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The Einsteinian gauge approach and the stress field of non-singular screw dislocation in the second order approximation

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Abstract

The stress field of non-singular screw dislocation is investigated in the second order approximation by means of the translational gauge approach of the Einstein-type. The stresses of second order around the screw dislocation are classically known for the hollow circular cylinder with traction-free external and internal boundaries. The inner boundary surrounds the dislocation's core, which is not captured by the conventional solution. The present gauge approach enables the classically known quadratic stresses be continued inside the core. The corresponding gauge equation is chosen in the Hilbert–Einstein form. The gauge equation plays a role of non-conventional incompatibility law, and the stress function method is used for its solution. The modified stress potential is obtained as a sum of two parts: conventional, say, 'background' and a short-ranged gauge contribution. The latter just causes additional stresses localized within the dislocation's core. The asymptotic properties of the resulting stresses are studied. Since the gauge contributions are short-ranged, the background stress field dominates at the distances sufficiently large with respect to the core. The outer cylinder's boundary is traction-free. At sufficiently moderate distances, the second order stresses acquire regular continuation within the core region, and the cut-off at the core is absent. Expressions for the asymptotically far stresses provide us self-consistently with new length parameters of the unconventional origin. These lengths are given by means of the elastic moduli and could characterize an exteriority of the dislocation core.

1 Introduction

Translationally-gauge model based on the Hilbert–Einstein gauge Lagrangian has been proposed in [1] for description of solid continua with static dislocations. The group of translations of three-dimensional Euclidean space $T(3) \approx \mathbb{R}^3$ is the gauge group in [1]. In linear approximation, the approach [1] gave a modification of the ordinary screw and edge dislocations of theoretical elasticity: two gauge defects presented eventually in [1] are missing the axial singularities inherent to the classical dislocations. The present paper is concerned with the non-singular screw dislocation obtained in [1] and is devoted to investigation of its stress field in the second order approximation.

The approach [1] is governed by the non-homogeneous Einstein-type gauge equation which has initially been written in [1] as driven by the elastic stress tensor. In linear approximation, this equation results in two, so-called, *modified stress potentials* which coincide, at small distances, with the Prandtl and Airy stress functions of the ordinary dislocation theory though are decaying at infinity. Therefore, the modified stress potentials provide opportunity to improve the short-distance stresses of conventional dislocations. Indeed, let the classical dislocations be considered as *background* (i.e., pre-imposed) sources of distortion. If so, their stresses can be smoothed out near the defect lines by means of superpositions with the stresses due to the modified potentials. Appropriate improvement to the gauge equation straightforwardly requires its driving source to be written as difference of two stresses: actual stress tensor minus a background one [1].

Two solutions obtained in [1] in the superposed form are characterized by compact core regions, where singularities of the classical screw and edge dislocations are smoothed out. In other words, the gauge approach [1] "generates" a length scale in the continuous description. Outside the core, the background stresses become dominating sufficiently far from the dislocation lines, while the stress components tend to zero within the core regions. In fact, the stress component $\sigma_{\phi z}$ of the screw dislocation, and $\sigma_{\rho\rho}$, $\sigma_{\rho\phi}$ of the edge dislocation are reproduced in [1] at sufficiently large distances; however, the stresses $\sigma_{\phi\phi}$ and σ_{zz} of the modified edge defect [1] behave unconventionally.

It should be reminded that the linear gauge approach [2], which precedes [1], also results in the modified screw dislocation. However, the solution [2] is based on the gauge-invariant Lagrangian, which differs from that in [1] since is quadratic in $T(3)$-gauge field strength. This Lagrangian has been initially proposed in [3], [4] in the framework of $T(3) \otimes SO(3)$-gauge approach to defects in solids. The modified screw dislocation was reproduced also in [6] for quadratic gauge Lagrangian of a structure more general than in [2] [1]. A common technical idea behind the gauging in [1], [2], [6] requires the driving sources of the gauge equations to be written as the differences of two stresses. As a result, additional gauge contributions to the stress fields enable smoothening, within compact core regions, of the classical singularities of the ordinary dislocations. At sufficiently large distances, the modified defects reproduce the ordinary ones, while transition to a non-singular behavior occurs within the cores.

Before than in [2], the same non-singular screw dislocation has already been obtained in [8] in the framework of non-local elasticity. More information on the singularity-less dislo-

[1] A gauge approach close to ours has been proposed in [5], [6]. It is based on the translational gauge Lagrangian $\mathcal{L}_T$ written as a general combination of the terms quadratic in the torsion components. For special choice of its parameters, $\mathcal{L}_T$ is equivalent either to the Hilbert-Einstein Lagrangian [1], or to the Maxwell-type Lagrangian considered in [2]. On the other hand, after [7] it is known that extensions of the Hilbert–Einstein Lagrangian by the terms quadratic in the torsion (and curvature) lead to more general Lagrangians, and, finally, to the most general eight-parameter three-dimensional Lagrangian.

cations obtainable by means of the non-local elasticity can be found in [9]. Besides, a range of other related investigations devoted to elimination of the axial singularities has to be pointed out. For instance, the first strain gradient elasticity is adopted in [10], [11], [12], [13], [15] to produce the corresponding non-singular solutions. See [14] for overview of the nanoscopics of the dislocations and disclinations (including the behavior near interfaces) in the gradient elasticity. Non-singular solutions for the dislocations in the framework of the generalized elasticity and generalized micropolar elasticity of the Helmholtz type are discussed in [16]. The non-local elasticity and the second strain gradient elasticity are investigated, respectively, in [17] and [18] in a more involved, so-called, bi-Helmholtz form. The gradient elasticity as well as the non-local elasticity based models enable to smooth out the conventional linear solutions for dislocations (and disclinations), and they demonstrate the core regions characterized by the corresponding length parameters.

Conventional, i.e., non-gauge attempts [19], [20], [21], [22] are known to go beyond the linear elasticity in description of the screw and edge dislocations in crystals. However, in these approaches (of second order, in fact) it is still impossible to reach the axes of the line defects due to a failure of the traditional elasticity. For instance, the second order stresses have been found by means of the stress function method in [19], [21], and by the displacement function approach in [20]. These fields are valid within a hollow cylinder with the outer radius, say, $\rho_{\rm e}$ and the inner one $\rho_{\rm c}$. Arbitrary constants in these models are fixed in the final answers by the traction-free boundary conditions: $\sigma_{\rho\rho}\Big|_{\rho=\rho_{\rm e}} = 0$, $\sigma_{\rho\rho}\Big|_{\rho=\rho_{\rm c}} = 0$, where $\sigma_{\rho\rho}$ is the radial stress component. Besides, vanishing of the stress σ_{zz} averaged over bulk's cross-section is also used for avoiding arbitrariness of the constants. A narrow vicinity of the defect axes corresponding to the segment $[0, \rho_{\rm c}\,] \ni \rho$ drops out from the consideration due to the cut-off at the inner boundary. It is natural to treat the radius $\rho_{\rm c}$ (which is classically free) as a boundary of the core region discussed in [1], [2], [6].

Relevance of second order effects in theoretical elasticity for physics of imperfections in crystals, namely for modelling dislocations, is discussed in [23]. Thus, it clearly would be interesting to investigate the model proposed in [1] in the second order also. This is just the problem to be studied in the present paper. However, its formal consistency is of main interest here.

As to the gauge approaches, it has already been attempted in [24] to follow [3] in order to obtain the second order stresses around the screw dislocation. To this purpose, the quadratic $T(3)$-gauge Lagrangian [3] is used in [24]. However, according to [1], the translational Lagrangian [3], [4] does not allow the modified stress potential to behave as the Airy stress function of the edge dislocation. Therefore, it is inappropriate to use the same Lagrangian to describe the second order stresses of the screw dislocation: the Kröner ansatz for the screw dislocation in the quadratic approximation is just of the same structure as for the edge dislocation in the first order. Besides, the elastic energy is taken in [24] incompletely in comparison with that of non-gauge approaches [19], [20], [21]. Therefore, the experience of [24] looks unsatisfactory.

The present paper demonstrates that the second order consideration can be carried out for the gauge model [1] along the line of [19], [20], [21]. In what follows, a straight screw dislocation in infinite circular cylinder constitutes the background defect, and its stress field is taken up to the second order terms found in [21]. Similarly to the linear approach, the Einstein-type gauge equation is again responsible for additional stresses of the short-ranged character. The second order solution considered below depends on the arbitrary constants which are adjusted differently than in [20], [21]. For instance, the traction-free condition is imposed only for the outer cylinder's surface. Instead, vanishing of certain combinations in the short-distance expansion of the second order stresses is required. It is appropriate to relate the occurence of the short-ranged contributions to the existence of the core region – just like this is naturally adopted for the corresponding solutions of the first order [1], [2], [6]. Therefore, the cut-off around the defect's axis is avoided, and the stress field is continued inside the core. At sufficiently large distances, the modified stresses of the second order reproduce the conventional stress field around the screw dislocation. At the same time, new length scales of the "gauge" origin are provided by expressions for the asymptotically far stresses. These lengths are given by means of the elastic parameters and could characterize an exterior structure of the dislocation core.

In the present paper we do not discuss in specific details such a complicated field as description of the core structures of the crystal dislocations. Instead, only a list (inevitably incomplete) of further references is proposed: for instance, one should be referred to [25], [26], [27], [28] for the first attempts to incorporate discreteness for description of the core structures; for the non-linear elasticity, crystallography, etc., one should be addressed to [29], [30], [31], [32], [33]. For theory and experiment concerning the dislocation core structures and for their influence on the physics of solids, one should be referred to [34], [35].

The paper is written in six sections. Section 1 is introductory (see also [1], [2], [6] for the motives and for more refs.). Section 2 outlines the Einstein-type gauge equation. Section 3 is devoted to specification of the gauge equation, and a perturbative scheme is set up. Solution to the gauge equation, which describes the modified stress potential of the second order, is obtained in Section 4. The corresponding stress fields and their asymptotic behavior are investigated in Section 5. Discussion in Section 6 concludes the paper. Details of the calculation are placed in Appendices A and B. The present paper constitutes an improved (concerning the large distance behavior of the resulting stresses) version of [36]. Notice, that bold-faced letters are reserved for second rank tensors, and appropriate indices can easily be restored.

2 The Einstein-type gauge equation

The Einstein-type gauge equation is only outlined in the present section, while for more differential–geometric details one should refer to [33], [37], [38]. As to generalities of theoretical elasticity, the Eulerian picture is chosen in the present approach instead of the Lagrangian one accepted in [1]. Initial and final states of the dislocated body can be referred to the (curvilinear) coordinate systems, say, $\{x^i\}$ and $\{x^a\}$, respectively. The corresponding squared length elements are expressed as $g_{ij}dx^i dx^j$ or $\eta_{ab}dx^a dx^b$ with respect to $\{x^i\}$ or $\{x^a\}$, accordingly (throughout the paper up and down indices repeated imply summation). Let us define the frame components $e^a_{\ i}$ by means of the relation $\partial_i = e^a_{\ i}\partial_a$ (here and below the partial derivatives $\partial/\partial x^i$ are denoted as ∂_i), and the co-frame components $\mathcal{E}_a^{\ i}$ are given by one-form $dx^i = \mathcal{E}_a^{\ i}dx^a$. The components $\mathcal{E}_a^{\ i}$ and their duals $e^a_{\ i}$ are orthogonal in the following sense: $e^a_{\ i}\mathcal{E}_b^{\ i} = \delta^a_b$, $e^a_{\ i}\mathcal{E}_a^{\ j} = \delta^j_i$. Thus, for the metric η_{ab} we get: $\eta_{ab} = g_{ij}\mathcal{E}_a^{\ i}\mathcal{E}_b^{\ j}$.

The Eulerian strain tensor is referred to the deformed (final) state, and it measures a deviation between the final and initial configurations [33]. Indeed, let us consider a mapping of initial state to the deformed one: $\mathbf{x} \longmapsto \boldsymbol{\xi}(\mathbf{x})$. Then the difference

$$\eta_{ab}d\xi^a d\xi^b - g_{ij}dx^i dx^j = 2e_{ab}d\xi^a d\xi^b, \tag{2.1}$$

where

$$2e_{ab} \equiv \eta_{ab} - g_{ab}, \qquad g_{ab} \equiv g_{ij}\mathcal{B}_a^{\ i}\mathcal{B}_b^{\ j}\,, \tag{2.2}$$

defines the Eulerian strain tensor $(\mathbf{e})_{ab} \equiv e_{ab}$. The metric tensor g_{ab} (2.2) is called the Cauchy deformation tensor. Here $\mathcal{B}_a^{\,i}$ are given by one-forms $dx^i = \mathcal{B}_a^{\,i} d\xi^a$ as follows [33]:

$$\mathcal{B}_a^{\,i} \equiv \frac{\partial x^i}{\partial \xi^a} = \mathcal{E}_a^{\,i} - \mathcal{E}_b^{\,i} \overset{(\eta)}{\nabla}_a u^b , \tag{2.3}$$

provided the displacement $\boldsymbol{\xi} - \mathbf{x}$ is expanded as $\xi^i - x^i = u^a \mathcal{E}_a^{\,i}$. The covariant derivative $\overset{(\eta)}{\nabla}_a$ in (2.3) is defined by the requirement that the components $\mathcal{E}_a^{\,i}$ are covariantly constant:

$$\overset{(\eta)}{\nabla}_a \mathcal{E}_b^{\,i} \equiv \partial_a \mathcal{E}_b^{\,i} - \left\{ \begin{array}{c} c \\ ab \end{array} \right\}_\eta \mathcal{E}_c^{\,i} = 0 , \tag{2.4}$$

where the notation $\left\{ \begin{array}{c} c \\ ab \end{array} \right\}_\eta$ implies the Christoffel symbol of second kind. The metric $\eta_{ab} = \mathcal{E}_a^{\,i} \mathcal{E}_{b\,i}$ is also covariantly constant provided (2.4) is fulfilled. In turn, Eq. (2.4) allows to express $\left\{ \begin{array}{c} c \\ ab \end{array} \right\}_\eta$ through the metric η_{ab} as follows:

$$\left\{ \begin{array}{c} c \\ ab \end{array} \right\}_\eta = \frac{1}{2} \eta^{ce} \left(\partial_a \eta_{be} + \partial_b \eta_{ae} - \partial_e \eta_{ab} \right) . \tag{2.5}$$

To implement the gauging of the group $T(3)$, we consider the co-frame components $\mathcal{B}_a^{\,i}$ as ("minimal") extensions of (2.3) by means of so-called compensating fields:

$$\mathcal{B}_a^{\,i} \equiv \frac{\partial x^i}{\partial \xi^a} - \varphi_a^{\,i} = \mathcal{E}_a^{\,i} - \left(\mathcal{E}_b^{\,i} \overset{(\eta)}{\nabla}_a u^b + \varphi_a^{\,i} \right) . \tag{2.6}$$

Here the entries $\varphi_a^{\,i}$ play the role of translational gauge potentials which behave under the local transformations $x^i \longrightarrow x^i + \eta^i(x)$ as follows:

$$\begin{aligned} \frac{\partial x^i}{\partial \xi^a} &\longrightarrow \frac{\partial x^j}{\partial \xi^a} \left(\delta^i_j + \frac{\partial \eta^i}{\partial x^j} \right) , \\ \varphi_a^{\,i} &\longrightarrow \varphi_a^{\,i} + \frac{\partial x^j}{\partial \xi^a} \frac{\partial \eta^i}{\partial x^j} . \end{aligned} \tag{2.7}$$

The shifts (2.7) ensure the gauge invariance of $\mathcal{B}_a^{\,i}$ (2.6) (and of (2.2), as well): the behavior of $\mathcal{B}_a^{\,i}$ at non-homogeneous transformations is the same as that of $\partial x^i / \partial \xi^a$ (2.3) at homogeneous ($\partial \eta^i / \partial x^j \equiv 0$) ones. Motivation of (2.6) should be traced back to $T(3) \otimes SO(3)$-gauging and to the corresponding Cartan structure equations (see Refs. in [1]). We associate occurence of non-trivial core region for the screw dislocation in question (i.e., appearance of the short-ranged stresses) with the gauge variables $\varphi_a^{\,i}$. Provided (2.6) takes place, the Eulerian strain (2.2) takes the form:

$$2e_{ab} = \overset{(\eta)}{\nabla}_a u_b + \varphi_{ab} + \overset{(\eta)}{\nabla}_b u_a + \varphi_{ba} - \left(\overset{(\eta)}{\nabla}_a u_b + \varphi_{ab} \right) \left(\overset{(\eta)}{\nabla}_b u_a + \varphi_{ba} \right) , \tag{2.8}$$

where the indices are lowered by means of the metric η_{ab}. When $\boldsymbol{\varphi}$ is zero, (2.8) is reduced to the conventionally-looking background strain tensor in the Eulerian picture [33].

Differential-geometrically, the framework of the present paper is the same as in [1], i.e., a viewpoint of so-called *teleparallel* description (see [39] for a quick remind) is adopted as a most suitable to describe the dislocations [37]. Let us consider the Riemann–Christoffel curvature tensor $\mathrm{R}_{abc}{}^{d}$:

$$\mathrm{R}_{abc}{}^{d} = \partial_a \left\{ \begin{array}{c} d \\ bc \end{array} \right\}_g - \partial_b \left\{ \begin{array}{c} d \\ ac \end{array} \right\}_g + \left\{ \begin{array}{c} d \\ ae \end{array} \right\}_g \left\{ \begin{array}{c} e \\ bc \end{array} \right\}_g - \left\{ \begin{array}{c} d \\ be \end{array} \right\}_g \left\{ \begin{array}{c} e \\ ac \end{array} \right\}_g , \tag{2.9}$$

where the Christoffel symbols are calculated analogously to (2.5) but for the metric g_{ab} (2.2) (the subscript 'g') with $\mathcal{B}^{\,i}_{a}$ given by (2.6). The metric g_{ab} is covariantly constant, i.e., the equation $\overset{(g)}{\nabla}_a \, g_{bc} = 0$ is fulfilled where the covariant derivative $\overset{(g)}{\nabla}_a$ is defined similar to (2.4). We obtain from $\overset{(\eta)}{\nabla}_a \, \eta_{bc} = 0$ the following relation between the Christoffel symbols [33]:

$$\left\{ \begin{array}{c} c \\ ab \end{array} \right\}_\eta - \left\{ \begin{array}{c} c \\ ab \end{array} \right\}_g = 2e_{ab}{}^{c} , \tag{2.10.1}$$

$$2e_{ab}{}^{c} \equiv g^{ce} \Big(\overset{(\eta)}{\nabla}_a \, e_{be} + \overset{(\eta)}{\nabla}_b \, e_{ae} - \overset{(\eta)}{\nabla}_e \, e_{ab} \Big) . \tag{2.10.2}$$

where (2.2) and (2.5) are taken into account. We use (2.10.1) in (2.9) to obtain another representation for the curvature:

$$\mathrm{R}_{abc}{}^{d} = \overset{(\eta)}{\mathrm{R}}_{abc}{}^{d} - 2 \Big(\overset{(\eta)}{\nabla}_a \, e_{bc}{}^{d} - \overset{(\eta)}{\nabla}_b \, e_{ac}{}^{d} + 2e_{be}{}^{d} e_{ac}{}^{e} - 2e_{ae}{}^{d} e_{bc}{}^{e} \Big) , \tag{2.11}$$

where $\overset{(\eta)}{\mathrm{R}}_{abc}{}^{d}$ is the Riemann curvature calculated analogously to (2.9) but for the metric η_{ab}.

Now we are ready to write the Hilbert–Einstein gauge equation. Let us define the Einstein tensor $G^{ef} \equiv \dfrac{1}{4} \mathcal{E}^{eab} \mathcal{E}^{fcd} \mathrm{R}_{abcd}$, where $\mathcal{E}^{abc}$ is the Levi–Civita tensor defined by means of the metric η_{ab} [33]. Then, the gauge equation proposed in Section 6 of Ref. [1] takes the form:

$$G^{ef} = (2s)^{-1} \big(\sigma^{ef} - (\sigma_{\mathrm{bg}})^{ef} \big) . \tag{2.12}$$

In the Eulerian approach, we assume that the geometry of the deformed state is Euclidean, and so we equate $\overset{(\eta)}{\mathrm{R}}_{abc}{}^{d}$ to zero in (2.11). Therefore, using (2.10.2) and (2.11) to re-express G^{ef}, we obtain from (2.12):

$$-\mathcal{E}^{eab} \mathcal{E}^{fcd} \, \overset{(\eta)}{\nabla}_a \overset{(\eta)}{\nabla}_c \, e_{bd} = (2s)^{-1} \big(\sigma^{ef} - (\sigma_{\mathrm{bg}})^{ef} \big) + 2\mathcal{E}^{eab} \mathcal{E}^{fcd} e_{adl} \, e_{bc}{}^{l} . \tag{2.13}$$

According to [1], the differential-geometric torsion is not considered as independent degree of freedom. It is just the Riemann–Christoffel curvature that is subjected to the Einstein-type gauge equation (2.12). If so, any attempt to take into account an independent contribution due to the torsion would imply that the framework of the teleparallel description should be abandoned (see the notes on the geometry in [1]). Therefore, it appears that the gauge equation (2.13) governs only the variables related to the metric tensor, i.e., the corresponding strains.

The variational derivation of (2.12) can be discussed along the line of [1]. Right-hand side of (2.12) is given by $(2s)^{-1}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathrm{bg}})$, where $\boldsymbol{\sigma}_{\mathrm{bg}}$ implies the background stress tensor. The difference $\boldsymbol{\sigma} - \boldsymbol{\sigma}_{\mathrm{bg}}$, i.e., just the deviation of the stress tensor of the model from $\boldsymbol{\sigma}_{\mathrm{bg}}$, plays the role of the driving source in the Einstein equation. The parameter s characterizes an energy scale intrinsic to the gauge field $\boldsymbol{\varphi}$ [1], and it makes right-hand side in (2.12) dimensionless.

In the present paper, $\boldsymbol{\sigma}_{\rm bg}$ implies the stress field of the straight screw dislocation lying along an infinite cylindric body. Besides, both $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_{\rm bg}$ are assumed to respect the equilibrium equations:

$$\overset{(\eta)}{\nabla}_a \sigma^{ab} = 0, \qquad \overset{(\eta)}{\nabla}_a \left(\sigma_{\rm bg}\right)^{ab} = 0\,. \tag{2.14}$$

More detailed consideration of the gauge geometry behind the model in question should be done elsewhere. However, Refs. [40], [41] should be mentioned in addition to those listed in [1] since they contain useful indications concerning the translational gauge geometry. It is also appropriate to mention [42], where a topological picture is proposed which includes dislocations (through the torsion) and extra-matter (through the non-metricity; see also [43]). Besides, [44] provides a further development of the geometric approach to defects [7].

3 Specification of the gauge equation

3.1 The stress function method

We shall investigate the gauge equation (2.13) using the method of stress functions proposed in [19], [21] for solving the internal stress problems in incompatible elasticity. An exposition of [21] can be found in Ref. [33] devoted to a review of the dislocation problems in non-linear elasticity. Certain details omitted below can be restored with the help of [21] and [33].

We shall use the stress function approach in a successive approximation form. To this end, let us represent the strain and the stress tensors as the perturbative expressions:

$$\mathbf{e} \approx \overset{(1)}{\mathbf{e}} + \overset{(2)}{\mathbf{e}}\,, \qquad\qquad \boldsymbol{\sigma} \approx \overset{(1)}{\boldsymbol{\sigma}} + \overset{(2)}{\boldsymbol{\sigma}}\,, \tag{3.1}$$

where $\overset{(2)}{\mathbf{e}}$, $\overset{(2)}{\boldsymbol{\sigma}}$ are of second order smallness in comparison with $\overset{(1)}{\mathbf{e}}$, $\overset{(1)}{\boldsymbol{\sigma}}$. The representation (2.8) implies that each contribution in (3.1) consists of two parts: one is due to the background and another – of the non-conventional origin. Following [21], we shall investigate only the stress problem. To trace connection between the stresses of the non-conventional origin and the gauge potentials explicitly – this is outside the scope of the present paper. Substituting (3.1) into (2.13) and (2.14), we obtain two groups of equations:

(first order)

$$\nabla_a \overset{(1)}{\sigma}{}^{ab} = 0\,, \tag{3.2.1}$$

$$\left(\mathrm{Inc}\ \overset{(1)}{\mathbf{e}}\right)^{ab} = (2s)^{-1}\left(\delta\ \overset{(1)}{\boldsymbol{\sigma}}\right)^{ab}; \tag{3.2.2}$$

(second order)

$$\nabla_a \overset{(2)}{\sigma}{}^{ab} = 0\,, \tag{3.3.1}$$

$$\left(\mathrm{Inc}\ \overset{(2)}{\mathbf{e}}\right)^{ab} = (2s)^{-1}\left(\delta\ \overset{(2)}{\boldsymbol{\sigma}}\right)^{ab} + \overset{(1)}{Q}{}^{ab}\,. \tag{3.3.2}$$

The following notations are used in (3.2) and (3.3):

$$\left(\mathrm{Inc}\ \overset{(i)}{\mathbf{e}}\right)^{ab} \equiv -\mathcal{E}^{acd}\mathcal{E}^{bfe}\nabla_c\nabla_f\ \overset{(i)}{e}_{de}\,, \tag{3.4.1}$$

$$\left(\delta\ \overset{(i)}{\boldsymbol{\sigma}}\right)^{ab} \equiv \overset{(i)}{\sigma}{}^{ab} - \left(\overset{(i)}{\boldsymbol{\sigma}}_{\rm bg}\right)^{ab}, \tag{3.4.2}$$

$$\overset{(1)}{Q}{}^{ab} \equiv 2\,\mathcal{E}^{acd}\mathcal{E}^{bfe}\,\overset{(1)}{e}{}_{cel}\,\overset{(1)}{e}{}_{df}{}^{l}\,, \tag{3.4.3}$$

where $i = 1, 2$. Besides, 'Inc' denotes the double curl *incompatibility operator* acting on the tensor argument, and ∇_a implies the covariant derivative $\overset{(\eta)}{\nabla}_a$. Indices in (3.2)–(3.4) are raised and lowered by means of the metric $\boldsymbol{\eta}$.

Equations (3.2.1) and (3.3.1) appear from the equilibrium equation (2.14) in the first and second orders, respectively. Analogously, (3.2.2) and (3.3.2) result from the gauge equation (2.13) and look similar to the conventional incompatibility conditions: see, for instance, Eqs. (622) and (623) in [33]. However, there exists a distinction that is due the terms $(2s)^{-1}\delta\,\overset{(i)}{\boldsymbol{\sigma}}$ in (3.2.2), (3.3.2). These terms are just responsible for the short-ranged behavior of the gauge parts of the resulting stress functions. Besides, $\overset{(1)}{\mathbf{Q}}$ (3.4.3) is free from the torsion tensor since our gauge equation is expressed by means of the Riemann–Christoffel curvature only. These are the gauge equations (3.2.2) and (3.3.2) which play the role of non-conventional incompatibility conditions in the present approach, and the deviations (3.4.2) play the role of their driving sources.

The elastic energy potential is chosen in the Eulerian representation as follows [21]:

$$W(\mathbf{e}) \;=\; jI_1^2(\mathbf{e}) \;+\; kI_2(\mathbf{e}) \;+\; l'I_1^3(\mathbf{e}) \;+\; m'I_1(\mathbf{e})\,I_2(\mathbf{e}) \;+\; n'I_3(\mathbf{e})\,, \tag{3.5}$$

where $j = \mu + \lambda/2$, $k = -2\mu$ are the elastic moduli of second order (λ and μ are the Lamé constants), while l', m', n' are the elastic moduli of third order. Provided $W(\mathbf{e})$ (3.5) is chosen, the constitutive law relates $\overset{(i)}{\mathbf{e}}$ to $\overset{(i)}{\boldsymbol{\sigma}}$ as follows (see [21], [33] for the derivation):

$$\begin{aligned} \overset{(1)}{\mathbf{e}} &= C_1 I_1\big(\overset{(1)}{\boldsymbol{\sigma}}\big)\boldsymbol{\eta} \;+\; C_4\,\overset{(1)}{\boldsymbol{\sigma}}\,, \\ \overset{(2)}{\mathbf{e}} &= C_1 I_1\big(\overset{(2)}{\boldsymbol{\sigma}}\big)\boldsymbol{\eta} \;+\; C_4\,\overset{(2)}{\boldsymbol{\sigma}} \;+\; \overset{(1)}{\boldsymbol{\Psi}}\,, \end{aligned} \tag{3.6}$$

where

$$\overset{(1)}{\boldsymbol{\Psi}} \equiv \Big(C_2 I_1^2\big(\overset{(1)}{\boldsymbol{\sigma}}\big) \;+\; C_3 I_2\big(\overset{(1)}{\boldsymbol{\sigma}}\big)\Big)\boldsymbol{\eta} \;+\; C_5 I_1\big(\overset{(1)}{\boldsymbol{\sigma}}\big)\,\overset{(1)}{\boldsymbol{\sigma}} \;+ C_7 I_3\big(\overset{(1)}{\boldsymbol{\sigma}}\big)\Big(\overset{(1)}{\boldsymbol{\sigma}}\Big)^{-1}, \tag{3.7}$$

and $\boldsymbol{\eta}$ is the metric. The functions I_m ($m = 1, 2, 3$) of a tensor argument, say, $\mathbf{t}$ are defined as follows:

$$I_1(\mathbf{t}) \;\equiv\; \mathrm{tr}\,(\mathbf{t})\,, \qquad I_2(\mathbf{t}) \;\equiv\; \frac{1}{2}\big(I_1^2(\mathbf{t}) \;-\; I_1(\mathbf{t}^2)\big)\,, \qquad I_3(\mathbf{t}) \;\equiv\; \det\,(\mathbf{t})\,. \tag{3.8}$$

The Cayley–Hamilton theorem is to express the inverse $\Big(\overset{(1)}{\boldsymbol{\sigma}}\Big)^{-1}$ in (3.7). The numerical coefficients in (3.6), (3.7) are:

$$\begin{aligned} C_1 &\equiv -\frac{\nu}{E}\,, & C_2 &\equiv \frac{1}{E^2}\big(3\nu(1-\nu)-1\big) + 3L + M\,, \\ C_3 &\equiv \frac{3}{E^2}\big(1-\nu^2\big) + M\,, & C_4 &\equiv \frac{1+\nu}{E}\,, \\ C_5 &\equiv \frac{1}{E^2}\big(3\nu-2\big)\big(1+\nu\big) - M\,, & C_7 &\equiv -\frac{1}{E^2}\big(1+\nu\big)^2 + N\,, \end{aligned} \tag{3.9}$$

where $\nu = \lambda/(2(\lambda + \mu))$ is the Poisson ratio, and $E = \mu(3\lambda + 2\mu)/(\lambda + \mu)$ is the elastic modulus. Three linear relationships between the elastic parameters of third order l', m', n'

in (3.5) and the corresponding parameters L, M, N in (3.9) can be found in [21] but not used in the present investigation.

Two pairs of equations, (3.2) and (3.3), are to be solved. The first equations in each pair should be fulfilled by means of the Kröner ansatz:

$$\overset{(i)}{\boldsymbol{\sigma}} = \operatorname{Inc} \overset{(i)}{\boldsymbol{\chi}} \,. \tag{3.10}$$

Substituting the first and second equations (3.6) into (3.2.2) and (3.3.2), respectively, and using (3.10), we obtain equations to determine the components of the stress potential $\overset{(i)}{\boldsymbol{\chi}}$ (down indices):

$$\begin{aligned} &\Delta\Delta \overset{(i)}{\chi}_{ab} + a\Big(\nabla_a\nabla_b - \eta_{ab}\Delta\Big)\Delta I_1(\overset{(i)}{\boldsymbol{\chi}}) + \Big((1-a)\nabla_a\nabla_b + a\,\eta_{ab}\Delta\Big)\nabla^c\nabla^d \overset{(i)}{\chi}_{cd} - \\ &\quad - \Delta\Big(\nabla_a\nabla_c \overset{(i)}{\chi}{}^c{}_b + \nabla_b\nabla_c \overset{(i)}{\chi}{}^c{}_a\Big) = \kappa^2\Big(\delta\,\overset{(i)}{\boldsymbol{\sigma}}\Big)_{ab} + 2\mu \overset{(i)}{S}_{(ab)}\,, \qquad i = 1, 2\,, \end{aligned} \tag{3.11}$$

where

$$\overset{(1)}{S}_{ab} \equiv 0\,, \qquad \overset{(2)}{S}_{(ab)} \equiv \overset{(1)}{Q}_{(ab)} - \Big(\operatorname{Inc} \overset{(1)}{\boldsymbol{\Psi}}\Big)_{(ab)}\,, \tag{3.12}$$

Δ implies the Laplacian, and $\delta\,\overset{(i)}{\boldsymbol{\sigma}}$ is expressed by means of (3.4.2) and (3.10). Besides, $\overset{(1)}{\boldsymbol{\Psi}}$ is given by (3.7), and $\overset{(1)}{Q}_{(ab)}$ is expressed by means of (3.4.3). The curly brackets around the indices imply symmetrization. Throughout the paper we use the dimensionless parameters $\kappa^2 \equiv \mu/s$ and a [3], [4]:

$$a \equiv \frac{\lambda}{3\lambda + 2\mu} = \frac{1}{1 + \nu^{-1}}\,, \qquad 1 - a = \frac{2(\lambda+\mu)}{3\lambda + 2\mu} = \frac{1}{1+\nu}\,. \tag{3.13}$$

3.2 The gauge equations in the first and second orders

Let us adjust (3.11) to the special problem in question. We replace the derivatives ∇_a by the partial derivatives ∂_a, where x^a are the Cartesian coordinates in the final state [21], [33], and assume $\partial_3 \equiv 0$. Conventional notations are adopted for those components of the stress potential which are non-trivial [37]:

$$\begin{aligned} &\mu\,\overset{(i)}{\phi} \equiv \partial_2 \overset{(i)}{\chi}_{13} - \partial_1 \overset{(i)}{\chi}_{23}\,, \qquad\qquad i = 1,\, 2\,, \\ &f \equiv \overset{(2)}{\chi}_{33}\,, \quad p \equiv -\partial^2_{11} \overset{(2)}{\chi}_{22} - \partial^2_{22} \overset{(2)}{\chi}_{11} + 2\,\partial^2_{12} \overset{(2)}{\chi}_{12}\,. \end{aligned} \tag{3.14}$$

Other components of $\overset{(i)}{\boldsymbol{\chi}}$ are zero. The background stress tensor $\boldsymbol{\sigma}_{\rm bg}$ is also given by (3.10), but the corresponding stress potential is labelled appropriately: $\overset{(i)}{\boldsymbol{\chi}}_{\rm bg}$.

3.2.1 The first order

In the first order, only the component of the stress potential $\overset{(1)}{\phi}$ (3.14) is non-zero for the screw dislocation. The corresponding governing equation appears from (3.11) as follows:

$$\Delta \overset{(1)}{\phi} = \kappa^2\big(\overset{(1)}{\phi} - \overset{(1)}{\phi}_{\rm bg}\big)\,, \tag{3.15}$$

where $\overset{(1)}{\phi}_{\rm bg} \equiv (-b/2\pi)\log\rho$ is the background stress potential. It is appropriate [1] to re-express (3.15):

$$(\Delta\, -\, \kappa^2)\,\big(\overset{(1)}{\phi}\, -\, \overset{(1)}{\phi}_{\rm bg}\big)\, =\, b\,\delta^{(2)}(x)\,, \tag{3.16}$$

where b implies the Burgers vector length. Remind that the Burgers vector is parallel to the line of the screw dislocation, and its length is the perturbative expansion parameter. Axial symmetry of the final state stimulates [21] to use the cylindric coordinates ρ, φ, z instead of $x^a, a = 1, 2, 3$ (ρ and φ are chosen in (x^1, x^2)-plane and $z \equiv x_3$). Then, we obtain from (3.16) the stress potential of the first order:

$$\overset{(1)}{\phi} = \overset{(1)}{\phi}_{\rm bg}\, - f_S\,, \qquad f_S\, \equiv\, (b/2\pi) K_0(\kappa\rho)\,, \tag{3.17}$$

where K_0 is the modified Bessel function [45].

Only the stress component $\sigma_{\phi z}$ is non-trivial for the modified screw dislocation in the first order, and we express it using (3.10) and (3.14) [1]:

$$\sigma_{\phi z}\, =\, -\,\mu\,\partial_\rho\, \overset{(1)}{\phi} = \frac{b\mu}{2\pi}\,\rho^{^{-1}}\big(1 - \kappa\rho K_1(\kappa\rho)\big)\,. \tag{3.18}$$

Equation (3.18) witnesses about existence of a core region at $\rho \lesssim \kappa^{^{-1}}$: the gauge correction to the classical long-ranged law $1/\rho$ is exponentially small outside this region. Inside it, the law $1/\rho$ is replaced by another non-singular one. More detailed information on the numerical behavior of (3.18) (including a treatment of κ^{-1} in terms of interatomic spacing) can be found in [2] and [6]. In the present paper, it is simply assumed that $b = O(\kappa^{-1})$. Solution (3.18) is in agreement with [2], [6] (the translational gauging), [8] (the non-local elasticity) and [10], [14], [15] (the strain gradient elasticity). This is because the Helmholtz-type governing equations of the same type as (3.15) are essential in the approaches mentioned.

3.2.2 The second order

Taking into account the definitions (3.14), we obtain from (3.11) the gauge equations of the second order in the following form [2]:

$$\begin{aligned}
-\partial^2_{22}\Big[(1-a)p\, +\, a\Delta f\, -\, \kappa^2(f - f_{\rm bg})\Big] &= 2\mu\Big[Q_{11}\, +\, \partial^2_{22}\Psi_{33}\Big]\,,\\
-\partial^2_{11}\Big[(1-a)p\, +\, a\Delta f\, -\, \kappa^2(f - f_{\rm bg})\Big] &= 2\mu\Big[Q_{22}\, +\, \partial^2_{11}\Psi_{33}\Big]\,,\\
\partial^2_{12}\Big[(1-a)p\, +\, a\Delta f\, -\, \kappa^2(f - f_{\rm bg})\Big] &= -2\mu\,\partial^2_{12}\Psi_{33}\,,
\end{aligned} \tag{3.19.1}$$

$$\begin{aligned}
(1-a)\Delta\Delta f\, +\, a\Delta p\, &-\, \kappa^2(p - p_{\rm bg})\, = \\
&= 2\mu\Big[Q_{33}\, -\, \partial^2_{12}\big(\Psi_{12}\, +\, \Psi_{21}\big)\, +\, \partial^2_{11}\Psi_{22}\, +\, \partial^2_{22}\Psi_{11}\Big]\,.
\end{aligned} \tag{3.19.2}$$

Equations (3.19) describe the stress potentials of second order f and p (3.14). Besides, there are equations to determine $\overset{(2)}{\phi}$. But since $\overset{(2)}{\phi}_{\rm bg}$ is zero in [21], we put consistently $\overset{(2)}{\phi} \equiv 0$. Further, it is necessary to specialize the components of $\mathbf{Q}$ and $\mathbf{\Psi}$, i.e., to express them

[2]Here and below, we use $\mathbf{Q}$, $\mathbf{\Psi}$, $\mathbf{e}$ without the superscript.

through the first order solution (3.17) by means of (3.4.3) and (3.7), respectively. Remind that only the strains e_{13} and e_{23} are non-zero in the first order: $e_{13} = \frac{1}{2}\partial_2 \overset{(1)}{\phi}$, $e_{23} = -\frac{1}{2}\partial_1 \overset{(1)}{\phi}$. Taking into account that $\partial_3 = 0$ and using (2.10.2) for $e_{ab}{}^c$, we obtain the following non-zero components of $\mathbf{Q}$:

$$Q_{11} = Q_{22} = \big(\partial_1 e_{23} - \partial_2 e_{13}\big)^2 = \frac{\big(\Delta \overset{(1)}{\phi}\big)^2}{4}\,, \tag{3.20.1}$$

$$\begin{aligned} Q_{33} &= 4\,\big(\partial_1 e_{23}\,\partial_2 e_{13} - \partial_2 e_{23}\,\partial_1 e_{13}\big) + \big(\partial_1 e_{23} - \partial_2 e_{13}\big)^2 \\ &= \Big(\partial^2_{12}\overset{(1)}{\phi}\Big)^2 - \partial^2_{11}\overset{(1)}{\phi}\,\partial^2_{22}\overset{(1)}{\phi} + \frac{\big(\Delta \overset{(1)}{\phi}\big)^2}{4}\,. \end{aligned} \tag{3.20.2}$$

In turn, the corresponding components of $\boldsymbol{\Psi}$ are expressed standardly (see [21], [33]), and using (3.20.2) we obtain the combination we are interested in:

$$\begin{aligned} Q_{33} - 2\partial^2_{12}\Psi_{12} + \partial^2_{11}\Psi_{22} + \partial^2_{22}\Psi_{11} &= \\ = \Delta\Psi_{33} + \big(1 - 2\mu^2 C_7\big)\,\Phi + \frac{\big(\Delta \overset{(1)}{\phi}\big)^2}{4}\,, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \Psi_{33} &\equiv -\,\mu^2\,C_3\Big(\big(\partial_1\overset{(1)}{\phi}\big)^2 + \big(\partial_2\overset{(1)}{\phi}\big)^2\Big)\,, \\ \Phi &\equiv \Big(\partial^2_{12}\overset{(1)}{\phi}\Big)^2 - \partial^2_{11}\overset{(1)}{\phi}\,\partial^2_{22}\overset{(1)}{\phi} \end{aligned} \tag{3.22}$$

in right-hand side of (3.21).

Equations (3.20), (3.21) differ from the analogous ones in [21], where the components Q_{11}, Q_{22} are zero, while Q_{33} appears without the term $(\partial_1 e_{23} - \partial_2 e_{13})^2$. Indeed, properly defined $\mathbf{Q}$ includes a dependence on the differential-geometric torsion, and this just causes the disappearance of $(\partial_1 e_{23} - \partial_2 e_{13})^2$. Remind that the conventional incompatibility law is formulated as the vanishing requirement of the Einstein tensor calculated by means of so-called Riemann–Cartan curvature [21]. The latter includes, apart from the Riemann–Christoffel part (2.9), a contribution due to the torsion tensor which is identified as the dislocation density. Therefore, the dislocation density contributes into the driving source of the incompatibility law. Thus, a distribution of dislocations must be postulated before going over to an explicit solution of the elastic problem. Instead, according to [1], a stress field due to the background defect is chosen first. Then, the gauge contribution to the elastic stress is determined by the gauge equation (2.13). Since the torsion is not considered as independent variable in the present investigation, the components Q_{11}, Q_{22}, Q_{33} acquire their form (3.20).

Equations (3.19) can be straightforwardly reduced to their classical version which is respected by the background stress potentials of second order $f_{\rm bg}$, $p_{\rm bg}$. Let us pass to the classically-looking Q_{11}, Q_{22}, Q_{12}:

$$Q_{11} = \partial^2_{22}Q'\,, \quad Q_{22} = \partial^2_{11}Q'\,, \quad Q_{12} = -\partial^2_{12}Q'\,, \tag{3.23}$$

where a specific numerical value of the constant Q' arises from a condition that a mean value of $\overset{(2)}{\sigma}_{33}$ is zero [21], [33]. Then, equations which define $f_{\rm bg}$ and $p_{\rm bg}$ take in our notations the form (after the limit $\kappa \to 0$):

$$\begin{aligned} \Delta\Delta f_{\rm bg} \,&=\, k\left[\Delta(\Psi^{\rm bg}_{33} + Q') \,+\, (1-a)\,(1\,-\,2\mu^2 C_7)\,\Phi_{\rm bg}\right], \\ (1-a)\,p_{\rm bg} \,+\, a\,\Delta f_{\rm bg} \,&=\, -2\mu\left(\Psi^{\rm bg}_{33} + Q'\right), \end{aligned} \tag{3.24}$$

where $\Psi^{\rm bg}_{33}$, $\Phi_{\rm bg}$ are given by (3.22) provided $\overset{(1)}{\phi}$ is replaced by $\overset{(1)}{\phi}_{\rm bg}$. Besides, Eq. (3.21) (without the last term in right-hand side) was used, and we defined

$$k \,\equiv\, \frac{2\mu}{1-2a} \,=\, 2\mu\,\frac{1+\nu}{1-\nu} \tag{3.25}$$

in (3.24). The first and second equations (3.24) correspond, respectively, to Eqs. (657) and (655) in [33], provided $\overset{(2)}{\sigma}_{33}$ and $F_{(2)}$ therein are identified as $p_{\rm bg}$ and $-f_{\rm bg}$. As well, the sign at Q' is originally opposite [21].

Let us turn to (3.20). In the present approach, $\big(\Delta\,\overset{(1)}{\phi}\big)^2$ represents the square of the density distribution of the modified screw dislocation [1], [2], [6]:

$$\big(\Delta\,\overset{(1)}{\phi}\big)^2 \,=\, \Big(\frac{b}{2\pi}\,\kappa^2\,K_0(\kappa\rho)\Big)^2 . \tag{3.26}$$

For sufficiently large $\kappa\rho$, $K_0(\kappa\rho)$ decays exponentially, and $\big(\Delta\,\overset{(1)}{\phi}\big)^2$ (3.26) can approximately be replaced by zero. Therefore, Q_{11}, Q_{22}, and Q_{33} (3.20) take, also approximately, their conventionally appropriate form. Then, Eqs. (3.19) are in agreement with the classical situation, since three equations (3.19.1) are reducible to a single one similar to the second in (3.24). However, $\Delta\,\overset{(1)}{\phi}$ is not negligible at sufficiently moderate $\kappa\rho$, and Eqs. (3.19.1) require some care inside the core region $\rho \lesssim \kappa^{-1}$. In other words, it might be in order to extend the geometric framework by means of the differential-geometric torsion. But this would imply that the teleparallel description chosen is no longer valid.

Instead, we shall establish an "effective" picture which should be viewed as still embraced by the teleparallel framework. However, this description is expected to incorporate certain features of approach extended differential-geometrically. Indeed, an independent non-trivial contribution due to the torsion would lead, in the first order, to the following equation instead of (3.15):

$$\Delta\,\overset{(1)}{\phi}_{\mathcal T} = \kappa^2\big(\overset{(1)}{\phi}_{\mathcal T} \,-\, \overset{(1)}{\phi}_{\rm bg}\big) \,+\, \mathcal{T}\,, \tag{3.27}$$

where $\mathcal{T}$ is a suitable density caused by non-triviality of the torsion components, say, $T^3_{12} = -T^3_{21}$ (this is just appropriate for the screw dislocation along Oz, [37]). The notation $\overset{(1)}{\phi}_{\mathcal T}$ implies the corresponding solution at given $\mathcal{T}$. Being considered as independent source of incompatibility, the density $\mathcal{T}$ can be assigned to possess a series expansion near the defect's axis with arbitrary numerical coefficients (to be eventually adjusted). As a result, the solution $\overset{(1)}{\phi}_{\mathcal T}$ itself can also acquire, in comparison with $\overset{(1)}{\phi}$ (3.17), certain modifications near $\rho = 0$ (if so, attractive features of $\overset{(1)}{\phi}$ should be kept).

Bearing in mind that $\big(\Delta\,\overset{(1)}{\phi}\big)^2$ is transformed, at an extended treatment, into $\big(\Delta\,\overset{(1)}{\phi}_{\mathcal T}\,\big)^2$, let us it replace now by another "renormalized" density. Since details of such density can not be captured by the present approach, it is suggestive to choose it piece-wisely constant [36]:

$$(\Delta \overset{(1)}{\phi})^2 \sim \Big(\frac{b}{\pi\rho_*^2}\Big)^2 h_{[0,\,\rho_*]}(\rho)\,, \tag{3.28}$$

where $h_{[0,\,\rho_*]}(\rho)$ is equal to unity at $\rho \in [0, \rho_*]$ (i.e., within a disc bounded at some $\rho = \rho_*$), and zero otherwise. It is crucial that (3.28) should not be treated as a relation to be iteratively improved. Provided (3.28) is used in Q_{11}, Q_{22} (3.20.1), the representations $Q_{11} = \partial^2_{22}\widetilde{g}$, $Q_{22} = \partial^2_{11}\widetilde{g}$, $Q_{12} = -\partial^2_{12}\widetilde{g}$, analogous to (3.23), are valid with the corresponding "potential" $\widetilde{g} \equiv \widetilde{g}(\rho)$ in the form:

$$\widetilde{g} = \frac{b^2}{8\pi^2\rho_*^2}\Big(\frac{\rho^2}{\rho_*^2} - 1\Big)\, h_{[0,\,\rho_*]}(\rho) + \mathcal{C}\,, \tag{3.29}$$

where the constants $\mathcal{C}$ and ρ_* should be specified later. Both the functions, $\dfrac{b}{2\pi}\kappa^2 K_0(\kappa\rho)$ and $\dfrac{b}{\pi\rho_*^2}h_{[0,\,\rho_*]}(\rho)$, exhibit a δ-like behavior provided, respectively, κ and $1/\rho_*$ are large enough. Notice that these functions, being considered as the surface densities, are properly normalized. Therefore the replacement (3.28) looks better, provided its δ-like character is sharper.

Using (3.23) with $\widetilde{g}$ (3.29) instead of Q', we reduce (3.19) to the following couple of equations [36]:

$$(\Delta - \kappa^2)\Big(\Delta + \frac{\kappa^2}{1-2a}\Big)(f - f_{\rm bg}) = k\,\mathcal{R}\,, \tag{3.30.1}$$

$$p = -\frac{a}{1-a}\,\Delta f - \frac{2\mu}{1-a}\,(\Psi_{33} + \widetilde{g}) + \frac{\kappa^2}{1-a}\,(f - f_{\rm bg})\,, \tag{3.30.2}$$

where $f_{\rm bg}$ respects the first equation in (3.24), and k is given by (3.25). The driving source in (3.30.1) is defined as follows:

$$\begin{aligned}\mathcal{R} \;\equiv\;& (\Delta - \kappa^2)\big(\Psi_{33} + \widetilde{g} - \Psi^{\rm bg}_{33} - Q'\big)\\ &+ (1-a)\big(1 - 2\mu^2 C_7\big)\big(\Phi - \Phi_{\rm bg}\big) - \frac{1-a}{4}\big(\Delta \overset{(1)}{\phi}\big)^2\,,\end{aligned} \tag{3.31}$$

where Ψ_{33}, Φ, or $\Psi^{\rm bg}_{33}$, $\Phi_{\rm bg}$, are given by (3.22) with $\overset{(1)}{\phi}$ or $\overset{(1)}{\phi}_{\rm bg}$ substituted appropriately. Besides, $\big(\Delta \overset{(1)}{\phi}\big)^2$ is kept in right-hand side of (3.31) only formally, as a symbol. The first equation (3.30.1) (would be called, after [17], [18], the non-homogeneous *bi-Helmholtz equation* [3]) is just to be solved in what follows to determine the stress function f (3.14), while (3.30.2) is simply to express the resulting stress component $\sigma_{zz} = p$.

It can be realized that the governing equations (3.30) are essentially similar to those which should be expected for a consideration extended by means of the torsion (see also explanations in Section 5.3). Since their actual derivation incorporates (3.20), (3.21) (though being subjected to (3.28)), it is suggestive to treat the teleparallel framework as valid approximately. Although we do not pass (at a same footing with (3.28)) to another first order

[3] See [17], [18] for appropriate refs. and more examples of non-homogeneous bi-Helmholtz-type equations, which are crucial for the non-local elasticity and for the second strain gradient elasticity. A δ-driven differential equation analogous to (3.30.1) has already appeared also in [1] to describe a gauge modification of the edge dislocation.

solution, the similarity noticed is favored by the free parameter ρ_*, which comes to play through (3.28). Indeed, arbitrariness due to ρ_* is reminiscent to a freedom at extension of the model by means of appropriate torsion density. Technical complexities, which might appear in a description based on (3.27), should be avoided now since deserve a separate studying. Therefore, the strategy based on (3.28) looks appropriate. Equations (3.30) lead, eventually, to a self-consistent stress tensor without an artificial cut-off around the core.

3.3 The driving source of the gauge equation

Before to solve (3.30.1), attention should be paid to its driving source $\mathcal{R}$ (3.31). It is convenient to consider $\mathcal{R}$ multiplied by ρ^2. Writing Φ and $\Delta\Psi_{33}$ (see (3.22)) in the cylindrical coordinates, and keeping the same notation for the driving source after the multiplication, we obtain [36]:

$$\mathcal{R} \,=\, \rho^2 W(\rho) \,+\, \sum_{a=1}^{4} \mathcal{R}_a \,, \tag{3.32}$$

where

$$\begin{aligned} \mathcal{R}_1 &= \big(\tilde{c} \,+\, c(\kappa\rho)^2\big)\Big(\big(\partial_\rho \overset{(1)}{\phi}\big)^2 \,-\, \big(\partial_\rho \overset{(1)}{\phi}_{\rm bg}\big)^2\Big)\,, \\ \mathcal{R}_2 &= \tilde{c}\,\rho\,\Delta \overset{(1)}{\phi}\;\partial_\rho f_S \,-\, \tilde{c}\,\rho\left(\Delta \overset{(1)}{\phi} \,-\Delta \overset{(1)}{\phi}_{\rm bg}\right)\partial_\rho \overset{(1)}{\phi}_{\rm bg}\,, \\ \mathcal{R}_3 &= 2c\,\rho^2\,\partial_\rho\Big(\Delta \overset{(1)}{\phi}\Big)\partial_\rho f_S \,-\, 2c\,\rho^2\,\partial_\rho\Big(\Delta \overset{(1)}{\phi} \,-\Delta \overset{(1)}{\phi}_{\rm bg}\Big)\partial_\rho \overset{(1)}{\phi}_{\rm bg}\,, \\ \mathcal{R}_4 &= 2c\,\rho^2\,\Big(\big(\Delta \overset{(1)}{\phi}_{\rm bg}\big)^2 \,-\, \big(\Delta \overset{(1)}{\phi}\big)^2\Big)\,. \end{aligned} \tag{3.33}$$

The following notations are accepted in (3.32), (3.33):

$$\begin{aligned} &\tilde{c} \,\equiv\, (1-a)(1-2\mu^2 C_7) \,-\, 4c\,, \qquad\qquad c \equiv \mu^2 C_3\,, \\ &W(\rho) \,\equiv\, \frac{1+a}{4}\,\big(\Delta \overset{(1)}{\phi}\big)^2 \,-\, \kappa^2\big(\widetilde{g} \,-\, Q'\big)\,, \end{aligned} \tag{3.34}$$

where a and $1-a$ are given by (3.13). Besides, we used (3.17) to write $\mathcal{R}_2$ and $\mathcal{R}_3$ (3.33).

Remind that the density $\Delta \overset{(1)}{\phi}_{\rm bg}$ is given, classically, by the Dirac delta-function. If so, at the same footing with (3.28), it seems inappropriate to account for the contributions due to the difference $\Delta \overset{(1)}{\phi} -\Delta \overset{(1)}{\phi}_{\rm bg}$ in $\mathcal{R}_2$, $\mathcal{R}_3$ (3.33). Indeed, the non-homogeneous equation for the stress function $f_{\rm bg}$ is conventionally considered only for strictly positive ρ so that the corresponding driving source is not influenced by $\Delta \overset{(1)}{\phi}_{\rm bg}$. Therefore, the proposal to neglect $\Delta \overset{(1)}{\phi} -\Delta \overset{(1)}{\phi}_{\rm bg}$ in (3.33) should be viewed as a "persistence" of the regularization allowed classically into the non-conventional approach.

We re-scale the radial coordinate $\kappa\rho \mapsto z$ and obtain $\mathcal{R}$ in the following approximate form:

$$\mathcal{R} \,\approx\, z^2 w(z) \,+\, \sum_{a=1}^{3} \mathcal{R}_a \,, \tag{3.35}$$

where

$$\begin{aligned}
X^{-2}\,\mathcal{R}_1 &= \left(\tilde{c} + cz^2\right) K_1(z) \left(K_1(z) - \frac{2}{z}\right), \\
X^{-2}\,\mathcal{R}_2 &\approx -\tilde{c} z K_1(z) \Delta_z \widetilde{\phi} \sim \frac{2\tilde{c}}{z_*^2}\, z K_1(z)\, \widetilde{h}_{[0,\, z_*]}(z)\,, \\
X^{-2}\,\mathcal{R}_3 &\approx -2c\, z^2 K_1(z) \frac{d}{dz}\Big(\Delta_z \widetilde{\phi}\Big) = -2c\,\Big(z K_1(z)\Big)^2,
\end{aligned} \tag{3.36}$$

and $\mathcal{R}_4$ is neglected. The replacement (3.28) is taken into account in $\mathcal{R}_2$ (3.36). We assume that $\widetilde{\phi}$ in (3.36) implies $\overset{(1)}{\phi}$ with removed multiple $\frac{b}{2\pi}$, and the factor $X^2 \equiv \left(\frac{b\,\kappa}{2\pi}\right)^2$ just points out that we are concerned with the contributions quadratic in b. Besides, Δ_z is $z^{-1}\frac{d}{dz}\left(z\,\frac{d}{dz}\right)$, and K_0, K_1 are the modified Bessel functions [45]. The notation $z^2\, w(z)$ in (3.35) appears after re-expressing $\rho^2 W(\rho)$ in terms of the new variable:

$$z^2\, w(z) \equiv X^2\, \frac{z^2}{z_*^4}\Big[(1+a) - \frac{1}{2}\big(z^2 - z_*^2\big)\Big]\, \widetilde{h}_{[0,\, z_*]}(z)\,, \tag{3.37}$$

where $z_* \equiv \kappa\rho_*$, and $\widetilde{h}_{[0,\, z_*]}(z)$ is equal to unity at $z \in [0,\, z_*]$ or zero, otherwise (we put $\mathcal{C} = Q'$).

Now the asymptotical properties of the driving source $\mathcal{R}$ (3.35) are characterized as follows. First of all, $\mathcal{R}$ is well localized at large z since it contains the Bessel functions. At small z, $\mathcal{R}$ is given by the series:

$$\begin{aligned}
\mathcal{R}(z) &\simeq p_1\, z^{-2} + p_2 \log z + p_3 \\
&+ p_4\, z^2 \log^2 z + p_5\, z^2 \log z + p_6\, z^2 + o(z^2)\,,
\end{aligned} \tag{3.38}$$

where the numerical coefficients $p_1, p_2, \ldots, p_6$ are:

$$\begin{aligned}
p_1 &= -X^2\, \tilde{c}\,, \qquad\qquad p_2 = 0\,, \\
p_3 &= X^2\left(\frac{2\tilde{c}}{z_*^2} - \alpha c\right), \quad p_4 = X^2\, \frac{\tilde{c}}{4}\,, \\
p_5 &= X^2\Big(\frac{\tilde{c}}{z_*^2} - \frac{\tilde{c}}{4}\Big(1 - 2\log\frac{\gamma}{2}\Big) + (1-\alpha)\, c\Big), \\
p_6 &= w(0) + X^2\Big(1 - 2\log\frac{\gamma}{2}\Big)\Big(\Big(1 - 2\log\frac{\gamma}{2} - \frac{8}{z_*^2}\Big)\frac{\tilde{c}}{16} + (\alpha - 1)\,\frac{c}{2}\Big),
\end{aligned} \tag{3.39}$$

with $\alpha = 1$ or $\alpha = 3$ in p_3, p_5, p_6. Indeed, calculation of $\mathcal{R}_3$ requires, as soon as (3.28) is imposed, to differentiate the step-function. But the replacement (3.28) should not lead to singularities in the driving source which would destroy consistency of our regularized scheme. We use p_3, p_5, p_6 with $\alpha = 1$, when $\mathcal{R}_3$ is simply omitted to express neglect of the corresponding differentiation. Otherwise (for a comparison), $\alpha = 3$ when $\mathcal{R}_3$ is formally calculated according to (3.36). However, α is left unspecified to indicate at both possibilities. Besides, in spite of $p_2 \equiv 0$, it is convenient to keep the corresponding term in (3.38), which is to be equated to zero at the end.

4 Solution of the gauge equation

We are going to solve (3.30.1) in two steps. First, we shall consider the non-homogeneous Bessel equation:

$$\left[z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - z^2\right]y(z) = \mathcal{R}(z)\,. \tag{4.1}$$

The method of variation of parameters provides a general solution to (4.1), denoted for a moment as $y(z)$, as follows [46]:

$$\begin{aligned} y(z) &= A\,y_1(z) + B\,y_2(z) \\ &- \int\limits_{t_0}^{z} \frac{y_1(z)\,y_2(t) - y_2(z)\,y_1(t)}{y_1(t)\,y_2'(t) - y_2(t)\,y_1'(t)}\,\frac{\mathcal{R}(t)}{t^2}\,dt\,. \end{aligned} \tag{4.2}$$

Here $y_1(z)$ and $y_2(z)$ are two linearly independent solutions of the associated homogeneous Bessel equation (i.e, the fundamental solutions), and t_0 is to be adjusted.

We choose the modified Bessel functions I_0 and K_0 [45] as the fundamental solutions $y_1(z) = I_0(z)$ and $y_2(z) = K_0(z)$, and we define an auxiliary function $G(z,\epsilon)$:

$$\begin{aligned} G(z,\epsilon) &= I_0(z)\Big(A(\epsilon) + \int\limits_{\epsilon}^{z} K_0(t)\mathcal{R}(t)\,\frac{dt}{t}\Big) \\ &+ K_0(z)\Big(B(\epsilon) - \int\limits_{\epsilon}^{z} I_0(t)\mathcal{R}(t)\,\frac{dt}{t}\Big)\,, \end{aligned} \tag{4.3.1}$$

where

$$\begin{aligned} A(\epsilon) &\equiv -\int\limits_{\epsilon}^{\infty} K_0(t)\mathcal{R}(t)\,\frac{dt}{t}\,, \\ B(\epsilon) &\equiv const + \int\limits_{\epsilon}^{1}\left(\frac{p_1}{t^3} + \frac{p_2}{t}\log t + \Big(\frac{p_1}{4} + p_3\Big)\frac{1}{t}\right)dt\,. \end{aligned} \tag{4.3.2}$$

As a result, the solution to (4.1) appears as follows:

$$G(z) = \lim_{\epsilon\to 0} G(z,\epsilon) \equiv G(z,0)\,. \tag{4.4}$$

Behavior of the integrands $t^{-1}I_0(t)\mathcal{R}(t)$ and $t^{-1}K_0(t)\mathcal{R}(t)$ at small and large t is important for justification of the formulas (4.3). When t is small, a regularization is required for the integrals containing $t^{-1}I_0(t)\mathcal{R}(t)$ and $t^{-1}K_0(t)\mathcal{R}(t)$ (see Appendix A for the corresponding asymptotical expansions (A1) and (A2)). Eventually, the regularization in $G(z)$ (4.4) is ensured by means of the specially constructed functions $A(\epsilon)$ and $B(\epsilon)$ (4.3.2), which behave at $\epsilon \to 0$ appropriately. When t is large, $K_0(t)\mathcal{R}(t)$ decays as e^{-2t}, whereas $I_0(t)\mathcal{R}(t)$ behaves, mainly, as a constant. Therefore, infinity as an upper bound in $A(\epsilon)$ (4.3.2) is appropriate. The prescribed choice of the integration boundaries implies that $G(z)$ decays exponentially at large z, i.e., $G(z)$ is well localized. Important arbitrariness in $G(z)$ (4.4) should be stressed, which is due to the parameter *const* in $B(\epsilon)$ (4.3.2).

Using Appendix A, it is straightforward to establish the behavior of $G(z)$ at small z:

$$
\begin{aligned}
G(z) \;&\simeq\; q_0 \,+\, q_1\, z^{-2} \,+\, q_2\, \log^3 z \,+\, q_3\, \log^2 z \,+\, q_4\, \log z \\
&+\; q_5\, z^2 \log^3 z \,+\, q_6\, z^2 \log^2 z \,+\, q_7\, z^2 \log z \,+\, q_8\, z^2 \,+\, \ldots\,.
\end{aligned}
\tag{4.5}
$$

The numerical coefficients $q_0, q_1, \ldots, q_6$ in (4.5) are expressed by means of the parameters p_1, p_2, p_3, p_4 (3.39) as follows:

$$
\begin{aligned}
& q_0 \,=\, -const \times \log\frac{\gamma}{2} \,-\, \mathcal{I}_K \,-\, \frac{p_1}{16}\,, \quad q_1 \,=\, \frac{p_1}{4}\,, \\
& q_2 \,=\, \frac{p_2}{6}\,, \quad q_3 \,=\, \frac{p_1}{8} \,+\, \frac{p_3}{2}\,, \quad q_4 \,=\, -\,const \,+\, \frac{3p_1}{4}\,, \\
& q_5 \,=\, \frac{p_2}{24}\,, \quad q_6 \,=\, \frac{p_1}{32} \,+\, \frac{p_3 - p_2}{8} \,+\, \frac{p_4}{4}\,,
\end{aligned}
\tag{4.6}
$$

where $\mathcal{I}_K$ is given by (A8) in Appendix A, and *const* in q_0, q_4 is introduced by the definition of $B(\epsilon)$ (4.3.2). Besides, it is seen that $p_2 \log z$ (3.38) is just responsible for the highest powers of the logarithm in (4.5) (and similarly for the logarithms multiplied by higher powers of z^2). Moreover, the coefficients q_7 and q_8 in (4.5) are practically too complicated. We omit them in (4.6), since their precise form is not of importance below (see Appendices for details).

As a second step, we express the difference $f - f_{\rm bg} \equiv k\,\mathcal{F}$, where $\mathcal{F}$ respects the non-homogeneous Bessel equation

$$
\Big[z^2 \frac{d^2}{dz^2} \,+\, z \frac{d}{dz} \,+\, z^2\Big] \mathcal{F}\Big(\frac{z}{\mathcal{N}}\Big) \;=\; \frac{z^2}{\mathcal{N}^2}\, G\,\Big(\frac{\kappa}{\mathcal{N}}\, z\Big)\,.
\tag{4.7}
$$

The modified stress potential of the second order appears as $f = f_{\rm bg} + k\,\mathcal{F}$. For convenience, Eq. (4.7) is also written in terms of the new variable z, which is however defined differently: $z \equiv \mathcal{N}\rho$, $\mathcal{N}^2 \equiv \dfrac{\kappa^2}{1-2a}$. The driving source in (4.7) is expressed by means of (4.3) and (4.4). Using again (4.2), we put $y_1(z) = Y_0(z)$ and $y_2(z) = J_0(z)$ (Y_0 and J_0 solve the homogeneous Bessel equation associated to (4.7), [45]), and obtain:

$$
\begin{aligned}
& \mathcal{F}(\rho) \;=\; C\,\widetilde{Y}_0(\mathcal{N}\rho) \,+\, D\,J_0(\mathcal{N}\rho) \,+\, I_{\mathcal{F}}(\rho)\,, \\
& I_{\mathcal{F}}(\rho) \;\equiv\; J_0(\mathcal{N}\rho) \int\limits_{\rho}^{\infty} \widetilde{Y}_0(\mathcal{N}t) G(\kappa t)\, t\, dt \,-\, \widetilde{Y}_0(\mathcal{N}\rho) \int\limits_{\rho}^{\infty} J_0(\mathcal{N}t) G(\kappa t)\, t\, dt\,,
\end{aligned}
\tag{4.8}
$$

where $\widetilde{Y}_0(z) \equiv (\pi/2) Y_0(z)$. Here the integrals are convergent at their upper bounds.

In the preceding consideration [36], the choice $C \neq 0$, $D = 0$ has been made for the coefficients in $\mathcal{F}(\rho)$ (4.8). However, in this case the asymptotical behavior of the stress field looks non-conventionally at sufficiently large distances. In the present paper, we put C and D in (4.8) equal to zero, and thus $\mathcal{F}(\rho) = I_{\mathcal{F}}(\rho)$. The integral $I_{\mathcal{F}}(\rho)$ decays exponentially at large ρ, while at small $\mathcal{N}\rho$ we obtain (see Appendix B):

$$
\begin{aligned}
\mathcal{F}(\rho) \;&\simeq\; r_0 \,+\, r_1\, \log\rho \,+\, r_2\, \log^2\rho \\
&+\; r_3\, \rho^2 \log^3\rho \,+\, r_4\, \rho^2 \log^2\rho \,+\, r_5\, \rho^2 \log\rho \,+\, r_6\, \rho^2 \,+\, r_7\, \rho^4 \log^3\rho \,+\, \ldots\,,
\end{aligned}
\tag{4.9}
$$

where

$$r_0 \,=\, -\widetilde{\mathcal{I}}_Y \,+\, \log\Big(\frac{\gamma}{2}\mathcal{N}\Big)\,\widetilde{\mathcal{I}}_J\,, \qquad r_1 \,=\, \widetilde{\mathcal{I}}_J\,, \qquad r_2 \,=\, \frac{p_1}{8\kappa^2}\,,$$

$$r_3 \,=\, \frac{p_2}{24}\,, \quad r_4 \,=\, \Big(1 \,-\, \frac{\mathcal{N}^2}{\kappa^2}\Big)\,\frac{p_1}{32} \,-\, (1 \,-\, \log\kappa)\frac{p_2}{8} \,+\, \frac{p_3}{8}\,, \tag{4.10}$$

$$r_7 \,=\, \frac{\kappa^2 - \mathcal{N}^2}{384}p_2\,.$$

The numerical coefficients r_2, r_3, r_4, r_7 (4.10) are expressed by means of p_1, p_2, p_3 given by (3.39). Besides, the parameters $\widetilde{\mathcal{I}}_Y$, $\widetilde{\mathcal{I}}_J$ are defined by (B12) in Appendix B. The coefficients r_5, r_6 are more complicated, and their values are given by (B14) in Appendix B.

5 The stress tensor

5.1 The components $\sigma_{\rho\rho}$ and $\sigma_{\phi\phi}$

Therefore the modified stress potential of second order f takes the form:

$$f \,=\, f_{\rm bg} \,+\, kI_{\mathcal{F}}\,, \tag{5.1.1}$$

$$f_{\rm bg} \,=\, -\,\widetilde{n}\,\log^2\rho \,+\, d_1\,\rho^2 \,+\, d_2\,\log\rho\,, \qquad \widetilde{n} \,\equiv\, k\,\frac{p_1}{8\kappa^2}\,, \tag{5.1.2}$$

where $I_{\mathcal{F}}$ is given by (4.8), and the background stress potential $f_{\rm bg}$ is written in agreement with [21]. The potential $f_{\rm bg}$ respects the following equation, which results from (3.24) at $\rho \neq 0$:

$$\Delta\Delta f_{\rm bg} \,=\, \frac{-8\widetilde{n}}{\rho^4}\,. \tag{5.2}$$

Taking into account (3.10) and (3.14), we obtain the second order stress tensor of the modified screw dislocation in the cylindrical coordinates as follows:

$$\overset{(2)}{\sigma}_{\rho\rho} = \,-\,\frac{1}{\rho}\,\frac{d}{d\rho}\,f\,, \qquad \overset{(2)}{\sigma}_{\phi\phi} = \,-\,\frac{d^2}{d\rho^2}\,f\,, \qquad \overset{(2)}{\sigma}_{zz} = \,p\,, \tag{5.3}$$

where f (5.1) and p (3.30.2) are substituted. Other components of $\overset{(2)}{\boldsymbol{\sigma}}$ are zero. In the classical problem, we use (5.3) with $f_{\rm bg}$ (5.1.2) and straightforwardly obtain:

$$\begin{aligned} (\overset{(2)}{\sigma}_{\rm bg})_{\rho\rho} \;&=\; 2\widetilde{n}\,\frac{\log\rho}{\rho^2} \,-\, \frac{d_2}{\rho^2} \,-\, 2d_1\,, \\ (\overset{(2)}{\sigma}_{\rm bg})_{\phi\phi} \;&=\; -2\widetilde{n}\,\frac{\log\rho}{\rho^2} \,+\, \frac{2\widetilde{n} + d_2}{\rho^2} \,-\, 2d_1\,. \end{aligned} \tag{5.4}$$

Although some of our signs are opposite to those in [21], essential correspondence can always be verified. Free parameters d_1 and d_2 in (5.4) are restricted by vanishing of $(\overset{(2)}{\sigma}_{\rm bg})_{\rho\rho}$ at the boundaries of hollow cylinder, e.g., at $\rho = \rho_{\rm c}$ and $\rho = \rho_{\rm e} > \rho_{\rm c}$ [21]:

$$(\overset{(2)}{\sigma}_{\rm bg})_{\rho\rho}\Big|_{\rho=\rho_{\rm e}} \,=\, (\overset{(2)}{\sigma}_{\rm bg})_{\rho\rho}\Big|_{\rho=\rho_{\rm c}} \,=\, 0\,. \tag{5.5}$$

Substituting (5.4) into (5.5), one defines d_1 and d_2 thus arriving at the classical answers found in [21]. For our purposes it is more suitable to consider the limit when $\rho_{\rm e}$ is sufficiently large in comparison to $\rho_{\rm c}$ ($\rho_{\rm e}/\rho$ is moderate). We approximately obtain that

$$d_1 \approx \frac{\widetilde{n}}{\rho_{\rm e}^2} \log \frac{\rho_{\rm e}}{\rho_{\rm c}} , \qquad d_2 \approx 2\widetilde{n} \log \rho_{\rm c} , \tag{5.6}$$

and

$$\begin{aligned} (\overset{(2)}{\sigma}_{\rm bg})_{\rho\rho} &\approx \frac{2\widetilde{n}}{\rho^2}\left(\log\frac{\rho}{\rho_{\rm c}} - \left(\frac{\rho}{\rho_{\rm e}}\right)^2 \log\frac{\rho_{\rm e}}{\rho_{\rm c}}\right), \\ (\overset{(2)}{\sigma}_{\rm bg})_{\phi\phi} &\approx \frac{2\widetilde{n}}{\rho^2}\left(1 - \log\frac{\rho}{\rho_{\rm c}} - \left(\frac{\rho}{\rho_{\rm e}}\right)^2 \log\frac{\rho_{\rm e}}{\rho_{\rm c}}\right). \end{aligned} \tag{5.7}$$

Notice, with regard at p_1 (3.39), that $\widetilde{n}$ is proportional to b^2, and $f_{\rm bg}$ (5.1.2) is indeed of second order in b. From (3.36), (4.3), (4.4), and (4.8) it is seen that $I_{\mathcal{F}}$ is also quadratic in b.

Let us turn to the stress components (5.3) with f (5.1). The modified solution in question depends on several free parameters: ρ_*, *const*, d_1, and d_2 (see (3.28), (4.3), and (5.1), respectively). Besides, accordingly to (3.29) and (3.30.2), there exists a dependence of $\overset{(2)}{\sigma}_{zz}$ on $\mathcal{C}$ (on Q', in fact, since $\mathcal{C} = Q'$). Arbitrariness caused by these parameters can be avoided by subjecting the asymptotical stresses to appropriate conditions at $\kappa\rho \gg 1$ and $\kappa\rho \ll 1$. For instance, the contribution due to $I_{\mathcal{F}}$ is exponentially small in f (5.1) at $\kappa\rho \gg 1$. Therefore, we obtain that $f \simeq f_{\rm bg}(\rho)$, and, consequently, $\overset{(2)}{\sigma}_{\rho\rho} \simeq (\overset{(2)}{\sigma}_{\rm bg})_{\rho\rho}$, $\overset{(2)}{\sigma}_{\phi\phi} \simeq (\overset{(2)}{\sigma}_{\rm bg})_{\phi\phi}$, where the background stresses can be thought of as given by (5.4). However, d_1, d_2 in the gauge case are defined below.

In the limit $\kappa\rho \ll 1$, the behavior of $\overset{(2)}{\sigma}_{\rho\rho}$ and $\overset{(2)}{\sigma}_{\phi\phi}$ is expressed by means of (4.9) and (5.1) as follows:

$$\begin{aligned} \overset{(2)}{\sigma}_{\rho\rho} - (\overset{(2)}{\sigma}_{\rm bg})_{\rho\rho} \simeq &- 2r_2\frac{\log\rho}{\rho^2} - \frac{r_1}{\rho^2} - 2r_3\log^3\rho - (3r_3 + 2r_4)\log^2\rho \\ &- 2(r_4 + r_5)\log\rho - (r_5 + 2r_6) - 4r_7\rho^2\log^3\rho + \ldots , \\ \overset{(2)}{\sigma}_{\phi\phi} - (\overset{(2)}{\sigma}_{\rm bg})_{\phi\phi} \simeq &\; 2r_2\frac{\log\rho}{\rho^2} - \frac{2r_2 - r_1}{\rho^2} \\ &- 2r_3\log^3\rho - (9r_3 + 2r_4)\log^2\rho - 2(3r_3 + 3r_4 + r_5)\log\rho \\ &- (3r_5 + 2r_4 + 2r_6) - 12r_7\rho^2\log^3\rho + \ldots , \end{aligned} \tag{5.8}$$

where the background stresses in left-hand side are given by (5.4), and now the coefficients r_1, r_2, ..., r_7 include, for notational compactness, the multiple k prescribed by (5.1.1). It is crucial that the contributions $\propto \rho^{-2}\log\rho$ disappear at both sides of (5.8) since r_2 is equal to $\widetilde{n}$. As noticed in Section 3.3, the short-distance expansion of $\mathcal{R}$ (3.38) is missing the term corresponding to p_2. As it is seen from (4.10), the vanishing of p_2 implies that the contributions due to r_3 and r_7 disappear in right-hand side of (5.8). Besides, the constant term r_0 (4.9) is irrelevant for the differentiations in (5.2). However, the terms which are either divergent or constant at $\rho \to 0$ still are contained in (5.8).

The number of the parameters in the present approach is smaller by a unit in comparison with [36] since now $C = 0$ is adopted in (4.8). Therefore, inappropriate terms in (5.8) should be handled differently than in [36]. In our approach, the core region corresponds to $\rho \lesssim \kappa^{-1}$ [1]. Let us introduce a fictitious boundary at $\rho = \rho_\varepsilon$, $\rho_\varepsilon \ll \kappa^{-1}$, and let us assume that the argument ρ of our gauge solution varies only within the segment $[\rho_\varepsilon, \rho_{\rm e}]$, where ρ_ε is allowed to be arbitrarily small but strictly non-zero. Then we arrive at the following set of equations:

$$\begin{aligned} &\frac{r_1 + d_2}{\rho_\varepsilon^2} + 2r_4 \log^2 \rho_\varepsilon + 2(r_4 + r_5) \log \rho_\varepsilon + r_5 + 2(r_6 + d_1) = 0\,, \\ &\frac{r_1 + d_2}{\rho_\varepsilon^2} - 2r_4 \log^2 \rho_\varepsilon - 2(3\,r_4 + r_5) \log \rho_\varepsilon - 3\,r_5 - 2(r_4 + r_6 + d_1) = 0\,, \\ &2\widetilde{n} \log \rho_{\rm e} - d_2 - 2\rho_{\rm e}^2\, d_1 = 0\,. \end{aligned} \tag{5.9}$$

The first two equations in (5.9) imply that the sums of the terms, which are either divergent or constant in $\overset{(2)}{\sigma}_{\rho\rho}$, $\overset{(2)}{\sigma}_{\phi\phi}$ (5.8), vanish at $\rho = \rho_\varepsilon$. Then, since r_7 (4.10) is zero at $p_2 = 0$, the stresses $\overset{(2)}{\sigma}_{\rho\rho}$, $\overset{(2)}{\sigma}_{\phi\phi}$ at $\rho = \rho_\varepsilon$ are mainly given by those terms in (5.8) which are $\propto \rho_\varepsilon^2 \log^2 \rho_\varepsilon$ and clearly go to zero at $\rho_\varepsilon \to 0$. The third equation (5.9) reads that the radial stress is zero at the outer boundary $\rho = \rho_{\rm e}$. It is crucial that arbitrariness of the parameters d_1, d_2, and *const* can be used to fulfill (5.9). Provided d_1, d_2, and *const* are thus determined (as functions of ρ_ε, in fact), we can use them in (5.1) to produce, by means of (5.3), the stress distribution valid on the ring (i.e., on the hollow disc), say, $\mathcal{D}_{[\rho_\varepsilon, \rho_{\rm e}]}$ confined between $\rho = \rho_\varepsilon$ and $\rho = \rho_{\rm e}$.

The conditions (5.9) enable the following viewpoint. Indeed, let us consider an infinite sequence of concentric circles of monotonically decreasing radii $\rho = \rho_\varepsilon$ which however remain non-zero. Then, an appropriate sequence of solutions defined on the rings $\mathcal{D}_{[\rho_\varepsilon, \rho_{\rm e}]}$ can also be considered. As ρ_ε's are decreasing, the boundary values $\overset{(2)}{\sigma}_{ij}\Big|_{\rho=\rho_\varepsilon}$ of the corresponding stress distributions are also gradually going to zero. Notice that each ring $\mathcal{D}_{[\rho_\varepsilon, \rho_{\rm e}]}$ is completely embedded into any other one possessing a strictly smaller internal radius. The whole sequence of embedded rings is covered by a disc with the same external boundary at $\rho = \rho_{\rm e}$ but with a puncture at $\rho = 0$, i.e., by the disc without a single point. The sequence of solutions defined on the rings can be naturally considered as convergent to an appropriate stress distribution defined on the punctured disc. Since the corresponding internal boundary stresses tend to zero, such limit should be expected to exist.

The present approach is concerned with the stress distributions valid on the rings $\mathcal{D}_{[\rho_\varepsilon, \rho_{\rm e}]}$ with the internal holes of sufficiently small radii. As it will be argued in Section 5.3, this framework is just appropriate to discuss the stress distribution continued inside the core region. Notice that even the definition of $G(z)$ given by (4.3), (4.4) is already concerned with a hollow domain possessing a sufficiently small internal radius. In turn, it is just the regularization adopted in (4.3) which nevertheless allows to send this radius in (4.3) to zero immediately.

To extract the dependence on *const* in (5.9), let us begin with the coefficients q_0, q_4 (4.6): $q_0 = -const \times \log(\gamma/2) + \ldots$, $q_4 = -const + \ldots$, where the dots correspond to contributions which depend on the material parameters but not on *const*. Using Appendix B, one can establish that r_5, r_6 (B14) depend on *const* only through the differences $\mathcal{Y}_6 - \mathcal{J}_6$, $\mathcal{Y}_7 - \mathcal{J}_7$: see (B16) and (B17). Eventually, r_1 (4.10), and r_5, r_6 (B14) are re-expressed as follows:

$$r_1 \equiv -\widetilde{C} + \widetilde{r}_1\,, \quad r_5 \equiv -\frac{\kappa^2}{4}\widetilde{C} + \widetilde{r}_5\,, \quad r_6 \equiv \frac{q\,\kappa^2}{4}\widetilde{C} + \widetilde{r}_6\,, \tag{5.10}$$

where

$$\widetilde{C} \equiv const \times \frac{k}{\kappa^2 + \mathcal{N}^2}\,, \qquad q \equiv 1 - \log\Big(\frac{\gamma}{2}\,\kappa\Big)\,. \tag{5.11}$$

We substitute (5.10) into (5.9) and rewrite the resulting equations as a single 3×3 matrix equation:

$$\begin{pmatrix} a & 0 & 1 \\ b & 2\rho_\varepsilon^2 & 1 \\ 0 & 2\rho_{\rm e}^2 & 1 \end{pmatrix} \begin{pmatrix} \widetilde{C} \\ d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}, \tag{5.12}$$

where

$$\begin{aligned} a &\equiv -1 + \frac{(\kappa\rho_\varepsilon)^2}{4}\,, \\ b &\equiv -1 - \frac{(\kappa\rho_\varepsilon)^2}{2}\Big(\frac{1}{2} - q + \log\rho_\varepsilon\Big)\,, \\ l_1 &\equiv 2r_4\rho_\varepsilon^2\log\rho_\varepsilon + (r_4 + \widetilde{r}_5)\rho_\varepsilon^2 - \widetilde{r}_1\,, \\ l_2 &\equiv -2r_4\rho_\varepsilon^2\log^2\rho_\varepsilon - 2(r_4 + \widetilde{r}_5)\rho_\varepsilon^2\log\rho_\varepsilon - (\widetilde{r}_5 + 2\widetilde{r}_6)\rho_\varepsilon^2 - \widetilde{r}_1\,, \\ l_3 &\equiv 2\,\widetilde{n}\log\rho_{\rm e}\,. \end{aligned} \tag{5.13}$$

Equations (5.12) are solved for d_1, d_2, $\widetilde{C}$ by the Cramer's rule:

$$\begin{aligned} \widetilde{C} &= \frac{2}{\mathcal{D}}\big(\rho_{\rm e}^2\,(l_2 - l_1) + \rho_\varepsilon^2(l_1 - l_3)\big)\,, \\ d_1 &= \frac{1}{\mathcal{D}}\big(al_2 - bl_1 + l_3(b-a)\big)\,, \\ d_2 &= \frac{2}{\mathcal{D}}\big(\rho_{\rm e}^2\,(bl_1 - al_2) + \rho_\varepsilon^2 a l_3\big)\,, \end{aligned} \tag{5.14}$$

where $\mathcal{D} \equiv 2a\rho_\varepsilon^2 + 2(b-a)\rho_{\rm e}^2$.

Most important for us is to represent (5.14) using, at sufficiently small ρ_ε, so-called *logarithmic approximation*:

$$1 \ll \kappa \ll \frac{1}{\rho_\varepsilon} \ll \frac{1}{\rho_\varepsilon}\log\frac{1}{\rho_\varepsilon}\,. \tag{5.15}$$

Then, the ratios (5.14) can be represented in the perturbative form:

$$\begin{aligned} \widetilde{C} &\simeq -\frac{4r_4}{\kappa^2}\log\frac{1}{\rho_\varepsilon} + \frac{4}{\kappa^2}\Big(\widetilde{r}_5 + (1+q)\,r_4\Big) + \ldots\,, \\ d_1 &\simeq \frac{1}{\rho_{\rm e}^2}\Big(\widetilde{n}\log\frac{\rho_{\rm e}}{\rho_1} + 2\frac{r_4}{\kappa^2}\log\frac{1}{\rho_\varepsilon}\Big) + \ldots\,, \\ d_2 &\simeq -4\frac{r_4}{\kappa^2}\log\frac{1}{\rho_\varepsilon} - 2\widetilde{n}\log\frac{1}{\rho_1} + \ldots\,, \end{aligned} \tag{5.16}$$

where dots imply irrelevant terms, which tend to zero at $\rho_\varepsilon \to 0$. Besides, the following new length parameter ρ_1 is defined to express the contributions in d_1, d_2 (5.16), which are constant at $\rho_\varepsilon \to 0$:

$$2\widetilde{n}\,\log\frac{1}{\rho_1} \equiv \widetilde{r}_1 - \frac{4}{\kappa^2}\Big(\widetilde{r}_5 + (1+q)\,r_4\Big)\,, \tag{5.17}$$

where q is given by (5.11). Using (5.4) and (5.16), we obtain at sufficiently far distances:

$$
\begin{aligned}
\overset{(2)}{\sigma}_{\rho\rho} &\approx \frac{2\widetilde{n}}{\rho^2}\left(\log\frac{\rho}{\rho_1} - \Big(\frac{\rho}{\rho_{\rm e}}\Big)^2 \log\frac{\rho_{\rm e}}{\rho_1}\right) - 4\frac{r_4}{\kappa^2}\log\rho_\varepsilon\Big(\frac{1}{\rho^2} - \frac{1}{\rho_{\rm e}^2}\Big)\,, \\
\overset{(2)}{\sigma}_{\phi\phi} &\approx \frac{2\widetilde{n}}{\rho^2}\left(1 - \log\frac{\rho}{\rho_1} - \Big(\frac{\rho}{\rho_{\rm e}}\Big)^2 \log\frac{\rho_{\rm e}}{\rho_1}\right) + 4\frac{r_4}{\kappa^2}\log\rho_\varepsilon\Big(\frac{1}{\rho^2} + \frac{1}{\rho_{\rm e}^2}\Big)\,.
\end{aligned}
\tag{5.18}
$$

As far as the asymptotics (5.18) are concerned, it is crucial that r_4 can be made equal to zero. Indeed, using r_4 in the form (4.10) (where we put $p_2 = 0$ and use p_1, p_3 (3.39)), we obtain ρ_*^2 from equation $r_4 = 0$ as follows (see [36]):

$$
\begin{aligned}
\rho_*^2 = \frac{B(\eta,\alpha)}{\kappa^2}\,, \qquad B(\eta,\alpha) &\equiv \frac{8(1-\nu)(1-\eta)}{\alpha(1-\nu)\eta - 2\nu(1-\eta)}\,, \\
\eta &\equiv 4(1+\nu)\,\frac{\mu^2 C_3}{1 - 2\mu^2 C_7}\,,
\end{aligned}
\tag{5.19}
$$

where c and $\widetilde{c}$ (3.34) are used, while $\alpha = 1$ or $\alpha = 3$, according to (3.39). Equations (5.19) lead to a positivity of the coefficient-function $B(\eta,\alpha)$, and it turns out that admissible η's are strictly between zero and unity. Validity of the replacement (3.28) holds better provided κ^{-1} and ρ_* are smaller. Once the scale κ^{-1} is chosen as small enough, a boundary B_0 can be imposed to restrict possible variations of $B(\eta,\alpha)$: $0 < B(\eta,\alpha) \lesssim B_0$, so that ρ_* (5.19) is estimated as $\rho_* = O(\kappa^{-1})$. In terms of η, the restriction takes the form:

$$
\Big(1 + \frac{\alpha}{2}\frac{1-\nu}{\nu}\Big)^{-1} < \big(1+U\big)^{-1} \lesssim \eta < 1\,, \qquad U \equiv \frac{\alpha(1-\nu)B_0}{8(1-\nu) + 2\nu B_0}\,,
\tag{5.20}
$$

where it is used that the Poisson ratio of isotropic materials is restricted: $0 < \nu \le 1/2$.

Using (5.19), it is straightforward to rewrite (5.20) in terms of the elastic parameters C_3, C_7:

$$
\begin{aligned}
&a) \quad C_7 + 2(1+\nu)\,C_3 < \frac{1}{2\mu^2} \lesssim C_7 + 2\,(1+\nu)\big(1 + U\big)\,C_3\,, \qquad C_3 > 0\,, \\
&b) \quad C_7 + 2\,(1+\nu)\big(1 + U\big)\,C_3 \lesssim \frac{1}{2\mu^2} < C_7 + 2(1+\nu)\,C_3\,, \qquad C_3 < 0\,.
\end{aligned}
\tag{5.21}
$$

Further, Eqs. (3.9) can be applied to go over from C_3, C_7 to the parameters M, N, which are, in turn, expressible (see, for instance, [32]) through the coefficients m', n'. The lasts characterize the elastic energy (3.5) but can also be related to the third order elastic moduli of the cubic crystals $C_{\rm ijk}$ (labelled by three indices i, j, k). In other words, the relations (5.21) restrict a choice of material. Numerical data provided by [32] witness that these conditions look realistically for certain materials. However, to discuss a fitting of (5.21) by the crystallographic data is beyond of the present investigation.

Provided ρ_*^2 is given by (5.19) and r_4 is thus zero, the limit $\rho_\varepsilon \to 0$ is suitable in (5.16) to eliminate ρ_ε in *const*, d_1, d_2. Then, d_1 and d_2 acquire the classically known form (5.6), though ρ_1 is present now instead of $\rho_{\rm c}$. The length ρ_1 (5.17) is purely of the "gauge" origin. The large-distance stresses (5.18) also take at $r_4 = 0$ their classical form (5.7). The values thus obtained for *const*, d_1, d_2 should be used in f (5.1). Eventually, the second order stresses calculated accordingly to (5.3) appear without a cut-off around the dislocation's core. Since appropriate differentiations are straightforward, we shall not elaborate the corresponding general formulas explicitly. Notice that the beginning of coordinates is excluded from the consideration since situated within a set of concentric circles of monotonically decreasing radii ρ_ε. Therefore, the axis Oz is not captured by the present approach. This is in agreement with the regularization of the density profile at $\rho = 0$ in Section 3.

5.2 The component σ_{zz}

Let us consider the stress component $\overset{(2)}{\sigma}_{zz} = p$. Classically, the second equation in (3.24) implies the following representation [21]:

$$\overset{(2)}{\sigma}_{zz} = \nu \left(\overset{(2)}{\sigma}_{\rho\rho} + \overset{(2)}{\sigma}_{\phi\phi} \right) - 2\mu(1+\nu)\left(\Psi^{\rm bg}_{zz} + Q' \right), \tag{5.22}$$

where the potential $f_{\rm bg}$ (5.1.2) is used for the stresses in right-hand side (the corresponding subscript is omitted for brevity), and $\overset{(1)}{\phi}_{\rm bg} = (-b/2\pi)\log\rho$ is substituted to express $\Psi^{\rm bg}_{zz}$ [4]. The argument ρ in (5.22) varies within the segment $[\rho_{\rm c}\,, \rho_{\rm e}\,]$, and Q' is determined from a requirement that $\overset{(2)}{\sigma}_{zz}$ averaged over cross-section of the bulk is zero (so-called, 'mean stress' theorem [32]):

$$\int\limits_0^{2\pi}\int\limits_{\rho_{\rm c}}^{\rho_{\rm e}} \overset{(2)}{\sigma}_{zz}\; \rho\, d\rho\, d\varphi \;=\; 0\,. \tag{5.23}$$

Provided Q' is thus found, the representation (5.22) reads at $\rho_{\rm e} \gg \rho_{\rm c}$:

$$\overset{(2)}{\sigma}_{zz} \approx 2\left[\nu\widetilde{n} + \left(\frac{b}{2\pi}\right)^2 \mu^3(1+\nu)C_3 \right] \left[\frac{1}{\rho^2} - \frac{2}{\rho_{\rm e}^2}\log\frac{\rho_{\rm e}}{\rho_{\rm c}} \right]. \tag{5.24}$$

The integral $\displaystyle\int_{\rho_{\rm c}}^{\rho_{\rm e}} \Delta f_{\rm bg}\,\rho\, d\rho$ is zero under the conventional choice of d_1, d_2, and therefore (5.22) can also be written at $\rho_{\rm e} \gg \rho_{\rm c}$ in the following form:

$$\overset{(2)}{\sigma}_{zz} \approx \nu \left(\overset{(2)}{\sigma}_{\rho\rho} + \overset{(2)}{\sigma}_{\phi\phi} \right) - 2\mu(1+\nu)\left[\Psi^{\rm bg}_{zz} - \frac{2}{\rho_{\rm e}^2}\int\limits_{\rho_{\rm c}}^{\rho_{\rm e}} \Psi^{\rm bg}_{zz}\,\rho\, d\rho \right]. \tag{5.25}$$

Now let us turn to $\overset{(2)}{\sigma}_{zz}$ of the modified defect given by (3.30.2):

$$\overset{(2)}{\sigma}_{zz} = \nu \left(\overset{(2)}{\sigma}_{\rho\rho} + \overset{(2)}{\sigma}_{\phi\phi} \right) - 2\mu\,(1+\nu)(\Psi_{zz} + \widetilde{g}) + \kappa^2\,(1+\nu)\,(f - f_{\rm bg})\,. \tag{5.26}$$

In this case, the argument ρ varies within $[\rho_\varepsilon, \rho_{\rm e}\,]$. From the previous considerations it is clear that $\overset{(2)}{\sigma}_{\rho\rho} + \overset{(2)}{\sigma}_{\phi\phi}$ and Ψ_{zz}, taken at $\rho = \rho_\varepsilon$, decrease when $\rho_\varepsilon \to 0$. Nevertheless, the behavior of $\overset{(2)}{\sigma}_{zz}$ sufficiently close to Oz appears a bit artificial, since the logarithmic term $r_2\log^2(\kappa\rho)$ dominates in $f - f_{\rm bg}$ at small enough ρ. However, this is not an obstacle to consider the rings $\mathcal{D}_{[\rho_\varepsilon,\rho_{\rm e}]}$, as well as (eventually) the punctured disc, as the domains of definition of the corresponding stress distributions. Indeed, the integral analogous to (5.23)

[4] In the polar coordinates, Ψ_{zz} (3.22) takes the form: $\Psi_{zz} = -c(\partial_\rho \overset{(1)}{\phi})^2$.

is convergent for $\overset{(2)}{\sigma}_{zz}$ (5.26) provided its lower bound ρ_ε goes to zero. In other words, $\overset{(2)}{\sigma}_{zz}$ (5.26) averaged over cross-section of the bulk is finite.

Let us obtain $\overset{(2)}{\sigma}_{zz}$ (5.26) at sufficiently large distances, where $f - f_{\rm bg}$ is exponentially small, and the sum $\Psi_{zz} + \widetilde{g}$ is given mainly by $\Psi^{\rm bg}_{zz} + Q'$. Using (5.18) at $r_4 = 0$ to estimate $\overset{(2)}{\sigma}_{\rho\rho} + \overset{(2)}{\sigma}_{\phi\phi}$, we obtain:

$$\overset{(2)}{\sigma}_{zz} \approx 2\nu\widetilde{n}\left[\frac{1}{\rho^2} - \frac{2}{\rho_{\rm e}^2}\log\frac{\rho_{\rm e}}{\rho_1}\right] - 2\mu(1+\nu)\left[Q' - \left(\frac{b}{2\pi}\right)^2\frac{c}{\rho^2}\right]. \tag{5.27}$$

We determine Q' from the condition that (5.26) integrated over cross-section is zero. Approximately, Q' takes the form:

$$Q' \approx \left(\frac{b}{2\pi}\right)^2 2c\,\frac{\log(\rho_{\rm e}/\rho_2)}{\rho_{\rm e}^2}, \tag{5.28}$$

where another length ρ_2 is introduced:

$$\begin{aligned}\left(\frac{b}{2\pi}\right)^2 2c\,\log\frac{1}{\rho_2} \;\equiv\; &\left(\frac{b}{4\pi}\right)^2 - 2\int_0^1 \Psi_{zz}\,\rho\,d\rho - 2\int_1^\infty\left[\Psi_{zz} + \left(\frac{b}{2\pi}\right)^2\frac{c}{\rho^2}\right]\rho\,d\rho\\ &+ \frac{\kappa^2}{\mu}\int_0^\infty \bigl(f - f_{\rm bg}\bigr)\,\rho\,d\rho\,.\end{aligned} \tag{5.29}$$

The estimate (5.28) is written under assumption that the upper integration bound $\rho_{\rm e}$ is replaced approximately by infinity, and $\rho_\varepsilon \to 0$ as the lower one. Therefore, the large distance asymptotic (5.27) depends on two scales which are, in principle, different in our consideration: ρ_1 (5.17) and ρ_2 (5.29). To re-write (5.27) more conventionally, it is appropriate to introduce another length, say $\rho_{\rm m}$, as follows:

$$\begin{aligned}&\left[\nu\widetilde{n} + \left(\frac{b}{2\pi}\right)^2\mu(1+\nu)c\right]\log\frac{1}{\rho_{\rm m}}\\ &\qquad\equiv \nu\widetilde{n}\,\log\frac{1}{\rho_1} + \left(\frac{b}{2\pi}\right)^2\mu(1+\nu)c\,\log\frac{1}{\rho_2}\,.\end{aligned} \tag{5.30}$$

Substituting Q' (5.28) into (5.27) and using (5.30), we obtain:

$$\overset{(2)}{\sigma}_{zz} \approx 2\left[\nu\widetilde{n} + \left(\frac{b}{2\pi}\right)^2\mu(1+\nu)c\right]\left[\frac{1}{\rho^2} - \frac{2}{\rho_{\rm e}^2}\log\frac{\rho_{\rm e}}{\rho_{\rm m}}\right]. \tag{5.31}$$

Equation (5.31) looks like (5.24) excepting that now $\rho_{\rm m}$ is present instead of $\rho_{\rm c}$.

The classical answers for d_1, d_2 are given at $\rho_{\rm e} \gg \rho_{\rm c}$ by (5.6). Since d_1, d_2 (5.16) include the constant contributions, it is appropriate to demonstrate agreement between (5.6) and (5.16) by means of the newly defined length ρ_1 (5.17). In its turn, the integral condition (5.23) implies that ρ_2 coincides, classically, with the lower integration boundary $\rho_{\rm c}$. Instead, Q' (5.28) is concerned with another length ρ_2 (5.29), since Ψ_{zz} is unconventional inside the core. In other words, the definitions (5.17) and (5.29) give classically the coinciding answers: $\rho_1 = \rho_{\rm c}$ and $\rho_2 = \rho_{\rm c}$, where $\rho_{\rm c}$ is the cut-off which remains unspecified. In the present approach, two different lengths arise instead of $\rho_{\rm c}$. The definition (5.30) is only to obtain the classically-looking representation (5.31).

We use $\widetilde{g}$ (3.30), where $\mathcal{C} = Q'$ with Q' given by (5.28), and thus we rewrite $\overset{(2)}{\sigma}_{zz}$ (5.26):

$$\begin{aligned}
\overset{(2)}{\sigma}_{zz} \;&=\; \nu\,\bigl(\overset{(2)}{\sigma}_{\rho\rho} \,+\, \overset{(2)}{\sigma}_{\phi\phi}\bigr) \,-\, 2\mu\,(1+\nu)\biggl[\Psi_{zz} \,-\, \frac{2}{\rho_{\rm e}^2}\int\limits_0^{\rho_{\rm e}} \Psi_{zz}\,\rho\,d\rho\biggr] \\
&-\; 2\mu\,(1+\nu)\Bigl(\frac{b}{4\pi}\Bigr)^2\biggl[\frac{1}{\rho_{\rm e}^2} \,+\, \frac{2}{\rho_*^2}\Bigl(\frac{\rho^2}{\rho_*^2} \,-\, 1\Bigr)\,h_{[0,\,\rho_*]}(\rho)\biggr] \\
&+\; \kappa^2\,(1+\nu)\,\biggl[f - f_{\rm bg} \,-\, \frac{2}{\rho_{\rm e}^2}\int\limits_0^{\rho_{\rm e}} \bigl(f - f_{\rm bg}\bigr)\,\rho\,d\rho\biggr]\,,
\end{aligned} \tag{5.32}$$

or, at sufficiently large $\rho_{\rm e}$,

$$\begin{aligned}
\overset{(2)}{\sigma}_{zz} \;&\approx\; \nu\,\bigl(\overset{(2)}{\sigma}_{\rho\rho} \,+\, \overset{(2)}{\sigma}_{\phi\phi}\bigr) \,-\, 2\mu\,(1+\nu)\Psi_{zz} \,+\, \kappa^2\,(1+\nu)\,(f - f_{\rm bg}) \\
&-\; 2\mu\,(1+\nu)\Bigl(\frac{b}{2\pi}\Bigr)^2\,\biggl[\frac{2c}{\rho_{\rm e}^2}\,\log\frac{\rho_{\rm e}}{\rho_2} \,+\, \frac{1}{2\rho_*^2}\Bigl(\frac{\rho^2}{\rho_*^2} \,-\, 1\Bigr)\,h_{[0,\,\rho_*]}(\rho)\biggr]\,,
\end{aligned}$$

where ρ_*^2 is given by (5.19).

5.3 Remarks on the logarithmic approximation

The form of the asymptotic relations (5.16) reminds that the ratios (5.14) should be rearranged perturbatively provided $\log(1/\rho_\varepsilon)$ and $\rho_{\rm e}$ are sufficiently large. It is instructive to re-derive the leading terms in (5.16) by means of a more straightforward use of the logarithmic approximation at $r_4 = 0$. Indeed, let us reconsider (5.12) with the parameters (5.13) taken in the main order as follows:

$$\begin{aligned}
&a \;\approx\; -1\,, &&b \;\approx\; -1 \,-\, \frac{(\kappa\rho_\varepsilon)^2}{2}\,\log\rho_\varepsilon\,, \\
&l_1 \;\approx\; -\widetilde{r}_1\,, &&l_2 \;\approx\; -\widetilde{r}_1 \,-\, 2\,\overset{\circ}{r}_5\rho_\varepsilon^2\log\rho_\varepsilon\,,
\end{aligned} \tag{5.33}$$

where $r_4 = 0$ in $\widetilde{r}_1$, while a special notation $\overset{\circ}{r}_5$ is chosen for $\widetilde{r}_5$ (5.10) at $r_4 = 0$ (see (B14)). Besides, l_3 is kept like in (5.13). We replace the entry equal to $2\rho_\varepsilon^2$ in (5.12) by zero. Then, in agreement with (5.16), the solution reads:

$$\widetilde{C} \;=\; \frac{4\,\overset{\circ}{r}_5}{\kappa^2}\,, \qquad d_1 \;=\; \frac{\widetilde{n}}{\rho_{\rm e}^2}\,\log\frac{\rho_{\rm e}}{\rho_1}\,, \qquad d_2 \;=\; 2\widetilde{n}\,\log\rho_1\,, \tag{5.34}$$

where ρ_1 corresponds to (5.17).

Moreover, assuming that $\log\kappa$ is also large (but probably not like $\log(1/\rho_\varepsilon)$ is), it is possible to estimate ρ_1. Let us separate out the logarithmic dependence of $\widetilde{r}_1$, $\widetilde{r}_6$ (5.10) as follows (r_4 is zero):

$$\widetilde{r}_1 \,-\, \frac{q_1}{\kappa^2}\,\log\kappa \;\equiv\; \overset{\circ}{r}_1\,, \qquad \widetilde{r}_6 \,-\, \overset{\circ}{r}_5\,\log\kappa \;\equiv\; \overset{\circ}{r}_6\,, \tag{5.35}$$

where $\overset{\circ}{r}_1$, $\overset{\circ}{r}_5$, and $\overset{\circ}{r}_6$ are free from $\log\kappa$. We can use (5.35) in (5.13) and again directly solve (5.12). Then, the leading terms in d_1 and d_2 appear just in the form (5.34) with $\rho_1 = B_1/\kappa$, where

$$2\widetilde{n}\log B_1 \;\equiv\; -\,\overset{\circ}{r}_1 \,+\, \frac{4}{\kappa^2}\,\overset{\circ}{r}_5\,. \tag{5.36}$$

In view of (5.35), it is seen that (5.36) is in agreement with (5.17). Further, the radius ρ_2 can also be estimated at sufficiently large $\log\kappa$. Indeed, we obtain from (5.29) that $\rho_2 = B_2/\kappa$, where

$$2c \log B_2 \equiv -\frac{1}{4} - 2c\int_0^1 (1 - s\,K_1(s))^2 \frac{ds}{s}\,, \tag{5.37}$$

and c is given by (3.34). Therefore, the logarithmic approximation allows to determine the leading behavior of $\widetilde{C}$, d_1, d_2, as well as to estimate ρ_1 and ρ_2. Note in passing that a certain influence of the regularization in (4.3) on B_1 (5.36), B_2 (5.37) should be considered, though elsewhere (since is concerned with specific numerical estimates).

The result (5.34) should be commented as follows. Let a sufficiently small positive $\in$ (so-called, *accuracy*) be prescribed so that absolute values of the stresses which are smaller than $\in$, say, twice (i.e., $|\overset{(2)}{\sigma}_{ij}| < \frac{\in}{2}$) should be treated, for engineering, as indistinguishable from zero. Then, the shrinking of ρ_ε can be ceased as soon as appropriate boundary values $\overset{(2)}{\sigma}_{ij}\Big|_{\rho=\rho_\varepsilon}$ become to fulfil this estimate. Since, at the same time, $\log(1/\rho_\varepsilon)$ is large (e.g., respects (5.15)), the results (5.34), (5.36), (5.37) are valid. In other words, the large distance behavior looks conventionally though includes now ρ_1, ρ_2, which are dictated by a choice of material. However, in the most part of the core region the stresses are determined by (5.3) with f (5.1). Provided ρ_ε is thus fixed, the stress $\overset{(2)}{\sigma}_{zz}$ remains bounded outside of an infinitesimally thin tube around Oz. It is essential that under prescribed accuracy $\in$, a distant "observer" outside the core is dealt only with the conventionally behaving stresses, which look physically (since are improved by means of ρ_1, ρ_2) and are insensitive to all those particular ρ_ε's which are "compatible" with a chosen $\in$.

Furthermore, it is also appropriate to turn again to our geometric interpretation. Remind that (3.28) is being applied to $\Delta\overset{(1)}{\phi}$, and the geometry is approximately treated as the teleparallel one. Let us, for a moment, imagine that the torsion is taken into account in order to make Q_{11}, Q_{22}, and Q_{33} (see (3.20)) conventionally-looking [21]. In a situation when $\Delta\overset{(1)}{\phi}_{\mathcal{T}}$ comes to play instead of $\Delta\overset{(1)}{\phi}$, while a specific shape of appropriatly localized density $\mathcal{T}$ is unclear, the replacement (3.28) can be applied, as a simplification, just to $\Delta\overset{(1)}{\phi}_{\mathcal{T}}$. Besides, as soon as Q_{11}, Q_{22}, and Q_{33} are thus corrected, the modification of $\mathcal{R}$ (3.31) consists in disappearance of the third term in its right-hand side. Also, the corresponding potential $\widetilde{g}$ is transformed into a number, and $w(z)$ does not occur in (3.35) provided $\widetilde{g}$ is equal to Q'.

The local properties of such new $\mathcal{R}$ can be estimated. Indeed, the modifications are expectable for p_4, p_5, p_6 (3.39). However, these changes do not influence the coefficients we consider in (4.9). Equation (3.30.1) is solved, in the extended case, in a way similar to that above. As a result, a change of shapes of the profiles of the stress components within the core is expectable but these shapes are outside the scope of our consideration. It is important that ρ_1 (5.17) and ρ_* (5.19) should not be affected at an extended treatment with $\Delta\overset{(1)}{\phi}_{\mathcal{T}}$ subjected to (3.28). Only ρ_2 (5.29) should be replaced since is defined "globally". Thus, the present picture is almost the same as that with the torsion admitted, though handled approximately. Since the usage of (3.28) simply allows to avoid a specification of $\mathcal{T}$, it is appropriate to keep the teleparallel interpretation, as discussed above.

6 Discussion

The stress tensor of the screw dislocation lying along elastically-isotropic cylinder is discussed in the present paper. Conventionally, the stress field is obtainable by means of the elasticity theory in two orders in the Burgers vector length. It is known that the stress distribution of the first order is singular on the dislocation line. The corresponding second order solution is peculiar due to occurence of the cut-off around the dislocation's axis [19], [20], [21], [22]. In other words, a traction-free condition on boundary of the defect's core comes to play. Notice that the 'core' is rather a crystalline notion since the theoretical elasticity has no appropriate length. In the given paper, the $T(3)$-gauge approach is investigated which enables the known second order stresses of the screw dislocation be continued inside the core. Although conventionally the cut-off radius is a free parameter, the continuation proposed manifests itself in appearance of special lengths which should characterize exteriority of the core region. These lengths are expressed through the elastic moduli of the cylinder's material.

The non-homogeneous Einstein-type gauge equation (which can be derived variationally by means of the Hilbert–Einstein gauge Lagrangian) plays the role of unconventional incompatibility law. It is just responsible for smearing of the first order stress distribution of the screw dislocation [1]. However, the other gauge solutions of the first order so far known [2], [5], [6] also look satisfactory as far as avoiding the singularity of the screw dislocation is concerned. But these solutions correspond to differently chosen translational Lagrangians [5]. Therefore, since the Einsteinian form of the gauge equation looks suggestively, it appears appropriate to investigate the stresses of the screw dislocation in the quadratic approximation also. Moreover, it seems interesting to handle the non-linearity of the gauge equation by means of the stress function approach [19], [21].

The Einsteinian gauge equation leads to the stress potential of second order which additionally includes, apart from the conventional terms, a non-conventional short-ranged contribution. The corresponding additional stresses are localized within the dislocation's core. To ensure self-consistency of the resulting stress distribution, arbitrariness of certain constants in the general solution is used. To adjust the parameters, we consider the stresses on the hollow disc with arbitrarily small internal radius which is kept nonzero but allowed to decrease monotonically. As a result, there is no cut-off at the distances comparable with the radius of the defect's core, but the points belonging to the axis Oz are eventually excluded from the consideration (the punctured disc as the cross-section).

Sufficiently far from the core, the stresses $\overset{(2)}{\sigma}_{\rho\rho}$, $\overset{(2)}{\sigma}_{\phi\phi}$, $\overset{(2)}{\sigma}_{zz}$ agree with those in [21] though include now the lengths ρ_1, ρ_2, which imply the lower bounds for a range of validity of the large distance asymptotical expressions. Besides, $\overset{(2)}{\sigma}_{\rho\rho}$ respects the free-surface condition at the outer cylinder's boundary. As to the short-distances, the components $\overset{(2)}{\sigma}_{\rho\rho}$, $\overset{(2)}{\sigma}_{\phi\phi}$ tend to zero, while $\overset{(2)}{\sigma}_{zz}$ grows logarithmically (though it is, as required [21], integrable) in a close vicinity of Oz. The solution obtained is characterized by one more length ρ_* which is a half-width of the effective density profile used in the present approach. The lengths ρ_1, ρ_2, ρ_* are expressed through the elastic moduli and demonstrate that the dislocation core appears to possess (in our description) a concentrically-layered external shell. However, possible numerical values for them are not discussed: the problem is strictly influenced by the crystallographic data and should be addressed separately. The "exterior" (in respect of the

[5]The author is grateful to Markus Lazar for calling attention to [51] where the modified screw dislocation has been also obtained for the gauge Lagrangian written as a restricted quadratic form.

core) stress distribution of the second order, which is improved by means of self-consistently arising radii ρ_1, ρ_2, plays a central role in the picture presented.

To avoid the use of the replacement for the density distribution, the geometric picture should be extended by means of the differential-geometric torsion taken into account. The torsion, as an independent localized source of incompatibility, could be expected to provide additional free parameters in the short-distance behavior of the stresses. The constant ρ_*, introduced by means of the "regularized" density, can be viewed as effectively playing a similar role. Additional arbitrariness in the short-distance behavior can be attempted to be used, for instance, to make $\overset{(2)}{\sigma}_{zz}$ less artificial near Oz.

The continuation proposed for the stresses of the screw dislocation is just achieved by means of the gauge equation considered as the incompatibility law. The constitutive equations which relate the stresses and the strains are non-linear in a usual sense, i.e., are due to the elastic energy written up to the third order in the strains. The present picture does not pretend to simulate the stress distribution within the core. For instance, extended treatment can probably affect details of the shapes of the localized parts of the stresses. Nevertheless, the model seems to provide effectively a self-contained description which accounts for, in two orders, the existence of the core region.

Certain features of an extended picture could be discovered in the present model which can, in turn, be used as a base of further, more involved, considerations. Estimations which account for the crystallographic data are also possible but should be done elsewhere. It is suffice to point out that validity of the solution proposed of the second order is concerned with the higher elastic moduli subjected to certain restrictions. But this point should be studied further after inclusion of the differential-geometric torsion.

A gauge approach to non-singular dislocations close to ours is developed in [5], [6], [47], [48], and it is based on the quadratic $T(3)$-gauge Lagrangian [6]. However, only the linear problems are studied therein. It is proposed in [48] to use the terms in the Lagrangian which are related to the energy of the rotation gradients. These terms allow [48] to improve the large distance behavior of the modified defect found in [1] by an analogy with the edge dislocation. Therefore, additional terms of the mechanical origin in the full Lagrangian also merit attention in the framework of the gauging proposed in the present paper.

Our investigation demonstrates that the Hilbert–Einsteinian gauge approach is flexible enough since it allows, in two orders, the description for non-singular screw dislocation. Moreover, it seemingly provides an interesting opportunity of further extended treatments: for instance, application to the edge dislocation would be also desirable.

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[6]Further applications to the disclinations are in [49], [50].

Appendix A

Appendices A and B contain intermediate results which are important to expand asymptotically the stress potential of second order f (5.1) (see also [36]). On starting with the series expansion (3.38) and using the Bessel functions in the series forms [45], we go over to the corresponding series $\mathcal{F}$ (4.9). First, let us estimate $G(z)$ given by (4.3), (4.4).

To begin with, we expand the products $t^{-1}K_0(t)\mathcal{R}(t)$ and $t^{-1}I_0(t)\mathcal{R}(t)$ at small arguments $t \ll 1$:

$$\begin{aligned} t^{-1}I_0(t)\mathcal{R}(t) \;&\simeq\; p_1\, t^{-3} \,+\, p_2\, t^{-1}\log t \,+\, \Big(\frac{p_1}{4} \,+\, p_3\Big)\, t^{-1} \\ &+\; \Big(p_4 \log^2 t \,+\, \Big(\frac{p_2}{4} \,+\, p_5\Big)\, \log t \,+\, \widehat{k}\Big)\, t \,+\, \ldots; \end{aligned} \tag{A1}$$

$$\begin{aligned} t^{-1}K_0(t)\mathcal{R}(t) \;&\simeq\; -p_1\, \log\Big(\frac{\gamma}{2}\, t\Big)\, t^{-3} \,-\, \big(p_2 \log^2 t \,+\, k_1\, \log t \,+\, k_2\big)\, t^{-1} \\ &-\; \big(p_4 \log^3 t \,+\, k_3 \log^2 t \,+\, k_4\, \log t \,+\, k_5\big)\, t \,+\, \ldots, \end{aligned} \tag{A2}$$

where $p_1, p_2, \ldots, p_5$ are given by (3.38), and the coefficients k_1, k_2, k_3 are expressed by means of $p_1, p_2, \ldots, p_5$ as follows:

$$\begin{aligned} k_1 \;&=\; \frac{p_1}{4} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, p_2 \,+\, p_3\,, \\ k_2 \;&=\; -\frac{p_1}{4} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, \Big(\frac{p_1}{4} \,+\, p_3\Big)\,, \\ k_3 \;&=\; \frac{p_2}{4} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, p_4 \,+\, p_5 \end{aligned} \tag{A3}$$

(γ is the Euler constant). The parameters k_4, k_5, and $\widehat{k}$ are not used in the present paper and can be found in [36].

The series representations for the modified Bessel functions I_0 and K_0, as well as the expansions (A1) and (A2), lead us to the following estimations at $z \ll 1$:

$$\begin{aligned} K_0(z)\Big(B(\epsilon) - \int\limits_{\epsilon}^{z} I_0(t)\mathcal{R}(t)\, \frac{dt}{t}\Big)\Big|_{s\to 0} \;&\simeq\; \mathcal{K}_0 \,+\, (\mathcal{K}_1\, \log z \,+\, \mathcal{K}_2)\; z^{-2} \\ &+\; \mathcal{K}_3\, \log^3 z \,+\, \mathcal{K}_4\, \log^2 z \,+\, \mathcal{K}_5\, \log z \\ &+\; \big(\mathcal{K}_6\, \log^3 z \,+\, \mathcal{K}_7\, \log^2 z \,+\, \mathcal{K}_8\, \log z \,+\, \mathcal{K}_9\big)\; z^2 \,+\, \ldots\,, \end{aligned} \tag{A4}$$

where

$$\begin{aligned} &\mathcal{K}_0 \;=\; -const \times \log\frac{\gamma}{2} \,+\, \Big(1 \,+\, 3\log\frac{\gamma}{2}\Big)\, \frac{p_1}{8}\,, \qquad \mathcal{K}_1 \;=\; \frac{-p_1}{2}\,, \\ &\mathcal{K}_2 \;=\; -\log\Big(\frac{\gamma}{2}\Big)\, \frac{p_1}{2}\,, \qquad \mathcal{K}_3 \;=\; \frac{p_2}{2}\,, \qquad \mathcal{K}_4 \;=\; \frac{p_1}{4} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, \frac{p_2}{2} \,+\, p_3\,, \\ &\mathcal{K}_5 \;=\; -const \,+\, \Big(\frac{3}{2} + \log\frac{\gamma}{2}\Big)\, \frac{p_1}{4} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, p_3\,, \qquad \mathcal{K}_6 \;=\; \frac{p_2}{8} \,+\, \frac{p_4}{2}\,, \\ &\mathcal{K}_7 \;=\; \frac{p_1}{16} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, \frac{p_2}{8} \,+\, \frac{p_3}{4} \,-\, \Big(1 \,-\, \log\frac{\gamma}{2}\Big)\, \frac{p_4}{2} \,+\, \frac{p_5}{2}\,. \end{aligned} \tag{A5}$$

Further, we get:

$$
\begin{aligned}
I_0(z) \int_z^\infty K_0(t)\mathcal{R}(t)\,\frac{dt}{t} \;&\simeq\; \mathcal{I}_0 \,+\, (\mathcal{I}_1 \,\log z \,+\, \mathcal{I}_2)\; z^{-2} \\
&+\;\; \mathcal{I}_3 \,\log^3 z \,+\, \mathcal{I}_4 \,\log^2 z \,+\, \mathcal{I}_5 \,\log z \\
&+\;\; \big(\mathcal{I}_6 \,\log^3 z \,+\, \mathcal{I}_7 \,\log^2 z \,+\, \mathcal{I}_8 \,\log z \,+\, \mathcal{I}_9\big)\; z^2 \,+\, \ldots\,,
\end{aligned}
\tag{A6}
$$

where

$$
\begin{aligned}
&\mathcal{I}_0 \,=\, \mathcal{I}_K \,+\, \Big(1+2\log\frac{\gamma}{2}\Big)\,\frac{3p_1}{16}\,, \qquad \mathcal{I}_1 \,=\, \frac{-p_1}{2}\,, \qquad \mathcal{I}_2 \,=\, -\Big(1+2\log\frac{\gamma}{2}\Big)\,\frac{p_1}{4}\,, \\
&\mathcal{I}_3 \,=\, \frac{p_2}{3}\,, \qquad \mathcal{I}_4 \,=\, \frac{k_1}{2}\,, \qquad \mathcal{I}_5 \,=\, \frac{-p_1}{8} \,+\, k_2\,, \\
&\mathcal{I}_6 \,=\, \frac{p_2}{12} \,+\, \frac{p_4}{2}\,, \qquad \mathcal{I}_7 \,=\, \frac{k_1}{8} \,-\, \frac{3p_4}{4} \,+\, \frac{k_3}{2}\,,
\end{aligned}
\tag{A7}
$$

and k_1, k_2, k_3 are provided by (A3). Besides, the numerical constant $\mathcal{I}_K$ in $\mathcal{I}_0$ (A7) is given by the regularized value of the integral:

$$
\begin{aligned}
\mathcal{I}_K \;\equiv\; &\int_1^\infty K_0(t)\mathcal{R}(t)\,\frac{dt}{t} \,+\, \int_0^1 \Big[K_0(t)\mathcal{R}(t) \,+\, p_1 \,\log\Big(\frac{\gamma}{2}\,t\Big)\,t^{-2} \\
&+\;\; k_2 \,+\, k_1 \,\log t \,+\, p_2 \,\log^2 t\Big]\,\frac{dt}{t}\,.
\end{aligned}
\tag{A8}
$$

Eventually, the series (4.5), which describes $G(z,0)$ at small z, is obtained by subtracting the series (A6) from the series (A4). The resulting coefficients (4.6) are calculated by means of (A5) and (A7):

$$
\begin{aligned}
q_0 \,&=\, \mathcal{K}_0 \,-\, \mathcal{I}_0\,, \\
q_i \,&=\, \mathcal{K}_{i+1} \,-\, \mathcal{I}_{i+1}\,, \qquad i \,=\, 1,2,\ldots,6\,.
\end{aligned}
$$

Since $\mathcal{K}_1 - \mathcal{I}_1 = 0$, the contribution $\propto z^{-2}\log z$ is absent in (4.5).

Appendix B

Now let us estimate $I_{\mathcal{F}}(\rho)$ (4.8). First, we expand the products $t\,J_0(\mathcal{N}t)G(\kappa t)$ and $t\,\widetilde{Y}_0(\mathcal{N}t)G(\kappa t)$ at small arguments $t \ll 1$ (see [45]):

$$
\begin{aligned}
&t\;\; J_0(\mathcal{N}t)\,G(\kappa t) \,\simeq\, \frac{q_1}{\kappa^2}\,t^{-1} \,+\, q_2\,t\log^3\big(\mathcal{N}t\big) \\
&+\, \Big(n_1\,\log^2\big(\mathcal{N}t\big) \,+\, n_2\,\log\big(\mathcal{N}t\big) \,+\, n_3\Big)\,t \\
&+\, \Big(n_4\,\log^3\big(\mathcal{N}t\big) \,+\, n_5\,\log^2\big(\mathcal{N}t\big) \,+\, n_6\,\log\big(\mathcal{N}t\big) \,+\, n_7\Big)\,t^3 \,+\, \ldots\,,
\end{aligned}
\tag{B1}
$$

$$
\begin{aligned}
t\ \widetilde{Y}_0(\mathcal{N}t)\, G(\kappa t) \,\simeq\, & \frac{q_1}{\kappa^2}\, t^{-1}\, \log\Big(\frac{\gamma}{2}\mathcal{N}t\Big) \\
+ & \Big(q_2 \log^4\big(\mathcal{N}t\big) \,+\, m_1 \log^3\big(\mathcal{N}t\big) \,+\, m_2 \log^2\big(\mathcal{N}t\big) \,+\, m_3 \log\big(\mathcal{N}t\big) \,+\, m_4\Big)\, t \\
+ & \Big(m_5 \log^4\big(\mathcal{N}t\big) \,+\, m_6 \log^3\big(\mathcal{N}t\big) \,+\, m_7 \log^2\big(\mathcal{N}t\big) \,+\, m_8 \log\big(\mathcal{N}t\big) \,+\, m_9\Big)\, t^3 \,+\, \ldots\,,
\end{aligned}
\tag{B2}
$$

where the parameters $n_1, n_2, \ldots, n_5$ are expressed by means of the previously found coefficients $q_1, q_2, \ldots, q_6$ (4.6):

$$
\begin{aligned}
n_1 \,&=\, 3\, \log\Big(\frac{\kappa}{\mathcal{N}}\Big)\, q_2 \,+\, q_3\,, \\
n_2 \,&=\, 3\, \log^2\Big(\frac{\kappa}{\mathcal{N}}\Big)\, q_2 \,+\, 2\, \log\Big(\frac{\kappa}{\mathcal{N}}\Big) q_3 \,+\, q_4\,, \\
n_3 \,&=\, q_0 \,-\, \frac{\mathcal{N}^2}{\kappa^2}\, \frac{q_1}{4} \,+\, \log^3\Big(\frac{\kappa}{\mathcal{N}}\Big)\, q_2 \,+\, \log^2\Big(\frac{\kappa}{\mathcal{N}}\Big) q_3 \,+\, \log\Big(\frac{\kappa}{\mathcal{N}}\Big) q_4\,, \\
n_4 \,&=\, -\mathcal{N}^2\, \frac{q_2}{4} \,+\, \kappa^2\, q_5\,, \\
n_5 \,&=\, -\mathcal{N}^2\, \frac{q_3}{4} \,+\, \kappa^2\, q_6 \,+\, 3\, \log\Big(\frac{\kappa}{\mathcal{N}}\Big)\, n_4\,,
\end{aligned}
\tag{B3}
$$

and the coefficients $m_1, m_2, \ldots, m_6$ are expressed by means of $n_1, n_2, \ldots, n_5$ and q_1, q_2:

$$
\begin{aligned}
& m_1 \,=\, \log\Big(\frac{\gamma}{2}\Big)\, q_2 \,+\, n_1\,, \qquad m_2 \,=\, \log\Big(\frac{\gamma}{2}\Big)\, n_1 \,+\, n_2\,, \\
& m_3 \,=\, \log\Big(\frac{\gamma}{2}\Big)\, n_2 \,+\, n_3\,, \qquad m_4 \,=\, \frac{\mathcal{N}^2}{\kappa^2}\, \frac{q_1}{4} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, n_3\,, \\
& m_5 \,=\, n_4\,, \qquad m_6 \,=\, \mathcal{N}^2\, \frac{q_2}{4} \,+\, \log\Big(\frac{\gamma}{2}\Big)\, n_4 \,+\, n_5\,.
\end{aligned}
\tag{B4}
$$

The numerical coefficients n_6, n_7, and m_7, m_8, m_9 are placed in (B1) and (B2) only formally: their explicit values are not of importance for the present investigation.

Using (B1)–(B4), we go over to the estimation of the integrals which constitute $I_{\mathcal{F}}(\rho)$ (4.8). First, we obtain:

$$
\begin{aligned}
-\,\widetilde{Y}_0(\mathcal{N}\rho) \int_{\rho}^{\infty} J_0(\mathcal{N}t)\, G(\kappa t)\, t\, dt \simeq\, & \mathcal{Y}_0 \,+\, \mathcal{Y}_1 \log^2\big(\mathcal{N}\rho\big) \,+\, \mathcal{Y}_2 \log\big(\mathcal{N}\rho\big) \\
& + \Big(\mathcal{Y}_3 \log^4\big(\mathcal{N}\rho\big) \,+\, \mathcal{Y}_4 \log^3\big(\mathcal{N}\rho\big) \,+\, \mathcal{Y}_5 \log^2\big(\mathcal{N}\rho\big) \,+\, \mathcal{Y}_6 \log\big(\mathcal{N}\rho\big) \,+\, \mathcal{Y}_7\Big)\, \rho^2 \\
& + \Big(\mathcal{Y}_8 \log^4\big(\mathcal{N}\rho\big) \,+\, \mathcal{Y}_9 \log^3\big(\mathcal{N}\rho\big) \,+\, \ldots\Big)\, \rho^4 \,+\, \ldots\,,
\end{aligned}
\tag{B5}
$$

where

$$
\begin{aligned}
&\mathcal{Y}_0 \equiv \log\Big(\frac{\gamma}{2}\Big)\,\mathcal{I}_J\,, \qquad \mathcal{Y}_1 = \frac{q_1}{\kappa^2}\,, \qquad \mathcal{Y}_2 = \mathcal{I}_J + \log\Big(\frac{\gamma}{2}\Big)\frac{q_1}{\kappa^2}\,, \qquad \mathcal{Y}_3 = \frac{q_2}{2}\,,\\
&\mathcal{Y}_4 = \frac{n_1}{2} - \Big(\frac{3}{2} - \log\frac{\gamma}{2}\Big)\frac{q_2}{2}\,, \qquad \mathcal{Y}_5 = -\frac{\mathcal{N}^2}{\kappa^2}\frac{q_1}{4} + \Big(1 - \log\frac{\gamma}{2}\Big)\Big(\frac{3q_2}{4} - \frac{n_1}{2}\Big) + \frac{n_2}{2}\,,\\
&\mathcal{Y}_6 = \frac{\mathcal{N}^2}{4}\Big(-\mathcal{I}_J + \Big(1 - \log\frac{\gamma}{2}\Big)\frac{q_1}{\kappa^2}\Big) + \log\Big(\frac{\gamma}{2}\Big)\widehat{\mathcal{Y}}_1 + \widehat{\mathcal{Y}}_2\,,\\
&\mathcal{Y}_7 = \frac{\mathcal{N}^2}{4}\Big(1 - \log\frac{\gamma}{2}\Big)\mathcal{I}_J + \log\Big(\frac{\gamma}{2}\Big)\widehat{\mathcal{Y}}_2\,, \quad \mathcal{Y}_8 = \frac{n_4}{4} - \mathcal{N}^2\frac{q_2}{8}\,,\\
&\mathcal{Y}_9 = \frac{\mathcal{N}^2}{8}\Big(\frac{5}{2} - \log\frac{\gamma}{2}\Big)q_2 - \mathcal{N}^2\frac{n_1}{8} - \Big(\frac{3}{4} - \log\frac{\gamma}{2}\Big)\frac{n_4}{4} + \frac{n_5}{4}\,.
\end{aligned}
\tag{B6}
$$

Next, we obtain the following estimation at small ρ:

$$
\begin{aligned}
-J_0(\mathcal{N}\rho)\int_\rho^\infty \widetilde{Y}_0(\mathcal{N}t)\,G(\kappa t)\,t\,dt \simeq\ & \mathcal{J}_0 + \mathcal{J}_1\log^2\big(\mathcal{N}\rho\big) + \mathcal{J}_2\log\big(\mathcal{N}\rho\big)\\
&+ \Big(\mathcal{J}_3\log^4\big(\mathcal{N}\rho\big) + \mathcal{J}_4\log^3\big(\mathcal{N}\rho\big) + \mathcal{J}_5\log^2\big(\mathcal{N}\rho\big) + \mathcal{J}_6\log\big(\mathcal{N}\rho\big) + \mathcal{J}_7\Big)\rho^2\\
&+ \Big(\mathcal{J}_8\log^4\big(\mathcal{N}\rho\big) + \mathcal{J}_9\log^3\big(\mathcal{N}\rho\big) + \ldots\Big)\rho^4 + \ldots\,,
\end{aligned}
\tag{B7}
$$

where

$$
\begin{aligned}
&\mathcal{J}_0 \equiv \mathcal{I}_Y\,, \quad \mathcal{J}_1 = \frac{q_1}{2\kappa^2}\,, \quad \mathcal{J}_2 = \log\Big(\frac{\gamma}{2}\Big)\frac{q_1}{\kappa^2}\,,\\
&\mathcal{J}_3 = \frac{q_2}{2}\,, \qquad \mathcal{J}_4 = \frac{m_1}{2} - q_2\,, \qquad \mathcal{J}_5 = -\frac{\mathcal{N}^2}{\kappa^2}\frac{q_1}{8} + \widehat{\mathcal{J}}_1\\
&\mathcal{J}_6 = -\frac{\mathcal{N}^2}{4}\mathcal{J}_2 + \widehat{\mathcal{J}}_2\,, \qquad \mathcal{J}_7 = -\frac{\mathcal{N}^2}{4}\mathcal{I}_Y + \widehat{\mathcal{J}}_3\,,\\
&\mathcal{J}_8 = \frac{m_5}{4} - \frac{\mathcal{N}^2}{8}q_2\,, \qquad \mathcal{J}_9 = \frac{m_6 - m_5}{4} + \frac{\mathcal{N}^2}{4}\Big(q_2 - \frac{m_1}{2}\Big).
\end{aligned}
\tag{B8}
$$

The following notations are accepted in (B6), (B8):

$$
\widehat{\mathcal{Y}}_1 = \frac{3q_2}{4} + \frac{n_2 - n_1}{2}\,, \qquad \widehat{\mathcal{Y}}_2 = -\frac{\widehat{\mathcal{Y}}_1}{2} + \frac{n_3}{2}\,,
\tag{B9}
$$

$$
\widehat{\mathcal{J}}_1 = \frac{3q_2}{2} - \frac{3m_1}{4} + \frac{m_2}{2}\,, \qquad \widehat{\mathcal{J}}_2 = -\widehat{\mathcal{J}}_1 + \frac{m_3}{2}\,, \qquad \widehat{\mathcal{J}}_3 = -\frac{\widehat{\mathcal{J}}_2}{2} + \frac{m_4}{2}\,.
\tag{B10}
$$

Besides, the following notations are valid in (B6) and (B8):

$$
\begin{aligned}
&\mathcal{I}_J \equiv \widetilde{\mathcal{I}}_J - \frac{q_1}{\kappa^2}\log\mathcal{N}\,,\\
&\mathcal{I}_Y \equiv \widetilde{\mathcal{I}}_Y - \frac{q_1}{\kappa^2}\log\mathcal{N}\left(\frac{\log\mathcal{N}}{2} + \log\frac{\gamma}{2}\right),
\end{aligned}
\tag{B11}
$$

where

$$
\begin{aligned}
\widetilde{\mathcal{I}}_J &\equiv \lim_{\epsilon\to 0+}\left(-\int_\epsilon^\infty J_0(\mathcal{N}t)\,G(\kappa t)\,t\,dt \,+\, \frac{q_1}{\kappa^2}\int_\epsilon^1 \frac{dt}{t}\right), \\
\widetilde{\mathcal{I}}_Y &\equiv \lim_{\epsilon\to 0+}\left(-\int_\epsilon^\infty \widetilde{Y}_0(\mathcal{N}t)\,G(\kappa t)\,t\,dt \,+\, \frac{q_1}{\kappa^2}\int_\epsilon^1 \log\!\Big(\frac{\gamma}{2}\mathcal{N}t\Big)\frac{dt}{t}\right).
\end{aligned}
\tag{B12}
$$

Eventually, we subtract (B7) from (B5) and, after an arrangement, obtain $I_{\mathcal{F}}(\rho)$ in the following form:

$$
\begin{aligned}
I_{\mathcal{F}}(\rho) \;&\simeq\; r_0 \,+\, r_1\log\rho \,+\, r_2\log^2\rho \,+\, r_3\,\rho^2\log^3\rho \\
&+\; r_4\,\rho^2\log^2\rho \,+\, r_5\,\rho^2\log\rho \,+\, r_6\,\rho^2 \,+\, r_7\,\rho^4\log^3\rho \,+\, \dots,
\end{aligned}
\tag{B13}
$$

where expressions for r_0, r_1, r_2, and r_3 are given in (4.10). Generally, the coefficients r_4, r_5, r_6, and r_7 appear as follows (though explicitly r_4 and r_7 are also given in (4.10)):

$$
\begin{aligned}
r_4 &\equiv 3\log\mathcal{N}\,\big(\mathcal{Y}_4 - \mathcal{J}_4\big) + \mathcal{Y}_5 - \mathcal{J}_5\,, \\
r_5 &\equiv 3\log^2\mathcal{N}\,\big(\mathcal{Y}_4 - \mathcal{J}_4\big) + 2\log\mathcal{N}\,\big(\mathcal{Y}_5 - \mathcal{J}_5\big) + \mathcal{Y}_6 - \mathcal{J}_6\,, \\
r_6 &\equiv \log^3\mathcal{N}\,\big(\mathcal{Y}_4 - \mathcal{J}_4\big) + \log^2\mathcal{N}\,\big(\mathcal{Y}_5 - \mathcal{J}_5\big) + \log\mathcal{N}\,\big(\mathcal{Y}_6 - \mathcal{J}_6\big) + \mathcal{Y}_7 - \mathcal{J}_7\,, \\
r_7 &\equiv \mathcal{Y}_9 - \mathcal{J}_9 = \frac{\kappa^2 - \mathcal{N}^2}{384}\,p_2\,.
\end{aligned}
\tag{B14}
$$

To express r_4 (as well as r_5 and r_6), we need the differences $\mathcal{Y}_4 - \mathcal{J}_4$ and $\mathcal{Y}_5 - \mathcal{J}_5$:

$$
\begin{aligned}
\mathcal{Y}_4 - \mathcal{J}_4 &= \frac{p_2}{24}\,, \\
\mathcal{Y}_5 - \mathcal{J}_5 &= \Big(1 - \frac{\mathcal{N}^2}{\kappa^2}\Big)\frac{p_1}{32} - \Big(1 + \log\frac{\mathcal{N}}{\kappa}\Big)\frac{p_2}{8} + \frac{p_3}{8}\,.
\end{aligned}
\tag{B15}
$$

However, we leave $\mathcal{Y}_6 - \mathcal{J}_6$ and $\mathcal{Y}_7 - \mathcal{J}_7$ undone:

$$
\begin{aligned}
\mathcal{Y}_6 - \mathcal{J}_6 &= \frac{\mathcal{N}^2}{4}\,\Big(-\mathcal{I}_J + \frac{q_1}{\kappa^2}\Big) + \log\!\Big(\frac{\gamma}{2}\Big)\,\widehat{\mathcal{Y}}_1 + \widehat{\mathcal{Y}}_2 - \widehat{\mathcal{J}}_2\,, \\
\mathcal{Y}_7 - \mathcal{J}_7 &= \frac{\mathcal{N}^2}{4}\,\Big(\mathcal{I}_Y + \Big(1 - \log\frac{\gamma}{2}\Big)\mathcal{I}_J\Big) + \log\!\Big(\frac{\gamma}{2}\Big)\,\widehat{\mathcal{Y}}_2 - \widehat{\mathcal{J}}_3\,.
\end{aligned}
\tag{B16}
$$

The terms $\propto \log^4\rho$ do not appear in (B13), since the differences $\mathcal{Y}_3 - \mathcal{J}_3$ and $\mathcal{Y}_8 - \mathcal{J}_8$ are equal to zero (according to (B4), $m_5 = n_4$). Using q_0 and q_4 (4.6), we determine the dependence of $\mathcal{Y}_6 - \mathcal{J}_6$, $\mathcal{Y}_7 - \mathcal{J}_7$ on *const* as follows:

$$
\mathcal{Y}_6 - \mathcal{J}_6 \,=\, -\kappa^2\,\frac{\widetilde{C}}{4k} + \dots, \qquad \mathcal{Y}_7 - \mathcal{J}_7 \,=\, \kappa^2(q + \log\mathcal{N})\frac{\widetilde{C}}{4k} + \dots, \tag{B17}
$$

where $\widetilde{C}$ and q are determined in (5.11).

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