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Non-degenerate colorings in the Brook's Theorem

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Abstract

Let $c \geq 2$ and $p \geq c$ be two integers. We will call a proper coloring of the graph G a (c,p)-*nondegenerate*, if for any vertex of G with degree at least p there are at least c vertices of different colors adjacent to it.

In our work we prove the following result, which generalizes Brook's Theorem. Let $D \geq 3$ and G be a graph without cliques on $D+1$ vertices and a degree of any vertex in this graph is not greater than D. Then for every integer $c \geq 2$ there is a proper (c,p)-nondegenerate vertex D-coloring of G, where $p = (c^3 + 8c^2 + 19c + 6)(c-1)$.

During the primary proof, some interesting corollaries are derived.

Key words: Brook's Theorem, conditional colorings, non-degenerate colorings, dynamic colorings.

Introduction

We follow the terminology and notations of the book [5] and consider finite and loopless graphs. As in [5], $\delta(G)$ and $\Delta(G)$ denote the minimal and the maximal degree of a graph G respectively. For a vertex $v \in V(G)$ the *neighborhood* of v in G is $N_G(v) = \{u \in V(G) : u \text{ is adjacent to } v \text{ in } G\}$. Vertices in $N_G(v)$ are called neighbors of v. Also $|S|$ denotes the cardinal number of a set S.

For an integer $k > 0$, let $\overline{k} = \{1, 2, \cdots, k\}$. A *proper k-coloring* of a graph G is a map $c : V(G) \mapsto \overline{k}$ such that if $u, v \in V(G)$ are adjacent vertices in G, then $c(u) \neq c(v)$. Let c is a proper k-coloring of G and a set $V' \subseteq V(G)$, then by $c(V')$ we denote a restriction of the map c to the set V', so we get a proper k-coloring of the induced graph $G(V')$.

A proper vertex k-coloring is a *proper conditional (k, c)-coloring*, if for any vertex of degree at least c there are at least c different colors in its neighborhood. This notion for $c = 2$ appeared in the works [3] and [4] as a dynamic coloring. But results obtained there were not the Brook's Theorem generalizations, because a number of colors in which graph was colored is bigger then it is in the Brook's Theorem.

Further development of this theme can be found in the work [6] where a definition of the conditional coloring has been given for the first time. In this paper authors remarked that it would be interesting to know an analogous of Brook's Theorem for conditional colorings. But the problem of finding such an analogous seems to be too hard in such formulations. Let us show consideration, which lets one to think about changing the statement. If there is a vertex of degree c in a graph, then in any (k, c)-coloring all its neighbors will be colored in different colors and it means that we can replace this vertex by c-hyperedge on its neighborhood. Repeating such transformations with the graph, we can obtain any c-hyperedge graph. So we would be able to extend all our results on such graphs. But a graph with hyperedges is a complicated object for investigation concerning proper colorings, so it seems natural to bring in a new definition instead of the one considered in previous papers.

The crucial consideration, which allowed us to get serious progress in this field, is that we demand another condition of non-degenerateness of a proper coloring. We will call this demand the (c, p)-nondegenerateness.

Definition. Let $c \geq 2$ and $p \geq c$ be positive integers. We call a vertex coloring of a graph G *(c, p)-nondegenerate* if for any vertex, with degree at least p, there are at least c vertices of different colors among all its neighbors.

So, speaking informally, we impose the requirement of nondegenerateness only to vertices of a large degree. But with such a weaker new requirement, we can state and prove stronger and more general theorem and receive several interesting corollaries during the proof.

Theorem 1. *Let $D \geq 3$ and G be a graph without cliques on $D + 1$ vertices and $\Delta_G \leq D$. Then for every integer $c \geq 2$ there is a proper (c, p)-nondegenerate vertex D-coloring of G, where $p = (c^3 + 8c^2 + 19c + 6)(c - 1)$.*

One of the main steps in the proof of the *theorem 1* is the following *theorem 2*, which by itself appears to be an interesting result.

Theorem 2. *Let G be a graph with no cliques on $D + 1$ vertices with $\Delta_G \leq D$. And let $D = \sum_{i=1}^{c+1} \alpha_i$, where $\alpha_i \geq 2$ are integer numbers. Then in the set Ξ of all colorings of G in $c + 1$ colors there is a coloring ξ such that:*

1) $\Phi(\xi) = \min\limits_{\psi \in \Xi} \Phi(\psi)$*, where* $\Phi = \sum\limits_{i=1}^{c+1} \frac{f_i}{\alpha_i}$ *and* f_i *is a number of edges in* G *connecting vertices of the i-th color.*
2) For any $1 \geq i \leq c+1$*, there are no cliques on* $\alpha_i + 1$ *vertices of the i-th color in* ξ*.*

In particular, there is a direct corollary from the *theorem 2*, which is similar to the result, obtained by L. Lovasz in the paper [2].

Corollary. *Let* G *be a graph with no cliques on* $D+1$ *vertices with* $\Delta_G \leq D$*. And let* $D = \sum\limits_{i=1}^{k} \alpha_i$*, where* $\alpha_i \geq 2$ *are integer numbers. Then the set* $V(G)$ *can be split into* k *subsets* $V_1, V_2, \dots, V_k$ *so that for any* $i \in [1, k]$ *there are no cliques on* $\alpha_i + 1$ *vertices in* $G(V_i)$ *and* $\Delta_{G(V_i)} \leq \alpha_i$*.*

Main theorem proof

Remark 1. The (c, p)-nondegenerateness of a coloring is a rather strong condition even in a case of a bipartite graph (and coloring it in p colors), since it is not easy to prove the statement analogous to the *theorem 1*. And if we want to get a (c, p)-nondegenerate proper D-coloring of a bipartite graph but do not bound the maximal degree of this graph, then the *theorem 1* doesn't hold for $c = 2$ and every p.

Contrary instance:

We take a set S_1 consisting of $(p-1)D + 1$ elements as the first part of G. As the second part of G, we take the set of all p-element samplings from S_1 and join every such sampling with all its elements in S_1 (see fig. 1). If we try to color G in D colors, then by the Dirichlet principle in the set S_1 one can find p vertices of the same color and this means that for correspondent p-element sampling in S_2 the $(2, p)$-nondegenerate condition does npt hold.

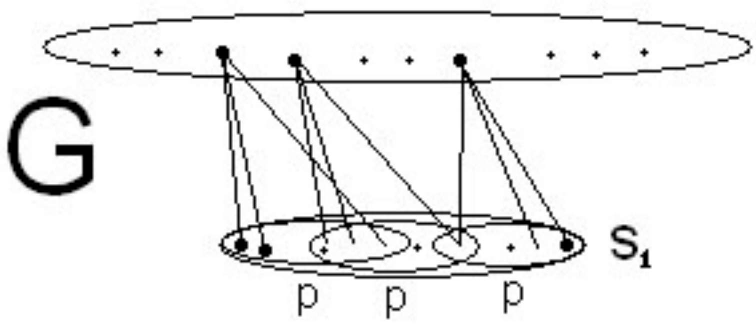

fig. 1.

Remark 2. Unfortunately our estimation $p(c) = (c^3 + 8c^2 + 19c + 6)(c - 1)$ gives rather large value for a small c. It is quite possible that using our proof method one can get a better estimation, but it is impossible to get an estimation asymptotically better than $c^4(1 + O(c^{-1}))$ using only our method.

Theorem 1. *Let* $D \geq 3$ *and* G *be a graph without cliques on* $D+1$ *vertices and* $\Delta_G \leq D$*. Then for every integer* $c \geq 2$ *there is a proper* (c, p)*-nondegenerate vertex* D*-coloring of* G*, where* $p = (c^3 + 8c^2 + 19c + 6)(c - 1)$*.*

Statement 1. *Without loss of generality graph G may be thought of as a graph containing no vertices of degree less than p.*

Proof. The following operation can be done with G: take two copies of G and join in this copies all pairs of similar vertices with degree less than p (see fig. 2).

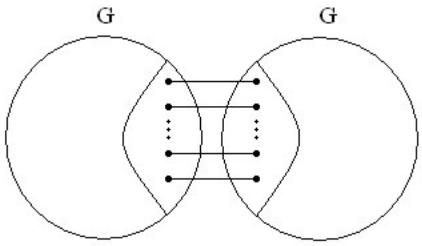

fig. 2.

Obtained graph satisfies all conditions of the *theorem 1*. Also notice that if we get a (c,p)-nondegenerate proper D-coloring of the obtained graph then we get the same for an every copy of G. We repeat this operation while there're vertices of degree less than p. We repeat this operation a finite number of times because, by every execution of such operation, we increase the smallest degree of a graph.

$\square$

Proof. The proof of *theorem 1* consists of two parts. In the first part we reduce our theorem to some lemma (see lemma 1). And in the second part we prove this lemma.

First part.

Choose such a number α_i for every $i \in \{1, 2, ..., c+1\}$, that $\alpha_i = \lceil \frac{D}{c+1} \rceil$ or $\alpha_i = [\frac{D}{c+1}]$ and $\sum\limits_{i=1}^{c+1} \alpha_i = D$ (it is clear that we can choose such a set of α_i). Consider for every coloring ξ in colors $\{1, 2, ..., c+1\}$ a function $\Phi(\xi)$ which is determined as follows: $\Phi(\xi) := \sum\limits_{i=1}^{c+1} \frac{f_i}{\alpha_i}$, where f_i is a number of edges connecting vertices of the i-th color in the coloring ξ. Then consider those colorings of the graph G in $c+1$ colors for which Φ reaches its minimum. Denote such a set of colorings as G_c. It is obvious that G_c is not empty. Then for any coloring ξ from the set G_c the following statements hold:

Statement 2. *For every color $i \in \{1, 2, ..., c+1\}$ in ξ and every i-th color vertex v of G a number of vertices of the i-th color adjacent to v of the i-th color does not exceed α_i.*

Proof. Suppose the statement is false. Then from condition that $\sum\limits_{j=1}^{c+1} \alpha_j = D$ there can be found a color j such that v is adjacent in the graph G to less than α_j vertices of the j-th color. So by recoloring v in the color j we arrive at a contradiction. $\square$

Statement 3. *If some vertex v of the i-th color in the coloring ξ of G is adjacent to exactly α_i vertices of the i-th color then v is adjacent to exactly α_j vertices of the j-th color for every color j.*

Proof. Assume the opposite to the *statement 3* assertion. Then by condition that $\sum\limits_{k=1}^{c+1} \alpha_k = D$ there can be found a color $j' \neq i$ such that v is adjacent in G to less than $\alpha_{j'}$ vertices of the j'-th color. So by recoloring v in the color j' we arrive at a contradiction. □

Statement 4. *If the vertex v of the i-th color in the coloring ξ of the graph G is adjacent to at least one vertex of the i-th color then it is adjacent to at least one vertex of any other color.*

Proof. Suggesting that statement fails we arrive at a contradiction with minimality of $\Phi(\xi)$ by recoloring v in the color with which v is not adjacent to. □

We prove now that there is a coloring in the colorings set G_c with no cliques in G on $\alpha_i + 1$ vertices of the i-th color. We will call such cliques the **large** cliques.

Due to the *statement 2* there can not be bigger cliques of the i-th color in G for any coloring from G_c.

For every coloring ξ from G_c denote as $\phi(\xi)$ a number of *large* cliques in ξ. Denote by Ω the set of all colorings from G_c with the smallest number of the *large* cliques. Let $\phi > 0$ for colorings from Ω.

Then using the *statement 3* we get:

Statement 5. *If we take a vertex v from some large clique in some coloring $g_c \in \Omega$ and recolor this vertex in any other color then an obtained coloring $g'_c \in G_c$ and $\phi(g'_c) \leq \phi(g_c)$.*

In *statement 5* we took ϕ to be the minimal on colorings from G_c, so a number of *large* cliques shouldn't change. And this means that the *large* clique should appear on vertices of the color in which we recolored v, besides we get that $g'_c \in \Omega$.

Statement 6. *Let coloring $\xi_1 \in \Omega$ and $\phi(\xi_1) > 0$. Let C_1 be the large clique of the i-th color. Consider the induced subgraph G_{ij} of G on all vertices of the i-th and j-th colors. Then connectivity component containing C_1 in the graph G_{ij} constitute a complete graph on $\alpha_i + \alpha_j + 1$ vertices.*

Proof. Recolor arbitrary vertex $v_1 \in C_1$ in the color j. According to *statement 5* we get a new coloring $\xi_2 \in \Omega$. At the same time v_1 should be included in some *large* clique C_2 of the j-th color. Recolor some distinct from v_1 vertex v_2 from the clique C_2 in the color i. Again according to *statement 5* we get a new coloring $\xi_3 \in \Omega$ in which v_2 necessarily should be included in some *large* clique C_3 of the i-th color. And so on: we recolor vertices in such a manner until we get a *large* clique a part of which we just have considered earlier (see fig. 3, where four recolorings have been done and $\alpha_i = \alpha_j = 3$).

1.a) At the end we return to a part of the clique C_1 and a number of recolorings is greater than two, i.e. the last coloring is ξ_k where $k \geq 3$. Recolor in the coloring ξ_1 some another than v_1 vertex v in the clique C_1 in j. According to the *statement 5* we get a *large* clique containing v_k and v of the color j and therefore the following holds: any vertex $v \in C_1$, where $v \neq v_1$, is adjacent to all vertices in C_k except v_{k-1}.

Draw the following conclusion:

Any vertex $u \in C_k$, where $u \neq v_{k-1}$, is adjacent to all vertices in C_1 except v_1.

Recolor vertex v in ξ_1 in the j-th color and then recolor some vertex $u \in C_k$ distinct from v_{k-1} and v_k in the i-th color (we can choose such a vertex u because of $\alpha_i \geq 2$ and $\alpha_j \geq 2$). So we get a coloring $\xi' \in G_c$ with a smaller value of ϕ as u is adjacent to all vertices in C_1

except v_1. The following figure 3 call upon to illustrate process of recolorings for $k = 4$ and $\alpha_i = \alpha_j = 3$.

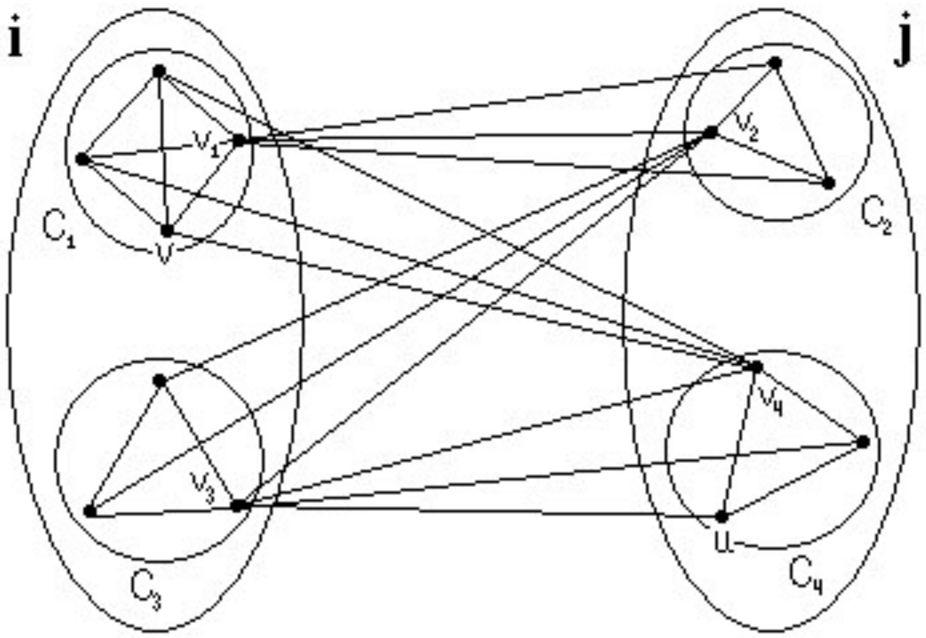

fig. 3.

1.b) Point out that if it was only two recolorings and we return to a part of the clique C_1 then the vertex v_2 is adjacent to all vertices in C_1 and so by recoloring of any vertex from the *large* clique C_1 in the j-th color we will get by *statement 5* a new *large* clique of the j-th color containing $C_2 \setminus \{v_1\}$. So $G(C_1 \cup C_2)$ is a complete graph. From arbitrariness of the v_1 and v_2 choice and from the fact that $G(C_1 \cup C_2)$ is a $\alpha_i + \alpha_j + 1$ size clique it follows that vertices from the set $C_1 \cup C_2$ are not adjacent to the rest vertices of the i-th and j-th colors.

2) If we stop the process of recolorings on a clique C_l where l not necessary equals to 1 then from the above reasoning it is clear that (we can assume that we start the process from ξ_l) $C_l \cup C_{l+1}$ constitute a clique in G. And so we get $l = 1$, because vertices from $C_l \cup C_{l+1}$ and the rest vertices of the i-th and j-th colors are not adjacent to.

□

Remark 3. Note that at the *statement 6* proof we make essentially use of $\alpha_i \geq 2$ and $\alpha_j \geq 2$. In other case we just could not chose a vertex distinct from all v_i.

Statement 7. *In any coloring $g_c \in \Omega$ there are no large cliques.*

Proof. There is a coloring $g_c \in \Omega$ with a *large* clique C on vertices of the i-th color. Without loss of generality suppose that $i = 1$. Apply the *statement 6* to the first and the second colors. We get a complete graph containing C on $\alpha_1 + \alpha_2 + 1$ vertices of the first and the second colors. We can split in arbitrary way this complete graph into two parts of the first and the second colors with correspondent sizes $\alpha_1 + 1$ and α_2 preserving remain coloring of the graph and an obtained coloring would also lay in Ω. By the *statement 6* and above consideration applying to the first and the i-th color ($i \in [2, c+1]$) there would be a complete subgraph of G on $1 + \sum\limits_{j=1}^{c+1} \alpha_j$ vertices, i.e. the complete subgraph on $D + 1$ vertices – contradiction with a condition of the *theorem 1*.

□

Remark 4. In fact we just now proved the *theorem 2*. Also note that desired in the *theorem 1* coloring ξ assign a partition of all vertices of the graph into required in the *corollary* sets.

Remark 5. Consider the particular coloring $g_c \in \Omega$. We just show that in g_c there is no *large* cliques. So using the Brook's theorem for any color in g_c we get a α_i proper coloring on i-th color vertices, so as a result we get a proper coloring of G in D colors ($\sum\limits_{j=1}^{c+1} \alpha_i = D$). If a vertex in the coloring g_c is adjacent to some vertex of its color, then by *statement 4* there should be at least $c+1$ vertices of different colors in the neighborhood of such a vertex. In other words a main problem is to color "singular" vertices, i.e. vertices not adjacent to its and some other colors in the coloring g_c. In fact, if G is a bipartite graph then the theorem about proper coloring it in D colors with the fulfilment of (c,p)-nondegenerateness would be none trivial fact. And a proof of the theorem for the case of a bipartite graph would show you a difficulty and specificity of the problem.

Consider a coloring $g_c \in \Omega$ and consider in it all vertices adjacent to less than $c-1$ different colors. Denote a set of all such vertices by Υ. Notice that every vertex $v \in \Upsilon$ has no adjacent vertices of the same as v color in the coloring g_c and there is some another color such that v has no adjacent vertices of this color. So we can change color $v \in \Upsilon$ by another such that obtained coloring as before would be from Ω. Moreover we can change color of any part of vertices from Υ of an i-th color so that obtained coloring will be in Ω (of course we would recolor this vertices probably in different colors). For every vertex $v \in \Upsilon$ there could be found a color in g_c such that v is adjacent to at least $\lceil \frac{p}{c-1} \rceil$ vertices of this color. So we can divide Υ into $c+1$ sets $\theta_1, \theta_2, ..., \theta_{c+1}$, in such a way that every vertex from θ_i is adjacent to at least $\lceil \frac{p}{c-1} \rceil$ vertices of the i-th color.

Denote by H_i for all $i \in [1, c+1]$ an induced subgraph of G on the vertices of the i-th color in the coloring g_c.

Statement 8. *For any vertex $v \in H_i$ the following inequality holds:*

$$\lceil d_{H_i}(v) + \frac{d_{G(\theta_i \cup \{v\})}(v)}{c+2} \rceil \leq \alpha_i.$$

Proof. Consider a set E_v of all edges in the graph G with one end at v. It's obviously that $|E_v| \leq D$. Consider a set E_1 of all edges from E_v which has the second end vertex distinct from v not laying in θ_i. Let from v there lead less than $\frac{\alpha_j}{\alpha_i} d_{H_i}(v)$ edges from the set E_1 into the color j distinct from i. Then we change the color of all vertices of the j-th color from the set $\theta_i \subseteq \Upsilon$ in such a way that an obtained coloring would lay in Ω. Clearly we recolored this vertices not in the color of v, so $d_{H_i}(v)$ doesn't change in the obtained coloring. If we recolor v in the new coloring in the j-th color then a magnitude less than $\frac{\frac{\alpha_j}{\alpha_i} d_{H_i}(v)}{\alpha_j} - \frac{d_{H_i}(v)}{\alpha_i} = 0$ is added to the value of Φ, thus we find a coloring with a smaller value of Φ and so we arrive at a contradiction.

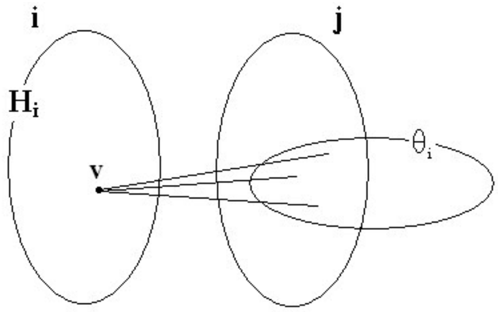

fig. 4.

So we can get the following lower bound on the number of edges coming from v:

$|E_v| \geq d_{H_i}(v) + d_{G(\theta_i \cup \{v\})}(v) + \sum\limits_{j \neq i} \frac{\alpha_j}{\alpha_i} d_{H_i}(v) = \sum\limits_{j=1}^{c+1} \frac{\alpha_j}{\alpha_i} d_{H_i}(v) + d_{G(\theta_i \cup \{v\})}(v) = \frac{D}{\alpha_i} d_{H_i}(v) + d_{G(\theta_i \cup \{v\})}(v)$.

By definition $|E_v| \leq D$. So we get:

$$D \geq \frac{D}{\alpha_i} d_{H_i}(v) + d_{G(\theta_i \cup \{v\})}(v) \Rightarrow \alpha_i \geq d_{H_i}(v) + \frac{\alpha_i}{D} d_{G(\theta_i \cup \{v\})}(v) \ .$$

Then by using fact that $\alpha_i \geq [\frac{D}{c+1}]$ and $D \geq (c^3 + 8c^2 + 19c + 6)(c-1)$ we get $\frac{\alpha_i}{D} > \frac{1}{c+2}$. So we get

$$\alpha_i \geq d_{H_i}(v) + \frac{d_{G(\theta_i \cup \{v\})}(v)}{c+2} \ .$$

□

Second part.

Lemma 1. *Let there are given two non empty sets A and B and a connected graph $H = (A \cup B, E)$. Denote G as the induced subgraph $H(B)$. Define $d_A(v), v \in B$ to be a number of edges coming from v to the set A. Let the graph H satisfy the following conditions:*

1) every two vertices of A are not join with an edge;

2) a degree of every vertex from A in the graph H is at least d, where $d = q^3 + 2q^2 - q - 8$ and $q \geq 4$;

3) for any vertex $v \in B$, the following inequality holds:

$$d_G(v) + \lceil \frac{d_A(v)}{q} \rceil \leq d. \tag{1}$$

Then the graph G could be properly colored into d colors in such a way, that for any vertex $v \in A$ among, all its neighbors in B there are vertices of at least q different colors.

Remark 6. In the *lemma 1*, the set B denotes H_i from the *first part*, the set of vertices A denotes θ_i from the *first part*. Also it makes no difference for us whether there are any edges between vertices from θ_i. We only need to know to which vertices from H_i vertices from θ_i adjacent, because we would color vertices only from H_i.

As q in *lemma 1*, we denoted the value of $c + 2$ from the *first part* and as d we denoted the value of α_i. Via H in the *lemma 1* we denoted the graph $G(\theta_i \cup H_i) - E(G(\theta_i))$. By definition of the set θ_i from any vertex $v \in \theta_i$ there comes at least $\frac{p}{c-1}$ edges in the set $V(H_i)$.

We will suppose in the *lemma 1* that the graph H is connected (in other case it is sufficient to prove for every connectivity component). Furthermore we can assume that θ_i is not empty, otherwise we prove the Brook's Theorem because of we need to color properly graph H_i into α_i colors, and we know that in H_i there are no complete subgraphs on $\alpha_i + 1$ vertices (in H_i there are no *large* cliques) and $d_{H_i} = \lceil d_{H_i}(v) + \frac{d_{G(\theta_i \cup \{v\})}(v)}{c+2} \rceil \leq \alpha_i$. Thus, all the conditions of *lemma 1* are satisfied for the sets $B = V(H_i)$ and $A = \theta_i$.

Suppose the *lemma 1* has been already proven. Then, if we color for every i the subgraph H_i in the coloring g_c of G in a proper way in a new α_i colors such that every vertex from θ_i would be adjacent to vertices of at least c different colors then we get a proper D-coloring of the whole graph G. At that the vertices from the set $\Upsilon = \bigcup\limits_{i=1}^{c+1} \theta_i$ would be adjacent to, at least, c vertices of different colors. Moreover in accordance with the definition of Υ all the vertices from the set $V(G) \setminus \Upsilon$ would be adjacent to at least c vertices of different colors. Thus, we reduce the *theorem 1* to the *lemma 1*.

Remark 7. The second part is devoted to the proof of *lemma 1*. So to avoid a misunderstanding for a coincidence of notations let us say that notations from the **first part** have no connection with notations from the **second part**.

Remark 8. In the assertion of the *lemma 1* it is possible to change q on $q - 2$, but we will not do this for convenience sake of calculations.

Remark 9. $(c+2)^3 + 2(c+2)^2 - (c+2) - 8 = c^3 + 8c^2 + 19c + 6.$

Proof of the Lemma 1. Suggest that assertion of the *lemma 1* fails. Then, consider the smallest for a number of vertices graph for which all the assumptions of the *lemma 1* holds but the statement of the *lemma 1* fails.

Definition 1. We will call a *permissible* the set $S_i \subseteq B$ if $S_i \subset N_H(v_i)$, where $v_i \in A$ and $|S_i| = q$. A set of all samplings of *permissible* sets for all $i \in \{1, 2, ..., |A|\}$ we will denote as Λ.

The assertion of out lemma abide by the following fact:

Fact. *For every vertex v_i from A we can choose a permissible set S_i in such a way that if we add to the edges set $E(G)$ all complete graphs on sets S_i where $i \in \{1, 2, ..., |A|\}$ then it is possible to color vertices of the obtained graph $\widetilde{G}$ properly in d colors.*

Remark 10. We would consider $\widetilde{G}$ as a graph with multiedges.

Remark 11. So we get an equivalent statement of the *lemma 1*.

Remark 12. In the new formula, it is convenient to make some reduction with a graph as follows:

Let there be a vertex $\hat{(v)}$ of degree d in a graph $\widetilde{G}$, then it is possible to "delete" this vertex from the graph $\widetilde{G}$ and prove a statement of the *fact* for the graph $\widetilde{G} \setminus \hat{v}$.

Definition 2. We will say that $\hat{v}$ is recursively deleted from $\widetilde{G}$ if there is a sequence of reductions described above with the last $\hat{v}$ reduction. We will call a graph $\widetilde{G}$ to be a recursive one, if it reduces to the empty graph.

Remark 13. Let us explain why we call such a reduction as a recursion. The matter is that if a graph reduces to the empty one then we will color it just by recursion.

Actually we will prove the following stronger fact:

Instead of the statement that $\widetilde{G}$ *properly colored into* d *colors, we will prove that* $\widetilde{G}$ *is a recursive with respect to coloring it in* d *colors.*

Return to the lemmas proof and more specifically to the proof of the stronger *fact.* Denote as S the set of vertices from B which are adjacent to at least one vertex in A.

Prove that for the graph H the strengthen *fact* will holds in assumption that H is minimal for the number of vertices a graph for which the statement of the *lemma 1* fails. Thus, we will arrive at a contradiction and so we will prove the *lemma 1.*

Definition 3. Define for any vertex v from the set B the magnitude

$$L(v) := d_{_G}(v) + \frac{d_{_A}(v)}{q+1}\ .$$

Remark 14. Notice that if we choose a sampling S_i at random (independently for any vertex v_i where all possible variants of a set S_i are equiprobable), then the distribution average of a variate of the degree in the graph $\widetilde{G}$ for any vertex from the set S would be not greater than $d_{_G}(v) + d_{_A}(v)(q-1)\frac{q}{q^3+2q^2-q-8}$, i.e. the degree would be not greater than $L(v)$ (since $q \geq 4$ then $q^3+2q^2-q-8 > q(q^2-1)$) and by the third condition of *lemma 1* would be less than d. Thus, at the average a degree of every vertex in $\widetilde{G}$ is less than d. And this gives us hope that the graph $\widetilde{G}$ turns out to be a recursive one, i.e. if we would successively deleted vertices from the $\widetilde{G}$ with degree less than d then we arrive to the empty graph.

For a lemma proof completion, we only need to chose successfully a sampling of S_i, i.e. to chose it in such a way that $\widetilde{G}$ become a recursive graph.

Definition 4. By the *change* of some *permissible* sets S_{i_1}, S_{i_2}, ...,S_{i_z} in a sampling $\lambda \in \Lambda$ to some other *permissible* sets S_{i_1}, ..., S_{i_z} we denote a substitution of λ for a $\lambda' \in \Lambda$, where λ' differ from λ only by that the all *permissible* sets S_{i_1}, ..., S_{i_z} in λ substitute for the other permissible sets S'_{i_1}, ..., S'_{i_z}. The sets S'_{i_1},..., S'_{i_z} we will call the *result of the change* of sets S_{i_1}, S_{i_2}, ..., S_{i_z}.

Denote as R the set $B \setminus S$. A degree of any vertex in R may be thought of as d because by the condition of *lemma 1* a degree of any vertex from B in the graph G is less or equal than d and if degree of a vertex is less than d, then it is possible to delete recursively this vertex from $\widetilde{G}$ for any *permissible* sampling.

Statement 9. *Let there be given a graph* F *such that* $V(F) = S_1 \cup S_2$ *and* $S_1 \cap S_2 = \emptyset$*, a degree of any vertex from* S_2 *in the graph* F *is less or equal than* D *and in* F *there is such a vertex* $v \in S_1$ *that the graph* $F(S_2 \cup \{v\})$ *is connected,* $d_F(v) < D$ *and the vertex* v *is adjacent to all the other vertices in* S_1*. Let the graph* $F(S_1)$ *be properly colored into* D *colors. Then it is possible to extend such a vertex coloring of* $F(S_1)$ *to the proper* D*-coloring of* F*.*

Remark 15. Unfortunately, in the formula of *statement 9* the coincidence of notations took place. Therefore, let us say that *statement 9* should be regarded as independent of the whole *theorem 1* statement.

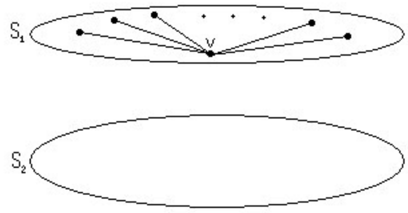

fig. 5.

Proof. Throw out from the graph F the vertex v, then we get a new graph F'. The set $S_1 \setminus \{v\}$ has already been properly colored into D colors. One by one we recursively color properly into D colors all the vertices from S_2, since $F(S_2 \cup \{v\})$ is a connected graph and a degree in the graph F of any vertex from S_2 is less or equal than D. Carry the obtained proper D-coloring of F' to F and then color v into some color distinct from all the colors of vertices from $N_F(v)$ (it is possible to do since $d_F(v) < D$), as a result we get a proper D-coloring of the graph F, but at that we probably change an initial color of the vertex v in the given coloring of S_1. Let us notice that all vertices from the set $S_1 \setminus \{v\}$ are colored into colors different from the color of v in the initial coloring of S_1, as initial coloring of S_1 was proper for the graph $F(S_1)$ and the vertex v is adjacent to all the other vertices in S_1, moreover all the colors of vertices from $S_1 \setminus \{v\}$ differ from the color of v in the obtained proper D-coloring of F. And now if the vertex v change its color in the obtained coloring in comparison with the given coloring on S_1 then we trade places of the current color of v with the color of v in the initial coloring. Thus, we get a proper D-coloring of F, but now equal on the set S_1 to the initial coloring.

□

Definition 5. By the *regular change* of the sets S_i of a sampling $\nu \in \Lambda$ with respect to a set S', we will call such a *change* of the sets S_i, where $i \in [1, |A|]$, to the sets S_i', $i \in [1, |A|]$, that for all $i \in [1, |A|]$ the set $S_i \cap S'$ will contain the set $S_i' \cap S'$. If there exists $i \in [1, |A|]$ such that $|S_i \cap S'|$ greater than $|S_i' \cap S'|$ then such a *regular change* we will call the *non-degenerate change.*

Remark 16. A *Regular change* with respect to some set is the *regular change* with respect to any subset of this set, but at that the *non-degeneracy* not necessarily preserves.

Statement 10. *There is a sampling of permissible sets $\eta = \{S_1, S_2, ..., S_{|A|}\}$ of the graph H — the smallest for the number of vertices graph which is contrary instance for the lemma 1 and let there are such sets $S' \subseteq S, R' \subseteq R$ that the all vertices from $B \setminus (S' \cup R')$ are recursively deleted from the graph $\widetilde{G}$, for all $u \in R$ $d_{\widetilde{G}}(u) = d$ and for all $u \in R'$ $d_{\widetilde{G}(S' \cup R')}(u) = d$.*

Let $\widetilde{H'} := \widetilde{G}(S' \cup R')$ and

$$\sum_{u \in \widetilde{H'}} d_{\widetilde{H'}}(u) > \sum_{u \in \widetilde{H'}} L(u). \tag{2}$$

Then it is possible to make a regular non-degenerate change of sets S_i with regard to the set $S' \cup R'$ so that all the set $B \setminus (S' \cup R')$ as before will be recursively deleted from the graph $\widetilde{G'}$ obtained from $\widetilde{G}$ in a result of this change.

Proof. We will prove this statement by induction on the set $B \setminus (S' \cup R')$ size.

The basis: the case when $|B \setminus (S' \cup R')| = 0$ obviously can not take place since by virtue of *remark 14* the condition (2) doesn't hold.

The inductive step: let the statement hold for all numbers less than k, then let us prove that it holds for the k.

Let $Z := B \setminus (S' \cup R')$.

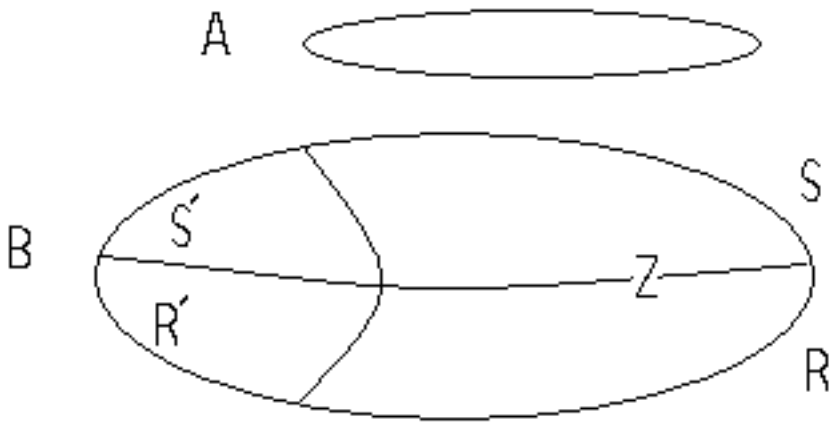

fig. 6.

Consider those sets S' and R' such that $|Z| = k$ and the assertion of the statement fails.

Let us show that there is a vertex $v_i \in A$ and correspondent to it the set S_i such that it is possible to make a *regular non-degenerate change* of S_i in relation to S'. If it is false then for any $v_j \in A$ and correspondent to it the set S_j only two possibilities can occurred:

1) the set $S_i \cap S' = \emptyset$ (see fig. 7);

2) the set $N_G(v_j) \setminus S' \subseteq S_i$ (see fig. 8).

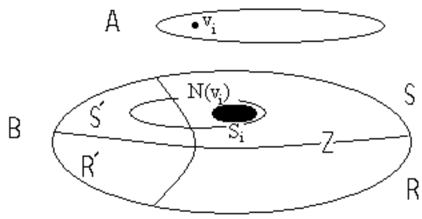

fig. 7.

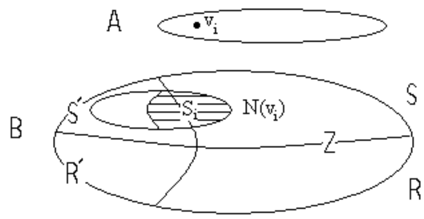

fig. 8.

In both of these cases the number of edges added to the graph G with two ends in S' reaches its minimum. Thus, for every vertex $v \in S' \cup R'$ the following chain of inequalities take place: $d_{\widetilde{H'}}(v) \le E(d_{\widetilde{H'}}(v)) \le E(d_{\widetilde{G}}(v)) \le L(v)$, where by the $E(\cdot)$, we denote the average of distribution of a variate with the distribution specified in the *remark 14*. We know from the condition (2) that $\sum\limits_{u \in \widetilde{H'}} d_{\widetilde{H'}}(u) > \sum\limits_{u \in \widetilde{H'}} L(u)$. So by a substitution of the inequality $d_{\widetilde{H'}}(v) \le L(v)$ in the previous inequality we get $\sum\limits_{u \in \widetilde{H'}} d_{\widetilde{H'}}(u) > \sum\limits_{u \in \widetilde{H'}} d_{\widetilde{H'}}(u)$ — a contradiction.

Hence, there is such a vertex $v_i \in A$, that a part of its neighborhood is contained in Z but the set $S_i \cap S' \neq \emptyset$ and S_i does not contain this part.

Consequently, we can consider such a vertex $v \in N_G(v_i)$, that it does not lay neither in the set S' nor in the set S_i, but some nonempty part of S_i is contained in the set S'. We know that Z can be recursively deleted from $\widetilde{G}$, so begin to *recursively delete* vertices from Z, but do it while it is possible to delete vertex distinct from v. At some moment we should stop this process. It means that we could not delete vertex except v and so we have only $v, u_1, u_2, ..., u_l \in S, w_1, w_2, ..., w_m \in R$ vertices remained in Z.

Denote by P the set of all remaining vertices in Z, and denote by $\widetilde{I}$ induced subgraph $\widetilde{G}(S' \cup R' \cup P)$ of G.

Let us notice that a degree in the graph $\widetilde{I}$ for any u_k vertex, where $k \in [1, l]$, or for any w_j, where $j \in [1, m]$, is at least d.

Let us notice also that a degree of v in $\widetilde{I}$ is less than d.

If the degree of v is less than $d - q + 1$ in $\widetilde{I}$, then let us make a change of S_i to a set S_i' in the following way: we take a vertex x in S_i which also is contained in the set $S' \cap S_i$ (those vertex necessarily turns up as $S' \cap S_i \neq \emptyset$), then $S_i' := \{(S_i \setminus \{x\}) \cup \{v\}\}$, the remaining sets of the sampling η we do not change. Let us notice that the change described above is a *regular* and *non-degenerate* one in regard to S' also it is clear that set Z will be *recursively deleted* in the obtained graph (it is clear that we can recursively delete as earlier vertices from $Z \setminus P$ then we can recursively delete v, as it has degree less than d, because before the change it has degree less than $d - q + 1$ and after the change the degree became not greater than $d - 1$, and then we can recursively delete all remaining vertices from Z, since Z has been recursively deleted from $\widetilde{G}$ and we drew no new edges in the graph $\widetilde{I}(V(\widetilde{I}) \setminus \{v\})$. So in this case we have proved an inductive step.

Thus we get that a degree of v is less than d but at least $d - q + 1$ in $\widetilde{I}$.

Let us prove that for the graph $\widetilde{I}$ the following condition holds:

$$\sum_{u \in \widetilde{I}} d_{\widetilde{I}}(u) > \sum_{u \in \widetilde{I}} L(u). \tag{2'}$$

With the proof, we can make use of an induction assumption for the sets S_0' and R_0', where $S_0' := (S' \cup R' \cup P) \cap S$ and $R_0' := (S' \cup R' \cup P) \cap R$, i.e. we can make a *regular non-degenerate change* of η in regard to $S' \cup R' \cup P$ in such a way that the set $B \setminus (S' \cup R' \cup P)$ will be recursively deleted in obtained graph. If a sampling had *regularly changed* in relation to $S' \cup R' \cup P$, then it is *regularly changed* in regard to $S' \cup R'$, also a composition of *regular changes* in regard to some a set is also the *regular change* in regard to this very set. Besides let us notice that in the graph obtained by this *change* all vertices from the set $B \setminus (S' \cup R')$ will be *recursively deleted*, as we can *recursively delete* at first the all vertices from $B \setminus (S' \cup R' \cup P)$ and then we can *recursively delete* as before all vertices from P since by the *change* we do not add new edges to P.

So we will do such *changes* until either S_i will be *regularly changed* in *non-degenerate* way in regard to $S' \cup R'$, or the degree of v in the graph $\widetilde{G}$ will become less than $d - q + 1$, or a degree of any vertex from $P \setminus \{v\}$ will become less than d. In the last case we can *recursively delete* some more vertices from Z and for the smaller graph $\widetilde{I}$ apply the same arguments. Here, it needs to be emphasized that some time or other we necessarily arrive at one of this cases else we will do an infinite number of *non-degenerate regular changes* in regard to the set $S' \cup R' \cup P$ and, hence, we will infinitely decrease a value of the sum $\sum\limits_{i=1}^{|A|} |S_i \cap (S' \cup R' \cup P)|$.

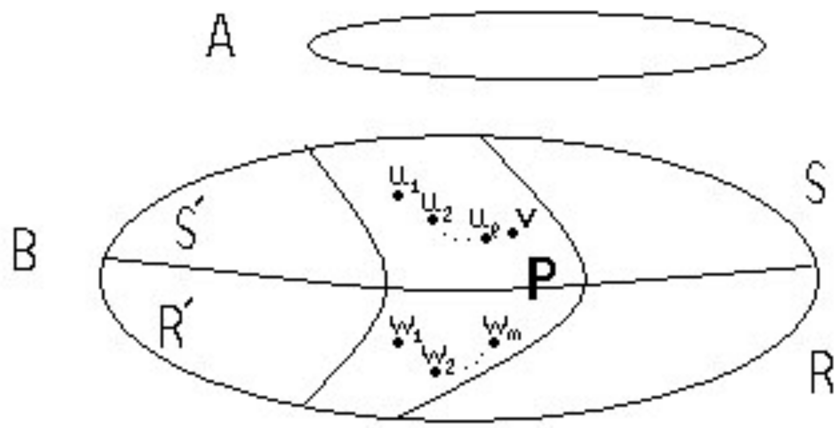

fig. 9.

Denote by l' the number of edges coming to the vertex set S' from P in the graph $\widetilde{G}$.

By the conditions of *statement 10* that for all $u \in R'$ $d_{\widetilde{G}(S' \cup R')}(u) = d$ and for all $u \in R$ $d_{\widetilde{G}}(u) = d$, there are no edges between P and R'.

So to end the proof of *statement 10* we only need to prove, that for the graph $\widetilde{I} = \widetilde{G}(S' \cup R' \cup P)$ the inequality (2') holds. Assume the contrary, then

$$\sum_{u \in \widetilde{I}} L(u) \geq \sum_{u \in \widetilde{I}} d_{\widetilde{I}}(u) \geq \sum_{u \in \widetilde{H'}} d_{\widetilde{H'}}(u) + l' + \sum_{u \in P} d_{\widetilde{I}}(u) > \sum_{u \in \widetilde{H'}} L(u) + \sum_{u \in P} d_{\widetilde{I}}(u) + l' \ .$$

So we get the following:

$$\sum_{u \in P} L(u) > l' + \sum_{u \in P} d_{\widetilde{I}}(u)' \ .$$

Hence, we get the inequality:

$$l' + \sum_{u \in P} d_{\widetilde{I}}(u) - L(u) < 0 \ . \qquad (3)$$

Let us bound the magnitude $d_{\widetilde{I}}(u_i) - L(u_i)$ for all $i \in [1, l]$.

By definition of $L(u)$ and by virtue of $d_{\widetilde{I}}(u_i) \geq d$ we get that for all $i \in [1, l]$ the following inequality holds: $d_{\widetilde{I}}(u_i) - L(u_i) \geq d - d_G(u_i) - \frac{d_A(u_i)}{q+1}$. Using the inequality (1) we get:

$$d_{\widetilde{I}}(u_i) - L(u_i) \geq d - d_G(u_i) - \frac{d_A(u_i)}{q+1} \geq d_G(u_i) + \lceil \frac{d_A(u_i)}{q} \rceil - d_G(u_i) - \frac{d_A(u_i)}{q+1} \ . \qquad (*)$$

Thus $d_{\widetilde{I}}(u_i) - L(u_i) \geq \lceil\frac{d_A(u_i)}{q}\rceil - \frac{d_A(u_i)}{q+1}$.

Also $d_A(u_i) > 0$ for all $i \in [1, l]$, as $u_i \in S$ for all $i \in [1, l]$. Let us consider two following cases:

a) $0 < d_A(u_i) \leq q$;

b) $d_A(u_i) \geq q + 1$.

In both of this cases the following inequality holds:

$$d_{\widetilde{I}}(u_i) - L(u_i) \geq \lceil\frac{d_A(u_i)}{q}\rceil - \frac{d_A(u_i)}{q+1} \geq \frac{1}{q+1} . \tag{4}$$

Let $q_1 := d_{\widetilde{I}}(v) - d + q$ then, as we have just showed it above, $q_1 > 0$. Let us notice that for the vertex v analogously to calculations $(*)$ we can get the following inequality:

$$d_{\widetilde{I}}(v) - L(v) \geq q_1 - q + \frac{1}{q+1} . \tag{5}$$

Since $w_i \in R$, where $i \in [1, m]$, $d_G(w_i) = L(w_i)$, moreover we can not *recursively delete* any vertex from the set $P \cap R$ in the graph $\widetilde{I}$. In addition using the *statement 10* condition, that for any vertex $u \in R$ $d_G(u) = d$, we get $d_{\widetilde{I}}(w_j) = d_G(w_j) = d$. And so for all $i \in [1, m]$ the we have

$$d = d_{\widetilde{I}}(w_i) = L(w_i). \tag{6}$$

It now follows from (4), (5), (6), (3) that:

$$l' + l\frac{1}{q+1} - q + q_1 + \frac{1}{q+1} < 0 .$$

Recall now that l is a number of vertices in the set $(P \cap S) \setminus \{v\}$, i.e. the number of u_i. We know that $q_1 \geq 1$. Then $l'(q+1) + l < (q-1)(q+1) - 1$, i.e.

$$(q+1)l' + l \leq q^2 - 3 . \tag{7}$$

From the inequality (7) we get two inequalities

$$l \leq q^2 - 3 \tag{8}$$

and

$$l' \leq q - 2 . \tag{9}$$

Denote by b_j (see fig. 10), where $j \in [1, r]$, the all vertices from the set $R \cap V(\widetilde{I})$, which are adjacent to v (r can be equal to 0). Let us consider some cases.

1) $r \geq q^2 - 3$.

By C_v we denote the union of all connectivity components of the graph $\widetilde{G}(R)$, which is minimal and contains all the vertices b_j, where $j \in [1, 2, ..., r]$. As we remark earlier, between sets P and R' there are no edges, so $C_v \subseteq R \setminus R'$. By equality (6) we have $d_{\widetilde{I}}(w_j) = d$, where $j \in [1, m]$. Thus, vertices from the set $P \cap R$ and from the set $Z \setminus P$ are not adjacent, and so $C_v \subseteq \{w_1, w_2, ..., w_m\}$.

Consider, in the vertex set S of the graph $\widetilde{G}$ all adjacent to C_v vertices and denote it by W. It is clear by virtue of $d_{\widetilde{I}}(w_i) = d$ and $d_G(w_i) = d$ that, firstly $W \subseteq V(\widetilde{I})$, secondly $v \in W$, and thirdly for all vertices $u \in C_v$ the equality $d_{\widetilde{G}(W \cup C_v)}(u) = d$ holds.

1.1) $|W| \geq q^2 - 1$.

Then $|W \cap S'| = |W| - |W \cap P \cap S| \geq q^2 - 1 - (l + 1)$. It is clear that $l' \geq |W \cap S'|$. Thus $l' \geq q^2 - 2 - l$, i.e. $l' + l \geq q^2 - 2$. So we arrive at a contradiction with inequality (7).

Thus $|W| \leq q^2 - 2$.

1.2) $|W| \leq q^2 - 2$.

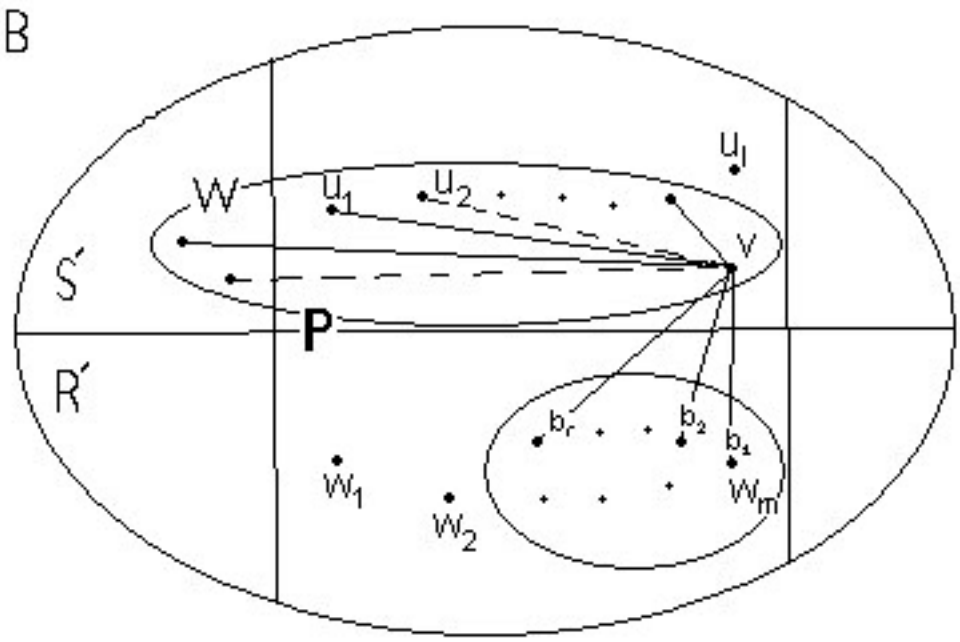

fig. 10.

Let us draw in the graph H all the edges of the type (u, v), where $u \in W$ and $(u, v) \notin E(H)$, denote by Θ the obtained graph. Let us verify all conditions of the *lemma 1* for the graph $\hat{H} := \Theta(V(H) \setminus C_v)$ and sets $\hat{A} := A$, $\hat{B} := B \setminus C_v$ and value $\hat{d} := d$.

Condition 1) is clear as A became the same.

Condition 2) is clear, since any vertex from C_v are not adjacent in H to any vertex from the set A.

Condition 3). It is sufficient to verify inequality (1) only for vertices from the set B, from which we draw any new edges, in other words we need to verify (1) only for the set W. By definition $\hat{G} := \hat{H}(\hat{B})$. For any vertex $u \in W$, $u \neq v$ we added not more than one edge with the end at u but also we deleted at least one edge coming from u to the set C_v (W is by definition the set of all vertices from S, which are adjacent with at least one vertex in C_v). Thus (1) remains true for all $u \in W$, $u \neq v$. The inequality (1) for v holds, as $|W| \leq q^2 - 2$, so we drew from v not greater than $q^2 - 3$ edges. On the other hand the set C_v by definition contains all the b_j, where $j \in [1, r]$, $r \geq q^2 - 3$, so we deleted at least $q^2 - 3$ edges with the end in v.

As we suppose H to be a minimal by the number of vertices graph for which the *lemma 1* doesn't hold, then *lemma 1* holds for $\hat{H}$ which has the smaller number of vertices. Then we can color properly the graph $\hat{G}$ into d colors in such a way that for any vertex $u \in A$ among its neighbors in $\hat{B}$ there would be at least q different colors. Denote by ξ such a proper d-coloring. It is clear that all assumptions of *statement 9* are satisfied for the graph $\Phi := \Theta(W \cup C_v)$, sets $S_1 := W$ and $S_2 := C_v$ and vertex v. Consider a proper d-coloring $\xi(W)$ of the graph $\Theta(W)$. By the *statement 9* we can extend $\xi(W)$ to a proper d-coloring ζ of the graph $\Theta(W \cup C_v)$. Let us notice that there are no edges in the graph Θ between the vertex set C_v and the vertex set

$V(H) \setminus (W \cup C_v)$, so we can combine ξ and ζ into one proper d-coloring of the graph G, also the condition, that for any vertex $u \in A$ there are at least q vertices of different colors in its neighborhood, holds for this combined coloring. Thus we get a coloring of the graph H we had seeking for in the *lemma 1*, so we arrive at a contradiction with assumption of the *statement 10*.

2) $r \leq q^2 - 4$

So from the vertex v in the graph $\widetilde{I}$ it outcomes no more than $q^2 - 4$ edges to the vertex set $R \cap V(\widetilde{I})$. The degree of v in the graph $\widetilde{I}$ is $d - q + q_1$. So from the vertex v it comes at least $d - q + q_1 - r - l'$ edges to the set $\{u_1, u_2, ..., u_l\}$. Let us notice that if a vertex $u \in S$ in the graph $\widetilde{G}$ has an edge of multiplicity k, then $d_A(u) \geq k - 1$. We know that from v, the outcome is at least $q^3 + 2q^2 - q - 8 - q + q_1 - r - l'$ edges to the vertices $u_1, u_2, ..., u_l$. Denote for all $i \in [1, l]$ as d_i the multiplicity of the edge (v, u_i) in the graph $\widetilde{H}$. Thus we know that $\sum\limits_{i=1}^{l} d_i \geq q^3 + 2q^2 - 2q - 8 + q_1 - r - l'$. Then we get

$$\sum_{i=1}^{l} d_A(u_i) \geq \sum_{i=1}^{l} (d_i - 1) \geq q^3 + 2q^2 - 2q - 8 + q_1 - r - l' - l \ . \tag{10}$$

By substituting inequality (5) and equality (6) into inequality (3) we get

$$l' - q + q_1 + \frac{1}{q+1} + \sum_{i=1}^{l} d_{\tilde{I}}(u_i) - L(u_i) < 0 \ . \tag{11}$$

We know that $d_{\tilde{I}}(u_i) - L(u_i) \geq d - d_G(u_i) - \frac{d_A(u_i)}{q+1}$ for all $i \in [1, l]$. By applying inequality (1) we get $d - d_G(u_i) - \frac{d_A(u_i)}{q+1} \geq \lceil \frac{d_A(u_i)}{q} \rceil - \frac{d_A(u_i)}{q+1} \geq d_A(u_i) \frac{1}{q(q+1)}$. Thus

$$d_{\tilde{I}}(u_i) - L(u_i) \geq d_A(u_i) \frac{1}{q(q+1)} \ . \tag{12}$$

Substitute (12) into (11) we get:

$$l' - q + q_1 + \frac{1}{q+1} + \frac{1}{q(q+1)} \sum_{i=1}^{l} d_A(u_i) < 0 \ . \tag{13}$$

Substitute (10) into (13):

$$l' - q + q_1 + \frac{1}{q+1} + \frac{q^3 + 2q^2 - 2q - 8 + q_1 - r - l' - l}{q(q+1)} < 0 \ .$$

We know that $l' \geq 0$ and $q_1 \geq 1$. Hence we have

$$l'(1 - \frac{1}{q(q+1)}) - (q-1) + \frac{1}{q+1} + \frac{q^3 + 2q^2 - 2q - 8 + 1 - r - l}{q(q+1)} < 0 \ .$$

We also know that $r \leq q^2 - 4$ and by inequality (8) $l \leq q^2 - 3$. Then

$$\frac{1}{q+1} + \frac{q^3 + 2q^2 - 2q - 8 + 1 - (q^2 - 4) - (q^2 - 3) - (q^3 - q)}{q(q+1)} < 0 \ .$$

So we get $\frac{1}{q+1} - \frac{q}{q(q+1)} < 0$, i.e. we arrive at a contradiction.

Thus we proved inequality ($2'$) for the graph $\widetilde{I}$ and so we proved the *statement 10.*

$\square$

Return to the *lemma 1* proof. Let us begin for the given sampling of sets $\lambda \in \Lambda$ to *delete recursively* vertices from $\widetilde{G}$ while it is possible. If graph $\widetilde{G}$ is not a *recursive* one, then a graph $\widetilde{G}(S' \cup R')$ will remain from $\widetilde{G}$, where $S' \subseteq S$, $R' \subseteq R$ and $S' \neq \emptyset$. Let us chose among all samplings from Λ such a sampling λ that the value of $|S'|+|R'|$ achieves minimum on it. Let us check up all assumptions of the *statement 10* for sets S' and R'. The unique non-trivial place in this check is to verify inequality (2).

Since we can not *delete recursively* from the graph $\widetilde{H'} := \widetilde{G}(S' \cup R')$ any vertex, then a degree of any vertex there is at least d. So by inequality (1) $L(u) = d_G(u) + \frac{d_A(u)}{q+1} \leq d$ for all $u \in B$, at that $L(u) < d$ for all vertices $u \in S$. Since $S' \neq \emptyset$ it is clear that

$$\sum_{u \in \widetilde{H'}} d_{\widetilde{H'}}(u) \geq d|\widetilde{H'}| > \sum_{u \in \widetilde{H'}} L(u).$$

So we will apply the *statement 10* to sets S' and R', while we get such a vertex such, that it degree in $\widetilde{H'}$ is less than d (let us notice that we wan't do an infinite number of *regular non-degenerate changes* in regard to the set $S' \cup R'$, since by any such a change we decrease the value of $\sum\limits_{i=1}^{|A|} |S_i \cap S'|$). Due to the *statement 10* we can as before to *delete recursively* all vertices from B except $S' \cup R'$, and then we can to delete recursively one extra vertex of degree less than d from the set $S' \cup R'$. Thus we arrive at a contradiction with minimality of $|S'| + |R'|$. Hence there is such a sampling of *permissible* sets that the graph $\widetilde{G}$ would be a *recursive* one. Thus we proved the *lemma 1* and finally we proved the *theorem 1* (see *remark 6*). $\square$

$\square$

Bibliography

[1] R. L. Brooks, *On coloring the nodes of network*, Proc. Cambridge Philos. Soc. 37 (1941) 194-197.

[2] L. Lovasz, *On decomposition of graphs*, Studia Sci. Math. Hungar. 1 (1966) 237-238.

[3] L. Hong-Jian, B. Montgomery, Hoifung Poon, *Upper Bounds of Dynamic Chromatic Number*, Ars. Combinatoria 68(2003), pp. 193-201.

[4] B. Montgomery, *Dynamic Coloring*, Ph.D. Dissertation, West Virginia University, 2001.

[5] J. A. Bondy, U. S. R. Murty, *Graph Theory with Applications*, American Elsevier, New York, 1976.

[6] X. Meng, L. Miao, Z. R. Li, B. Su, *The Conditional coloring numbers of Pseudo-Halin graphs*, Ars Combinatoria 79(2006), pp. 3-9.

[7] Suohai Fan, Hong-Jian Lai, Jianliang Lin, Bruce Montgomery, Zhishui Tao, *Conditional colorings of graphs*, Discrete Mathematic 16(2006), 1997-2004.