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REACHABLE AND UNREACHABLE SETS IN THE SCATTERING PROBLEM FOR THE ACOUSTICAL EQUATION IN $\mathbb{R}^3$

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The scattering problem is to find $u = u^f(x,t)$ satisfying

$$
\begin{aligned}
&u_{tt} - \Delta u + qu = 0, \qquad (x,t) \in \mathbb{R}^3 \times (-\infty, \infty)\\
&u\,|_{|x|<-t} = 0, \qquad t < 0\\
&\lim_{s\to\infty} su((s+\tau)\omega, -s) = f(\tau,\omega), \qquad (\tau,\omega) \in [0,\infty) \times S^2
\end{aligned}
$$

for a real smooth compactly supported potential $q = q(x)$ and a control $f \in \mathcal{F} = L_2([0,\infty); L_2(S^2))$. The corresponding control problem is: given $y \in \mathcal{H} = L_2(\mathbb{R}^3)$ find $f \in \mathcal{F}$ providing $u^f(\cdot,0) = y$; the reachable set is $\mathcal{U} = \{u^f(\cdot,0) \mid f \in \mathcal{F}\}$; the subspace of unreachable states is $\mathcal{D} = \mathcal{H} \ominus \mathcal{U}$. The main subject of the paper is the structure of $\mathcal{U}$ and $\mathcal{D}$. We present an example of the finite energy solution u^f satisfying $u^f|_{|x|<|t|} = 0$, i.e., vanishing simultaneously in the past and future cones (reversing wave) and introduce the set of points at which such a "revers effect" occurs. The existence of the reversing waves turns out to be equivalent to the lack of controllability $\mathcal{D} \neq \{0\}$. Cauchy data of such waves belong to the classes $D_\mp$ of the incoming and outgoing data simultaneously, providing $D_- \cap D_+ \neq \{0\}$. We also describe the simple conditions on f ensuring $\|u^f(\cdot,t)\|_{\mathcal{H}} \leq c\|f\|_{\mathcal{F}}$ for all $t \in (-\infty,\infty)$.

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0 Introduction

0.1 Dynamical system

We denote $B_r := \left\{x \in \mathbb{R}^3 \mid |x| < r\right\}, S^2 := \left\{\theta \in \mathbb{R}^3 \mid |\theta| = 1\right\}, \Sigma := (0, \infty) \times S^2$. Let $q = q(x)$, $x \in \mathbb{R}^3$ be a real smooth [1] compactly supported function (potential), supp $q \subset \overline{B}_{r_*}$. The system under consideration is

$$u_{tt} - \Delta u + qu = 0, \qquad (x,t) \in \mathbb{R}^3 \times (0, \infty) \tag{0.1}$$

$$u\mid_{|x|<-t} = 0, \qquad t < 0 \tag{0.2}$$

$$\lim_{s\to\infty} su((s+\tau)\theta, -s) = f(\tau, \theta), \qquad (\tau, \theta) \in \overline{\Sigma} \tag{0.3}$$

where f is a control, $u = u^f(x,t)$ is a solution (wave). The value $t = 0$ is referred as a final moment; however, in the sequel we also deal with the problem (0.1)–(0.3) for all $t \in (-\infty, \infty)$.

0.2 Spaces, sets, operators

With a dynamical system (0.1)–(0.3) one associates:

- an outer space of controls $\mathcal{F} := L_2([0, \infty); L_2(S^2))$ and its subspaces

  $$\mathcal{F}^\xi := \{f \in \mathcal{F} \mid \text{supp}\, f \subset [\xi, \infty)\}, \quad \xi \geq 0$$

  consisting of the delayed controls;

- an inner space of states $\mathcal{H} := L_2(\mathbb{R}^3)$ (we consider a wave $u^f(\cdot, t)$ as a time dependent element of $\mathcal{H}$) and its subspaces

  $$\mathcal{H}^\xi := \left\{y \in \mathcal{H} \mid \text{supp}\, y \subset \mathbb{R}^3 \setminus B_\xi\right\}, \quad \xi \geq 0;$$

- a control operator $W : \mathcal{F} \to \mathcal{H}$

  $$Wf := u^f(\cdot, 0)$$

  and the subspace of the null-controls

  $$\mathcal{N} := \text{Ker}\, W = \left\{f \in \mathcal{F} \mid u^f(\cdot, 0) = 0\right\};$$

- the reachable sets

  $$\mathcal{U}^\xi := W\mathcal{F}^\xi = \left\{u^f(\cdot, 0) \mid f \in \mathcal{F}^\xi\right\}, \quad \xi \geq 0$$

  (we also denote $\mathcal{U}^0$ by $\mathcal{U}$); by hyperbolicity of the problem (0.1)–(0.3) one has $\mathcal{U}^\xi \subset \mathcal{H}^\xi$;

[1] everywhere in the parer "smooth" means C^∞-smooth.

- the defect subspaces (unreachable sets)

$$\mathcal{D}^\xi := \mathcal{H}^\xi \ominus \mathcal{U}^\xi, \quad \xi \geq 0$$

(we denote $\mathcal{D}^0 =: \mathcal{D} = \mathcal{H} \ominus \mathcal{U}$).

We also introduce the polyharmonic subspaces $\mathcal{A}^\xi \subset \mathcal{H}^\xi$,

$$\mathcal{A}^\xi := \text{clos } \left\{ a \in \mathcal{H}^\xi \mid \exists \, \text{integer} \, p \geq 1 : \ (-\Delta + q)^p a = 0 \ \text{ in } \mathbb{R}^3 \setminus \overline{B}_\xi \right\}, \quad \xi \geq 0,$$

(the closure in $\mathcal{H}$; $\overline{B}_0 := \{0\}$) and denote $\mathcal{A} := \mathcal{A}^0$. It is easy to see that $\dim \mathcal{A}^\xi = \infty$ for $\xi > 0$.

Evolution of the system (0.1)–(0.3) is governed by the self-adjoint operator $H : \mathcal{H} \to \mathcal{H}$, $\text{Dom } H = H^2(\mathbb{R}^3)$, $Hy = (-\Delta + q)y$. This operator can have at most a finite number of negative eigenvalues of finite multiplicity, whereas its absolutely continuous spectrum fills $[0, \infty)$. The spectral point $\lambda = 0$ also can belong to the point spectrum $\sigma_p(H)$, so that $\text{Ker } H$ can be a nonzero finite dimensional subspace in $\mathcal{H}$ (see [13]).

0.3 The results

The control problem for the system (0.1)–(0.3) is set up as follows: given $y \in \mathcal{H}$ find $f \in \mathcal{F}$ providing

$$u^f(\cdot, 0) = y. \tag{0.4}$$

Loosely speaking, we deal with a "wave shaping": managing the control one needs to create the incoming (from infinity) wave of the prescribed shape. We also consider the same problem for delayed controls $f \in \mathcal{F}^\xi$.

To study the solvability of (0.4) is to investigate the structure of the reachable sets $\mathcal{U}, \mathcal{U}^\xi$ or, equally, of the defect subspaces $\mathcal{D}, \mathcal{D}^\xi$. Our main result is

Theorem 1 *(i) The sets $\mathcal{U}^\xi$ are closed. The relations*

$$\mathcal{A}^\xi \subseteq \mathcal{D}^\xi, \quad \xi \geq 0; \quad \text{Ker} H \subseteq \mathcal{A} \subseteq \mathcal{D} \tag{0.5}$$

and

$$\dim \text{Ker} H \leq \dim \mathcal{D} = \dim \mathcal{N} < \infty \tag{0.6}$$

hold.

(ii) If $||q||_{L^\infty(B_{r_*})}$ is small enough then $\mathcal{U} = \mathcal{H}$ and $\mathcal{D} = \{0\}$.

(iii) If the potential is radially symmetric, i.e., $q = q(|x|)$ then

$$\mathcal{D}^\xi = \mathcal{A}^\xi, \quad \xi \geq 0; \quad \mathcal{D} = \text{Ker } H \ . \tag{0.7}$$

In control theory the equality $\mathcal{U} = \mathcal{H}$ is interpreted as exact controllability of the system (0.1)–(0.3). So, for small potentials the controllability occurs, whereas in the case of radial q the only obstacle for controllability is $0 \in \sigma_p(H)$. The latter is not true in the general case: we construct an example of (not radial) potential such that $0 \notin \sigma_p(H)$ (i.e., $\operatorname{Ker} H = \{0\}$) but $\mathcal{D} \neq \{0\}$, realizing the strict inequality in (0.6).

The presence of the defect $\mathcal{D} \neq \{0\}$ is equivalent to the existence of the null-controls producing the waves, which exhibit rather curious behavior. If $f \in \mathcal{N}$, then u^f is an L_2- solution of (0.1)–(0.3) satisfying $u^f(\cdot, 0) = 0$. As a consequence, the function

$$w^f(x,t) := \begin{cases} \int\limits_{-\infty}^{t} u^f(\cdot, s)\, ds, & t \leq 0 \\ \int\limits_{-\infty}^{-t} u^f(\cdot, s)\, ds, & t > 0 \end{cases}$$

turns out to be a finite (positive) energy solution of the acoustical equation (0.1) in $\mathbb{R}^3 \times (-\infty, \infty)$ satisfying

$$w^f \mid_{|x|<|t|} = 0, \quad w^f_t \mid_{t=0} = 0, \quad w^f(\cdot, -t) = w^f(\cdot, t)$$

i.e., vanishing in the past and future cones simultaneously. So, w^f is a wave, which comes from infinity, stops at the moment $t = 0$ and then returns back to infinity along the same trajectory. The amazing peculiarity of such a behavior is that this wave possesses a back front: it leaves supp q remaining no residual perturbation. The noticeable fact is that existence of such reversing waves is equivalent to lack of controllability of the system.

The Cauchy data of the reversing wave $\{w^f(\cdot, 0), w^f_t(\cdot, 0)\} = \{y, 0\}$ belong to the energy class (see [10], [12]). At the same time, since $w^f \mid_{|x|<|t|} = 0$, the pair $\{y, 0\}$ belongs to the classes D_- and D_+ of the incoming and outgoing data simultaneously and, thus, gives an example of $D_+ \cap D_- \neq \{0\}$. The noticeable fact is that such an intersection is possible even in the case Ker $H \neq \{0\}$. Perhaps, this example is new: we didn't succeed in finding analogs in literature.

The origin $x = 0$ plays the role of a point, which stops the reversing waves. We introduce the set $\Upsilon[q] \subset \mathbb{R}^3$ of such points and hope that it prove to be an interesting object for further investigations.

At the end of the paper we present the orthogonality conditions on a control f ensuring L_2-stability of the trajectory: $\|u^f(\cdot, t)\|_{\mathcal{H}} \leq c\|f\|_{\mathcal{F}}$ holds for all $t \in (-\infty, \infty)$ if and only if $(f, g_k)_{\mathcal{F}} = 0,\ k = 1, \ldots, p$, where g_k are determined by the negative spectrum of H in explicit way.

0.4 Comments

- Dealing with the scattering problem of such a kind, one can't avoid parallels to the Lax-Phillips theory (LPT) and we notice such parallels in the text. However, our presentation is independent. One of the reasons is that at the moment we don't fully recognize the place of our central object $\mathcal{D}^\xi$ in the context of LPT. We also deal mainly with L_2-solutions u^f, – not with a complete state $\{u^f, u^f_t\}$, – since for the incoming waves u^f_t

is determined by u^f. We prefer L_2-norms avoiding the indefinite energy $\int_{\mathbb{R}^3} u_t^2 + |\nabla u|^2 + qu^2$.

- We plan to apply the results of this paper to the inverse problem, which is to recover the potential q from the response operator R of the system (0.1)–(0.3). The recovering procedure must be local in the following sense: given $R\mid_{\mathcal{F}^\xi}$ it has to recover $q\mid_{\mathbb{R}^3\setminus B_\xi}$. In other words, knowing the respond of the system on the delayed controls $f \in \mathcal{F}^\xi$ one has to determine the potential in the domain $\mathbb{R}^3 \setminus B_\xi$ filled with the delayed incoming waves $u^f(\cdot, 0)$. Such a locality is a specific feature of the boundary control method which is an approach to the inverse problems based upon their connections with control theory [2]. It is the approach which motivates the study of $\mathcal{U}^\xi$ and $\mathcal{D}^\xi$.

- Our paper extends the results of the article [5], which deals with $q = 0$, to the perturbed case.

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1 Unperturbed system

Here, dealing mainly with the case $q = 0$, we present some results of [5] and supplement them with relations from geometrical optics.

1.1 Expansions over spherical harmonics

Let $Y_l^m(\theta), \quad l \geq 0,\ m = -l, -l+1, \dots, 0, \dots, l-1, l$ be the standard spherical functions (harmonics) satisfying

$$-\Delta_\omega Y_l^m = l(l+1)Y_l^m \quad \text{on} \quad S^2$$

(Δ_ω is the Beltrami-Laplace operator) and forming an orthonormal basis in $L_2(S^2)$. The unperturbed system is

$$u_{tt} - \Delta u = 0, \qquad (x,t) \in \mathbb{R}^3 \times (-\infty, 0) \tag{1.1}$$

$$u\mid_{|x|<-t} = 0 \tag{1.2}$$

$$\lim_{s\to\infty} su((s+\tau)\theta, -s) = f(\tau, \theta), \qquad (\tau,\theta) \in [0,\infty) \times S^2; \tag{1.3}$$

its outer and inner spaces can be represented in the form

$$\mathcal{H} = \oplus \sum_{l,m} \mathcal{H}_{lm}\ , \quad \mathcal{H}_{lm} := \left\{ y = \frac{v(r)}{r} Y_l^m(\omega) \middle| \int_0^\infty v^2(r)dr < \infty \right\} \tag{1.4}$$

(we denote $\sum_{l,m} := \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l}$, $r := |x|$, $\omega := \frac{x}{|x|}$ and keep these notations in the sequel) and

$$\mathcal{F} = \oplus \sum_{l,m} \mathcal{F}_{lm} \ , \quad \mathcal{F}_{lm} = \left\{ f = g(\tau) Y_l^m(\omega) \Big| \int_0^\infty g^2(\tau) dr < \infty \right\} . \qquad (1.5)$$

1.2 Partial problems

Let $f \in \mathcal{F}$; separating variables in (1.1)–(1.3) in accordance with decompositions (1.4),(1.5), we represent

$$f(\tau, \theta) = \sum_{l,m} f_{lm}(\tau) Y_l^m(\theta) \qquad (1.6)$$

and look for the solution $u = u^f(x,t)$ in the form

$$u^f(x,t) = \sum_{l,m} \frac{u_{lm}(r,t)}{r} Y_l^m(\omega) \qquad (1.7)$$

that reduces (1.1)–(1.3) to a series of the partial problems

$$u_{tt} - u_{rr} + \frac{l(l+1)}{r^2} u = 0, \qquad (r,t) \in (0,\infty) \times (-\infty, 0) \qquad (1.8)$$

$$u\,|_{r<-t} = 0 \qquad (1.9)$$

$$\lim_{s\to\infty} u(s+\tau, -s) = f(\tau), \qquad \tau \in [0,\infty) \ . \qquad (1.10)$$

Solving them for $f = f_{lm}(\tau)$, we get the solution $u = u_{lm}(r,t)$ and arrive at the following results (see e.g. [5]).

(i) The partial problem can be solved explicitly: for any $\phi \in C_0^\infty(0,\infty)$ the function

$$u^f(r,t) = r^l \left(\frac{\partial}{\partial r} \frac{1}{r} \right)^l \phi(r+t) \qquad (1.11)$$

satisfies (1.8)–(1.10) with

$$f(\tau) = \left(\frac{d}{d\tau} \right)^l \phi(\tau) \ . \qquad (1.12)$$

For controls of the form (1.12), the embedding $\operatorname{supp} f \subset [a,b] \subset (0,\infty)$ implies

$$\operatorname{supp} u^f(\cdot, t) \subset [a-t, b-t]$$

and, hence, the solution is compactly supported for all times t. One more important fact is that the class of controls

$$L_l := \left(\frac{d}{d\tau} \right)^l C_0^\infty(0,\infty), \quad l = 0, 1, \ldots$$

as well as each of its subclasses

$$L_l^{(k)} := \left(\frac{d}{d\tau}\right)^k L_l\ , \quad k = 0, 1, \cdots$$

are dense in $L_2(0,\infty)$. Note also the density of the classes $\left(\int\limits_0^\tau\right) L_l^{(k)}$ for $l+k>0$.

(ii) Representations (1.11)–(1.12) determine the map $W^l : f \mapsto u^f(\cdot, 0)$, which is an isometric operator from $L_2(0,\infty)$ to $L_2(0,\infty)$. Its extension to $L_2(0,\infty)$ turns out to be a unitary operator, which we also denote by W^l. The L_2-solutions of (1.8)–(1.10) can be introduced through this operator by

$$u^f(\cdot, t) = W^l T_t f, \quad t \le 0\ , \tag{1.13}$$

where T_t is a shift operator,

$$(T_t f)(\tau) := \begin{cases} 0, & 0 \le \tau \le |t| \\ f(\tau - |t|), & \tau \ge |t|\ . \end{cases}$$

(iii) The reachable sets of the system (1.8)–(1.10) are

$$U_l^\xi := W^l L_2(\xi, \infty)\ , \quad \xi \ge 0\ ,$$

where $L_2(\xi,\infty) := T_\xi L_2(0,\infty)$ is the class of delayed controls; the defect subspaces (unreachable sets) are

$$D_l^\xi := L_2(\xi,\infty) \ominus U_l^\xi\ , \quad \xi \ge 0\ .$$

The following characterization of the defect is obtained in [5].

Assign a function $a = a(r) \in L_2(0,\infty)$ to a class A_l^ξ if $\operatorname{supp} a \subset [\xi, \infty)$ and there exists an integer $p \ge 1$ such that

$$\left[-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2}\right]^p a = 0, \quad r > \xi\ . \tag{1.14}$$

For an $a \in A_l^\xi$, define $\operatorname{rank} a$ as the minimal p providing (1.14), and put

$$\operatorname{rank} A_l^\xi := \max_{a \in A_l^\xi} \operatorname{rank} a\ .$$

Then, one has

$$D_0^\xi = A_0^\xi = \{0\}\ , \quad \xi \ge 0 \tag{1.15}$$

(that is commented as "$s-$waves are controllable"), whereas for $l \ge 1$ the set A_l^ξ turns out to be a finite dimensional subspace,

$$\dim A_l^\xi = \operatorname{rank} A_l^\xi = d(l) := \frac{l}{2} + \frac{1-(-1)^l}{4}\ ,$$

spanned on a basis

$$\frac{\chi(r-\xi)}{r^l},\ \frac{\chi(r-\xi)}{r^{l-2}},\ \ldots\ ,\frac{\chi(r-\xi)}{r^{\frac{3+(-1)^l}{2}}}\ , \tag{1.16}$$

where $\chi(s) := \frac{1}{2}[1 + \text{sign } s]$ is the Heavyside function. As is shown in [5], the equality

$$D_l^\xi = A_l^\xi\ , \quad l \geq 0, \xi > 0 \tag{1.17}$$

holds. For $\xi = 0$, the functions (1.16) do not belong to $L_2(0,\infty)$ and, hence,

$$D_l^0 = A_l^0 = \{0\}\ . \tag{1.18}$$

1.3 Control operator

Returning from the partial problems to the system (1.1)–(1.3), we define its L_2-solutions in accordance with (1.6),(1.7),(1.13): for $f \in \mathcal{F}$ we set

$$u^f(x,t) := \sum_{l,m} \frac{1}{r}\left(W^l T_t \Pi_{lm} f\right)(r) Y_l^m(\omega) \tag{1.19}$$

where $\Pi_{lm} : \mathcal{F} \to L_2(0,\infty)$,

$$\left(\Pi_{lm} f\right)(\tau) := \int\limits_{S^2} f(\tau,\theta) Y_l^m(\theta)\, d\theta\ , \quad \tau \geq 0\ .$$

By this, the control operator of the unperturbed system $W_0 : \mathcal{F} \to \mathcal{H}, \quad W_0 f = u^f(\cdot,0)$ is unitary [2] and the relation

$$u^f(\cdot,t) = W_0 T_t f\ , \quad t \leq 0 \tag{1.20}$$

holds (see (1.13)). It is easy to verify that an L_2-solution u^f satisfies (1.1) also in the sense of distributions and, as an $\mathcal{H}$-valued function of time (trajectory), belongs to the class $C((-\infty,0];\mathcal{H})$. Moreover, considering (1.1)–(1.3) for all times, one has $u^f \in C((-\infty,\infty);\mathcal{H})$. For a smooth control f vanishing near $\tau = 0$, the solution u^f is smooth and can be represented in the well-known form: assuming f extended to $\tau < 0$ by zero one has

$$u^f(x,t) = \frac{1}{2\pi}\int\limits_{S^2} \frac{\partial f}{\partial \tau}(t + x\cdot\theta, \theta)\, d\theta \tag{1.21}$$

(see e.g. [10]). This formula yields

$$(W_0 f)(x) = \frac{1}{2\pi}\int\limits_{S^2} \frac{\partial f}{\partial \tau}(x\cdot\theta, \theta)\, d\theta\ . \tag{1.22}$$

[2] this fact also easily follows from LPT.

For the later use, it is convenient to introduce one more special class of controls and the corresponding solutions. Define

$$\mathcal{L}_{lm} := \{f \in \mathcal{F} \mid \; f = g(\tau) Y_l^m(\theta), \quad g \in L_l\}$$

(see sec. 1.2, (i)) and consider the class of controls

$$\mathcal{L} := \left[\oplus \sum_{l,m} \mathcal{L}_{lm} \right] \cap C_0^\infty(\Sigma) \; ,$$

which is dense in $\mathcal{F}$. Its subclasses

$$\mathcal{L}^{(k)} := \left(\frac{\partial}{\partial \tau} \right)^k \mathcal{L} \; , \quad k = 1, 2, \ldots$$

also are dense. Note in addition that $\mathcal{L}^{(k)} \cap \mathcal{F}^\xi$ is dense in $\mathcal{F}^\xi$ for every $\xi \geq 0$.

If $f \in \mathcal{L}$ and $\operatorname{supp} f \subset [a, b] \times S^2 \subset \Sigma$ then, in accordance with the results of sec. 1.2, (i), the representation (1.7) yields

$$\operatorname{supp} u^f(\cdot, t) \subset \overline{B}_{b-t} \setminus B_{a-t} \; , \quad t \leq 0 \; . \tag{1.23}$$

So, the controls $f \in \mathcal{L}$ produce the waves compactly supported in $\mathbb{R}^3$.

The space of controls contains the Sobolev class

$$\mathcal{F}_1 := \left\{ f \in \mathcal{F} \left| \; \frac{\partial f}{\partial \tau} \in \mathcal{F} \right. \right\}$$

equipped with the norm $\|f\|_1^2 = \|f\|_{\mathcal{F}}^2 + \|\frac{\partial f}{\partial \tau}\|_{\mathcal{F}}^2$ and its subclass

$$\mathcal{F}_{1,0} := \left\{ f \in \mathcal{F}_1 \mid \; f|_{\tau=0} = 0 \right\} \; ,$$

the set $\mathcal{L}$ being dense in $\mathcal{F}_{1,0}$.

Take $f \in \mathcal{L}$; integrating by parts with regard to isometry of the control operator and the property (1.23), we have

$$\begin{aligned} \|f\|_1^2 &= \|f\|_{\mathcal{F}}^2 + \left(\frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial \tau} \right)_{\mathcal{F}} = \|f\|_{\mathcal{F}}^2 - \left(\frac{\partial^2 f}{\partial \tau^2}, f \right)_{\mathcal{F}} \\ &= \|u^f(\cdot, 0)\|_{\mathcal{H}}^2 - \left(u^{\frac{\partial^2 f}{\partial \tau^2}}(\cdot, 0), u^f(\cdot, 0) \right)_{\mathcal{H}} = \|u^f(\cdot, 0)\|_{\mathcal{H}}^2 - \left(u_{tt}^f(\cdot, 0), u^f(\cdot, 0) \right)_{\mathcal{H}} \\ &= \|u^f(\cdot, 0)\|_{\mathcal{H}}^2 - \left(\Delta u^f(\cdot, 0), u^f(\cdot, 0) \right)_{\mathcal{H}} = \|u^f(\cdot, 0)\|_{H^1(\mathbb{R}^3)}^2 \; . \end{aligned}$$

Extending to $f \in \mathcal{F}_{1,0}$, we obtain an isometrical embedding

$$W_0 \mathcal{F}_{1,0} \subset H^1(\mathbb{R}^3) \; . \tag{1.24}$$

This simple straightforward derivation demonstrates the utility of the class $\mathcal{L}$. However, (1.24) also can be derived from LPT (see [10]).

1.4 Observation operator

The adjoint operator $W_0^* : \mathcal{H} \to \mathcal{F}$ plays the role of an observation operator of the system (1.1)–(1.3). Since W_0 is unitary, one has $W_0^* = W_0^{-1}$ that relates W_0^* to the Radon transform

$$(\mathcal{R}y)\,(\tau,\theta) := \int\limits_{\sigma\cdot\theta=\tau} y(\sigma)\,d\sigma, \quad (\tau,\theta)\in\Sigma$$

as follows. For any $y \in C_0^\infty(\mathbb{R}^3)$, by the Radon inversion formula one has

$$y(x) = -\frac{1}{8\pi^2}\Delta_x \int\limits_{S^2} (\mathcal{R}y)(x\cdot\theta,\theta)\,d\theta = -\frac{1}{8\pi^2}\int\limits_{S^2}\left[\frac{\partial^2}{\partial\tau^2}\mathcal{R}y\right](x\cdot\theta,\theta)\,d\theta =$$

$$= \frac{1}{2\pi}\int\limits_{S^2}\frac{\partial}{\partial\tau}\left[-\frac{1}{4\pi}\frac{\partial}{\partial\tau}\mathcal{R}y\right](x\cdot\theta,\theta)\,d\theta = \langle see\ (1.22)\rangle =$$

$$\left(W_0\left[-\frac{1}{4\pi}\frac{\partial}{\partial\tau}\mathcal{R}y\right]\right)(x)\ ;$$

hence,

$$W_0^* = W_0^{-1} = -\frac{1}{4\pi}\frac{\partial}{\partial\tau}\mathcal{R} \tag{1.25}$$

whereas the isometry of W_0^* corresponds to the classical Plancherelle formula for $\mathcal{R}$ (see [7], [8],[10]).

The embedding

$$W_0^* H^1(\mathbb{R}^3) \subset \mathcal{F}_1\ . \tag{1.26}$$

will be used below. It easily follows from the properties of the Radon transform and (1.25).

1.5 Controllability

To describe the structure of the sets $\mathcal{U}^\xi$ and $\mathcal{D}^\xi$ in the unperturbed case one needs merely to reformulate the results of sec. 1.2, (iii).

Recall that the polyharmonic subspaces $\mathcal{A}^\xi$ are defined at the end of sec. 0.2; in the case $q=0$, by this definition and in accordance with (1.4), one easily gets

$$\mathcal{A}^\xi = \oplus\sum_{l,m}\left\{ y\in\mathcal{H}_{lm}\ \middle|\ y = \frac{a(r)}{r}Y_l^m(\omega), \quad a\in A_l^\xi\right\}\ .$$

By virtue of (1.15)–(1.18), the equalities

$$\mathcal{D}^\xi = \mathcal{A}^\xi,\ \xi>0; \quad \mathcal{D}^0 = \mathcal{A}^0 = \{0\} \tag{1.27}$$

hold (see [5]). Thus, for $\xi>0$ the defect sets are infinite dimensional, whereas the reachable sets $\mathcal{U}^\xi = W_0\mathcal{F}^\xi$ are the closed subspaces determined by infinite number of orthogonality conditions: $y = \sum_{l,m}\frac{y_{lm}(r)}{r}Y_l^m(\omega)$ belongs to $\mathcal{U}^\xi$ if and only if $y_{lm}\perp A_l^\xi$.

1.6 Geometrical optics

Here we modify the representations (1.21) and (1.22), extending them to the case of controls with $f\mid_{\tau=0}\neq 0$. Such controls with a jump at $\tau=0$ produce the waves with a jump at the characteristic cone $\{(x,t)\mid\ |x|=|t|\}$, i.e, at the forward front of the wave. The geometrical optics formulae express the amplitude of the jump of u^f through $f\mid_{\tau=0}$.

Fix $\omega\in S^2$ and define

$$\pi_b(\omega):=\begin{cases}\left\{\theta\in S^2\mid \omega\cdot\theta=b\right\}, & b\in[-1,1]\\ \emptyset, & |b|>1\end{cases}$$

The set $\pi_b(\omega)$ is a parallel on the unit sphere with the North Pole ω, the length of the parallel is equal to $2\pi\sqrt{1-b^2}$; $\pi_0(\omega)$ is the equator; $\pi_{\pm1}(\omega)=\pm\omega$. For a function g on S^2, denote by

$$[g]_b(\omega):=\begin{cases}\frac{1}{2\pi\sqrt{1-b^2}}\int\limits_{\pi_b(\omega)} g(\theta)\,d\theta, & b\in[-1,1]\\ g(-\omega), & b=-1\\ g(\omega), & b=1\\ 0, & |b|>1\end{cases}$$

the meanvalue of g on the parallel.

Lemma 1 *For $f\in\mathcal{F}_1$, the representation*

$$u^f(x,t)=\frac{1}{2\pi}\int\limits_{S^2} f_\tau(t+r\omega\cdot\theta,\theta)\,d\theta+\frac{1}{r}[f(0,\cdot)]_{-\frac{t}{r}}(\omega) \tag{1.28}$$

holds, where $r=|x|$, $\omega=\frac{x}{|x|}$, and f_τ is understood as a classical (Sobolev) derivative of $f|_{\tau>0}$, the derivative being extended to $\tau<0$ by zero.

Proof Let us begin with $f\in\mathcal{F}_1\cap C^1(\overline{\Sigma})$. Fix $\varepsilon>0$ and denote

$$f^\varepsilon(\tau,\omega):=\begin{cases}0, & \tau<0\\ \frac{f(0,\omega)}{\varepsilon}\tau, & 0\le\tau<\varepsilon\\ f(\tau-\varepsilon,\omega), & \varepsilon\le\tau<\infty\end{cases}$$

Approximating $f^\varepsilon\in\mathcal{F}_{1,0}$ by smooth controls, we easily extend (1.21) to $f=f^\varepsilon$ and get

$$u^{f^\varepsilon}(x,t)=\frac{1}{2\pi}\int\limits_{S^2}\frac{\partial f^\varepsilon}{\partial\tau}(t+x\cdot\theta,\theta)\,d\theta=\frac{1}{2\pi\varepsilon}\int\limits_{0\le t+x\cdot\theta<\varepsilon} f(0,\theta)\,d\theta+$$

$$+\frac{1}{2\pi}\int\limits_{t+x\cdot\theta\ge\varepsilon} f_\tau(t+x\cdot\theta-\varepsilon,\theta)\,d\theta=:\ I_\varepsilon+II_\varepsilon\ . \tag{1.29}$$

To analyze I_ε let us introduce the polar coordinates: choose $\omega', \omega'' \in S^2$ such that $\{\omega, \omega', \omega''\}$ form a Cartesian basis in $\mathbb{R}^3$ and define $\varphi = \varphi(\theta) \in [0, 2\pi)$, $\psi = \psi(\theta) \in [0, \pi]$ by $\cos\varphi = \omega' \cdot \theta, \quad \sin\varphi = \omega'' \cdot \theta, \quad \cos\psi = \omega \cdot \theta$. Since the set of integration in I_ε is determined by the inequalities

$$\frac{-t}{r} \le \omega \cdot \theta \le \frac{-t}{r} + \frac{\varepsilon}{r} \ ,$$

one has

$$I_\varepsilon = \frac{1}{2\pi\varepsilon} \int\limits_0^{2\pi} d\varphi \int\limits_{\arccos[\frac{-t}{r} + \frac{\varepsilon}{r}]}^{\arccos\frac{-t}{r}} f\left(0, \theta(\varphi, \psi)\right) \sin\psi \, d\psi$$

$$= \frac{1}{2\pi} \int\limits_0^{2\pi} d\varphi \ \frac{1}{\varepsilon} \int\limits_{\frac{-t}{r}}^{\frac{-t}{r} + \frac{\varepsilon}{r}} f\left(0, \theta(\varphi, \arccos\eta)\right) \, d\eta \ .$$

Fix x, t such that $|x| > -t$; tending $\varepsilon \to 0$ we get

$$I_\varepsilon \to \frac{1}{2\pi r} \int\limits_0^{2\pi} f\left(0, \theta(\varphi, \arccos\frac{-t}{r})\right) \, d\varphi = \frac{1}{r}[f(0, \cdot)]_{-\frac{t}{r}}(\omega) \ .$$

By continuity of f_τ, the limit passage in II_ε for fixed x, t gives

$$II_\varepsilon \to \frac{1}{2\pi} \int\limits_{t + x\cdot\omega \ge 0} f_\tau(t + x \cdot \theta, \theta) \, d\theta = \int\limits_{S^2} f_\tau(t + r\omega \cdot \theta, \theta) \, d\theta \ .$$

Hence, the r.h.s. of (1.29) tends point wise in $\{(x, t) \mid 0 \le -t < |x|\}$ to the r.h.s. of (1.28). At the same time, $f_\varepsilon \to f$ in $\mathcal{F}$ that implies $u^{f^\varepsilon}(\cdot, t) \to u^f(\cdot, t)$ in $\mathcal{H}$, and we arrive at the representation (1.28). For $(x, t) : |x| < -t$ the representation takes the trivial form $0 = 0$.

If $f \in \mathcal{F}_1$ then the value $f|_{\tau=0}$ is well defined as an element of $L_2(S^2)$, as well as $[f(0, \cdot)]_b(\cdot) \in L_2(S^2)$. [3] Therefore, approximating f by elements of $\mathcal{F}_1 \cap C^1(\overline{\Sigma})$ in $\mathcal{F}_1$-norm and passing to the limit, we justify (1.28). QED

The peculiarity of the formula (1.28) is that, for a fixed t, the summands in its r.h.s. not necessarily belong to $\mathcal{H} = L_2(\mathbb{R}^3)$ separately. The formula represents the wave $u^f(\cdot, t)$ in a neighborhood of $x = 0$ and allows to analyze its behavior near the forward front $\{x \in \mathbb{R}^3 \mid \ |x| = -t\}$. Namely, if x approaches to the front ($|x| \to -t + 0$) then the first summand vanishes (since mes $\{\theta \in S^2 \mid t + x \cdot \omega \ge 0\} \to 0$), whereas the second summand tends to the limit $r^{-1}[f(0, \cdot)]_1(\omega) = r^{-1} f(0, \omega)$. Summarizing, we get the relation

$$u^f(x, t)|_{x = (-t+0)\omega} = -\frac{f(0, \omega)}{t} \tag{1.30}$$

[3] see the Comments below

that is a kind of geometrical optics formulae.

Fixing $x \neq 0$ and taking $t = 0$ in (1.28), we represent the unperturbed control operator in the form

$$(W_0 f)(x) = \frac{1}{2\pi} \int\limits_{S^2} f_\tau(r\omega \cdot \theta, \theta)\, d\theta + \frac{1}{r}[f(0,\cdot)]_0(\omega)\ , \quad x \in \mathbb{R}^3 . \tag{1.31}$$

The first summand belongs to $H^1_{\rm loc}(\mathbb{R}^3)$. Indeed, choose an (arbitrarily large) $r' > 0$ and $\eta = \eta(\tau) \in C^\infty[0,\infty)$ such that $\eta \mid_{[0,r']} = 1, \eta \mid_{[r'+1,\infty)} = 0$. Then, representing

$$\begin{aligned} f(\tau,\cdot) &= \eta(\tau)[f(\tau,\cdot) - f(0,\cdot)] + \{\eta(\tau)f(0,\cdot) + [1-\eta(\tau)]f(\tau,\cdot)\} = \\ &= f_1(\tau,\cdot) + f_2(\tau,\cdot) \end{aligned}$$

with $f_1 \in \mathcal{F}_{1,0}$ and supp $(f_2)_\tau \subset [r',\infty)$, we have

$$\int\limits_{S^2} f_\tau(r\omega\cdot\theta,\theta)\, d\theta = \int\limits_{S^2} f_{1\tau}(r\omega\cdot\theta,\theta)\, d\theta + \int\limits_{S^2} f_{2\tau}(r\omega\cdot\theta,\theta)\, d\theta = I_1(x) + I_2(x)\ ,$$

where $I_1 = W_0 f_1 \in H^1(\mathbb{R}^3)$ (see (1.24)) and $\operatorname{supp} I_2 \subset \mathbb{R}^3 \setminus B_{r'}$. The second summand in (1.31) is singular at $x = 0$ and doesn't belong to $H^1_{\rm loc}(\mathbb{R}^3)$.

1.7 Comments

- In [5] we introduce one more representation of W_0^* through the so-called $M-$transform. This transform is constructed from the jumps appearing in result of projecting the functions on te reachable sets (see also [3]).
- The map $g \mapsto [g]_b(\cdot)$ is known as the Minkowski-Funk transform [7] ; for $|b| < 1$ it is a compact operator in $L_2(S^2)$.
- Most probably, the formulae (1.28), (1.30) are known but we didn't succeed in finding exact references.

2 Perturbed system

We study the system (0.1)–(0.3) and prepare auxiliary results required for further investigation of its controllability.

2.1 Representation of u^f

Recall that the potential is assumed compactly supported: supp $q \subset \overline{B}_{r_*}$.

One of possible ways to establish the solvability of the problem (0.1)–(0.3) for all $t \in (-\infty,\infty)$ is to present it in the form

$$(u - u_0^f)_{tt} - \Delta(u - u_0^f) = -qu, \qquad (x,t) \in \mathbb{R}^3 \times (-r_*,\infty) \tag{2.1}$$

$$(u - u_0^f) \mid_{t<-r_*} = 0, \tag{2.2}$$

where u_0^f is the solution of (1.1)–(1.3) and, applying the Poisson-Kirchoff formula, to reduce to a Volterra type equation

$$u(x,t) = u_0^f(x,t) - \frac{1}{4\pi} \int\limits_{B_{t+r_*}(x)} \frac{q(\eta)u(\eta, t+r_* - |x-\eta|)}{|x-\eta|} d\eta \ , \qquad (2.3)$$

which can be analyzed by standard methods (see e.g. [14]). If $f \in C_0^\infty(\Sigma)$, the solution $u = u^f(x,t)$ is classical (smooth).

The second representation is derived as follows. Analyzing (2.3), one establishes the well-known property of finiteness of the domain of influence, which leads to

$$\text{supp}(u^f - u_0^f) \subset \{(x,t) \mid t \geq -r_*, -t \leq |x| \leq 2r_* + t\}$$

(on Fig 1 this support is shadowed):
Therefore, for times $t \leq 0$ the problem for the difference $u^f - u_0^f$ can be reduced to the cylinder $B_{2r_*} \times [-r_*, 0]$ and written in the form

$$\begin{aligned}
&(u - u_0^f)_{tt} - \Delta(u - u_0^f) = -qu \ , \qquad (x,t) \in \ B_{2r_*} \times (-r_*, 0) \\
&(u - u_0^f) \mid_{t=-r_*} = (u - u_0^f)_t \mid_{t=-r_*} = 0 \ , \qquad x \in \overline{B}_{2r_*} \\
&(u - u_0^f) \mid_{\Sigma^{r_*}} = 0,
\end{aligned}$$

where $\Sigma^{r_*} := \partial B_{2r_*} \times [-r_*, 0]$. Then, by an operator version of the Duhamel

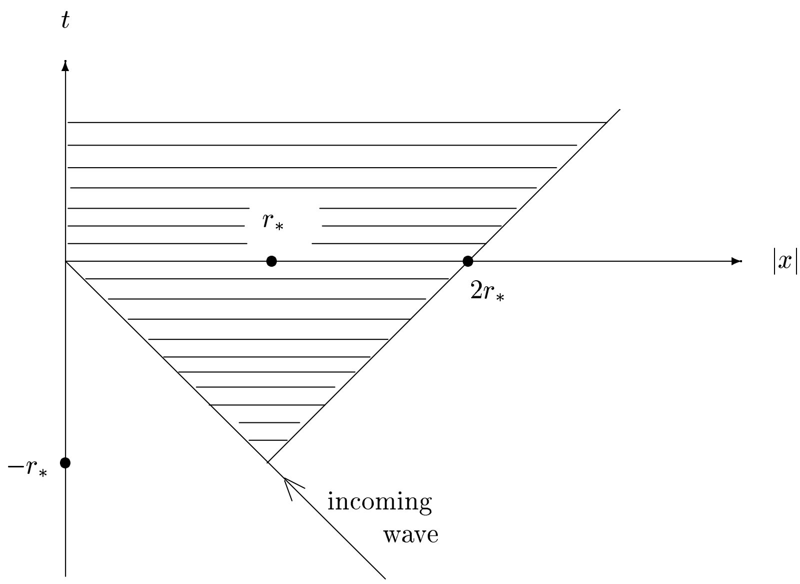

Figure 1: The sets supp $(u^f - u_0^f)$

formula, one has

$$\left[u(\cdot,t)-u_0^f(\cdot,t)\right]\Big|_{B_{2r_*}\times[-r_*,0]} \;=\; -L^{-\frac{1}{2}}\int\limits_{-r_*}^{t}\sin\left[(t-s)L^{\frac{1}{2}}\right]Y\left[q(\cdot)u(\cdot,s)\right]\,ds\;,$$

where $L: L_2(B_{2r_*})\to L_2(B_{2r_*})$, Dom $L=H^2(B_{2r_*})\cap H_0^1(B_{2r_*})$, $Ly=-\Delta y$; $Y: L^2(\mathbb{R}^3)\to L^2(B_{2r_*})$ is a reduction, $Yy:=y\,|_{B_{2r_*}}$. Returning to $\mathbb{R}^3$, we get

$$u(\cdot,t)=u_0^f(\cdot,t)-Y^*L^{-\frac{1}{2}}\int\limits_{-r_*}^{t}\sin\left[(t-s)L^{\frac{1}{2}}\right]Y\left[q(\cdot)u(\cdot,s)\right]\,ds. \tag{2.4}$$

The third representation uses a spectral decomposition of the operator $H=-\Delta+q$ (see the end of sec. 0.2). Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}: E_\lambda=E_{\lambda-0}$ be its spectral measure, $\mathcal{H}_-:=E_0\mathcal{H}$ be the subspace of negative spectrum, $\mathcal{H}_0:=(E_{+0}-E_0)\mathcal{H}=\operatorname{Ker}H$, $\mathcal{H}_+:=(I-E_{+0})\mathcal{H}$, so that $\mathcal{H}=\mathcal{H}_-\oplus\mathcal{H}_0\oplus\mathcal{H}_+$. Let $\varphi_1,\varphi_2\cdots\varphi_p$ and $\varphi_1^0,\varphi_2^0\cdots\varphi_n^0$ be the orthonormal bases in $\mathcal{H}_-$ and $\mathcal{H}_0$ respectively, $H\varphi_k=-\varkappa_k^2\varphi_k$, $\varkappa_k>0$, $H\varphi_k^0=0$. Solving the problem

$$\begin{aligned} &u_{tt}+Hu=0, \qquad t\in(-\infty,\infty)\\ &u\,|_{t=0}=u^f(\cdot,0), \quad u_t\,|_{t=0}=u_t^f(\cdot,0)\end{aligned}$$

by the Fourier method, we get

$$\begin{aligned} u^f(\cdot,t)=&\sum_{k=1}^{p}\left\{\cosh\varkappa_k t\,\left(u^f(\cdot,0),\varphi_k\right)_{\mathcal{H}}+\frac{\sinh\varkappa_k t}{\varkappa_k}\left(u_t^f(\cdot,0),\varphi_k\right)_{\mathcal{H}}\right\}\varphi_k+\\ &+\sum_{k=1}^{n}\left\{\mathbf{1}(t)\,\left(u^f(\cdot,0),\varphi_k^0\right)_{\mathcal{H}}+t(u_t^f(\cdot,0),\varphi_k^0)_{\mathcal{H}}\right\}\varphi_k^0+\\ &+\int\limits_{+0}^{\infty}\left\{\cos\sqrt{\lambda}t\;dE_\lambda u^f(\cdot,0)+\frac{\sin\sqrt{\lambda}\,t}{\sqrt{\lambda}}dE_\lambda u_t^f(\cdot,0)\right\}. \end{aligned}\tag{2.5}$$

Each of the representations mentioned above can be used for introducing a generalized solution of (0.1)–(0.3) and proving its existence and uniqueness. For instance, considering (2.4) as a second-order Volterra equation in the space $C([-r_*,0];\mathcal{H})$, it is easy to establish its solvability and the estimates

$$\begin{aligned} &\|u^f(\cdot,t)\|_{\mathcal{H}}\;\le\;\|u^f\|_{C([-r_*,0];\mathcal{H})}\;\le\;c_q\|u_0^f\|_{C([-r_*,0];\mathcal{H})}\;=\\ &\langle\text{see }(1.20)\rangle\;=\;c_q\sup_{t\in[-r_*,0]}\|W_0T_tf\|_{\mathcal{H}}=c_q\|f\|_{\mathcal{F}} \end{aligned}\tag{2.6}$$

(with regard to isometry of W_0 and T_t), where a constant c_q is determined by r_* and $\|q\|_{L_\infty(B_{r_*})}$. Then the solution of (2.4) is claimed to be an L_2-solution of (0.1)–(0.3) for $f\in\mathcal{F}$.

Note in addition that for $f \in \mathcal{L}$

$$\operatorname{diam}\operatorname{supp} u^f(\cdot,t) < \infty, \quad t \in (-\infty,\infty). \tag{2.7}$$

Indeed, for $t < -r_*$ one has $u^f(\cdot,t) = u_0^f(\cdot,t)$ and, hence, $\operatorname{supp} u^f(\cdot,t)$ is a compact set (see (1.23)). For $t \in [-r_*,\infty)$ this property is preserved by finiteness of domain of influence. Sure, the diameter tends to infinity as $t \to \infty$.

2.2 Control operator

Along with L_2-solution u^f of (0.1)–(0.3), the operator $W : \mathcal{F} \to \mathcal{H}$, $Wf = u^f(\cdot,0)$ is well defined and bounded: $\|W\| \le c_q$ (see (2.6)). The solution can be represented through the control operator in the form

$$u^f(\cdot,t) = WT_t f, \qquad t \le 0. \tag{2.8}$$

Putting $t = 0$ in (2.4) and changing $s \to -s$ in the integral, we obtain

$$u^f(\cdot,t) = u_0^f(\cdot,t) - Y^* L^{-\frac{1}{2}} \int\limits_0^{r_*} \sin\left[sL^{\frac{1}{2}}\right] Y\left[q(\cdot)u(\cdot,-s)\right]\, ds$$

that, with regard to (2.8), is equivalent to an operator relation

$$W = W_0 - K\ , \tag{2.9}$$

where

$$K = Y^* L^{-\frac{1}{2}} \int\limits_0^{r_*} \sin\left[sL^{\frac{1}{2}}\right] Y\hat{q}WT_s\, ds$$

and $\hat{q}$ multiplies functions by $q(\cdot)$. Since $L^{-\frac{1}{2}}$ maps $L_2(B_{2r_*})$ onto $H_0^1(B_{2r_*})$ continuously, the operator $K : \mathcal{F} \to \mathcal{H}$ is compact, $\operatorname{Ran} K \subset Y^* H_0^1(B_{2r_*}) \subset H^1(\mathbb{R}^3)$, and the evident estimate

$$\|K\| \le \lambda_*^{-\frac{1}{2}} r_* c_q \|q\|_{L_\infty(B_{r_*})} \tag{2.10}$$

holds, where $\lambda_* > 0$ is the lowest eigenvalue of L.

Recall the definitions

$$\mathcal{N} := \operatorname{Ker} W, \quad \mathcal{U} := W\mathcal{F} = \operatorname{Ran} W, \quad \mathcal{D} := \mathcal{H} \ominus \mathcal{U}$$

(see sec. 0.2). Since W_0 is unitary and K is compact, (2.9) and Fredholm's theorems follow to

$$\dim \mathcal{N} = \dim \mathcal{D} < \infty\ ,$$

proving a part of (0.6). Recall also that $\mathcal{F}^\xi = T_\xi \mathcal{F}$ is the subspace of delayed controls. Later we'll use the following property of the control operator.

Lemma 2 *The relation*

$$\mathcal{F}^\xi \cap \mathcal{N} = \{0\}, \quad \xi > 0$$

holds.

Proof Fix $\xi > 0$ and choose $f \in \mathcal{F}^\xi \cap \mathcal{N}$. We'll show that $f = 0$.

Since $f \in \mathcal{N}$, one has $u^f(\cdot, 0) = 0$ and, hence, the odd extension

$$u(\cdot, t) = \begin{cases} u^f(\cdot, t), & t < 0 \\ -u^f(\cdot, -t), & t \geq 0 \end{cases}$$

turns out to be a solution of the class $C((-\infty, \infty); \mathcal{H})$ of the equation

$$u_{tt} - \Delta u + qu = 0, \qquad (x, t) \in \mathbb{R}^3 \times (-\infty, \infty) \ . \tag{2.11}$$

Since $f \in \mathcal{F}^\xi$ implies $\operatorname{supp} u^f \subset \{(x, t) \mid |x| \geq -t + \xi\}$, such a solution satisfies $\operatorname{supp} u \subset \{(x, t) \mid |x| \geq |t| + \xi\}$ and, hence,

$$u = 0, \qquad (x, t) \in B_\xi \times (-\infty, \infty). \tag{2.12}$$

Applying the Fourier transform in (2.11),(2.12), we see that the function

$$\tilde{u}(x, k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikt} u(x, t)\, dt$$

satisfies

$$-k^2 \tilde{u} - \Delta \tilde{u} + q\tilde{u} = 0, \qquad x \in \mathbb{R}^3$$

and

$$\tilde{u}(x, k) = 0, \quad x \in B_\xi \tag{2.13}$$

for all $k \in (-\infty, \infty)$. By the well-known uniqueness theorem for homogeneous second-order elliptic equations [9], (2.13) implies $\tilde{u} = 0$ everywhere in $\mathbb{R}^3$. Hence, $u = 0,\ u^f = 0$ and, finally, $f = 0$. QED

2.3 Observation operator

A dynamical system

$$v_{tt} - \Delta v + qv = 0, \qquad (x, t) \in \mathbb{R}^3 \times (-\infty, 0) \tag{2.14}$$

$$v\, |_{t=0} = 0, \quad v_t\, |_{t=0} = y, \qquad x \in \mathbb{R}^3 \tag{2.15}$$

is called dual to the system (0.1)–(0.3). For $y \in \mathcal{H}$, its L^2-solution $v = v^y(x, t)$ can be introduced through an equivalent integral equation

$$v(x, t) = -\frac{1}{4\pi|t|} \int_{\partial B_{|t|}(x)} y(\sigma)\, d\sigma - \frac{1}{4\pi} \int_{B_{|t|}(x)} \frac{q(\eta) v(\eta, |t| - |x - \eta|)}{|x - \eta|}\, d\eta \tag{2.16}$$

($B_r(x)$ is a ball centered at x) obtained by applying the Poisson-Kirchhoff formula to (2.14), (2.15). This Volterra type equation has a unique solution $v^y \in C_{loc}((-\infty, 0]; \mathcal{H})$.

With the dual system one associates the observation operator $O : \mathcal{H} \to \mathcal{F}$, $\operatorname{Dom} O = \mathcal{S}$ (the Schwartz class of the smooth rapidly decaying functions)

$$(Oy)(\tau, \omega) := \lim_{s\to\infty} s \left[\frac{\partial v^y}{\partial t} + \frac{\partial v^y}{\partial r}\right]((s+\tau)\omega, -s) \ ,$$

where $\frac{\partial}{\partial r} := \frac{\partial}{\partial |x|} = \frac{x}{|x|} \cdot \nabla$. The well-posedness of this definition (i.e., the existence of the pointwise limit belonging to $\mathcal{F}$) can be seen from (2.16): for smooth y, the solution v^y is classical (smooth) and straightforward calculations lead to

$$\lim_{s\to\infty} s \left[\frac{\partial v^y}{\partial t} + \frac{\partial v^y}{\partial r}\right]((s+\tau)\omega, -s) = \\ -\frac{1}{4\pi}\frac{\partial}{\partial \tau} \int\limits_{\omega\cdot\sigma=\tau} y(\sigma)\, d\sigma + \frac{1}{4\pi} \int\limits_{\omega\cdot\eta\geq\tau} q(\eta) v^y_t(\eta, \tau - \omega\cdot\eta)\, d\eta \ , \tag{2.17}$$

where the first summand in the r.h.s. is $W_0^* y$ (see (1.25)), and the second one is smooth and compactly supported on $\overline{\Sigma}$. Since $|y|, |\nabla y|$ rapidly decrease at infinity, the limit turns out to be uniform w.r.t. $(\tau, \omega) \in \overline{\Sigma}$.

The following result clarifies a duality of the systems (0.1)–(0.3) and (2.14), (2.15).

Lemma 3 *The relation $O^* = W$ is valid.*

Proof For $y \in \operatorname{Dom} O$ and $f \in \mathcal{L}$ the solutions u^f and v^f are smooth and (2.7) holds. Integrating by parts, one has

$$0 = \int\limits_{B_s\times[-s,0]} \left[u^f_{tt} - \Delta u^f + q u^f\right] v^y\, dxdt = \int\limits_{B_s} dx \left[u^f_t v^y - u^f v^f_t\right] \Big|_{t=-s}^{t=0} - \\ \int\limits_{\partial B_s\times[-s,0]} \left[\frac{\partial u^f}{\partial r} v^y - u^f \frac{\partial v^y}{\partial r}\right] d\sigma dt + \int\limits_{B_s\times[-s,0]} u^f \left[v^y_{tt} - \Delta v^y + q v^y\right] dxdt \ = \\ \langle \text{see } (2.14), (2.15)\rangle = \\ -\left\{\int\limits_{B_s} u^f(\cdot, 0) y\, dx + \int\limits_{B_s} \left[u^f_t v^y - u^f v^y_t\right] \Big|_{t=-s} dx\right\} - \\ \left\{\int\limits_{-s}^{0} dt \int\limits_{\partial B_s} \left[\frac{\partial u^f}{\partial r} v^y - u^f \frac{\partial v^y}{\partial r}\right] d\sigma\right\} \ =: \ -I(s) - II(s) \ , \tag{2.18}$$

where $s > 0$ is a large parameter. Since $u^f|_{|x|<-t} = 0$, one has $u^f(\cdot,-s)|_{B_s} = u^f_t(\cdot,-s)|_{B_s} = 0$ and, hence,

$$I(s) \to \int\limits_{\mathbb{R}^3} u^f(\cdot,0)y\,dx = (Wf,y)_{\mathcal{H}} \tag{2.19}$$

as $s \to \infty$. Changing variables $t \to \tau - s,\ \sigma \to s^2\omega$, one gets

$$II(s) = \int\limits_{[0,s]\times S^2} s^2 \left[\frac{\partial u^f}{\partial r}v^y - u^f\frac{\partial v^f}{\partial r}\right](s\omega,\tau-s)\,d\tau d\omega\ . \tag{2.20}$$

For $y \in \mathcal{L}$, the set supp $f \subset \overline{\Sigma}$ is a compact and one can show that the limit in (0.3) is uniform w.r.t. $(\tau,\omega) \in \overline{\Sigma}$, whereas the asymptotic

$$su^f(s\omega,\tau-s) = f(\tau,\omega) + o(1) \tag{2.21}$$

is differentiable:

$$\frac{d}{ds}[su^f(s\omega,\tau-s)] = o(1).$$

This implies

$$u^f(s\omega,\tau-s) + s\left[\frac{\partial u^f}{\partial r} - \frac{\partial u^f}{\partial t}\right](s\omega,\tau-s) = o(1)$$

and, combining these asymptotics, we get

$$\begin{aligned} s\frac{\partial u^f}{\partial r}(s\omega,\tau-s) &= s\frac{\partial u^f}{\partial t}(s\omega,\tau-s) + o(1) = \\ &= \frac{\partial}{\partial\tau}u^f(s\omega,\tau-s) + o(1) \end{aligned}$$

uniformly w.r.t. $(\tau,\omega) \in \overline{\Sigma}$. Substituting the asymptotics in (2.20), we have

$$\begin{aligned} II(s) = \int\limits_{[0,s]\times S^2} &\left\{\left[s\frac{\partial}{\partial\tau}u^f(s\omega,\tau-s)\right][sv^y(s\omega,\tau-s)] - \right. \\ &\left.[su^f(s\omega,\tau-s)]\left[s\frac{\partial v^y}{\partial r}(s\omega,\tau-s)\right]\right\}d\tau d\omega + o(1). \end{aligned}$$

Integrating by parts in the first summand in $\{\cdots\}$ and taking into account the relations

$$u^f(s\omega,0)\big|_{s>>1} = 0, \quad u^f(s\omega,-s) = 0$$

provided by $f \in \mathcal{L}$, one obtains

$$II(s) = -\int\limits_{[0,s]\times S^2} \left[su^f(s\omega,\tau-s)\right]s\left[\frac{\partial v^y}{\partial t} + \frac{\partial v^y}{\partial r}\right](s\omega,\tau-s)\,d\tau d\omega + o(1).$$

Passing to the limit with regard to (2.21) we get

$$II(s) \to -\int\limits_{[0,s]\times S^2} f(\tau,\omega)\left\{\lim_{s\to\infty} s\left[\frac{\partial v^y}{\partial t}+\frac{\partial v^y}{\partial r}\right](s\omega,\tau-s)\right\}d\tau d\omega = -(f,Oy)_{\mathcal{F}}. \tag{2.22}$$

Finally, tending $s \to \infty$ in (2.18) and taking into account (2.19), (2.22), we arrive at

$$0 = -(Wf,y)_{\mathcal{H}} + (f,Oy)_{\mathcal{F}}$$

that follows to the assertion of the lemma in view of the density of the used f's and y's in $\mathcal{F}$ and $\mathcal{H}$ respectively. QED

Let us discuss some consequences of this result. The eigenfunctions φ_k of the negative spectrum of H possess the well-known asymptotic as $|x| \to \infty$:

$$\varphi_k(x) = \alpha_k(\omega)\frac{e^{-\varkappa_k r}}{r}\left[1+o(1)\right]. \tag{2.23}$$

Put $y = \varphi_k$ in (2.14), (2.15); the solution of the dual problem is

$$v^{\varphi_k}(x,t) = \frac{\sinh \varkappa_k t}{\varkappa_k}\varphi_k(x). \tag{2.24}$$

Using (2.23), (2.24), one can find the limit, which defines the observation operator, by the straightforward calculation and then, applying Lemma 3, obtain

$$(W^*\varphi_k)(\tau,\omega) = \alpha_k(\omega)e^{-\varkappa_k \tau}, \quad (\tau,\omega)\in\Sigma. \tag{2.25}$$

In the case of $0 \in \sigma_p(H)$ the eigenfunctions φ_k^0 have the asymptotic

$$\varphi_k^0(x) = \frac{\alpha_k^0(\omega)}{r^2}\left[1+o(1)\right]. \tag{2.26}$$

Considering the dual problem with $y = \varphi_k^0$ we have

$$v^{\varphi_k^0}(x,t) = t\varphi_k^0(x). \tag{2.27}$$

Using (2.26), (2.27) one can obtain

$$(W^*\varphi_k^0)(\tau,\omega) = 0, \quad (\tau,\omega)\in\Sigma$$

that yields

$$\operatorname{Ker} H \subset \operatorname{Ker} W^*. \tag{2.28}$$

One more general fact is the embedding $W^*H^1(\mathbb{R}^3) \subset \mathcal{F}_1$, which can be derived from (1.26), (2.17) and Lemma 3.

2.4 Response operator

Dealing with the system (0.1)–(0.3) for all times $t \in (\infty, \infty)$, one associates with it a response operator $R : \mathcal{F} \to \mathcal{F}$, $\quad$ Dom $R = \mathcal{L}$,

$$(Rf)(\tau, \omega) := \lim_{s\to\infty} s u^f((s+\tau)\omega, s), \quad (\tau, \omega) \in \Sigma.$$

The well-possedness of this definition can be verified as follows. If $f \in \mathcal{L}$ and $\operatorname{supp} f \subset [a,b] \times S^2$ $(a > 0)$, then the property (1.23) and the finiteness of the domain of influence imply

$$u_0^f\big|_{0\le t\le |x|+a} = 0, \quad u^f\big|_{0\le t\le |x|-2r_*} = 0,$$

for a large enough $|x|$ (see Fig 2: $\operatorname{supp} u_0^f$ and $\operatorname{supp} u^f$ are shadowed; $l_\tau = \big\{(x,t) \;\big|\; x = (s+\tau)\omega,\, t = s; \;\; s \ge 0\big\}$ is a space-time ray; the dotted line is the cone $|x| = |t|$).

In result, $s u_0^f((s+\tau)\omega, s)\,|_{\tau\ge 0} = 0$ for large enough s and, taking into account the asymptotic

$$s - |(s+\tau)\omega - \eta| \to \omega\cdot\eta - \tau + o(1),$$

the limit passage in (2.3) easily leads to

$$\begin{aligned}(Rf)(\tau,\omega) &= \lim_{s\to\infty} s u^f((s+\tau)\omega, s) = \\ &-\frac{1}{4\pi}\int\limits_{\omega\cdot\eta \ge \tau - r_*} q(\eta) u^f(\eta, \omega\cdot\eta - \tau + r_*)\, d\eta\ . \end{aligned} \tag{2.29}$$

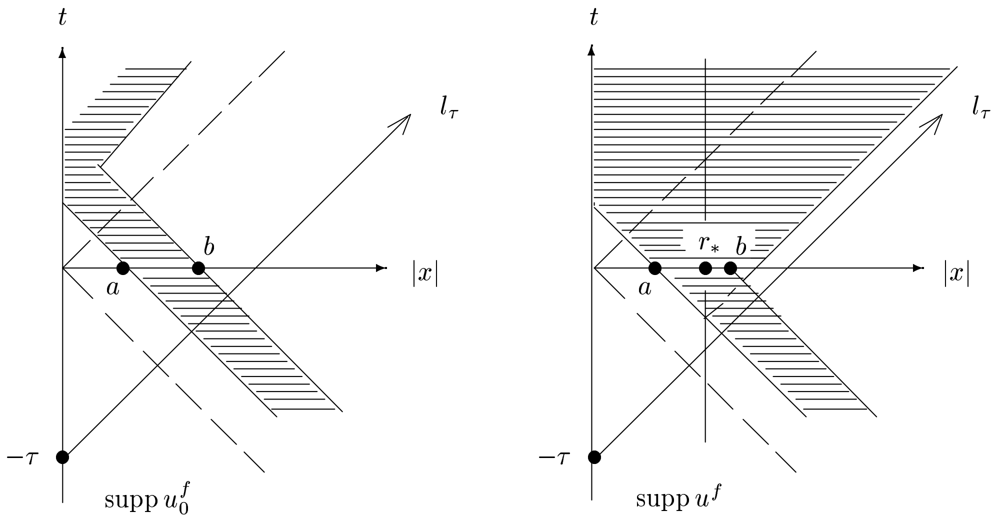

Figure 2: Supports

This justifies the definition and also gives a representation of the response operator. Note a simple consequence of (2.29): since for $\tau > 2r_*$ the domain of integration doesn't intersect $\operatorname{supp} q$ one has $Rf\,|_{\tau>2r_*}=0$, i.e.,

$$\operatorname{supp} Rf \subset [0, 2r_*]\ .$$

Using the same representation, one can show that R acts continuously from $\mathcal{F}$ to $C([0,2r_*]\times S^2)$ and is a compact operator in $\mathcal{F}$. Hence, R can be extended to $\mathcal{F}$, representation (2.29) remaining in force for $f \in \mathcal{F}$.

The important property of R is the local character of its dependence on the potential: the respond of the system on delayed controls $R\mathcal{F}^\xi$ is determined by $q\,|_{\mathbb{R}^3\setminus B^\xi}$, $\xi \geq 0$. One more general property, following from the time independence of q is the relation

$$RT_\xi = T^*_\xi R, \quad \xi \geq 0$$

and its consequence

$$R\frac{\partial}{\partial\tau} f = -\frac{\partial}{\partial\tau} Rf, \quad f \in \mathcal{F}_{1,0}\ . \tag{2.30}$$

Note in addition that the operator R can be identified with some block of the scattering operator of LPT in the translation representation.

2.5 Connecting operator

A map $C:\mathcal{F}\to\mathcal{F}$,

$$C := W^*W$$

is called a connecting operator of the system (0.1)–(0.3). By this definition, for $f, g \in \mathcal{F}$ one has

$$(Cf,g)_{\mathcal{F}} = (Wf,Wg)_{\mathcal{H}} = \left(u^f(\cdot,0), u^g(\cdot,0)\right)_{\mathcal{H}}\ , \tag{2.31}$$

i.e, C connects the Hilbert metrics of the outer and inner spaces.

Lemma 4 *The relation*

$$C = I + R \tag{2.32}$$

holds.

Proof Take $f, g \in \mathcal{L}$; let u^f, u^g be the corresponding solutions of the problem (0.1)–(0.3) considered for all times $t \in (-\infty,\infty)$. The Blagovestchenskii function

$$b(s,t) := \left(u^f(\cdot,s),\, u^g(\cdot,s)\right)_{\mathcal{H}}, \quad -\infty < s,t < \infty$$

is well defined by (2.7) and possesses the following properties.

(i) Function b satisfies the wave equation

$$b_{tt} - b_{ss} = 0\ . \tag{2.33}$$

Indeed

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial s^2}\right] b(s,t)=\int\limits_{\mathbb{R}^3}\left[u^f(x,s)u^g_{tt}(x,t)-u^f_{ss}(x,s)u^g(x,t)\right]\,dx=$$

$$\int\limits_{\mathbb{R}^3}\left\{u^f(x,s)\left[\Delta u^g(x,t)-q(x)u^g(x,t)\right]-\left[\Delta u^f(x,s)-q(x)u^f(x,s)\right]u^g(x,t)\right\}\,dx=$$

$$\int\limits_{\mathbb{R}^3}\left[u^f(x,s)\Delta u^g(x,t)-\Delta u^f(x,s)u^g(x,t)\right]\,dx=0$$

by compactness of $\operatorname{supp} u^f(\cdot,t)$ and $\operatorname{supp} u^g(\cdot,t)$.

(ii) The relation

$$\lim_{s\to\infty} b(-s,-s)=(f,g)_{\mathcal{F}} \tag{2.34}$$

holds. Indeed, let $\operatorname{supp} f, \operatorname{supp} g\subset(0,\beta]\times S^2$ $(\beta<\infty)$. Since u^f and u^g vanish in the past cone one has

$$b(-s,-s)=\int\limits_{\mathbb{R}^3}u^f(x,-s)u^g(x,-s)\,dx=\int\limits_{|x|>-s}=$$

$$\langle\text{see (1.23)}\rangle\;=\int\limits_s^{s+\beta}dr\;r^2\int\limits_{S^2}u^f(r\omega,-s)u^g(r\omega,-s)\,d\omega=$$

$$\int\limits_0^{\beta}d\tau(s+\tau)^2\int\limits_{S^2}u^f((s+\tau)\omega,-s)u^g((s+\tau)\omega,-s)\,d\omega=$$

$$\int\limits_0^{\beta}d\tau(1+\frac{\tau}{s})^2\int\limits_{S^2}\left[su^f((s+\tau)\omega,-s)\right]\left[su^g((s+\tau)\omega,-s)\right]\,d\omega=$$

$$=\langle\text{see (0.3)}\rangle\to(f,g)_{\mathcal{F}}\quad\text{as}\quad s\to\infty\;.$$

(iii) The relation

$$\lim_{s\to\infty} b(s,-s)=(Rf,g)_{\mathcal{F}} \tag{2.35}$$

holds. Indeed,

$$b(s,-s)=\int\limits_{\mathbb{R}^3}u^f(x,s)u^g(x,-s)\,dx= \langle\text{see (1.23)}\rangle =$$

$$\int\limits_s^{s+\beta}dr\;r^2\int\limits_{S^2}u^f(r\omega,s)u^g(r\omega,-s)\,d\omega=$$

$$\int\limits_0^{\beta}d\tau\left[1+\frac{\tau}{s}\right]^2\int\limits_{S^2}[su^f((s+\tau)\omega,s)][su^g((s+\tau)\omega,-s)]\,d\omega\to$$

$$(Rf,g)_{\mathcal{F}}\quad\text{as } s\to\infty.$$

(iv) The relation

$$\lim_{s\to\infty}b(0,-s)=0 \tag{2.36}$$

is valid. Indeed,

$$b(0,-s)=\int\limits_{\mathbb{R}^3}u^f(x,0)u^g(x,-s)\,dx\Big|_{s\gg 1}=0$$

since $\operatorname{supp}u^f(\cdot,0)\subset B_\beta$ and $\operatorname{supp}u^f(\cdot,-s)\subset\mathbb{R}^3\setminus B_s$.

Since b satisfies (2.33), it is a sum of two D'Alembert solutions:

$$b(s,t)=\Phi(s+t)+\Psi(s-t).$$

At the same time, relations (2.34)–(2.36) imply

$$(f,g)_{\mathcal{F}}=\Phi(-\infty)+\Psi(0),\quad (Rf,g)_{\mathcal{F}}=\Phi(0)+\Psi(\infty),\quad 0=\Phi(-\infty)+\Psi(\infty)\;.$$

Combining these equalities, one gets

$$(f,g)_{\mathcal{F}}+(Rf,g)_{\mathcal{F}}=\Phi(0)+\Psi(0)=b(0,0)=$$
$$\left(u^f(\cdot,0),u^g(\cdot,0)\right)_{\mathcal{H}}= \langle\text{see (2.31)}\rangle =(Cf,g)_{\mathcal{F}}$$

and, hence, $Cf=f+Rf$ for $f\in\mathcal{L}$. By density of $\mathcal{L}$ we arrive at (2.32). QED

The result of the lemma connects null-controls $f\in\mathcal{N}$ with the response operator: since $\operatorname{Ker}C=\operatorname{Ker}W=:\mathcal{N}$, the representation (2.32) implies

$$\mathcal{N}=\operatorname{Ker}\,(I+R)\,. \tag{2.37}$$

Therefore $f=-Rf$ and, by virtue of the embedding supp $Rf\subset[0,2r_*]$, the equality (2.37) implies supp $f\subset[0,2r_*]$.

One more consequence is the representation

$$R=-W_0^*K-K^*W_0+K^*K$$

following from (2.32) and (2.9).

3 Controllability

Here we prove Theorem 1 and study the structure of the defect subspaces in more detail.

3.1 Sets $\mathcal{U}^\xi$

Recall that the objects in use are defined in sec. 0.2.

Fix $\xi \geq 0$. Since $W = W_0 - K$, where W_0 is unitary and K is compact, it follows from the result of Lemma 2 and Fredholm theory that the operator W maps $\mathcal{F}^\xi$ onto $W\mathcal{F}^\xi = \mathcal{U}^\xi$ isomorphically. Hence, the reachable sets $\mathcal{U}^\xi$ are the closed subspaces and $\mathcal{U}^\xi \subset \mathcal{H}^\xi$ holds.

Let us prove the first embedding of (0.5). Let $a \in \mathcal{A}^\xi$ be such that $(-\Delta + q)^p a = 0$ in $\mathbb{R}^3 \setminus \overline{B}_\xi$; take $g \in \mathcal{F}^\xi \cap \mathcal{L}$ and put $f = (\frac{\partial}{\partial \tau})^{2p} g$. Since the potential q doesn't depend on time, the solution u^f of (0.1)–(0.3) satisfies

$$u^f(\cdot, 0) = u^{(\frac{\partial}{\partial \tau})^{2p} g}(\cdot, 0) = \left(\frac{\partial^2}{\partial t^2}\right)^p u^g(\cdot, 0) = (\Delta - q)^p\, u^g(\cdot, 0).$$

Therefore

$$\begin{aligned} \left(a, u^f(\cdot, 0)\right)_{\mathcal{H}} &= \int\limits_{\mathbb{R}^3} a(x)\,(\Delta - q)^p\, u^g(x, 0)\, dx = \\ \langle \text{see } (2.7) \rangle &= \int\limits_{\mathbb{R}^3} \left[(\Delta - q)^p\, a(x)\right] u^g(x, 0)\, dx = 0 \ . \end{aligned} \tag{3.1}$$

Since the set of f's used in this calculation is dense in $\mathcal{F}^\xi$ (see sec. 1.3), the set of the corresponding waves $u^f(\cdot, 0)$ is dense in $\mathcal{U}^\xi = W\mathcal{F}^\xi$. So, $a \perp \mathcal{U}^\xi$ and, hence, $\mathcal{A}^\xi \subset \mathcal{H}^\xi \ominus \mathcal{U}^\xi = \mathcal{D}^\xi$. Also, if $\xi > 0$ then $\dim \mathcal{A}^\xi = \infty$ and, hence, $\dim \mathcal{D}^\xi = \infty$.

Proving the second relation of (0.5), it is enough to take $p = 1$ in (3.1). Another argument is the relation (2.28) implying $\operatorname{Ker} H \subset \operatorname{Ker} W^* = \mathcal{H} \ominus \operatorname{Ran} W = \mathcal{H} \ominus \mathcal{U} = \mathcal{D}$. So, part (i) of the theorem is proved.

By (2.10), for small enough $\|g\|_{L_\infty(B_{r^*})}$ one has $\|K\| < 1$ and $W = W_0 - K$ turns out to be an isomorphism. This implies $\mathcal{U} = \mathcal{H}$, $\mathcal{D} = \{0\}$, and proves the part (ii).

Postponing the further proof, let us comment the results obtained. So, at the moment $t = 0$ the delayed incoming waves fill the subdomain $\mathbb{R}^3 \setminus B_\xi$ but the set of such waves $\mathcal{U}^\xi$ is not complete in $\mathcal{H}^\xi = L_2(\mathbb{R}^3 \setminus B_\xi)$ as $\xi > 0$ and, moreover, $\dim \mathcal{H}^\xi \ominus \mathcal{U}^\xi = \infty$. In the mean time, such a completeness occurs in the case of a bounded domain. Namely, consider the problem

$$u_{tt} - \Delta u + q = 0 \quad (x, t) \in B_T \times (-T, 0) \tag{3.2}$$

$$u\,|_{t=-T} = u_t\,|_{t=-T} = 0 \quad \text{in } B_T \tag{3.3}$$

$$u\,|_{\Sigma^T} = f \ , \tag{3.4}$$

where $\Sigma^T = \partial B_T \times [-T, 0]$, $u = u^f(x,t)$ is a solution. Denote

$$\mathcal{F}^T := L_2(\Sigma^T), \ \mathcal{F}^{T,\xi} := \left\{ f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \partial B_T \times [-T+\xi, 0] \right\}$$

and

$$\mathcal{H} := L_2(B_T), \ \mathcal{H}^\xi := \left\{ y \in \mathcal{H} \mid \operatorname{supp} y \subset \overline{B}_T \setminus B_\xi \right\};$$

introduce the reachable sets

$$\mathcal{U}^\xi := \left\{ u^f(\cdot, 0) \mid f \in \mathcal{F}^{T,\xi} \right\}.$$

By finiteness of the domains of influence, the embedding $\mathcal{U}^\xi \subset \mathcal{H}^\xi$ holds. The well-known fact derived from the fundamental Holmgren–John–Tataru theorem (see e.g. [2]) is the density of this embedding:

$$\operatorname{clos} \mathcal{U}^\xi = \mathcal{H}^\xi, \quad 0 \le \xi \le T \tag{3.5}$$

that is interpreted as an approximate controllability of system (3.2)–(3.5) in subdomains filled with waves. Problem (0.1)–(0.3) may be considered as a limit case of (3.2)–(3.4) as $T \to \infty$: loosely speaking, the system (0.1)–(0.3) is governed by controls located at the sphere of infinite radius. However, the properties of the reachable sets turn out to be different: comparing (3.5) and (0.5), we state a lack of controllability in the limit case[4]. One more noteworthy difference is that in a finite ball equality (3.5) fails without the closure (see [1]) whereas in the limit case $\mathcal{U}^\xi$ are closed.

The following result shows that, studying the general properties of the reachable sets, it is hardly possible to find a simple characterization of their elements.

Lemma 5 *If $\xi > 0$ and u is a compactly supported element of $\mathcal{U}^\xi$, then the set $\mathbb{R}^3 \setminus \operatorname{supp} u$ is disconnected.*

Proof (sketch) Assume the opposite: there exists a nonzero compactly supported $u \in \mathcal{U}^\xi$ such that $\mathbb{R}^3 \setminus \operatorname{supp} u$ is an (open) connected set.

Since $\mathcal{U}^\xi$ is closed, one has $u = u^f(\cdot, 0)$ for some $0 \ne f \in \mathcal{F}^\xi$. The function

$$w(x,t) = \begin{cases} u^f(x,t), & t \le 0 \\ -u^f(x,-t), & t > 0 \end{cases}$$

is a solution of

$$w_{tt} - \Delta w + qw = -2u^f(\cdot, 0)\delta'(t)$$

($\delta(t)$ is the Dirac function) satisfying

$$w(\cdot, t)\big|_{B_\xi} = 0, \quad t \in (-\infty, \infty).$$

[4]This lack, in a sense, is partially compensated by the following property. It can be shown that for any $\xi, \xi' : 0 \le \xi < \xi' < \infty$ one has

$$\operatorname{clos} \left\{ u^f(\cdot, 0)\Big|_{B_{\xi'} \setminus B_\xi} \;\Big|\; f \in \mathcal{F}^\xi \right\} = L_2(B_{\xi'} \setminus B_\xi),$$

i.e., the heads of the delayed incoming waves possess a local completeness.

Its Fourier transform $\tilde{w}^f(\cdot, k)$ satisfies

$$\Delta\tilde{w} + (k^2 - q)\tilde{w} = 2iku^f(\cdot, 0)$$

and

$$\tilde{w}\big|_{B_\xi} = 0, \quad k \in (-\infty, \infty).$$

Hence, out of supp $u^f(\cdot, 0)$ the function $\tilde{w}$ is a solution of a homogeneous elliptic equation, which vanishes identically on B_ξ. By [9], $\tilde{w} = 0$ everywhere in $\mathbb{R}^3 \setminus$ supp $u^f(\cdot, 0)$. In particular,

$$\tilde{w}(\cdot, k)\big|_{\mathbb{R}^3 \setminus B_{\xi'}} = 0, \quad k \in (-\infty, \infty)$$

for large enough ξ' providing supp $u^f(\cdot, 0) \subset \overline{B}^{\xi'} \setminus B^\xi$. This leads to

$$w(\cdot, t)\big|_{\mathbb{R}^3 \setminus B_{\xi'}} = 0, \quad t \in (-\infty, \infty),$$

yields $u^f = 0$ in $[\mathbb{R}^3 \setminus B_{\xi'}] \times (-\infty, 0]$ and follows to $f = 0$ that contradicts the assumption $f \neq 0$. QED

This rather subtle result is of general character for systems governed by hyperbolic equations [1], [3], [4].

3.2 Radial potential

Here we prove the relations (0.7); so, in this section we deal with the case $q = q(r)$.

In this case, the sets $\mathcal{A}^\xi, \mathcal{D}^\xi$ and $\mathcal{U}^\xi$ are invariant w.r.t. rotations and, hence, are reduced by spherical harmonics. Namely,

$$\mathcal{A}^\xi = \oplus \sum_{l,m} \mathcal{A}^\xi{}_{lm} \ , \quad \mathcal{A}^\xi{}_{lm} := \left\{ y \in \mathcal{H}^\xi \mid y(x) = \frac{a(r)}{r} Y_l^m(\omega), \quad a \in A_l^\xi \right\},$$

where A_l^ξ are defined in the same way as in unperturbed case, replacing (1.14) by

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + q(r) \right)^p a \ = \ 0, \quad r > \xi.$$

As well as in the unperturbed case, the ODE theory implies rank $A_l^\xi = \dim A_l^\xi = d(l)$ as $\xi > 0$. Quite analogously,

$$\mathcal{U}^\xi = \oplus \sum_{l,m} \mathcal{U}^\xi{}_{lm} \ , \quad \mathcal{U}^\xi{}_{lm} = W \mathcal{F}^\xi{}_{lm} \ ,$$

$$\mathcal{D}^\xi = \oplus \sum_{l,m} \mathcal{D}^\xi{}_{lm} \ , \quad \mathcal{D}^\xi{}_{lm} = \mathcal{H}^\xi{}_{lm} \ominus \mathcal{U}^\xi{}_{lm} \supset \mathcal{A}^\xi{}_{lm} \ ,$$

where

$$\mathcal{F}^{\xi}{}_{lm} := \mathcal{F}_{lm} \cap \mathcal{F}^{\xi}, \quad \mathcal{H}^{\xi}{}_{lm} := \mathcal{H}_{lm} \cap \mathcal{H}^{\xi}$$

(see (1.4), (1.5)). Marking the objects corresponding to $q = 0$ by the index $\overset{\circ}{\{\}}$, one evidentally has

$$\mathcal{A}^{\xi}{}_{lm} \mid {}_{\xi > r_*} = \mathring{\mathcal{A}}^{\xi}_{lm}, \quad \mathcal{U}^{\xi}{}_{lm} \mid {}_{\xi > r_*} = \mathring{\mathcal{U}}^{\xi}_{lm}, \quad \mathcal{D}^{\xi}{}_{lm} \mid {}_{\xi > r_*} = \mathring{\mathcal{D}}^{\xi}_{lm} \ .$$

We say a family of subspaces $\{\mathcal{B}^{\xi}\}$ to be continuous w.r.t. ξ if the orthogonal projection on $\mathcal{B}^{\xi}$ is continuous in the strong operator topology. By Lemma 2, as $\xi > 0$ the control operator W maps $\mathcal{F}^{\xi}{}_{lm}$ onto $\mathcal{U}^{\xi}{}_{lm}$ isomorphically. Therefore, since $\{\mathcal{F}^{\xi}{}_{lm}\}_{\xi>0}$ is continuous, the family $\{\mathcal{U}^{\xi}{}_{lm}\}_{\xi>0}$ also is continuous. Correspondingly, $\{\mathcal{D}^{\xi}{}_{lm}\}_{\xi>0}$ is continuous that implies $\dim \mathcal{D}^{\xi}{}_{lm} = \text{const} \leq \infty$. In the mean time,

$$\begin{aligned}
&\dim \mathcal{D}^{\xi}{}_{lm} \mid {}_{\xi > r_*} = \dim \mathring{\mathcal{D}}^{\xi}_{lm} = \langle \text{see } (1.27) \rangle = \dim \mathring{\mathcal{A}}^{\xi}{}_{lm} = \\
&(2l+1) \dim A^{\xi}_{l} = \langle \text{see sec. 1.2, (iii)} \rangle = (2l+1) d(l) < \infty;
\end{aligned}$$

hence, $\dim \mathcal{D}^{\xi}{}_{lm} = (2l+1)d(l)$ for all $\xi > 0$. From the other hand, by ODE theory one has $\dim \mathcal{A}^{\xi}{}_{lm} = \dim \mathring{\mathcal{A}}^{\xi}_{lm}$ for all $\xi > 0$. Therefore, the evident embedding $\mathcal{D}^{\xi}{}_{lm} \supset \mathcal{A}^{\xi}{}_{lm}$ leads to $\mathcal{D}^{\xi}{}_{lm} = \mathcal{A}^{\xi}{}_{lm}$ and, finally, to $\mathcal{D}^{\xi} = \mathcal{A}^{\xi}, \quad \xi > 0$.

Let us prove that $\mathcal{D} = \mathcal{A} = \operatorname{Ker} H$. If $h \in \mathcal{D}$ then $h\big|_{\mathbb{R}^3 \setminus B_\xi} \in \mathcal{D}^{\xi}$ and, hence, for $\xi > 0$ we have $h\big|_{\mathbb{R}^3 \setminus B_\xi} \in \mathcal{A}^{\xi}$. Therefore, representing $h(x) = \sum_{l,m} \frac{h_{lm}(r)}{r} Y_l^m(\omega)$, for $g := h_{lm}$ we have

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} + q(r) \right)^p g = 0, \quad r > 0 \tag{3.6}$$

with an integer $p > 0$. If $\xi > r_*$ then $g\big|_{(\xi,\infty)} \in \mathring{A}^{\xi}{}_{l}$ and, hence, $g\big|_{(\xi,\infty)}$ satisfies

$$\left(-\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} \right)^{d(l)} g = 0, \quad r > \xi \tag{3.7}$$

(see (1.14)). Comparing (3.6) with (3.7), we conclude that (3.6) holds for $p = d(l)$.

Since g is an L_2-solution of (3.6) with $p = d(l)$, by the ODE theory it has the differentiable asymptotic $g|_{r\to 0} = O(r^{l+1})$ whereas $g|_{r > r_*}$ is a linear combination of the solutions (1.16). From this it easily follows that $\frac{g(r)}{r} Y_l^m(\omega) \in \operatorname{Dom} H^j$ as $j = 1, 2, \ldots, d(l)$ and, hence, $\frac{g(r)}{r} Y_l^m(\omega) \in \operatorname{Ker} H^{d(l)}$. Since H is a self-adjoint operator, the last inclusion implies $\frac{g(r)}{r} Y_l^m(\omega) \in \operatorname{Ker} H$ and, returning to the element h, we get $h \in \operatorname{Ker} H$. Thus, $\mathcal{D} \subset \operatorname{Ker} H$ whereas the opposite embedding has been proved above in sec. 3.1. So, we arrive at $\mathcal{D} = \operatorname{Ker} H$.

The proof of Theorem 1 is completed.

3.3 Example of $\mathcal{D} \neq \operatorname{Ker} H = \{0\}$

The result of this section shows that, in general (not radial) case, the defect subspace is not exhausted by Ker H. Here, dealing with operator H for different q's, we denote it by H_q.

Take a smooth compactly supported q' such that $\operatorname{Ker} H_{q'} \neq \{0\}$ [5] and choose $\varphi' \in \operatorname{Ker} H_{q'}$, so that

$$(-\Delta + q')\varphi' = 0 \quad \text{in } \mathbb{R}^3.$$

By ellipticity one has $\varphi' \in C^\infty(\mathbb{R}^3)$. Let x_0 and a (small) $\varepsilon > 0$ be such that $\varphi'(x) \geq \text{const} > 0, \quad x \in B_\varepsilon(x_0)$. Construct a function

$$\varphi''(x) = \begin{cases} \varphi'(x), & x \in \mathbb{R}^3 \setminus B_\varepsilon(x_0) \\ \geq \frac{\text{const}}{2}, & x \in B_\varepsilon(x_0) \setminus B_{\frac{\varepsilon}{2}}(x_0) \\ \frac{1}{|x-x_0|}, & x \in B_{\frac{\varepsilon}{2}}(x_0) \setminus \{x_0\} \end{cases}$$

such that $\varphi'' \in C^\infty(\mathbb{R}^3 \setminus \{x_0\})$ and define

$$q''(x) = \begin{cases} q'(x), & x \in \mathbb{R}^3 \setminus B_\varepsilon(x_0) \\ \frac{\Delta\varphi''(x)}{\varphi''(x)} & x \in B_\varepsilon(x_0) \setminus \{x_0\} \\ 0 & x = x_0. \end{cases}$$

It is easily to see that $q'' \in C_0^\infty(\mathbb{R}^3)$ and φ'' satisfies

$$(-\Delta + q'')\varphi'' = 0 \quad \text{in } \mathbb{R}^3 \setminus \{x_0\} \ .$$

Note also that $q'' = 0$ in $B_{\frac{\varepsilon}{2}}(x_0)$.

Defining $\varphi(x) := \varphi''(x - x_0),\ q(x) := q''(x - x_0)$, we get

$$\varphi \in \mathcal{H}, \quad (-\Delta + q)\varphi = 0 \quad \text{in } \mathbb{R}^3 \setminus \{0\}$$

that obviously implies $\varphi \in \mathcal{A} \subset \mathcal{D}$. In the mean time, $\varphi \notin \operatorname{Ker} H_q$ because φ is singular at $x = 0$ whereas $\operatorname{Ker} H_q \subset C^\infty(\mathbb{R}^3)$.

If $\operatorname{Ker} H_q = \{0\}$ then the required example is presented. If, occasionally, $\operatorname{Ker} H_q \neq \{0\}$ then one can deform q, preserving non-trivial defect but gaining $\operatorname{Ker} H_q = \{0\}$. This can be done in the following way. For simplicity we assume that $\lambda = 0$ is an ordinary eigenvalue: $\dim \operatorname{Ker} H_q = 1$.

Let $0 \neq \psi \in \operatorname{Ker} H_q$; such an element satisfies $(-\Delta + q)\psi = 0$ and, hence,

$$\psi(x) = -\frac{1}{4\pi} \int\limits_{\mathbb{R}^3} \frac{q(y)\psi(y)}{|x-y|}\, dy \ , \qquad x \in \mathbb{R}^3 \ . \tag{3.8}$$

If $\operatorname{supp} q \subset \overline{B}_{r_*}$ then $\Delta\psi = 0$ in $\mathbb{R}^3 \setminus \overline{B}_{r_*}$; therefore, $\psi \in L_2(\mathbb{R}^3)$ implies $\psi = O(r^{-2})$ as $r \to \infty$.

[5] it exists: see [13].

Take a ball $B \subset B_{2r_*} \setminus \overline{B}_{r_*}$ (ensuring $q|_B = 0$) such that $\varphi|_B \neq 0$, where φ has been constructed above. Choose $\eta \in C_0^\infty(\mathbb{R}^3)$, supp $\eta \subset B$; define a function $\varphi_\varepsilon := \varphi + \varepsilon\eta$ and a deformation of q by

$$q_\varepsilon := \frac{\Delta\varphi_\varepsilon}{\varphi_\varepsilon} = \frac{\Delta\varphi + \varepsilon\Delta\eta}{\varphi + \varepsilon\eta} = \frac{\Delta\varphi}{\varphi} + \varepsilon\frac{\Delta\eta}{\varphi} + o(\varepsilon) = q + \varepsilon\frac{\Delta\eta}{\varphi} + o(\varepsilon) \ .$$

Note that q_ε is analytic w.r.t. ε and $q_\varepsilon = q$ out of B. We shall show that the assumption $\operatorname{Ker} H_{q_\varepsilon} \neq \{0\}$, $\varepsilon \in (0, \varepsilon_0)$ leads to a contradiction.

Reducing (3.8) to B_{2r_*}, we get an integral equation

$$\psi(x) = (G_q\psi)(x) := -\frac{1}{4\pi}\int\limits_{B_{2r_*}} \frac{q(y)\psi(y)}{|x-y|}\,dy \ , \qquad x \in B_{2r_*}.$$

By the same arguments, for $\psi_\varepsilon \in \operatorname{Ker} H_{q_\varepsilon}$ we have

$$\psi_\varepsilon(x) = -\frac{1}{4\pi}\int\limits_{B_{2r_*}} \frac{q_\varepsilon(y)\psi_\varepsilon(y)}{|x-y|}\,dy \ , \qquad x \in B_{2r_*}.$$

The analyticity implies $\psi_\varepsilon = \psi + \varepsilon h + o(\varepsilon)$ as $\varepsilon \to 0$; substituting this asymptotic and the asymptotic of q_ε in the last equation, we easily obtain

$$h(x) + \frac{1}{4\pi}\int\limits_{B_{2r_*}} \frac{q(y)h(y)}{|x-y|}\,dy = -\frac{1}{4\pi}\int\limits_{B_{2r_*}} \frac{\frac{\Delta\eta}{\varphi}(y)\psi(y)}{|x-y|}\,dy =: g(x), \quad x \in B_{2r_*}$$

so that h solves this integral equation. Note also the asymptotic $g|_{r\to\infty} = r^{-1}$).

The solvability of the last equation is equivalent to the orthogonality condition $g \perp \operatorname{Ker}(I - G_q{}^*)$ in $L_2(B_{2r_*})$. In the mean time, writing $(-\Delta + q)\psi = 0$ in the form

$$\Delta\psi(x) = -\frac{q(x)}{4\pi}\int\limits_{\mathbb{R}^3} \frac{\Delta\psi(y)}{|x-y|}\,dy \ = -\frac{q(x)}{4\pi}\int\limits_{B_{2r_*}} \frac{\Delta\psi(y)}{|x-y|}\,dy \ , \qquad x \in \mathbb{R}^3 \ ,$$

we conclude that $\Delta\psi|_{B_{2r_*}} \in \operatorname{Ker}(I - G_q{}^*)$. Therefore, by the orthogonality one has

$$0 = (\Delta\psi, g)_{L_2(B_{2r_*})} = \int\limits_{B_{2r_*}} \Delta\psi(x)g(x)\,dx = \int\limits_{\mathbb{R}^3} \Delta\psi(x)g(x)\,dx \ .$$

Integrating by parts with regard to the asymptotics of ψ and g (as $r \to \infty$), we have

$$0 = \int\limits_{\mathbb{R}^3} \psi(x)\Delta g(x)\,dx = \int\limits_{\mathbb{R}^3} dx \ \psi(x)\left[\frac{\Delta\eta}{\varphi}\psi\right](x) \ dx = \int\limits_{B} =$$

$$\int\limits_{B} \eta(x)\left[\Delta\frac{\psi^2}{\varphi}\right](x)\,dx = 2\int\limits_{B} \eta(x)\frac{|\psi(x)\nabla\varphi(x) - \varphi(x)\nabla\psi(x)|^2}{\varphi^3(x)}\,dx \ .$$

In the last equality we've used the fact that φ and ψ are harmonic in B in view of $q|_B = 0$. By arbitrariness of η, we have $|\psi\nabla\varphi - \varphi\nabla\psi|\,|_B= 0$ that means φ and ψ are proportional in B and, by ellipticity, everywhere. The last is impossible since ψ is smooth whereas φ has a singularity at $x = 0$.

This contradiction shows that one can choose $\varepsilon > 0$ providing $\operatorname{Ker} H_{q_\varepsilon} = \{0\}$, and we assume such an ε chosen. Then, the corresponding function φ_ε belongs to $\mathcal{H}$, satisfies $(-\Delta + q_\varepsilon)\varphi_\varepsilon = 0$ in $\mathbb{R}^3 \setminus \{0\}$, and $\varphi_\varepsilon = \frac{1}{r}$ near $x = 0$. Therefore, $\varphi_\varepsilon \in \mathcal{A}_{q_\varepsilon} \subset \mathcal{D}_{q_\varepsilon}$ and we have $\mathcal{D}_{q_\varepsilon} \neq \{0\}$ although $\operatorname{Ker} H_{q_\varepsilon} = \{0\}$. Thus, the condition $0 \in \sigma_p(H)$ is sufficient but not necessary for the lack of controllability.

3.4 General properties of $h \in \mathcal{D}$

Here we detail the behavior of unreachable elements near $x = 0$.

Lemma 6 *For $h \in \mathcal{D}$, the representation*

$$h(x) = \frac{\alpha(\omega)}{r} + \tilde{h}(x) \tag{3.9}$$

holds with $\alpha \in L_2(S^2)$ and $\tilde{h} \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$.

Proof By virtue of (2.9), an element $h \in \mathcal{D} = \operatorname{Ker} W^*$ satisfies $(W_0 - K)^*\, h = 0$ that implies

$$h = W_0 K^* h. \tag{3.10}$$

The control $f = K^* h$ can be written in the form

$$f = \int\limits_0^{r^*} T_s^* W^*\ \hat{q}\ Y^* \sin\left[sL^{\frac{1}{2}}\right] L^{-\frac{1}{2}} Y h\, ds$$

(see (2.9)). Analyzing, step by step, this representation, we have: $L^{-\frac{1}{2}}Yh \in H^1(B_{2r^*})$; a function $\hat{q}\, Y^* \sin\left[sL^{\frac{1}{2}}\right] L^{-\frac{1}{2}}Yh$ belongs to $H^1(\mathbb{R}^3)$ and is supported in B_{2r^*} for any s; W^* transfers this function to a control of the class $\mathcal{F}_1$ supported in $[0, 2r^*]$; this control is transferred by the left (reduced) shift T_s^* to a control of the same class supported on $[0, 2r^* - s]$. At last, the integration gives $f \in \mathcal{F}_1, \quad \operatorname{supp} f \subset [0, 2r^*]$.

Representing by (3.10) $h = W_0 f$, in accordance with (1.31) one obtains

$$h = \frac{\alpha(\omega)}{r} + \tilde{h}\ ,$$

where $\alpha = [f(0,\cdot)]_0$ and $\tilde{h} = \frac{1}{2\pi}\int\limits_{S^2} f_\tau(x\cdot\theta, \theta)\, d\theta \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$. QED

If $h \in \operatorname{Ker} H \subset \mathcal{D}$ then h is smooth and α in (3.9) equals zero. The singular element $\varphi \in \mathcal{D}$ constructed in sec. 3.3 is of the form $\varphi = \frac{1}{r} + \tilde{h}$, so that its $\alpha = 1$. In this connection the following question remains open: Do there exist the defect elements with variable α? We also don't know the simple conditions (besides a smallness of q) ensuring $\mathcal{D} = \{0\}$. For instance, does $q \geq 0$ imply $\mathcal{D} = \{0\}$? Is the presence of the negative spectrum of H necessary for $\mathcal{D} \neq \{0\}$? These natural questions are still open.

3.5 Reversing waves and s-points

Let the potential q be such that $\mathcal{D} \neq \{0\}$. In this case, $\mathcal{N} \neq \{0\}$ (see (0.6)) and there exists a null-control $f \neq 0$ providing $u^f(\cdot, 0) = Wf = 0$. It is easily proved that, due to the last equality, the function

$$u^f_-(\cdot, t) := \begin{cases} u^f(\cdot, t), & t \leq 0 \\ -u^f(\cdot, -t), & t > 0 \end{cases}$$

turns out to be an L_2-solution of the problem (0.1)–(0.3) in $\mathbb{R}^3 \times (-\infty, \infty)$. This solution is odd w.r.t. time and vanishes in $\{(x,t) \mid |x| < |t|\}$, i.e., in the past and future cones simultaneously.

As a consequence, the function

$$w^f(x,t) = \int_{-\infty}^{t} u^f(x,s)\, ds\ , \quad (x,t) \in \mathbb{R}^3 \times (-\infty, \infty)$$

is an even L_2-solution of the acoustical equation in $\mathbb{R}^3 \times (-\infty, \infty)$ also vanishing in both of the cones. As one can show, w^f is a finite energy solution:

$$\int_{\mathbb{R}^3} \left\{ \left[w^f_t(x,t) \right]^2 + \left| \nabla w^f(x,t) \right|^2 + \left[q(x) w^f(x,t) \right]^2 \right\} dx = \text{const} > 0$$

and, hence, its Cauchy data $\{w^f(\cdot, 0), w^f_t(\cdot, 0)\} = \{y, 0\}$ belong to the energy class (see [10], [12]). At the same time, since $w^f \mid_{|x|<|t|} = 0$, the pair $\{y, 0\}$ belongs to the classes D_- and D_+ of the incoming and outgoing data simultaneously. Therefore, we get an example of $\{y, 0\} \in D_+ \cap D_- \neq \{0\}$. Note that such an intersection is possible even in the case $\operatorname{Ker} H \neq \{0\}$.

From the physical point of view, the behavior of the waves u^f_- and w^f is very curious. They come from infinity, penetrate into supp q and interact with the potential. Then the waves return back to infinity, leaving supp q and remaining no trace. The absence of the residual perturbation in supp q is amazing and, probably, may be interpreted as a subtle interference effect. The existence of such reversing waves is equivalent to the lack of controllability of the system (0.1)–(0.3).

In the effect described above, the origin $x = 0$ plays the role of a point that, in a sense, stops the reversing waves. This observation motivates the following definition. For $s \in \mathbb{R}^3$, we denote the shift of q by $q_s(x) = q(x - s)$; recall that $\sigma(H_{q_s}) = \sigma(H)$. An $s \in \mathbb{R}^3$ is said to be a stop point (s-point) of the potential q if $\operatorname{Ker} H = \{0\}$ and $\mathcal{D}_{q_s} \neq \{0\}$. The set of s-points is denoted by $\Upsilon[q]$.

Loosely speaking, the potential q makes space inhomogeneous and the set $\Upsilon[q]$ consists of points, which can stop incoming waves. We hope this set is worthy of further investigation. At the moment our knowledge about $\Upsilon[q]$ is exhausted by the following observations:

- (*connection with the operator extensions*) Assuming again $\operatorname{Ker} H = \{0\}$, fix $s \in \mathbb{R}^3$ and consider the operators $L_p : \mathcal{H} \to \mathcal{H}$, $\operatorname{Dom} L_p = \{y \in C_0^\infty(\mathbb{R}^3) \mid y = 0 \text{ near } s\}$, $L_p y = (-\Delta + q)^p y$; $p = 1, 2, \ldots$. Every L_p is a densely defined symmetric semi-bounded operator and, hence, it has the self-adjoint semi-bounded extensions. If at least one L_p possesses an extension $\tilde{L_p}$ with $\operatorname{Ker} \tilde{L_p} \neq \{0\}$ then $s \in \Upsilon[q]$. Perhaps, this gives a characterization of s-points.

- (*model example*) Consider a singular potential $q = \delta(x)$. Understanding $-\Delta - \delta$ in a proper sense (see [6], [11]), one can show that this operator has only one eigenvalue $\lambda = -1$ and $\Upsilon[\delta] = S^2$. Other examples of this sort motivate the following hypothesis: any potential with negative spectrum has a nonempty set of stop-points.

3.6 Stability

Here we present the necessary and sufficient conditions providing a global L_2-boundness of trajectory of system (0.1)–(0.3). We use the same notations as in (2.5),(2.23), and denote $g_k := W^* \varphi_k \in \mathcal{F}$ (see (2.25)).

Lemma 7 *The estimate $\|u^f(\cdot, t)\|^2_{\mathcal{H}} \leq c \|f\|_{\mathcal{F}}$ holds for all $t \in (-\infty, \infty)$ if and only if $(f, g_k)_{\mathcal{F}} = 0,\ k = 1, 2, \ldots p$.*

Proof

(i) (*orthogonality $\Rightarrow$ stability*) For $t \leq 0$ one has

$$\|u^f(\cdot, t)\|_{\mathcal{H}} = \langle \text{see (2.8)} \rangle = \|W T_t f\|_{\mathcal{H}} \leq \|W\| \|f\|_{\mathcal{F}}$$

so that $u^f \mid_{t<0}$ is bounded. Let $t > 0$; take $f \in C_0^\infty(\Sigma)$ and find the Fourier coefficients in (2.5):

$$\cosh \varkappa_k t \left(u^f(\cdot, 0), \varphi_k\right)_{\mathcal{H}} + \frac{\sinh \varkappa_k t}{\varkappa_k} \left(u^f_t(\cdot, 0), \varphi_k\right)_{\mathcal{H}} =$$

$$\frac{e^{\varkappa_k t}}{2} \left(u^f(\cdot, 0) + \frac{1}{\varkappa_k} u^f_t(\cdot, 0), \varphi_k\right)_{\mathcal{H}} + \frac{e^{-\varkappa_k t}}{2} \left(u^f(\cdot, 0) - \frac{1}{\varkappa_k} u^f_t(\cdot, 0), \varphi_k\right)_{\mathcal{H}} =$$

$$\frac{e^{\varkappa_k t}}{2} \left(W \left[f + \frac{1}{\varkappa_k} f_\tau\right], \varphi_k\right)_{\mathcal{H}} + \frac{e^{-\varkappa_k t}}{2} \left(W \left[f - \frac{1}{\varkappa_k} f_\tau\right], \varphi_k\right)_{\mathcal{H}} =$$

$$\langle \text{see (2.25)} \rangle = \frac{e^{\varkappa_k t}}{2} \left(f + \frac{1}{\varkappa_k} f_\tau, g_k\right)_{\mathcal{F}} + \frac{e^{-\varkappa_k t}}{2} \left(f - \frac{1}{\varkappa_k} f_\tau, g_k\right)_{\mathcal{F}} = e^{\varkappa_k t} (f, g_k)_{\mathcal{F}}$$

(we've used $\left(\frac{1}{\varkappa_k} f_\tau, g_k\right)_{\mathcal{F}} = (f, g_k)_{\mathcal{F}}$) and

$$1(t)\left(u^f(\cdot,0),\varphi_k^0\right)_{\mathcal{H}} + t\left(u_t^f(\cdot,0),\varphi_k^0\right)_{\mathcal{H}} =$$
$$1(t)\left(Wf,\varphi_k^0\right)_{\mathcal{H}} + t\left(Wf_\tau,\varphi_k^0\right)_{\mathcal{H}} = \langle\text{see } (2.28)\rangle = 0\ .$$

Now (2.5) can be written in the form

$$u^f(\cdot,t) = \sum_{k=1}^{p} e^{\varkappa_k t}(f,g_k)_{\mathcal{F}}\ \varphi_k +$$
$$+\int_{+0}^{\infty}\left\{\cos\sqrt{\lambda}t\, dE_\lambda u^f(\cdot,0) + \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, dE_\lambda u_t^f(\cdot,0)\right\} \tag{3.11}$$

whereas the orthogonality condition leads to

$$u^f(\cdot,t) = \int_{+0}^{\infty}\left\{\cos\sqrt{\lambda}t\, dE_\lambda u^f(\cdot,0) + \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, dE_\lambda u_t^f(\cdot,0)\right\}, \tag{3.12}$$

so that $u^f(\cdot,t)$ belongs to the subspace $\mathcal{H}_+ = (I - E_{-0})\,\mathcal{H}$ of absolutely continuous spectrum of H. Since $f \perp g_k$ implies $f_\tau \perp g_k$, the same is valid for $u^{f_\tau(\cdot,t)} = u_t^f(\cdot,t) \in \mathcal{H}_+$.

To estimate the norm of the wave u^f represented by (3.12) let us assume in addition that $(Jf)(\tau,\omega) = \int_0^\tau f(s,\omega)\,ds \in C_0^\infty(\Sigma)$. The set of such f's is dense in $\mathcal{F}$ and $u^{Jf}(\cdot,0) \in \operatorname{Dom} H,\ u^{Jf}(\cdot,0) \perp \operatorname{Ker} H$. Now, we have the inequality

$$\left\|u^f(\cdot,t)\right\|_{\mathcal{H}}^2 \le 2\left[\left\|\int_{+0}^{\infty}\cos\sqrt{\lambda}t\, dE_\lambda u^f(\cdot,0)\right\|_{\mathcal{H}}^2 + \right.$$
$$\left.\left\|\int_{+0}^{\infty}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\, dE_\lambda u_t^f(\cdot,0)\right\|_{\mathcal{H}}^2\right] =: 2\,[I(t) + II(t)] \tag{3.13}$$

and the evident estimate

$$I(t) \le \|u^f(\cdot,0)\|_{\mathcal{H}}^2 \le \|W\|^2\|f\|_{\mathcal{F}}^2. \tag{3.14}$$

The part of the operator H in $\mathcal{H} \ominus \operatorname{Ker} H$ is injective; its part $H_+ := H\,(I - E_{-0})$ acting in $\mathcal{H}_+$ is a positive injective operator. This enables to represent

$$u^{Jf}(\cdot,0) = H^{-1}Hu^{Jf}(\cdot,0) = H^{-1}\left(-u_{tt}^{Jf}(\cdot,0)\right) =$$
$$-\,H^{-1}u^{(Jf)_{tt}}(\cdot,0) = -H^{-1}u^{f_\tau}(\cdot,0) = -H^{-1}u_t^f(\cdot,0)$$

and to get

$$\left(u^{Jf}(\cdot,0),u_t^f(\cdot,0)\right)_{\mathcal{H}}=-\left(H^{-1}u_t^f(\cdot,0),u_t^f(\cdot,0)\right)_{\mathcal{H}}=$$
$$-\left\|H_+^{-\frac{1}{2}}u_t^f(\cdot,0)\right\|_{\mathcal{H}}^2 \; .$$

From the other hand,

$$\begin{aligned}&\left(u^{Jf}(\cdot,0),u_t^f(\cdot,0)\right)_{\mathcal{H}}=\left(u^{Jf}(\cdot,0),u^{f_\tau}(\cdot,0)\right)_{\mathcal{H}}=\\&=\langle\text{see }(2.31),(2.32)\rangle=\left([I+R]\,Jf,f_\tau\right)_{\mathcal{F}}=\\&-\left(\frac{\partial}{\partial\tau}\,[I+R]\,Jf,f\right)_{\mathcal{F}}=\langle\text{see }(2.30)\rangle=\\&-\left([I-R]\,f,f\right)_{\mathcal{F}}=\langle\text{see }(2.32)\rangle=\\&-\left([2I-C]\,f,f\right)_{\mathcal{F}}\;,\end{aligned}$$

and we obtain

$$\begin{aligned}&\left\|H_+^{-\frac{1}{2}}u_t^f(\cdot,0)\right\|_{\mathcal{H}}^2=\left([2I-C]\,f,f\right)_{\mathcal{F}}\le\\&\left(2+\|C\|\right)\|f\|_{\mathcal{F}}^2=\left(2+\|W\|^2\right)\|f\|_{\mathcal{F}}^2\;.\end{aligned}$$

Thereafter, we can estimate the second norm in (3.13):

$$II(t)=\left\|\int\limits_{+0}^{\infty}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,dE_\lambda u_t^f(\cdot,0)\right\|_{\mathcal{H}}^2=\left\|\int\limits_{+0}^{\infty}\sin\sqrt{\lambda}t\,dE_\lambda H_+^{-\frac{1}{2}}u_t^f(\cdot,0)\right\|_{\mathcal{H}}^2\le$$
$$\left\|H_+^{-\frac{1}{2}}u_t^f(\cdot,0)\right\|_{\mathcal{H}}^2\le\left(2+\|W\|^2\right)\|f\|_{\mathcal{F}}^2\;.$$

Substituting this and (3.14) in (3.13), we get the estimate

$$\left\|u^f(\cdot,t)\right\|_{\mathcal{H}}^2\le\left(3+\|W\|^2\right)\|f\|_{\mathcal{F}}^2 \tag{3.15}$$

on a dense set of controls f. Extending (3.15) to $\mathcal{F}$, we conclude that $u^f(\cdot,t)$ is bounded in $\mathcal{H}$ uniformly w.r.t. $t\in(-\infty,\infty)$.

(ii) (*stability* $\Rightarrow$ *orthogonality*) If the orthogonality doesn't hold, by (3.11) we have

$$\left\|u^f(\cdot,t)\right\|_{\mathcal{H}}^2\ge\sum_{k=1}^{p}e^{2\varkappa_k t}(f,g_k)_{\mathcal{F}}^2\to\infty\quad\text{as }t\to\infty,$$

so that $u^f\,|_{t>0}$ is unbounded, i.e., the trajectory isn't stable. QED

3.7 Comments

- Returning to (0.5), our hypothesis is that $\mathcal{A}^\xi = \mathcal{D}^\xi$, $\xi > 0$ holds in the general case, - not only for radial potentials.

- Let us repeat once again the open question of the principal character: Is $\sigma_p(H) \neq \{\emptyset\}$ necessary for $\mathcal{D} \neq \{0\}$?

- As we saw in sec 3.5, if $0 \in \Upsilon[q]$ and $f \in \mathcal{N}$ then the solution u^f_- is a reversing wave satisfying $u^f_- \mid_{|x|<|t|} = 0$. Its Fourier transform $\tilde{u}^f_-(x,k) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{ikt} u^f_-(x,t)\,dt = \frac{1}{2\pi}\int\limits_{-|x|}^{|x|} e^{ikt} u^f_-(x,t)\,dt$ satisfies

  $$\left[-\Delta + (q - k^2)\right] \tilde{u}(\cdot, k) = 0 \quad \text{in } \mathbb{R}^3$$

  and, hence, is an eigenfunction of continuous spectrum of H, which is an *entire function* of k. Such eigenfunctions are associated with each point of $\Upsilon[q]$. It would be interesting to investigate their properties, - completeness, orthogonality, asymptotic as $|x|, |k| \to \infty$ etc.

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