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Subgroup schemes of a reductive group containing a split maximal torus

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Abstract. In the previous paper, we classified all connected subgroup schemes of a reductive group, that contain a split maximal torus, over an arbitrary field. In the present paper we study the normalizer of the subgroup schemes as above and classify all (not necessarily connected) intermediate subgroup schemes. The answer is given in terms of functions on the root system and subgroups of the Weyl group. We also describe the lattice of intermediate subgroup schemes.

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§ 1 Introduction

In the present paper we continue studying subgroup schemes of a reductive group. In the previous paper [8] we classified connected subgroup schemes in a reductiove group, that contain a split maximal torus. Now we use this result to classify all (not necessarily connected) subgroup schemes containing the torus, ant describe the lattice of intermediate group schemes.

Intermediate subgroups of a reductive groups were discussed in a number of papers. In their fundamental paper [1], A. Borel and J. Tits desribed algebraic subgroups normalized by a split maximal torus. To do this, they introduced the notion of quasi-closed sets. In context of our problem, algebraic subgroups correspond to reduced subgroup schemes.

These results were generalized to groups over some classes of rings, including semi-local ones, by Z. Borewicz, N. Vavilov, and others. On the other hand, G. Seitz described overgroups of all not necessarily split maximal tori over a finite field. Surveys of research in this direction and further references can be found in the papers [11, 10] by N. Vavilov.

In our paper we generalize Borel–Tits theorem in a slightly different direction. Namely, we describe not necessarily reduced subgroup schemes. By Cartier's theorem (see, for example, [3] II §6 no. 1 or [13] 11.4), any affine group scheme over a field of characteristic zero is reduced. So we assume that the characteristic of the base field is positive.

Non-reduced subgroup schemes of reductive groups were considered, in particular, in the papers by Ch. Wenzel (see [14, 15]) and F. Knop (see [5]). Ch. Wenzel classified parabolic subgroup schemes of a reductive group over an algebraically closed field under some mild restriction on characteristic. F. Knop described all subgroup schemes of SL_2.

In the present paper we describe subgroup schemes containing a split maximal torus in a reductive group. In the article [7], addressed this problem to the special case of GL_n. Using some results and ideas from the papers by Z. Borewicz and N. Vavilov (see, for example, [2]), we gave a complete description of overgroup schemes of a maximal tori in GL_n.

The results of the work [8] and of the present paper generalize simultaneously the classical result by A. Borel and J. Tits, the result by Ch. Wenzel on parabolic schemes, and the above result for GL_n. We develop the approach applied to the case of GL_n in [7] and use standard techniques and tools from algebraic geometry: reduction to an algebraically closed field, Frobenius morphism, functorial properties.

The rest of the article is organized as follows. In Section 2, we fix notation related with the structure of reductive groups. In Section 3, we recall necessary results from the paper [8] on connected subgroup schemes containing a split maximal torus and prove a useful lemma on the intersection of these schemes with the normalizer of the torus. Section 4 contains the classification of all (not necessarily connected) subgroup schemes in a reductive group that contain a split maximal torus. In Subsection 4.1 we formulate the main result of the section, Theorem 6. In Subsection 4.2 we construct all intermediate subgroup schemes and study their basic properties. In Subsection 4.3 we reduce to the case of algebraically closed base field and show that all intermediate subgroup schemes are exhausted by constructed ones, and this completes the proof of Theorem 6. Finally, in Section 5, we describe the intersection of two given intermediate subgroup schemes.

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§ 2 Preliminaries

In this section we fix notation and recall necessary results concerning reductive groups. We follow the books [4], [9], and [6] Exp. XXII.

Let G be a reductive group containing a split maximal torus T over a field K. Let Φ be the root system of the pair (G, T). Consider the adjoint representation of G on its Lie algebra L. The T-module L admits a decomposition

$$L = L_0 \oplus (\bigoplus_{\beta \in \Phi} L_\beta),$$

where each root subspace L_β has dimension 1.

For each root $\beta \in \Phi$ there is a homomorphism $x_\beta : G_a \to G$ such that the corresponding tangent map induces an isomorphism of the Lie algebra of the additive group G_a onto L_β. Such a root homomorphism is unique up to multiplication by an element of K^*. For each root β we fix a homomorphism x_β that comes from a homomorphism over $\mathbb{Z}$.

For root subgroups corresponding to roots α and β with $\beta \neq -\alpha$, we have the Chevalley commutator formula

$$[x_\alpha(a), x_\beta(b)] = \prod_{r,s>0} x_{r\alpha+s\beta}(N_{\alpha\beta rs} a^r b^s),$$

where $N_{\alpha\beta rs}$ are integral coefficients. The values of $N_{\alpha\beta rs}$ can be found in the paper [12].

Let $W(\Phi)$ be the Weyl group of the root system Φ. We denote by the same letter an etale group scheme and its group of K-points. So by $W(\Phi)$ we denote the etale group scheme $N(T)/T$ and the finite group $(N(T)/T)(K)$.

§ 3 Connected subgroup schemes containing a split maximal torus

In this section we recall necessary results of the paper [8] and prove a useful lemma on the intersection of the scheme H_φ and the normalizer $N(T)$.

Definition. Let S be a subset of a root system Φ. Suppose that for any two roots α and β in S and positive integers r and s such that $r\alpha + s\beta$ is a root and $p \nmid N_{\alpha\beta rs}$, the root $r\alpha + s\beta$ lies in S. Then the subset S is called *quasi-closed*.

For a quasi-closed set S, by $G(S)$ we denote the subgroup scheme of $G(\Phi)$ generated by T and root subgroups $x_\beta(G_a)$ for all $\beta \in S$.

Lemma 1. (Lemma 3, [8]; proved in [6] Exp. XXII 5.4.3).

$$x_\beta(G_a) \cap G(S) = \begin{cases} x_\beta(G_a), & \text{if } \beta \in S; \\ 1, & \text{otherwise.} \end{cases}$$

Corollary 2. (Corrolary 5, [8]).
The correspondence $S \mapsto G(S)$ gives a bijection between all quasi-closed subsets of Φ and all connected smooth subgroup schemes containing the torus T in $G(\Phi)$.

Let K be a field of characteristic $p > 0$. Let Φ be a root system, $G(\Phi)$ be a reductive group of type Φ over the field K, T be a split maximal torus in $G(\Phi)$.

Theorem 3. (Theorem 1, [8]).
There is a one-to-one correspondence between all **connected** *subgroup schemes of $G(\Phi)$ that contain T, and all functions $\varphi : \Phi \to \mathbb{N} \cup \{0, \infty\}$ satisfying*

$$\varphi(r\alpha + s\beta) \geq \min(\varphi(\alpha) - \log_p r, \varphi(\beta) - \log_p s) \qquad (*)$$

for any roots α, β, $r\alpha + s\beta \in \Phi$ such that $p \nmid N_{\alpha\beta rs}$.

By H_φ we denote the subgroup scheme corresponding to a given function φ. By the construction of the scheme H_φ (see [8]), the reduced subscheme $(H_\varphi)_{\mathrm{red}}$ is $G(S)$, where $S = \{\beta \in \Phi \mid \varphi(\beta) = \infty\}$ is a quasi-closed subset of Φ.

Lemma 4. (Lemma 12, [8]).
$x_\beta(G_a) \cap H_\varphi = x_\beta(\alpha_{p^{\varphi(\beta)}})$.
In particular, all schemes H_φ are distinct.

Lemma 5.
1. $G(S) \cap N(T) = N_{G(S^r)}(T)$.
2. $H_\varphi \cap N(T) = N_{G(S^r)}(T)$,
where $S = \{\beta \in \Phi \mid \varphi(\beta) = \infty\}$.

Proof. 1. The group $G(S)$ admits the decomposition $G(S) = G(S^r)U(S^u)$, where $S^r = \{\beta \in S \mid -\beta \in S\}$, $S^u = \{\beta \in S \mid -\beta \notin S\}$, $U(S^u) = \prod_{\beta \in S^u} x_\beta(G_a)$ is the unipotent radical of $G(S)$, the subgroup $G(S^r)$ normalizes the subgroup $U(S^u)$, and the intersection of $G(S^r)$ and $U(S^u)$ is trivial. In other words, the product $G(S^r)U(S^u)$ is a direct product of varieties.

Consider an element of the intersection $G(S) \cap N(T)$. It equals to a product gh, where $g \in G(S^r)$ and $h \in U(S^u)$. Since the element gh normalizes the torus T, we have $ghTh^{-1}g^{-1} = T$, hence $hTh^{-1} = g^{-1}Tg$. On the other hand, we have the inclusions $hTh^{-1} \subseteq G(S^r)$ and $g^{-1}Tg \subseteq TU(S^u)$. Since the group $G(S^r)$ contains the torus T and the intersection of $G(S^r)$ and $U(S^u)$ is trivial, we have $G(S^r) \cap TU(S^u) = T$. Therefore $hTh^{-1} = g^{-1}Tg = T$, and $g, h \in N(T)$. Since $N(T)$ contains no unipotent element, $h = 1$. Thus $gh \in N_{G(S^r)}(T)$. The first assertion of the lemma is proved.

2. The intersection $H_\varphi \cap N(T)$ contains the torus T and is contained in the normalizer $N(T)$. So the intersection $H_\varphi \cap N(T)$ is the inverse image of some subgroup scheme of the quotient $N(T)/T$ in $N(T)$. Since quotient $N(T)/T$ is etale and the torus T is reduced, the scheme $H_\varphi \cap N(T)$ is reduced. Therefore the

intersection $H_\varphi \cap N(T)$ is contained in the reduced subscheme $(H_\varphi)_{\mathrm{red}} = G(S)$. Thus

$$H_\varphi \cap N(T) = G(S) \cap N(T) = N_{G(S^r)}(T)$$

by the first assertion of the lemma. $\square$

§ 4 Normalizer

In this section we classify arbitrary (not necessarily connected) subgroup schemes of a reductive group that contain a split maximal torus over an arbitrary field.

Let H be a subgroup scheme of $G(\Phi)$ that contains the torus T. The connected component of the scheme H is a connected subgroup scheme of $G(\Phi)$ and contains T, because the torus T is connected. So we apply Theorem 3 and conclude that the connected component of the scheme H is a scheme H_φ for an appropriate function φ satisfying $(*)$. The connected component of any group scheme is a normal subgroup scheme (see [13] 6.7), therefore we have the following inclusions:

$$H_\varphi \leq H \leq N(H_\varphi).$$

Thus the classification of intermediate group schemes is equivalent to the classification of subgroup schemes of the quotients $N(H_\varphi)/H_\varphi$. It turns out that this quotient is an etale group scheme, which is isomorphic to the quotient $N_{W(\Phi)}(\varphi)/W(S^r)$, where $S = \{\alpha \in \Phi \mid \varphi(\alpha) = \infty\}$. Below we formulate this statement in a slightly different way and prove it.

§ 4.1 Statement of Main Theorem

In this section we formulate the main result of the section.

Let K be a field of characteristic $p > 0$. Let Φ be a root system, $G(\Phi)$ be a reductive group of type Φ over the field K, T be a split maximal torus in $G(\Phi)$.

Main Theorem 6. *There is a one-to-one correspondence between all subgroup schemes of $G(\Phi)$ that contain the torus T, and all pairs (W, φ), where the function $\varphi : \Phi \to \mathbb{N} \cup \{0, \infty\}$ satisfies*

$$\varphi(r\alpha + s\beta) \geq \min(\varphi(\alpha) - \log_p r, \varphi(\beta) - \log_p s) \qquad (*)$$

for any roots α, β, $r\alpha + s\beta \in \Phi$ such that $p \nmid N_{\alpha\beta rs}$, and W is a subgroup of the Weyl group $W(\Phi)$ such that W normalizes the function φ and contains all reflections w_α with $\varphi(\alpha) = \varphi(-\alpha) = \infty$.

Remarks. 1. A connected intermediate subgroup scheme H_φ corresponds to the pair $(W(S^r), \varphi)$, where $S = \{\beta \in \Phi \mid \varphi(\beta) = \infty\}$. The normalizer $N(H_\varphi)$ corresponds to the pair $(N_{W(\Phi)}(\varphi), \varphi)$.

2. The description of intermediate subgroup schemes in GL_n, e.g. the equations for them, can be found in the paper [7].

§ 4.2 Construction of intermediate group schemes

Let (W, φ) be a pair satisfying the conditions of Theorem 6, i.e. the function $\varphi : \Phi \to \mathbb{N} \cup \{0, \infty\}$ satisfies the inequality $(*)$ and W is a subgroup of the Weyl group $W(\Phi)$ such that W normalizes the function φ and contains $W(S^r)$, where $S = \{\alpha \in \Phi \mid \varphi(\alpha) = \infty\}$. Thus the condition for W can be written in the form

$$W(S^r) \leq W \leq N_{W(\Phi)}(\varphi).$$

Recall that, by definition, $W(\Phi)$ is the quotient $N(T)/T$. By $\widetilde{W}$ we denote the subgroup scheme of $N(T)$ that projects onto W, i.e. the inverse image of W in $N(T)$.

Lemma 7. *A subgroup W of the Weyl group $W(\Phi)$ normalizes the function φ if and only if the corresponding subgroup scheme $\widetilde{W}$ normalizes the scheme H_φ.*

Proof. Suppose that W normalizes φ. Let $\tilde{w}$ be an arbitrary element of $\widetilde{W}(K)$. Consider the scheme $H' = \tilde{w} H_\varphi \tilde{w}^{-1}$. We claim that H' is a connected subgroup scheme of $G(\Phi)$. Furthermore, H' contains T, because $\tilde{w}$ belongs to $N(T)$. Thus, by Theorem 3, $H' = H_\psi$ for some function ψ satisfying $(*)$.

Consider the intersection of H' and a root subgroup $x_\beta(G_a)$. We have the following relations:

$$H' \cap x_\beta(G_a) = \tilde{w} H_\varphi \tilde{w}^{-1} \cap x_\beta(G_a) = \tilde{w}(H_\varphi \cap \tilde{w}^{-1} x_\beta(G_a) \tilde{w}) \tilde{w}^{-1}.$$

Furthermore,

$$\tilde{w}^{-1} x_\beta(G_a) \tilde{w} = x_{w^{-1}(\beta)}(G_a),$$

where w is the image of $\tilde{w}$ in $W \leq N(T)(K)/T(K)$. By Lemma 4, we have

$$H_\varphi \cap x_{w^{-1}(\beta)}(G_a) = x_{w^{-1}(\beta)}(\alpha_{p^{\varphi(w^{-1}(\beta))}}).$$

Since w normalizes φ, we have $\varphi(w^{-1}(\beta)) = \varphi(\beta)$. Therefore

$$H' \cap x_\beta(G_a) = \tilde{w} x_{w^{-1}(\beta)}(\alpha_{p^{\varphi(\beta)}}) \tilde{w}^{-1} = x_\beta(\alpha_{p^{\varphi(\beta)}}).$$

On the other hand, by Lemma 4, we have

$$H' \cap x_\beta(G_a) = x_\beta(\alpha_{p^{\psi(\beta)}}).$$

Thus $\varphi = \psi$ and $\tilde{w}$ normalizes H_φ.

By the criteria for elements to be in the normalizer (see [3] II, §1, no 1, prop. 3.5), this means that any element of the group $\widetilde{W}(K)$ belongs to the normalizer of the torus T. Since the scheme $\widetilde{W}$ is reduced, it follows that $\widetilde{W} \leq N(H_\varphi)$, i.e. $\widetilde{W}$ normalizes H_φ.

Now we prove the other implication. Suppose that $\widetilde{W}$ normalizes H_φ. In particular, for any element $w \in W$, its inverse image $\tilde{w} \in \widetilde{W}$ satisfies the relation $\tilde{w} H_\varphi \tilde{w}^{-1} = H_\varphi$. The calculations as above show that $\varphi(w^{-1}(\beta)) = \varphi(\beta)$, i.e. the element w normalizes the function φ. Therefore the group W normalizes the function φ. The lemma is proved. $\square$

By the previous lemma, the inclusions for the group W are equivalent to

$$N_{G(S^r)}(T) \leq \widetilde{W} \leq N(T) \cap N(H_\varphi).$$

Define the group scheme $H_{W,\varphi}$ corresponding to the pair (W, φ) as the product $\widetilde{W} H_\varphi$. The group scheme $\widetilde{W}$ normalizes H_φ, therefore $H_{W,\varphi}$ is a subgroup scheme of $G(\Phi)$. The scheme $H_{W,\varphi}$ contains the torus T, because H_φ does. Thus the subgroup scheme $H_{W,\varphi}$ satisfies the conditions of Theorem 6.

Lemma 8.
1. The connected component of $H_{W,\varphi}$ is H_φ.
2. $\widetilde{W} = N(T) \cap H_{W,\varphi}$.

Proof. 1. The scheme H_φ is connected. Furthermore, H_φ is a normal subgroup scheme of $H_{W,\varphi}$ by the definition of the scheme $H_{W,\varphi}$. So it is enough to prove that the quotient $H_{W,\varphi}/H_\varphi$ is an etale group scheme.

Note that we have an isomorphism

$$H_{W,\varphi}/H_\varphi \cong \widetilde{W}/(\widetilde{W} \cap H_\varphi).$$

Furthermore, by Lemma 4,

$$\widetilde{W}/(\widetilde{W} \cap H_\varphi) = \widetilde{W}/N_{G(S^r)}(T) \cong W/W(S^r),$$

where $S = \{\beta \in \Phi \mid \varphi(\beta) = \infty\}$. Since W is etale, the quotient $H_{W,\varphi}/H_\varphi$ is etale.

2. By definition, we have $H_{W,\varphi} = \widetilde{W} H_\varphi$. Since $\widetilde{W}$ is contained in $N(T)$, the following relations hold:

$$H_{W,\varphi} \cap N(T) = \widetilde{W} H_\varphi \cap N(T) = \widetilde{W} \cdot (H_\varphi \cap N(T)).$$

Now we apply Lemma 5 and get

$$\widetilde{W} \cdot (H_\varphi \cap N(T)) = \widetilde{W} N_{G(S^r)}(T).$$

Since $\widetilde{W}$ contains $N_{G(S^r)}(T)$, we get $\widetilde{W} N_{G(S^r)}(T) = \widetilde{W}$, and this completes the proof. $\square$

§ 4.3 Description of intermediate group schemes

In this section we prove that any intermediate group scheme is $H_{W,\varphi}$ for an appropriate pair (W, φ) satisfying the conditions of Theorem 6.

First of all, we reduce the general situation to the case of algebraically closed field. For this, we use the following lemma.

Lemma 9. (Lemma 13, [8]). *Let H and H' be closed subschemes of any algebraic affine scheme G over a field K. If the schemes $H_{\overline{K}}$ and $H'_{\overline{K}}$ coincide as subschemes of the scheme $G_{\overline{K}}$ over the algebraic closure $\overline{K}$, then the schemes H and H' coincide.*

Now we assume that the field K is algebraically closed.

Consider an arbitrary (not necessarily connected) subgroup scheme H of $G(\Phi)$ that contains the torus T. The connected component of H is a connected subgroup scheme of $G(\Phi)$ and contains T, because the torus T is connected. Therefore we apply Theorem 3 and conclude that the connected component of H is equal to the subscheme H_φ for some function φ satisfying $(*)$. Recall that the connected component is a normal subgroup scheme (see [13] 6.7), i.e. H_φ is normal in H. By $\widetilde{W}$ we denote the intersection of the subgroup schemes H and $N(T)$. Define W as the image of $\widetilde{W}$ in the quotient $N(T)/T = W(\Phi)$. By Lemma 7, the group W normalizes the function φ.

Lemma 10.
1. $H(K) = \widetilde{W}(K)H_\varphi(K)$.
2. $H = \widetilde{W}H_\varphi$

Proof. 1. Let g be an element of $H(K)$. Then $g^{-1}T(K)g$ is a split maximal torus. Since $H_\varphi(K)$ is a normal subgroup of $H(K)$ and contains $T(K)$, the torus $g^{-1}T(K)g$ is contained in $H_\varphi(K)$. All split maximal tori in an algebraic group over an algebraically closed field are conjugate (see [13] 10.5). Therefore the tori $g^{-1}T(K)g$ and $T(K)$ are conjugate in the group $H_\varphi(K)$. In other words, $h^{-1}g^{-1}T(K)gh = T(K)$ for some element $h \in H_\varphi(K)$. So the element $\tilde{w} = gh$ belongs to the normalizer $N(T)$ of the torus T. On the other hand, $\tilde{w} \in H(K)$. Therefore we have $\tilde{w} \in (H \cap N(T))(K) = \widetilde{W}(K)$. Thus the element $g \in H(K)$ is equal to the product $\tilde{w}h^{-1}$, where $\tilde{w} \in \widetilde{W}(K)$ and $h^{-1} \in H_\varphi(K)$, i.e. the group $H(K)$ is equal to the product $\widetilde{W}(K)H_\varphi(K)$.

2. The quotient map $H \to \pi_0 H$ induces a map $\widetilde{W} \to \pi_0 H$. We know that the homomorphism $H(K) \to \pi_0 H(K)$ is surjective, the group $H_\varphi(K)$ is the kernel of this homomorphism, and, by the first assertion of the lemma, $H(K) = \widetilde{W}(K)H_\varphi(K)$. Therefore the homomorphism $\widetilde{W}(K) \to \pi_0 H(K)$ is surjective. Since the group schemes $\widetilde{W}$ and $\pi_0 H$ are reduced, this means that $\widetilde{W} \to \pi_0 H$ is a quotient map.

The subgroup scheme H_φ is the kernel of the quotient map $H \to \pi_0 H$, the subgroup scheme $\widetilde{W}$ maps onto the image of this map, therefore the product $\widetilde{W}H_\varphi$ is the whole group scheme H. The assertion is proved. $\square$

Proof of Main Theorem 6. In Subsection § 4.2, we defined a scheme $H_{W,\varphi}$ that corresponds to a given pair (W, φ) satisfying the conditions of Main Theorem 6. By the construction, the scheme $H_{W,\varphi}$ is a subgroup scheme of $G(\Phi)$ and contains the torus T. By Lemma 8, the connected component of the scheme $H_{W,\varphi}$ is the subscheme H_φ and the scheme $H_{W,\varphi}$ satisfies the relation $(N(T) \cap H_{W,\varphi})/T = W$. In particular, this means that all schemes $H_{W,\varphi}$ are distinct.

It remains to prove that any subgroup scheme H of $G(\Phi)$, that contains the torus T, is the scheme $H_{W,\varphi}$ for an appropriate pair (W, φ) satisfying the conditions of Main Theorem 6. By Lemma 9, we may assume that the base field K is algebraically closed. Under this assumption, we determined the pair (W, φ) corresponding to a given intermediate group scheme H, so that the function φ satisfies the condition $(*)$ by Theorem 3, the group W contains $W(S^r)$ by Lemma 5 and normalizes φ by Lemma 7. Finally, it follows from Lemma 10 that $H = H_{W,\varphi}$. The theorem is proved. $\square$

§ 5 Intersections

In this section we study the lattice of the intermediate group schemes classified in the previous sections. Proposition 12, which is the main result of this section, generalizes the following well-known result.

Lemma 11. *$(G(S) \cap G(S'))^0 = G(S \cap S')$.*

Proof. See [6] Exp. XXII 5.4.5 or [1] Prop. 3.22. $\square$

Proposition 12.
1. $(H_\varphi \cap H_{\varphi'})^0 = H_{\min(\varphi,\varphi')}$.
2. $H_{W,\varphi} \cap H_{W',\varphi'} = H_{W \cap W',\min(\varphi,\varphi')}$.

Proof. 1. The scheme $(H_\varphi \cap H_{\varphi'})^0$ is a connected subgroup scheme of $G(\Phi)$. Moreover, the schemes H_φ and $H_{\varphi'}$ contain the torus T and the torus is connected, therefore the connected component $(H_\varphi \cap H_{\varphi'})^0$ contains the torus T. So we apply Theorem 3 and conclude that $(H_\varphi \cap H_{\varphi'})^0 = H_\psi$ for some function ψ satisfying the condition $(*)$.

Now consider the intersections of the schemes H_φ, $H_{\varphi'}$, and H_ψ with root subgroups. By Lemma 4, we have the equalities $x_\beta(G_a) \cap H_\varphi = x_\beta(\alpha_{p^{\varphi(\beta)}})$, $x_\beta(G_a) \cap H_{\varphi'} = x_\beta(\alpha_{p^{\varphi'(\beta)}})$, and $x_\beta(G_a) \cap H_\psi = x_\beta(\alpha_{p^{\psi(\beta)}})$. Since any scheme α_{p^n} is connected and $(H_\varphi \cap H_{\varphi'})^0 = H_\psi$, we obtain $\psi = \min(\varphi, \varphi')$.

The first assertion of the proposition is proved.

2. First of all, the intersection $H_{W,\varphi} \cap H_{W',\varphi'}$ is a subgroup scheme of $G(\Phi)$ and contains the torus T. Therefore, by Main Theorem 6, $H_{W,\varphi} \cap H_{W',\varphi'} = H_{W'',\varphi''}$ for some pair (W'', φ'').

Consider the connected component of the intersection $H_{W,\varphi} \cap H_{W',\varphi'}$. It is the maximal connected subgroup scheme that is contained in the intersection $H_{W,\varphi} \cap H_{W',\varphi'}$, therefore we have

$$(H_{W,\varphi} \cap H_{W',\varphi'})^0 \subseteq (H_{W,\varphi})^0 \cap (H_{W',\varphi'})^0 = H_\varphi \cap H_{\varphi'}$$

and $(H_{W,\varphi} \cap H_{W',\varphi'})^0 \subseteq (H_\varphi \cap H_{\varphi'})^0$. On the other hand, we have the inclusion $(H_\varphi \cap H_{\varphi'}) \subseteq (H_{W,\varphi} \cap H_{W',\varphi'})$, therefore $(H_\varphi \cap H_{\varphi'})^0 \subseteq (H_{W,\varphi} \cap H_{W',\varphi'})^0$. So the connected components of the subgroup schemes $H_\varphi \cap H_{\varphi'}$ and $H_{W,\varphi} \cap H_{W',\varphi'}$ coincide. By the first assertion of the proposition, $(H_\varphi \cap H_{\varphi'})^0 = H_{\min(\varphi,\varphi')}$. Thus we get $(H_{W,\varphi} \cap H_{W',\varphi'})^0 = H_{\min(\varphi,\varphi')}$ and $\varphi'' = \min(\varphi, \varphi')$.

By the construction of intermediate group schemes, $W'' = (N(T) \cap H_{W'',\varphi''})/T$. On the other hand,

$$(N(T) \cap H_{W'',\varphi''})/T = ((N(T) \cap H_{W,\varphi}) \cap (N(T) \cap H_{W',\varphi'}))/T = W \cap W'.$$

Therefore $W'' = W \cap W'$. The proposition is proved. $\square$

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