PDMI PREPRINT — 19/2006

AN OPERADIC APPROACH TO DEFORMATION QUANTIZATION OF COMPATIBLE POISSON BRACKETS

Vladimir Dotsenko

Independent University of Moscow
Bolshoj Vlassievsky per. 11,
Moscow, Russia

dotsenko@mccme.ru

November 2006

Abstract. We construct an analogue of the Livernet–Loday operad for two compatible brackets. The Livernet–Loday operad can be used to define $\star$-products and deformation quantization for Poisson structures. We make use of our operad in the same way, introducing a definition of (operadic) deformation quantization of compatible Poisson structures.

The work is partially supported by the LIEGRITS fellowship (contract No.MRTN-CT-2003-505078).

1. Introduction

All algebras and operads throughout the text are defined over an arbitrary field $\Bbbk$ of characteristic zero (unless the ground field or ring is specified explicitly).

The general deformation quantization problem for Poisson structures is set, for example, by Bayen–Flato–Fronsdal–Lichnerowicz–Sternheimer [**1**]. Let us give a pure algebraic definition.

Definition 1. *A $\star$-product on a vector space A is a $\Bbbk[[\hbar]]$-linear associative product on $A[[\hbar]]$. Thus, for $a, b \in A$*

$$a \star b = a *_0 b + \hbar a *_1 b + \ldots + \hbar^k a *_k b + \ldots \qquad \text{with } a *_k b \in A, k \geq 0.$$

It is immediate to check that A with the operations $a \cdot b = a *_0 b$ and $\{a, b\} = a *_1 b - b *_1 a$ becomes a Poisson algebra, i.e. the product $a \cdot b$ is commutative and associative, and the bracket $\{a, b\}$ is a Lie bracket satisfying the Leibniz rule.

Definition 2. *For a Poisson algebra A, its deformation quantization is a $\star$-product on A such that the associated Poisson algebra structure coincides with the original structure on A.*

Some constructions of deformation quantization are known now for the case which was the most important for [**1**], namely the algebra of functions on a smooth Poisson manifold; see, for example, the work of Kontsevich [**7**]. It seems to be much more difficult to deal with deformation quantization of two compatible Poisson brackets — even on the level of introducing the problem and giving the necessary definitions. While for a Poisson bracket the quantum object should possess a simple natural structure (namely, an associative algebra structure), for two compatible brackets it is not clear at a glance which structures in principle one should expect. For example, an obvious idea that a pencil of Poisson structures can be quantized into a pencil of associative products is wrong. Using our previous results on the bihamiltonian operad from [**2**], [**3**], we generalize the Livernet–Loday idea of an operadic definition of deformation quantization from the Poisson structures to the bihamiltonian structures. According to our computation, the following algebraic structure should be used in the setting of the corresponding deformation quantization problem.

Definition 3. *The operad $\mathscr{A}ss_2$ is a quadratic operad generated by two binary operations $\star\colon a, b \mapsto a \star b$ and $[,]\colon a, b \mapsto [a, b]$ such that the following identities hold for each elements of any algebra over this operad:*

- *$a \star (b \star c) = (a \star b) \star c$;*
- *$[a, b] + [b, a] = 0$;*
- *$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$;*
- *$[a, b \star c + c \star b] = [a, b] \star c + b \star [a, c] + [a, c] \star b + c \star [a, b]$;*
- *$[a, b \star c - c \star b] + [b, c \star a - a \star c] + [c, a \star b - b \star a] +$
  $+ a \star [b, c] - [b, c] \star a + b \star [c, a] - [c, a] \star b + c \star [a, b] - [a, b] \star c = 0$.*

Thus the first operation is an associative product, and the second one is a Lie bracket which is compatible with the skew-symmetrization of the product and satisfies the Leibniz rule with the symmetrization of the product.

The paper is organized as follows. We begin with some known facts on the operad defined by Livernet and Loday which can be used to define $\star$-products operadically. We prove that this operad is related to the algebras of locally constant functions on the complements of hyperplane arrangements [**4**]. Then we define a family of associative algebras (depending on a point of $\Bbbk^2$) which provides a deformation for dual spaces of components of the bihamiltonian operad (which are double even Orlik–Solomon algebras from [**2**]); we call them double Gelfand–Varchenko algebras. The collection of these algebras possesses a cooperad structure, and it turns out that the dual operad is isomorphic to $\mathscr{A}ss_2$ everywhere except for the origin. We prove that this operad is Koszul and that our deformation of the bihamiltonian operad is flat. (The proof uses the theorem on filtered operads proved in [**6**].) Finally, we introduce the operadic problem of deformation quantization for compatible Poisson brackets. We hope to discuss examples of such a quantization and further results on it elsewhere.

Throughout the paper we use quadratic operads and the Koszul duality for operads. The corresponding definitions and notation can be found in [**3**], [**5**], [**10**].

The author is grateful to B.Feigin, A.Khoroshkin and L. Rybnikov for useful discussions.

2. The Livernet–Loday operad and related algebras

Definition 4. *The double Livernet–Loday operad $\mathscr{L}\mathscr{L}_\hbar$ is a quadratic operad generated by a commutative operation $a, b \mapsto a \cdot b$ and a skew-commutative operation $a, b \mapsto [a, b]$ satisfying the identities*

(i) $[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b$,
(ii) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$,
(iii) $(a \cdot b) \cdot c - a \cdot (b \cdot c) = \hbar^2 [b, [a, c]]$.

PROPOSITION 1 (M. Livernet, J.–L. Loday (published in [**9**])). *For each $\hbar \neq 0$ the operad $\mathscr{L}\mathscr{L}_\hbar$ is isomorphic to the associative operad. If $\hbar = 0$ then the operad $\mathscr{L}\mathscr{L}_\hbar$ is isomorphic to the Poisson operad.*

The relevance of this operad for deformation quantization is explained by the following proposition.

PROPOSITION 2 ([**9**]). *A $\star$-product on a vector space V is the same as an $\mathscr{L}\mathscr{L}_\hbar$-algebra structure on the $\Bbbk[[\hbar]]$-module $V[[\hbar]]$.*

The following is clear from the proof of the previous statement in [**9**].

PROPOSITION 3. *A deformation quantization of a Poisson algebra A is a $\star$-product on this algebra such that the operations on the vector space $A = A[[\hbar]]/\hbar A[[\hbar]]$ induced from the $\mathscr{L}\mathscr{L}_\hbar$-structure coincide with the original Poisson operations on A.*

Markl and Remm [**9**] proved that the operad $\mathscr{L}\mathscr{L}_\hbar$ is a Hopf operad. It follows that the dual spaces to its components are associative algebras. The following fact is quite surprising though.

THEOREM 1. *The associative algebra dual to the nth component of the Livernet–Loday operad can be presented by generators x_{ij}, $1 \leq i, j \leq n$, and relations*

$$\begin{gathered} x_{ij} + x_{ji} = \hbar, \\ x_{ij}x_{jk} + \hbar x_{ik} = x_{ij}x_{ik} + x_{jk}x_{ik}, \\ x_{ij}^2 = \hbar x_{ij}. \end{gathered}$$

*Thus for $\hbar \neq 0$ it is isomorphic to the Gelfand–Varchenko algebra of locally constant functions on the complement of the real hyperplane arrangement A_{n-1} (studied in [**4**]).*

PROOF. A straightforward calculation shows that the above relations are satisfied in the duals of the components. Thus it remains to prove that the mappings from the corresponding abstract algebras to the duals $\mathscr{L}\mathscr{L}_\hbar(n)^*$ are isomorphisms. It is easy to see that these abstract algebras are isomorphic to the Gelfand–Varchenko algebras. Thus the dimensions of these algebras are indeed equal to the dimensions of the duals (the number of regions for the arrangement A_{n-1} is equal to $n!$, which is also the dimension of the nth component of the associative operad), and it is enough to check the surjectivity, which is clear. $\square$

3. Double Gelfand–Varchenko algebras

Although the operad $\mathscr{A}ss_2$ can be (and, in fact, is) studied separately from the double Gelfand–Varchenko algebras, the construction of these algebras was crucial in our work and so we discuss them here. Moreover, one can easily check that — under reasonable restrictions — they give the essentially unique deformation of double even Orlik–Solomon algebras and thus lead to a distinguished approach to deformation quantization.

Definition 5. *The double Gelfand–Varchenko algebra $\mathscr{GV}_{2,\hbar_1,\hbar_2}(n)$ is an associative commutative algebra with generators x_{ij} and y_{ij}, $1 \le i, j \le n$, and relations*

$$\begin{gathered} x_{ij} + x_{ji} = \hbar_1, y_{ij} + y_{ji} = \hbar_2, \\ x_{ij}x_{jk} + \hbar_1 x_{ik} = x_{ij}x_{ik} + x_{jk}x_{ik}, \\ x_{ij}y_{jk} + y_{ij}x_{jk} + \hbar_2 x_{ik} + \hbar_1 y_{ik} = x_{ij}y_{ik} + y_{ij}x_{ik} + x_{jk}y_{ik} + y_{jk}x_{ik}, \\ y_{ij}y_{jk} + \hbar_2 y_{ik} = y_{ij}y_{ik} + y_{jk}y_{ik}, \\ x_{ij}^2 = \hbar_1 x_{ij}, y_{ij}^2 = \hbar_2 y_{ij}, 2x_{ij}y_{ij} = \hbar_2 x_{ij} + \hbar_1 y_{ij}. \end{gathered}$$

It turns out that the collection of double Gelfand–Varchenko algebras possesses a cooperad structure. The corresponding formulas coincide with the formulas from [**2**].[1] (One of the restrictions fixing the deformation which were mentioned above, is that the cocomposition remains the same. It looks too restrictive at a glance, but it is true in the case of the Poisson operad.)

Definition 6. *Let I, J be finite sets, $* \notin I$. Let us define an algebra homomorphism*

$$\rho_{IJ}\colon \mathscr{GV}_{2,\hbar_1,\hbar_2}(I \sqcup J) \to \mathscr{GV}_{2,\hbar_1,\hbar_2}(I \sqcup \{*\}) \otimes \mathscr{GV}_{2,\hbar_1,\hbar_2}(J)$$

by

$$\rho_{IJ}(x_{ij}) = \begin{cases} x_{ij} \otimes 1, \text{if } i, j \in I, \\ 1 \otimes x_{ij}, \text{if } i, j \in J, \\ x_{i*} \otimes 1, \text{if } i \in I, j \in J, \\ x_{*i} \otimes 1, \text{if } i \in J, j \in I, \end{cases}$$

[1] As usual, we define the algebra $\mathscr{GV}_{2,\hbar_1,\hbar_2}(I)$ for an arbitrary finite set I (indexing the generators); thus, $\mathscr{GV}_{2,\hbar_1,\hbar_2}(n) = \mathscr{GV}_{2,\hbar_1,\hbar_2}(\{1, \dots, n\})$, and cocompositions are some mappings from $\mathscr{GV}_{2,\hbar_1,\hbar_2}(I \sqcup J)$ to $\mathscr{GV}_{2,\hbar_1,\hbar_2}(I \sqcup \{*\}) \otimes \mathscr{GV}_{2,\hbar_1,\hbar_2}(J)$.

$$\rho_{IJ}(y_{ij}) = \begin{cases} y_{ij} \otimes 1, \text{if } i, j \in I, \\ 1 \otimes y_{ij}, \text{if } i, j \in J, \\ y_{i*} \otimes 1, \text{if } i \in I, j \in J, \\ y_{*i} \otimes 1, \text{if } i \in J, j \in I. \end{cases}$$

LEMMA 1. *For all I, J the homomorphism ρ_{IJ} is well-defined.*

It is easy to check that these mappings satisfy all the cocomposition axioms, and so define a cooperadic structure. To make the computations in the dual operad more simple and transparent, we used the following technical result.

PROPOSITION 4. *The algebra $\mathcal{GV}_{2,\hbar_1,\hbar_2}(n)$ is isomorphic to the associative commutative algebra with generators a_{ij}, b_{ij}, $1 \leq i, j \leq n$ and relations*

$$a_{ij} + a_{ji} = b_{ij} + b_{ji} = 0,$$
$$a_{ij}a_{jk} + a_{jk}a_{ki} + a_{ki}a_{ij} = \hbar_1^2,$$
$$a_{ij}b_{jk} + b_{ij}a_{jk} + a_{jk}b_{ki} + b_{jk}a_{ki} + a_{ki}b_{ij} + b_{ki}a_{ij} = 2\hbar_1\hbar_2,$$
$$b_{ij}b_{jk} + b_{jk}b_{ki} + b_{ki}b_{ij} = \hbar_2^2,$$
$$a_{ij}^2 = \hbar_1^2, b_{ij}^2 = \hbar_2^2, a_{ij}b_{ij} = \hbar_1\hbar_2.$$

4. The Double Livernet–Loday operad

DEFINITION 7. *The double Livernet–Loday operad $\mathcal{LL}_{2,\hbar_1,\hbar_2}$ is generated by a commutative operation $a, b \mapsto a \cdot b$ and two skew-commutative operations $a, b \mapsto \{a, b\}$ and $a, b \mapsto [a, b]$ satisfying the identities*

(i) $[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b,$
(ii) $\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b,$
(iii) $\{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0,$
(iv) $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0,$
(v) $\{a, [b, c]\} + \{b, [c, a]\} + \{c, [a, b]\} + {}$
$+ [a, \{b, c\}] + [b, \{c, a\}] + [c, \{a, b\}] = 0,$
(vi) $(a \cdot b) \cdot c - a \cdot (b \cdot c) =$
$= \hbar_1^2[b, [a, c]] + \hbar_1\hbar_2([b, \{a, c\}] + \{b, [a, c]\}) + \hbar_2^2\{b, \{a, c\}\}.$

PROPOSITION 5. *For each point $(\hbar_1, \hbar_2)$ except for the origin the operad $\mathcal{LL}_{2,\hbar_1,\hbar_2}$ is isomorphic to the operad $\mathcal{A}ss_2$. If $(\hbar_1, \hbar_2) = (0, 0)$ then the operad $\mathcal{LL}_{2,\hbar_1,\hbar_2}$ is isomorphic to the bihamiltonian operad.*

PROOF. The case of the origin is clear. Let $(\hbar_1, \hbar_2) \neq (0, 0)$. Consider a new operation $a, b \mapsto \hbar_1[a, b] + \hbar_2\{a, b\}$. It is clear that the operad generated by this operation and $\cdot$ is isomorphic to $\mathcal{LL}_\hbar$ (and thus, according to Proposition 1, to the associative operad). The remaining bracket gives the second operation of $\mathscr{A}ss_2$. $\square$

5. Filtrations and Koszulness

Here we recall the results of A. Khoroshkin that are crucial for proving the Koszulness of operads in several important cases.

Let Γ be a set-theoretic operad. It is called an ordered operad if each set $\Gamma(n)$ is ordered and this order is compatible with operadic compositions: for $a < b \in \Gamma(n)$, $c \in \Gamma(k)$, $i = 1, \ldots, k$, $j = 1, \ldots, n$

$$c \circ_i a < c \circ_i b, \quad a \circ_j c < b \circ_j c.$$

Suppose that for each n and each $\alpha \in \Gamma(n)$ we have a subspace $F_\alpha \subset \mathscr{O}(n)$. The collection of these subspace is called a Γ-valued filtration on $\mathscr{O}$, if

- For all $\alpha \leq \beta$ we have $F_\alpha \subset F_\beta$.
- For the maximal element $\delta_n \in \Gamma(n)$ we have $F_{\delta_n} = \mathscr{O}(n)$.
- Compositions are compatible with the filtration:

$$\gamma_{l,m_1,\ldots,m_l}(F_{\alpha_l} \otimes F_{\beta_1} \otimes \ldots \otimes F_{\beta_l}) \subset F_{\alpha_l(\beta_1,\ldots,\beta_l)}.$$

The associated graded operad $gr^F\mathscr{O}(n)$ is, by the definition, the sum

$$\bigoplus_{\alpha\in\Gamma(n)} \frac{F_\alpha}{\Sigma_{\alpha'<\alpha} F_{\alpha'}}.$$

A Γ-valued filtration on $\mathscr{O}$ is *generated by binary operations* if for each α the space F_α coincides with the span of compositions of all binary operations that belong to some $F_{\alpha'}$ with $\alpha' \leq \alpha$.

The following theorem is proved in [**6**]; both the theorem and its proof generalize Theorem 7.1 of [**12**] from the quadratic algebras to quadratic operads.

THEOREM 2. *Suppose that a Γ-valued filtration on a quadratic operad $\mathscr{A}$ (with generators $\mathscr{O}_{\mathscr{A}}$ and relations $\mathscr{R}_{\mathscr{A}}$) is generated by binary operations and satisfies the following conditions:*

(i) *the operad $gr^F\mathscr{A}$ does not have nontrivial relations of degree* 3 *(between the elements belonging to $gr^F\mathscr{A}(4)$);*
(ii) *the quadratic operad $qgr^F\mathscr{A}$ with generators $gr^F\mathscr{O}_{\mathscr{A}}$ and relations $gr^F\mathscr{R}_{\mathscr{A}}$ is Koszul.*

Then the operad $gr^F\mathscr{A}$ is quadratic (and thus isomorphic to $qgr^F(\mathscr{A})$), and the operad $\mathscr{A}$ is Koszul.

As and example we give a proof of the theorem on distributive laws between Koszul operads [**8**]. The proof of this statement in [**8**] is partially based on an erroneous formula for the composition in the symmetric case [**8**, Prop. 1.7] and is therefore incomplete. The proof below is essentially due to A. Khoroshkin [**6**].

Let $\mathscr{A}$ and $\mathscr{B}$ be two quadratic operads. Denote by $\mathscr{O}_{\mathscr{A}}$, $\mathscr{O}_{\mathscr{B}}$ the corresponding sets of generators and by $\mathscr{R}_{\mathscr{A}}$, $\mathscr{R}_{\mathscr{B}}$ the spaces of relations. Denote also by $\mathscr{O}_{\mathscr{A}} \bullet \mathscr{O}_{\mathscr{B}}$ the subspace in the free operad $\mathscr{F}_{\mathscr{O}_{\mathscr{A}} \oplus \mathscr{O}_{\mathscr{B}}}$ generated by all elements $\phi(1,\psi)$ with $\phi \in \mathscr{O}_{\mathscr{A}}$, $\psi \in \mathscr{O}_{\mathscr{B}}$ (and 1 stands for the identity unary operation). The notation $\mathscr{O}_{\mathscr{B}} \bullet \mathscr{O}_{\mathscr{A}}$ has the analogous meaning. Assume that there is an S_3-equivariant mapping

$$d\colon \mathscr{O}_{\mathscr{B}} \bullet \mathscr{O}_{\mathscr{A}} \to \mathscr{O}_{\mathscr{A}} \bullet \mathscr{O}_{\mathscr{B}}.$$

Let $\mathscr{C}$ be the quadratic operad with generators $\mathscr{O}_{\mathscr{C}} = \mathscr{O}_{\mathscr{A}} \oplus \mathscr{O}_{\mathscr{B}}$ and relations $\mathscr{R}_{\mathscr{C}} = \mathscr{R}_{\mathscr{A}} \oplus \mathscr{D} \oplus \mathscr{R}_{\mathscr{B}}$, where

$$\mathscr{D} = \{x - d(x) \mid x \in \mathscr{O}_{\mathscr{B}} \bullet \mathscr{O}_{\mathscr{A}}\}.$$

THEOREM 3 ([**8**],[**6**]). *Assume that the natural mapping*

$$\xi\colon (\mathscr{A} \circ \mathscr{B})(4) \to \mathscr{C}(4)$$

is an isomorphism. Then the operad $\mathscr{C}$ is Koszul, and $\mathbb{S}$-modules $\mathscr{A} \circ \mathscr{B}$ and $\mathscr{C}$ are isomorphic.

PROOF. The main ingredient of the proof is some particular set-theoretic operad Γ and a Γ-filtration on $\mathscr{C}$ for which the associated graded operad is isomorphic to $\mathscr{A} \circ \mathscr{B}$ (where the composition in the reverse order is equal to zero).

Consider the free set-theoretic operad generated by two binary operations a and b. Let T be an arbitrary element of this operad, i.e. a binary tree whose internal vertices are labeled by a and b. By the definition, the degree $\deg(T)$ is equal to the number of a's among the labels, and the number of inversions $\mathrm{inv}(T)$ is equal to the number of pairs (v_1, v_2) of internal vertices of T where v_2 belongs to the set of descendants of v_1, the label of v_2 is equal to a, and the label of v_1 is equal to b. We have

$$\mathscr{F}_2(n) = \bigsqcup_{k=0}^{n-1} \mathscr{F}_2(n)_k \quad \text{with } \mathscr{F}_2(n)_k := \{T \,|\, \deg(T) = k\}$$

Let $m_{n,k}$ be the maximal number of inversions for the elements of $\mathscr{F}_2(n)_k$.

Let us define the operad of inversions $\mathscr{F}_2^{\mathrm{inv}}$. Its component $\mathscr{F}_2^{\mathrm{inv}}(n)$ is a disjoint union of sets

$$\bigsqcup_{k=0}^{n-1}\{s_0^k(n),\ldots,s_{m_{n,k}}^k(n)\},$$

the ordering of this component is lexicographic (to compare $s_i^j(n)$ and $s_k^l(n)$, we use the lexicographic ordering for the pairs (j,i) and (l,k)), and the compositions are given by

$$s_i^l(n)(s_{j_1}^{k_1}(r_1),\ldots,s_{j_n}^{k_n}(r_n)) := s^{l+k_1+\ldots+k_n}_{\max(\mathrm{inv}(\alpha(\beta_1,\ldots,\beta_n)))}(r_1+\ldots+r_n),$$

(maximum is taken over all collections $(\alpha,\beta_1,\ldots,\beta_n)$ with $\mathrm{inv}(\alpha)=i$, $\mathrm{inv}(\beta_1) = j_1$,…,$\mathrm{inv}(\beta_n) = j_n$, $\deg(\alpha) = l$, $\deg(\beta_1) = k_1$, …, $\deg(\beta_n)=k_n$).

There exists a natural $\mathscr{F}_2^{\mathrm{inv}}$-valued filtration F on $\mathscr{C}$: it assigns the label a to the generators belonging to $\mathscr{A}$, and the label b to the generators belonging to $\mathscr{B}$. The associated graded operad $qgr^F\mathscr{C}$ is isomorphic to $\mathscr{A}\circ\mathscr{B}$ (with zero reverse composition). Indeed, the associate graded relations include $\mathscr{R}_{\mathscr{A}}$ and $\mathscr{R}_{\mathscr{B}}$ remaining as is, while the relations $x-d(x)$ are transformed into x, since $d(x)$ belongs to the previous level of the filtration with respect to the number of inversions.

The operad $qgr^F\mathscr{C}$ is Koszul, since for the corresponding Koszul complexes we have the following isomorphisms and quasi-isomorphisms:

$$K^\bullet(qgr^F\mathscr{C}) \simeq (\mathscr{A}\circ\mathscr{B})\circ((\mathscr{A}\circ\mathscr{B})^!)^\vee \simeq$$
$$\simeq \mathscr{A}\circ\left(\mathscr{B}\circ(\mathscr{B}^!)^\vee\right)\circ(\mathscr{A}^!)^\vee \xrightarrow{quis} \mathscr{A}\circ(\mathscr{A}^!)^\vee \xrightarrow{quis} \Bbbk.$$

Moreover, the operad $gr^F\mathscr{C}$ has no nontrivial relations of degree 3, since the corresponding components of $gr^F\mathscr{C}$ and $qgr^F\mathscr{C}$ are isomorphic. It follows that $\mathscr{C}$ is Koszul, and $qgr^F\mathscr{C}\simeq gr^F\mathscr{C}$. Taking the associated graded operad does not affect the $\mathbb{S}$-module structure, so $\mathscr{C}\simeq\mathscr{A}\circ\mathscr{B}$. $\square$

This proof can be easily generalized for the following problem.

Let again $\mathscr{A}$ and $\mathscr{B}$ be two quadratic operads (all notation remains the same). Assume that there are S_3-equivariant mappings

$$s\colon \mathscr{F}_{\mathscr{O}_{\mathscr{A}}}(3)\to \mathscr{O}_{\mathscr{B}}\bullet\mathscr{O}_{\mathscr{A}}\oplus\mathscr{O}_{\mathscr{A}}\bullet\mathscr{O}_{\mathscr{B}}\oplus\mathscr{F}_{\mathscr{O}_{\mathscr{V}}}(3).$$

and

$$d\colon \mathscr{O}_{\mathscr{B}}\bullet\mathscr{O}_{\mathscr{A}}\to\mathscr{O}_{\mathscr{A}}\bullet\mathscr{O}_{\mathscr{B}}.$$

Let $\mathscr{E}$ be the quadratic operad with generators $\mathscr{O}_{\mathscr{E}}=\mathscr{O}_{\mathscr{A}}\oplus\mathscr{O}_{\mathscr{B}}$ and relations $\mathscr{R}_{\mathscr{E}}=\mathscr{S}\oplus\mathscr{D}\oplus\mathscr{R}_{\mathscr{B}}$, where

$$\mathscr{S}=\{x-s(x)\mid x\in\mathscr{F}_{\mathscr{O}_{\mathscr{A}}}(3)\},\quad \mathscr{D}=\{x-d(x)\mid x\in\mathscr{O}_{\mathscr{B}}\bullet\mathscr{O}_{\mathscr{A}}\}.$$

Proposition 6. *Assume that the natural mapping*

$$\xi\colon (\mathscr{A}\circ\mathscr{B})(4)\to\mathscr{E}(4)$$

is an isomorphism. Then the operad $\mathscr{E}$ is Koszul, and $\mathbb{S}$-modules $\mathscr{A}\circ\mathscr{B}$ and $\mathscr{E}$ are isomorphic.

Proof. The same filtration can be used here as well. It turns out again that $qgr^F\mathscr{E}\simeq\mathscr{A}\circ\mathscr{B}$ (the associated graded relations included $\mathscr{R}_{\mathscr{B}}$ remaining as is, the relations $x-d(x)$ are transformed into x since $d(x)$ belongs to the previous level of filtration with respect to the number of inversions, while the relations $x-s(x)$ are transformed into x since $s(x)$ belongs to the previous level of filtration with respect to the degree), and the remaining proof is the same. □

Theorem 4. *The operad $\mathscr{LL}_{2,\hbar_1,\hbar_2}$ is Koszul, and the $\mathbb{S}$-module $\mathscr{LL}_{2,\hbar_1,\hbar_2}$ is isomorphic to $\mathscr{C}om\circ\mathscr{L}ie_2$. $\mathbb{S}$-modules $\mathscr{LL}_{2,\hbar_1,\hbar_2}$ and $\mathscr{P}_2$ are isomorphic, so we have constructed a flat deformation of the bihamiltonian operad.*

Proof. We apply the theorem to $\mathscr{A}=\mathscr{C}om$, $\mathscr{B}=\mathscr{L}ie_2$. A straightforward calculation shows that there are no nontrivial relations of degree 3 in $gr^F\mathscr{LL}_{2,\hbar_1,\hbar_2}$. Since $\mathscr{P}_2\simeq\mathscr{C}om\circ\mathscr{L}ie_2$ ([**3**]), the last statement follows. □

Similarly to [**2**], we can derive some information about double Gelfand–Varchenko algebras from the results about the double Livernet–Loday operad.

The following proposition is straightforward.

Proposition 7. *Consider the operad $\mathscr{GV}^*$ dual to the cooperad of double Gelfand–Varchenko algebras. Then*

(i) *the relations of $\mathscr{LL}_{2,\hbar_1,\hbar_2}$ hold in $\mathscr{GV}^*$;*
(ii) *quadratic relations of $\mathscr{GV}^*$ are exactly the relations of $\mathscr{LL}_{2,\hbar_1,\hbar_2}$.*

It follows that there exists a natural mapping of the operads

$$\pi\colon \mathscr{LL}_{2,\hbar_1,\hbar_2}\to\mathscr{GV}^*.$$

Theorem 5. *The operad $\mathscr{GV}^*$ is quadratic; π is an isomorphism.*

Proof. The proof is completely analogous to the proof in the case of double even Orlik–Solomon algebras in [**2**]. We already proved that the operad $\mathscr{LL}_{2,\hbar_1,\hbar_2}$ is a flat deformation of the bihamiltonian operad. In fact, it is clear that the monomials introduced in [**2**] give a basis for this operad as well. These monomials can be used to prove that the pairing between $\mathscr{LL}_{2,\hbar_1,\hbar_2}$ and the collection of double Gelfand–Varchenko algebras is nondegenerate (where we have to check the triangularity of a certain matrix), which proves the theorem. □

6. Deformation quantization: an operadic definition

We introduce the following definitions which are motivated by Propositions 2 and 3.

DEFINITION 8. *A $\star_{\hbar_1,\hbar_2}$-product on a vector space V is a $\mathscr{L}\mathscr{L}_{2,\hbar_1,\hbar_2}$-algebra structure on the $\Bbbk[[\hbar_1,\hbar_2]]$-module $V[[\hbar_1,\hbar_2]]$.*

DEFINITION 9. *A deformation quantization of a bihamiltonian algebra A is a $\star_{\hbar_1,\hbar_2}$-product on this algebra such that the operations on the vector space $A = A[[\hbar_1,\hbar_2]]/(\hbar_1,\hbar_2)A[[\hbar_1,\hbar_2]]$ induced from the $\mathscr{L}\mathscr{L}_{2,\hbar_1,\hbar_2}$-structure coincide with the original (bihamiltonian) operations on A.*

References

[1] Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D., Deformation theory and quantization, I, II. *Ann. Physics*, **111** (1978), 61–151.

[2] Bershtein M., Dotsenko V., Khoroshkin A., Quadratic algebras related to the bihamiltonian operad, *Int. Math. Res. Notices*, to appear; `math.RT/0607289`.

[3] Dotsenko V., Khoroshkin A., Character formulas for the operad of two compatible brackets and for the bihamiltonian operad, *Funct. Anal. Appl.*, to appear; `math.QA/0605233`.

[4] Gelfand I. M., Varchenko A. N., Heaviside functions of a configuration of hyperplanes. *Funktsional. Anal. i Prilozhen.*, **21** (1987), no. 4, 1–18 (in Russian).

[5] Ginzburg V., Kapranov M., Koszul duality for operads. *Duke mathematical journal*, **76** (1994), no. 1, 203–272.

[6] Khoroshkin A., Koszul operads and distributive lattices, preprint ITEP-TH-24/06.

[7] Kontsevich M., Deformation quantization of Poisson manifolds, I, `q-alg/9709040`.

[8] Markl M., Distributive laws and Koszulness, *Ann.Inst.Fourier*, **46** (1996), no. 2, 307–323.

[9] Markl M., Remm E., Algebras with one operation including Poisson and other Lie-admissible algebras, `math.AT/0412206`.

[10] Markl M., Shnider S., Stasheff J. D. Operads in Algebra, Topology and Physics, *Mathematical Surveys and Monographs*, vol. 96, AMS, Providence, RI, 2002.

[11] Mathieu O., The symplectic operad, In "Functional analysis on the eve of the 21st century", vol. 1 (New Brunswick, NJ, 1993), *Progr. Math.*, **131**, Birkhäuser Boston, Boston, MA, 1995, p. 223–243.

[12] Polishchuk A., Positselski L., Quadratic algebras, *University Lecture Series*, **37**, AMS, Providence, RI, 2005.