon \in \{\sigma, \tau\}$ and $\xi, \eta \in \Xi$, such that $\overline{\upsilon} \in X$ and $N(\overline{\upsilon}) = \{A_1, A_2\}$. Moreover, $N(A_1) = \{\overline{\xi}, \overline{\upsilon}, A_1'\}$ with $A_1' = \overline{\upsilon'\xi'}$ and $N(A_2) = \{\overline{\eta}, \overline{\upsilon}, A_2'\}$ with $A_2' = \overline{\upsilon'\eta'}$. Without loss of generality we assume that

$$\overline{\xi} = \overline{\eta}. \tag{24}$$

(Otherwise by statement (2) of Lemma 7.6 with $P = A_1\,\overline{\xi}\,\overline{\xi\eta}\,\overline{\eta}\,A_2$ the classes A_1 and A_2 are quasi-comparable. Then by the Generalized Degree Argument with $A = \overline{\upsilon}$ the configuration Γ is 2-reducible and we are done.) To complete the proof we consider two cases.

Let $X^* = \{A_1, A_2\}$. Then $A_1', A_2' \in \overline{\Omega}_0$. Equality (24) and statement (3) of Lemma 8.2 imply that $A_1' = \overline{\xi'\eta'} = A_2'$ (Type Argument). So $N(A_1) = N(A_2)$ and hence A_1 and A_2 are comparable by statement (1) of Lemma 7.6. From the Degree Argument with $A = \overline{\upsilon}$ it follows that the configuration Γ is 1-reducible. Contradiction.

Let $X^* \neq \{A_1, A_2\}$. By Lemma 8.2 (applied to an element of X^* other than A_1 and A_2) we conclude that $X^* = \{A_1, A_2, A_1', A_2'\}$ where $A_1' \neq A_2'$, and $\overline{\upsilon'} \in X$ and $N(\overline{\upsilon'}) = \{A_1', A_2'\}$. Moreover,

$N(A'_1) = \{\overline{\xi'}, \overline{\upsilon'}, A_1\}$ and $N(A'_2) = \{\overline{\eta'}, \overline{\upsilon'}, A_2\}$. By equality (24) (with $A_1 = A'_1$ and $A_2 = A'_2$) we can assume that $\overline{\xi'} = \overline{\eta'}$. Then

$$N(A_1) \setminus \{A'_1\} = \{\overline{\upsilon}, \overline{\xi}\} = N(A_2) \setminus \{A'_2\}, \quad N(A'_1) \setminus \{A_1\} = \{\overline{\upsilon'}, \overline{\xi'}\} = N(A'_2) \setminus \{A_2\}.$$

So by statement (2) of Lemma 7.6 with $P = A_1\, A'_1\, \overline{\xi'}\, A'_2\, A_2$ the classes A_1 and A_2 are quasi-comparable. Thus by the Generalized Degree Argument with $A = \overline{\upsilon}$, the configuration Γ is 2-reducible.■

9 Proof of Theorem 1.2

Let $\Gamma = (\Omega, E, t)$ be a configuration where $t = (t_1, t_2)$. Given $(\xi, \eta) \in E$ we set $t(\xi, \eta) = (t_1(\xi), t_2(\xi, \eta))$; a pair $(\mathfrak{t}_1, \mathfrak{t}_2) \in \mathfrak{T}$ where $\mathfrak{T} = \mathfrak{T}_1 \times \mathfrak{T}_2$ is denoted by $(\mathfrak{t}_2)_{\mathfrak{t}_1}$ (see Notation).

Let V be the set of elements of a projective plane $\mathcal{P}$. For $i, j, k, l \in \{1, 2, 3, 4\}$ set

$$\mathrm{Cyl}_{\mathfrak{t}}(i, j) = \{x \in V^4 : \ t(x_i, x_j) = \mathfrak{t}\}, \qquad \mathfrak{t} \in \mathfrak{T}, \tag{25}$$

$$\mathrm{Col}_{\mathfrak{s}}(i, j, k) = \{x \in V^4 : \ t(x_i, x_j x_k) = t(x_j, x_k x_i) = t(x_k, x_i x_j) = \mathfrak{i}_{\mathfrak{s}}\}, \qquad \mathfrak{s} \in \mathfrak{T}_1, \tag{26}$$

$$\begin{aligned} \mathrm{Col}^{(2)}_{\mathfrak{s}}(i, j, k, l) = &\{x \in V^4 : \ t(x_i x_j, x_k x_l) = \mathfrak{i}_{\mathfrak{s}'}, \\ &t(x_i, x_k x_l) = t(x_j, x_k x_l) = \mathfrak{e}'_{\mathfrak{s}}, t(x_k, x_i x_j) = t(x_l, x_i x_j) = \mathfrak{e}'_{\mathfrak{s}'}\}, \qquad \mathfrak{s} \in \mathfrak{T}_1. \end{aligned} \tag{27}$$

In (25) we have $\mathrm{Cyl}_{\mathfrak{t}}(i, j) = \mathrm{Cyl}_R(i, j)$ where $R = E_{\mathfrak{s}}$ (resp. $E'_{\mathfrak{s}}$, $I_{\mathfrak{s}}$, $I'_{\mathfrak{s}}$) for $\mathfrak{t} = e_{\mathfrak{s}}$ (resp. $e'_{\mathfrak{s}}$, $i_{\mathfrak{s}}$, $i'_{\mathfrak{s}}$), $\mathfrak{s} \in \mathfrak{T}_1$ (see (1), (2) and Section 3). Next, in (26) we have $t_1(x_i) = t_1(x_j) = t_1(x_k) = \mathfrak{s}$, and the elements x_i, x_j, x_k and so the indices i, j, k are pairwise distinct. This relation is defined by the condition that the elements x_i, x_j, x_k are collinear.[4] In (27) we have $t_1(x_i) = t_1(x_j) = \mathfrak{s}$ and $t_1(x_k) = t_1(x_l) = \mathfrak{s}'$ where $\mathfrak{s}'$ is the type dual to $\mathfrak{s}$. Besides, the elements x_i, x_j, x_k, x_l, and so the indices i, j, k, l are pairwise distinct. In fact, this relation is defined by the condition that the elements x_i, x_j, $x_k x_l$ as well as the elements x_k, x_l, $x_i x_j$ are collinear and pairwise distinct. The relations defined in (25), (26) and (27) are called *cylindricity*, *collinearity* and *double collinearity* relations of the plane $\mathcal{P}$ respectively.

Lemma 9.1 *The derivative partition $\Pi = \mathrm{Der}(\mathcal{P})$ associated with the plane $\mathcal{P}$ coincides with the partition of V^4 generated by all cylindricity, collinearity and double collinearity relations of the plane $\mathcal{P}$, i.e. with the coarsest partition of V^4 for which any of these relations is the union of classes.*

Proof. Denote by Π' the second partition from lemma's statement. It is easily seen that given $i, j, k, l \in \{1, 2, 3, 4\}$ and $\mathfrak{t} \in \mathfrak{T}$ (resp. $\mathfrak{s} \in \mathfrak{T}_1$) the relation $\mathrm{Cyl}_{\mathfrak{t}}(i, j)$ (resp. the relations $\mathrm{Col}_{\mathfrak{t}}(i, j, k)$

[4]Elements of $\mathcal{P}$ are called *collinear* if there exists an element incident to all of them; thus concurrent lines are also called collinear here.

and $\mathrm{Col}^{(2)}_{\mathfrak{t}}(i,j,k,l))$ are unions of the relations $R(\Gamma,\mathcal{P})$ where the configuration Γ runs over the classes $\mathcal{K}_0$ (resp. $\mathcal{K}_1$ and $\mathcal{K}_2$) defined as follows:

$$\begin{aligned}
&\mathcal{K}_0=\{\Gamma\in\mathcal{K}:\ t(\xi_i,\xi_j)=\mathfrak{t}\},\\
&\mathcal{K}_1=\{\Gamma\in\mathcal{K}:\ t(\xi_i,\xi_j\xi_k)=t(\xi_j,\xi_k\xi_i)=t(\xi_k,\xi_i\xi_j)=\mathfrak{i}_{\mathfrak{s}}\},\\
&\mathcal{K}_2=\{\Gamma\in\mathcal{K}:\ t(\xi_i\xi_j,\xi_k\xi_l)=\mathfrak{i}_{\mathfrak{s}'},t(\xi_i,\xi_k\xi_l)=t(\xi_j,\xi_k\xi_l)=\mathfrak{e}'_{\mathfrak{s}},\ t(\xi_k,\xi_i\xi_j)=t(\xi_l,\xi_i\xi_j)=\mathfrak{e}'_{\mathfrak{s}'}\}
\end{aligned}\tag{28}$$

where $\mathcal{K}$ is the class of all Ξ-configurations. Thus Π is coarser than Π'. Conversely, let $R\in\Pi$. Then $R=R(\Gamma,\mathcal{P})$ for some Ξ-configuration $\Gamma=(\Omega,\Omega^2,t)$. It is easily seen that

$$R=\bigcap_{\xi,\eta\in\Omega}R_{\xi,\eta,t(\xi,\eta)}$$

where for $\mathfrak{t}\in\mathfrak{T}$ and $\xi,\eta\in\Omega$ we set $R_{\xi,\eta,\mathfrak{t}}=\{x\in V^4:\ \xi,\eta\in\Omega_x,\ t_x(\xi,\eta)=\mathfrak{t}\}$ with Ω_x and t_x such that $\mathrm{Der}(x,\mathcal{P})=(\Omega_x,\Omega_x^2,t_x)$. So it suffices to verify that any relation $R_{\xi,\eta,\mathfrak{t}}$ is the union of elements of the partition Π'.

If $\xi=\xi_i$ and $\eta=\xi_j$ for some $i,j\in\{1,\dots,4\}$, then obviously $R_{\xi,\eta,\mathfrak{t}}=\mathrm{Cyl}_{\mathfrak{t}}(i,j)$. Let $\xi=\xi_i$ and $\eta=\xi_j\xi_k$ for some $i,j,k\in\{1,\dots,4\}$ (the case $\xi=\xi_j\xi_k$ and $\eta=\xi_i$ is treated similarly). Then

$$R_{\xi,\eta,\mathfrak{t}}=\begin{cases}\mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(i,j))\cap\mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(i,k)\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(j,k), & \text{if } \mathfrak{t}=\mathfrak{e}_{\mathfrak{s}},\\
(\mathrm{Cyl}_{\mathfrak{i}'_{\mathfrak{s}}}(i,j)\cup\mathrm{Cyl}_{\mathfrak{i}'_{\mathfrak{s}}}(i,k))\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(j,k), & \text{if } \mathfrak{t}=\mathfrak{e}'_{\mathfrak{s}},\\
((\mathrm{Cyl}_{\mathfrak{e}_{\mathfrak{s}}}(i,j)\cup\mathrm{Cyl}_{\mathfrak{e}_{\mathfrak{s}}}(i,k))\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(j,k))\cup S, & \text{if } \mathfrak{t}=\mathfrak{i}_{\mathfrak{s}},\\
(\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i,j)\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i,k)\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(j,k))\setminus S, & \text{if } \mathfrak{t}=\mathfrak{i}'_{\mathfrak{s}}\end{cases}$$

where $\mathfrak{s}\in\mathfrak{T}_1$ and $S=\mathrm{Col}_{\mathfrak{s}}(i,j,k)$ if i,j,k are pairwise distinct, and $S=\emptyset$ otherwise. Finally, if $\xi=\xi_i\xi_j$ and $\eta=\xi_k\xi_l$, then

$$R_{\xi,\eta,\mathfrak{t}}=\begin{cases}R_{\xi_i,\eta,\mathfrak{i}_{\mathfrak{s}'}}\cap R_{\xi_j,\eta,\mathfrak{i}_{\mathfrak{s}'}}, & \text{if } \mathfrak{t}=\mathfrak{e}_{\mathfrak{s}},\\
(\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(i,j)\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i,k))\setminus R_{\xi,\eta,\mathfrak{e}_{\mathfrak{s}}}, & \text{if } \mathfrak{t}=\mathfrak{e}'_{\mathfrak{s}},\\
\mathrm{Col}^{(2)}_{\mathfrak{t}}(i,j,k,l)\cup S_1\cup S_2\cup S_3\cup S_4, & \text{if } \mathfrak{t}=\mathfrak{i}_{\mathfrak{s}},\\
(\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i,j)\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(i,k))\setminus R_{\xi,\eta,\mathfrak{i}_{\mathfrak{s}}}, & \text{if } \mathfrak{t}=\mathfrak{i}'_{\mathfrak{s}}\end{cases}$$

where $\mathfrak{s}\in\mathfrak{T}_1$ and $S_1=R_{\xi_i,\eta,\mathfrak{e}_{\mathfrak{s}}}\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i,j)$, $S_2=R_{\xi_j,\eta,\mathfrak{e}_{\mathfrak{s}}}\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i,j)$, $S_3=R_{\xi,\xi_k,\mathfrak{e}_{\mathfrak{s}'}}\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(k,l)$, $S_4=R_{\xi,\xi_l,\mathfrak{e}_{\mathfrak{s}'}}\cap\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(k,l)$.■

Let us turn to the proof of Theorem 1.2. Let $\mathcal{C}'=(V^2,\Pi)$ be the derivative scheme of $\mathcal{P}$. It is easy to see that

$$\Delta(\Delta_2)=\bigcup_{\mathfrak{s}\in\mathfrak{T}_1}(\mathrm{Cyl}_{\mathfrak{e}_{\mathfrak{s}}}(1,2)\cap\mathrm{Cyl}_{\mathfrak{e}_{\mathfrak{s}}}(1,3)\cap\mathrm{Cyl}_{\mathfrak{e}_{\mathfrak{s}}}(1,4)),\quad R\otimes S=\mathrm{Cyl}_R(1,3)\cap\mathrm{Cyl}_S(2,4)$$

where $\Delta_2=\Delta(V)$ and R,S are basis relations of the scheme $\mathcal{C}$. So by Lemma 9.1 and the definition of 2-extension (see (10) we conclude that

$$\mathcal{C}'\geq\widehat{\mathcal{C}}.\tag{29}$$

To prove the converse inclusion by Lemma 9.1 it suffices to verify that all cylindricity, collinearity and double collinearity relations of the plane $\mathcal{P}$ are relations of the scheme $\widehat{\mathcal{C}}$. The cylindricity part of the statement follows from Theorem 3.2. To prove the collinearity part let $i, j, k \in \{1, \ldots, 4\}$ be pairwise distinct and $\mathfrak{s} \in \mathfrak{T}_1$. We note that $\mathrm{Col}_{\mathfrak{s}}(i, j, k)$ does not change under any permutation of i, j, k. So due to the closedness with respect to transposition of both $\mathcal{R}^*(\widehat{\mathcal{C}})$ and the set of all collinearity relations, without loss of generality we assume that $\{i, j\} = \{1, 2\}$. Then one can see that

$$\mathrm{Col}_{\mathfrak{s}}(i, j, k) = S \cdot \mathrm{Cyl}_{\mathfrak{e}_{\mathfrak{s}}}(1, 3)$$

where S is the intersection of all relations of the form $\mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(a, b)$ or $\mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(a, k')$ with $a, b = 1, 2, k$, $a \neq b$ and k' uniquely defined by the condition $\{k, k'\} = \{3, 4\}$. Thus the collinearity part follows from the cylindricity one. To prove the double collinearity part of the statement let $\{i, j, k, l\} = \{1, 2, 3, 4\}$ and $\mathfrak{s} \in \mathfrak{T}_1$. We note that $\mathrm{Col}^{(2)}_{\mathfrak{s}}(i, j, k, l)$ does not change under any permutation of i, j as well as k, l. So due to the closedness with respect to transposition of both $\mathcal{R}^*(\widehat{\mathcal{C}})$ and the set of all double collinearity relations, without loss of generality we can assume that $\{i, j\} = \{1, 2\}$ or $\{i, k\} = \{1, 2\}$. Then a straightforward check shows that

$$\mathrm{Col}^{(2)}_{\mathfrak{s}}(i, j, k, l) = \begin{cases} (S_1 \cdot S_2) \cap \mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i, j) \cap \mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(k, l) & \text{if } \{i, j\} = \{1, 2\}, \\ (S_3 \cdot S_4) \cap \mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}}}(i, j) \cap \mathrm{Cyl}_{\mathfrak{e}'_{\mathfrak{s}'}}(k, l) & \text{if } \{i, k\} = \{1, 2\} \end{cases}$$

where

$$S_1 = \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(i, 3) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(j, 3) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(4, 3), \quad S_2 = \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(2, k) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(2, l) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(2, 1),$$

$$S_3 = \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(i, 3) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(4, k) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(4, 3), \quad S_4 = \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(j, 1) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(2, l) \cap \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(2, 1).$$

So the double collinearity part follows from the cylindricity one. Thus due to (29) we conclude that $\mathcal{C}' \geq \widehat{\mathcal{C}}$, which proves statement (1) of the theorem. To prove statement (2) let $R \in \mathcal{R}(\mathcal{C})$. Then obviously the configuration $\Gamma = \mathrm{Der}(x_{u,v}, \mathcal{P})$ does not depend on the choice of $(u, v) \in R$ where $x_{u,v} = (u, u, v, v)$, and $R^\delta = \{x_{u,v} : (u, v) \in R\} = R(\Gamma, \mathcal{P})$ (see (11)). However, by statement (1) $R(\Gamma, \mathcal{P})$ is a basis relation of the scheme $\widehat{\mathcal{C}}$. Statement (2) is proved.■

10 Proof of Theorem 1.4

We deduce Theorem 1.4 from the following result also to be used in Section 11.

Theorem 10.1 *Let $\mathcal{C}$ and $\mathcal{D}$ (resp. $\mathcal{C}'$ and $\mathcal{D}'$) be the non-homogeneous and homogeneous schemes of a projective plane $\mathcal{P}$ (resp. $\mathcal{P}'$) respectively and let $\psi_0 : \mathcal{C} \to \mathcal{C}'$ and $\psi : \mathcal{D} \to \mathcal{D}'$ be the canonical similarities. Suppose that the nontrivial similarity φ of $\mathcal{C}$ is an m-similarity for some positive integer m. Then*

(1) $\widehat{\mathcal{D}} = \widehat{\mathcal{C}}^{\widehat{\Phi}}$ *where* $\widehat{\Phi} = \{1_{\widehat{\mathcal{C}}}, \widehat{\varphi}\}$ *with* $\widehat{\mathcal{C}} = \widehat{\mathcal{C}}^{(m)}$, $\widehat{\mathcal{D}} = \widehat{\mathcal{D}}^{(m)}$ *and* $\widehat{\varphi} = \widehat{\varphi}^{(m)}$,

(2) $\psi_0 \in \mathrm{Sim}_m(\mathcal{C}, \mathcal{C}')$ *iff* $\psi \in \mathrm{Sim}_m(\mathcal{D}, \mathcal{D}')$.

Proof. It is easily seen that $\widehat{\mathcal{C}} \geq \widehat{\mathcal{D}}$. Since obviously $\widehat{\Phi}$ leaves the set Δ_m as well as each basis relation of $\mathcal{D}^m$ fixed, it follows that

$$\widehat{\mathcal{D}} \leq \widehat{\mathcal{C}}^{\widehat{\Phi}} \leq \widehat{\mathcal{C}}. \tag{30}$$

Since $\mathcal{D}$ contains the equivalence relation $V_{\mathfrak{p}}^2 \cup V_{\mathfrak{l}}^2$, from Theorem 3.2 it follows that given $R \in \mathcal{R}(\widehat{\mathcal{D}})$ and $i, j \in \{1, \ldots, m\}$ we have either $t_1(x_i) = t_1(x_j)$ for all $x \in R$, or $t_1(x_i) \neq t_1(x_j)$ for all $x \in R$. So given $\mathfrak{s} \in \mathfrak{T}_1^m$ and $\delta \in \{0,1\}^m$ the scheme $\widehat{\mathcal{D}}$ contains both the set $X_{\mathfrak{s}} \cup X_{\mathfrak{s}'}$ where

$$X_{\mathfrak{s}} = \{x \in V^m : \ t_1(x_i) = \mathfrak{s}_i, \ i = 1, \ldots, m\} \tag{31}$$

and $\mathfrak{s}'$ is obtained from $\mathfrak{s}$ by replacing $\mathfrak{s}_i$ with its dual $\mathfrak{s}'_i$ for all i, and the relation

$$R_\delta = \{(x,y) : \ \delta_{x_i, y_i} = \delta_i, \ i = 1, \ldots, m\} = \bigcup_{\mathfrak{s} \in \mathfrak{T}_1^m} X_{\mathfrak{s}} \times X_{\overline{\mathfrak{s}}} \tag{32}$$

where $\overline{\mathfrak{s}} \in \mathfrak{T}_1^m$ is such that $\delta_{\overline{\mathfrak{s}}_i, \mathfrak{s}_i} = \delta_i$ for all i. Since $E = R_{\delta_0}$ with $\delta_0 = (0, \ldots, 0)$ is an equivalence relation on V^m with classes $X_{\mathfrak{s}}$, and the degree of the relation $(R_\delta)_{V^m/E}$ equals 1 for all δ, this implies that the scheme $\widehat{\mathcal{D}}_{V^m/E}$ is semiregular. By Theorem 2.1 with $\mathcal{C} = \widehat{\mathcal{D}}$ we conclude that

$$\widehat{\mathcal{D}} = (\widehat{\mathcal{D}}_E)^{\Psi} \tag{33}$$

where Ψ as in the theorem. In our case the homogeneity sets of the scheme $\widehat{\mathcal{D}}_{V^m/E}$ are of the form $\{X_{\mathfrak{s}}, X_{\mathfrak{s}'}\}$ where $\mathfrak{s} \in \mathfrak{T}_1^m$. So $|\Psi| = 2$.

Clearly, $\widehat{\mathcal{D}}$ contains any relation of the form $\otimes_{i=1}^m (R_i \cup R_i^{\varphi})$ where $R_i \in \mathcal{R}(\mathcal{C})$ for all i. Besides, due to (31) we have

$$\otimes_{i=1}^m R_i = (\otimes_{i=1}^m (R_i \cup R_i^{\varphi})) \cap (X_{\mathfrak{s}} \times X_{\overline{\mathfrak{s}}}) \tag{34}$$

where $\mathfrak{s}$ and $\overline{\mathfrak{s}}$ are uniquely defined from the condition $R_i \subset V_{\mathfrak{s}_i} \times V_{\overline{\mathfrak{s}}_i}$. So $\widehat{\mathcal{D}}_E \geq \mathcal{C}^m$. Taking into account that $\widehat{\mathcal{D}}_E$ also contains Δ_m, we conclude that

$$\widehat{\mathcal{D}}_E = \widehat{\mathcal{C}}. \tag{35}$$

Thus due to (30) and (33) we have $\widehat{\mathcal{D}} = \widehat{\mathcal{C}}^{\Psi} \leq \widehat{\mathcal{C}}^{\widehat{\Phi}}$. Since both the groups Ψ and $\widehat{\Phi}$ are of order 2 and act on $\mathcal{R}(\widehat{\mathcal{C}})$ without fixed points, this implies that $\Psi = \widehat{\Phi}$, whence statement (1) follows.

To prove statement (2) let $\psi_0 \in \mathrm{Sim}_m(\mathcal{C}, \mathcal{C}')$. Set $\widehat{\psi_0} = \widehat{\psi_0}^{(m)}$. Then given $R, S \in \mathcal{R}(\mathcal{C})$ we have

$$(\otimes_{i=1}^m (R_i \cup R_i^{\varphi}))^{\widehat{\psi_0}} = \otimes_{i=1}^m (R_i^{\psi_0} \cup R_i^{\varphi\psi_0}) = \otimes_{i=1}^m (R_i^{\psi_0} \cup R_i^{\psi_0 \varphi'})$$

where φ' is the nontrivial canonical similarity of $\mathcal{C}'$ (we used the straightforward fact that $\varphi\psi_0 = \psi_0\varphi'$). Since obviously $(\Delta_m)^{\widehat{\psi_0}} = \Delta'_m$, the restriction of $\widehat{\psi_0}$ to $\mathcal{R}(\widehat{\mathcal{D}})$ is the the m-extension of ψ. Thus $\psi \in \mathrm{Sim}_m(\mathcal{D}, \mathcal{D}')$. Conversely, let $\psi \in \mathrm{Sim}_m(\mathcal{D}, \mathcal{D}')$. Then from Theorem 3.2 it follows that

$$(X_{\mathfrak{s}} \cup X_{\mathfrak{s}'})^{\widehat{\psi}} = X'_{\mathfrak{s}} \cup X'_{\mathfrak{s}'}, \quad (R_\delta)^{\widehat{\psi}} = R'_\delta, \qquad \mathfrak{s} \in \mathfrak{T}_1^m, \ \delta \in \{0,1\}^m \tag{36}$$

where $\widehat{\psi} = \widehat{\psi}^{(m)}$ and $X'_{\mathfrak{s}}$ and R'_δ are defined by formulas (31) and (32) with $\mathcal{P}$ replaced by $\mathcal{P}'$. We observe that the scheme $\mathcal{D}_{V^m/E}$ being semiregular is generated by its homogeneity sets $\{X_{\mathfrak{s}}, X_{\mathfrak{s}'}\}$ and the relations $(R_\delta)_{V^m/E}$ having full support. So due to (34) and (36) the similarity $\widehat{\psi}_{V^m/E}$ is induced by the bijection $f : X_{\mathfrak{s}} \mapsto X'_{\mathfrak{s}}$. So by statement (4) of Theorem 2.1 the similarity $\widehat{\psi}$ can be extended to a similarity $\widehat{\psi}_E : \mathcal{D}_E \to \mathcal{D}'_{E'}$ such that $(\widehat{\psi}_E)_{V^m/E}$ is induced by f. Then from (34) it follows that

$$(\otimes_{i=1}^m R_i)^{\widehat{\psi}_E} = (\otimes_{i=1}^m (R_i \cup R_i^\varphi))^{\widehat{\psi}_E} \cap (X_{\mathfrak{s}} \times X_{\bar{\mathfrak{s}}})^{\widehat{\psi}_E} = \otimes_{i=1}^m (R_i \cup R_i^\varphi)^\psi \cap (X_{\mathfrak{s}}^f \times X_{\bar{\mathfrak{s}}}^f) =$$

$$\otimes_{i=1}^m (R_i \cup R_i^\varphi)^{\psi_0} \cap (X'_{\mathfrak{s}} \times X'_{\bar{\mathfrak{s}}}) = \otimes_{i=1}^m (R_i^{\psi_0} \cup R_i^{\psi_0\varphi'}) \cap (X'_{\mathfrak{s}} \times X'_{\bar{\mathfrak{s}}}) =$$

$$\otimes_{i=1}^m (R'_i \cup (R'_i)^{\varphi'}) \cap (X'_{\mathfrak{s}} \times X'_{\bar{\mathfrak{s}}}) = \otimes_{i=1}^m R'_i$$

where φ' is the nontrivial similarity of the scheme $\mathcal{C}'$. Besides, obviously $(\Delta_m)^{\widehat{\psi}_E} = \Delta'_m$, and $\mathcal{D}_E = \widehat{\mathcal{C}}$, $\mathcal{D}'_{E'} = \widehat{\mathcal{C}}'$ (see (35)). Thus $\widehat{\psi}_E$ is the m-extension of ψ_0 and hence $\psi_0 \in \mathrm{Sim}_m(\mathcal{C}, \mathcal{C}')$.■

Now statements (1) and (3) follows from Theorem 1.3 and Theorem 10.1 for $m = 2$, whereas statement (2) is a consequence of statement (1) and the 2-closedness of the scheme $\mathcal{C}$ (statement (2) of Theorem 1.2).■

11 Proof of Theorem 1.7

For a projective plane $\mathcal{P}$ and $\mathfrak{s} \in \mathfrak{T}_1$ we define a binary relation on V^3 where V is the set of elements of $\mathcal{P}$ as follows:

$$C_{\mathfrak{s}} = \{(x, y) \in T_{\mathfrak{s}} \times T_{\mathfrak{s}} :\ \exists v \in V :\ t_1(v) = \mathfrak{s} \text{ and } v, x_i, y_i \text{ are collinear } i = 1, 2, 3\} \tag{37}$$

where $T_{\mathfrak{s}} = \{x \in V^3 :\ t_1(x_1) = t_1(x_2) = t_1(x_3) = \mathfrak{s}$ and x_1, x_2, x_3 are not collinear$\}$ (see footnote on page 25). Treating the elements of $T_{\mathfrak{p}}$ and $T_{\mathfrak{l}}$ as triangles in $\mathcal{P}$ we see that $C_{\mathfrak{p}}$ and $C_{\mathfrak{l}}$ consist of pairs of triangles that are in perspective from some point and from some line respectively. Thus $\mathcal{P}$ is a Desarguesian plane if and only if

$$C_{\mathfrak{p}} = E \cdot C_{\mathfrak{l}} \cdot E^T \tag{38}$$

where

$$E = \{(x, y) \in T_{\mathfrak{p}} \times T_{\mathfrak{l}} :\ y_1 = x_2x_3,\ y_2 = x_3x_1,\ y_3 = x_1x_2\}. \tag{39}$$

We observe that the finite Desarguesian projective planes are exactly the Galois planes. On the other hand, it is easily seen that

$$C_{\mathfrak{s}} = R \cdot R^T, \quad \Delta(T_{\mathfrak{s}}) = \Delta(V_{\mathfrak{s}}^3) \setminus (S_{\mathfrak{s}} \cdot S_{\mathfrak{s}}^T), \tag{40}$$

$$E = (\bigcap_{1 \le i \ne j \le 3} \mathrm{Cyl}_{\mathfrak{i}_{\mathfrak{s}}}(i, j+3)) \cap (T_{\mathfrak{p}} \times T_{\mathfrak{l}}) \tag{41}$$

where

$$R = (\bigcap_{1\le i\le 3} \mathrm{Cyl}_{\mathfrak{i}_\mathfrak{s}}(i,i+3)) \cap (T_\mathfrak{s} \times (V_\mathfrak{s}^3 \setminus T_\mathfrak{s})), \qquad \Delta(V_\mathfrak{s}^3) = \bigcap_{1\le i\le 3} \mathrm{Cyl}_{\mathfrak{e}_\mathfrak{s}}(i,i+3)$$

$$S_\mathfrak{s} = \mathrm{Cyl}_{\mathfrak{i}_\mathfrak{s}}(1,4) \cap \mathrm{Cyl}_{\mathfrak{i}_\mathfrak{s}}(2,4) \cap \mathrm{Cyl}_{\mathfrak{e}_\mathfrak{s}}(3,4).$$

From now on we assume that $\mathcal{P}$ is a Galois plane. We observe that the nontrivial similarity of the non-homogeneous scheme $\mathcal{C}$ of $\mathcal{P}$ is induced by any polarity of this plane. So it is an m-similarity for all $m \ge 1$. Now let $\varphi : \mathcal{C} \to \mathcal{C}'$ be a 3-similarity. Then without loss of generality we assume that $\mathcal{C}'$ is the scheme of a projective plane $\mathcal{P}'$ and φ is canonical. From Theorem 3.2 it follows that $C_\mathfrak{s}$ and E (resp. $C'_\mathfrak{s}$ and E') are relations of the scheme $\widehat{\mathcal{C}}^{(3)}$ (resp. $\widehat{\mathcal{C}'}^{(3)}$) where $C'_\mathfrak{s}$ and E' are the binary relations defined for $\mathcal{P}'$ by (37) and (39) respectively. Moreover, the same theorem implies that given $\mathfrak{s} \in \{\mathfrak{p}, \mathfrak{l}\}$ we have

$$(C_\mathfrak{s})^{\widehat{\varphi}} = C'_\mathfrak{s}, \quad E^{\widehat{\varphi}} = E'$$

where $\widehat{\varphi} = \widehat{\varphi}^{(3)}$. On the other hand, since $\mathcal{P}$ is a Galois plane, equality (38) holds in $\mathcal{C}$ and hence in $\mathcal{C}'$. Thus $\mathcal{P}'$ is also a Galois plane. Since $\mathcal{P}$ and $\mathcal{P}'$ are of the same order, they are isomorphic. The similarity φ being the canonical one, is induced by any isomorphism of these planes. Thus $\mathrm{Sim}_3(\mathcal{C},\mathcal{C}') = \mathrm{Sim}_\infty(\mathcal{C},\mathcal{C}')$ and hence $s(\mathcal{C}) \le 3$.

Let $\mathcal{D}$ be the homogeneous scheme of the plane $\mathcal{P}$ and $\psi : \mathcal{D} \to \mathcal{D}'$ a 3-similarity. Without loss of generality we assume that $\mathcal{D}'$ is a homogeneous scheme of a projective plane $\mathcal{P}'$ and the similarity ψ is the canonical one. Since the nontrivial similarity of $\mathcal{C}$ is a 3-similarity (see above), by Theorem 10.1 for $m = 3$ so is the canonical similarity $\psi_0 : \mathcal{C} \to \mathcal{C}'$ where $\mathcal{C}$ and $\mathcal{C}'$ are non-homogeneous schemes of $\mathcal{P}$ and $\mathcal{P}'$ respectively. By the first part of this corollary ψ_0 is induced by some isomorphism. Clearly, it also induces ψ.■

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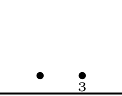
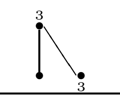
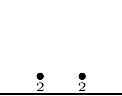
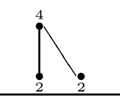
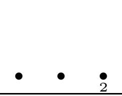
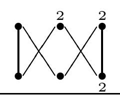
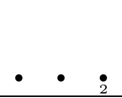
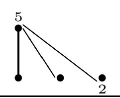
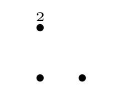
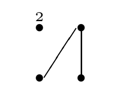
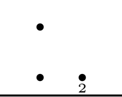
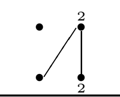
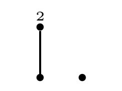
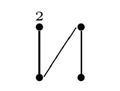
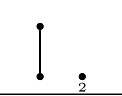
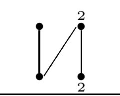
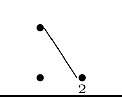
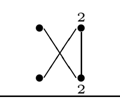
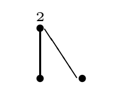
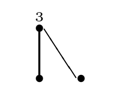
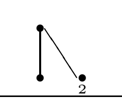
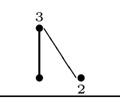
Appendix

In the table below we enumerate all Ξ-configurations Γ belonging to the set $\mathrm{Der}(\mathcal{P})$ where $\mathcal{P}$ is a projective plane of order q. Any such configuration satisfies the following properties: (a) any two vertices of Γ belonging to the same class are equal (the type of the corresponding edge is $\mathfrak{e}$), (b) any two vertices of Γ belonging to incident classes are incident, (c) there are no quadrangles in Γ (see Sections 4 and 5). We represent a configuration as a bipartite graph the parts of which consist of classes of the same type, and the edges correspond to incident classes (3th column); the restriction of Γ to Ξ is represented in the 2nd column. The label of a class is the cardinality of it; we omit the label if the class is a singleton. In the 5th column of the table we write down the number of Ξ-configurations represented by the graph in the corresponding row; the number in the 6th column shows that the corresponding row applies to a projective plane of order greater or equal than that number.

N	Γ_Ξ	Γ	Comment	n	q
1			4 equal elements of the same type	2	1
2			4 elements of the same type 3 of which are equal	8	1
3			2 pairs of equal elements of the same type	6	1
4			3 equal elements not incident to 4th	8	1
5			2 pairs of equal elements of distinct types that are not incident	6	1
6			3 equal elements incident to 4th	8	1
7			2 pairs of equal elements that are incident	6	1
8			3 elements of the same type in general position, 4th equals to one of them	12	1
9			3 collinear elements in general position, 4th equals to one of them	12	2
10			2 distinct elements of the same type not incident to 2 equal elements of the other type	12	2
11			3 elements with 2 of them being equal are not incident to 4th	24	2
12			2 distinct elements of the same type one of which incident to 2 equal elements of the other type	24	1
13			2 incident elements and 2 equal elements in general position	24	1
14			2 non-incident elements and 2 equal elements incident to one of them	24	1
15			2 equal elements of the same type are incident to 2 distinct elements of the other type	12	1
16			2 incident elements one of which is incident to 2 equal elements	24	2

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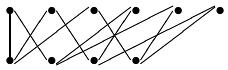
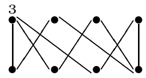
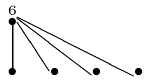
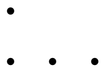
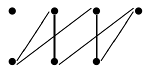
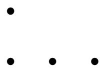
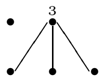
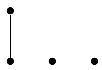
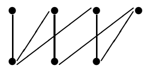
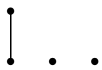
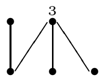
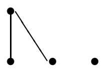
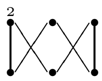
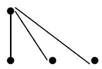
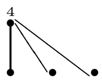
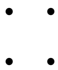
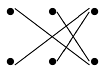
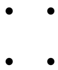
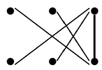
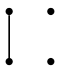
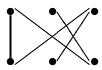
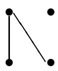
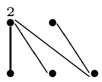
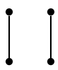
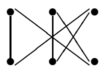
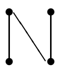
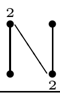
17			4 elements of the same type in general position	2	2
18			4 non-equal elements of the same type 3 of which are collinear	8	2
19			4 non-equal collinear elements	2	3
20			3 elements of the same type in general position not incident to the 4th	8	2
21			3 non-equal collinear elements not incident to the 4th	8	3
22			3 elements of the same type in general position one of which incident to the 4th	24	2
23			3 non-equal collinear elements one of which incident to the 4th	24	2
24			3 elements of the same type in general position 2 of which incident to the 4th	24	1
25			1 element incident to 3 distinct elements	8	2
26			2 elements of the same type and 2 elements of the other type in general position	6	3
27			the double collinearity	6	2
28			2 incident elements and 2 non-incident elements in general position	24	2
29			2 non-equal elements incident to the 3th element and not incident to the 4th	24	2
30			two pairs of 2 incident elements in general position	12	1
31			2 incident elements on of which incident to the 3th element and another one to the 4th	24	1