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DYNAMICAL INVERSE PROBLEM FOR A LAMÉ TYPE SYSTEM

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Abstract

The system under consideration is governed by the equation $u_{tt} = \nabla\kappa\text{div}u - \text{rot}\mu\text{rot}u$ in $\Omega \times (0,T)$; its response operator ("input $\mapsto$ output" map) R^T plays the role of the inverse data. It is shown that R^{2T} determines $\kappa\mid_{\Omega_p^T}$ and $\mu\mid_{\Omega_s^T}$, where Ω_p^T and Ω_s^T are the subdomains of Ω filled (at the moment T) with $p-$ and $s-$waves propagating from $\partial\Omega$ with velocities $c_p = \sqrt{\kappa}$ and $c_s = \sqrt{\mu}$ correspondingly. Due to the wave splitting $u = \nabla p + \text{rot}s$ the problem is reduced to the inverse problems for the acoustical and Maxwell subsystems governed by the equations $p_{tt} = \kappa\Delta p$ and $s_{tt} = -\mu\text{rotrot}s$ with the response operators R_p^{2T} and R_s^{2T} determined by R^{2T}. The first problem can be solved by the BC method (Belishev, 1986), the second one is solved by a version of the method based on the blow up effect. This version is the main subject of the paper.

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1 Introduction.

1.1 The Lamé type system

Let $\Omega \subset \mathbf{R^3}$ be a bounded domain with the smooth [2] connected boundary $\Gamma := \partial\Omega$. In its general form the Lamé equation written by components in the Cartesian coordinates is

$$\rho(u_i)_{tt} = (\lambda+\mu)\frac{\partial}{\partial x_i}\mathrm{div}u + \mu\Delta u + \frac{\partial\lambda}{\partial x_i}\mathrm{div}u + \sum_{k=1}^{3}\frac{\partial\mu}{\partial x_k}\Big(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}\Big) \quad (1.0)$$

with smooth variable coefficients provided $\rho > 0, \mu > 0, 3\lambda + 2\mu > 0$ in $\overline{\Omega}$. Assuming $\mu = \mathrm{const}$ we have

$$\rho u_{tt} = \nabla(\lambda+2\mu)\mathrm{div}u - \mathrm{rot}\mu\mathrm{rot}u\,;$$

taking $\rho = 1$ and admitting μ to be variable we arrive at a *model* (Lamé type) dynamical system α^T of the form

$$u_{tt} = \nabla\kappa\mathrm{div}u - \mathrm{rot}\mu\mathrm{rot}u \qquad \text{in } Q^T \quad (1.1)$$

$$u\,|_{t=0} = u_t\,|_{t=0} = 0 \qquad \text{in } \Omega \quad (1.2)$$

$$u = f \qquad \text{on } \Sigma^T. \quad (1.3)$$

It is the system we deal with in the paper; here $\kappa := \lambda + 2\mu,\ Q^T := \Omega \times (0,T),\ \Sigma^T := \Gamma \times (0,T)$; f is a boundary control, u^f is a solution (wave). The system is equipped with the outer space of controls

$$\mathcal{F}^T := L_2(\Sigma^T; \mathbf{R^3})$$

and the inner space of states

$$\mathcal{H} := L_2(\Omega; \mathbf{R^3})\ ;$$

the solution is considered as an $\mathcal{H}$–valued function of time: $u^f(\cdot,t) \in \mathcal{H},\ t \in [0,T]$. For controls of the class

$$\mathcal{M}^T := \Big\{f \in C^\infty(\Sigma^T; \mathbf{R^3}) \mid f = 0 \ \text{ near } \Gamma \times \{0\}\Big\}$$

the solution is classical: $u^f \in C^\infty(\overline{Q}^T; \mathbf{R^3})$.

[2]Everywhere in the paper "smooth" means C^∞–smooth.

1.2 Green formula

Fix $\gamma \in \Gamma$; a vector a at γ can be represented as

$$a \;=\; a^\nu \nu + a_\theta \tag{1.4}$$

where $\nu = \nu(\gamma)$ is an outward normal, $a^\nu = a \cdot \nu, \quad a_\theta = a - a^\nu \nu$ is a tangent component, whereas the inner product can be written in the form

$$a \cdot b \;=\; \begin{pmatrix} a^\nu \\ a_\theta \end{pmatrix} \cdot \begin{pmatrix} b^\nu \\ b_\theta \end{pmatrix} \;=\; a^\nu b^\nu + a_\theta b_\theta \ .$$

Denote $\mathcal{G} := L_2(\Gamma; \mathbf{R^3})$ and introduce the operator

$$L := \nabla \kappa \mathrm{div} - \mathrm{rot} \mu \mathrm{rot}$$

governing the evolution of the system α^T. Integrating by parts for smooth u, v one has

$$(Lu, v)_{\mathcal{H}} - (u, Lv)_{\mathcal{H}} = (Nu, Dv)_{\mathcal{G}} - (Du, Nv)_{\mathcal{G}} =$$

$$= \int_\Gamma \mathrm{d}\gamma \Big[\begin{pmatrix} \kappa \mathrm{div} u \\ \mu \Phi(\mathrm{rot} u)_\theta \end{pmatrix} \cdot \begin{pmatrix} v^\nu \\ v_\theta \end{pmatrix} \;-\; \begin{pmatrix} u^\nu \\ u_\theta \end{pmatrix} \cdot \begin{pmatrix} \kappa \mathrm{div} v \\ \mu \Phi(\mathrm{rot} v)_\theta \end{pmatrix} \Big] \tag{1.5}$$

where Φ transforms the tangent vectors by the rule $\Phi a = -\nu \times a$ (turns a onto the angle $-\frac{\pi}{2}$ with respect to ν) and

$$Du := \begin{pmatrix} u^\nu \\ u_\theta \end{pmatrix}, \quad Nu := \begin{pmatrix} \kappa \mathrm{div} u \\ \mu \Phi(\mathrm{rot} u)_\theta \end{pmatrix}$$

on Γ.

1.3 Response operator

The form of the response operator R^T ("input $\mapsto$ output" map of the system α^T) is imposed by the choice of the boundary operators D and N: it acts in $\mathcal{F}^T$ on Dom $= \mathcal{M}^T$ by the rule $Du^f \mapsto Nu^f$, i.e.,

$$R^T f := \begin{pmatrix} \kappa \mathrm{div} u^f \\ \mu \Phi(\mathrm{rot} u^f)_\theta \end{pmatrix} \qquad \text{on } \Sigma^T.$$

Note that R^T can be extracted from measurements of the stress appearing at Γ in result of interactions of the waves with the boundary.

1.4 Main result

In the inverse problem the response operator plays the of the data for determination of the coefficients κ and μ into Ω. To describe the domains of determination we introduce the metrics

$$\mathrm{d}\tau_p^2 := \frac{|\mathrm{d}x|^2}{\kappa}, \quad \mathrm{d}\tau_s^2 := \frac{|\mathrm{d}x|^2}{\mu}$$

and denote $\tau_p(x,y)$, $\tau_s(x,y)$ the corresponding distances in Ω. Also, we set

$$\tau_p(x) := \tau_p(x,\Gamma), \quad \tau_s(x) := \tau_s(x,\Gamma) \qquad x \in \overline{\Omega};$$

$$T_p := \max_\Omega \tau_p(\cdot), \quad T_s := \max_\Omega \tau_s(\cdot)$$

and

$$\Omega_p^T := \{x \in \overline{\Omega} \mid \tau_p(x) < T\}, \ \Omega_s^T := \{x \in \overline{\Omega} \mid \tau_s(x) < T\}\ .$$

We impose the following implicit condition on the coefficient μ (on the metric $\mathrm{d}\tau_s$):

(A) for each $T \in (0, T_s)$ excluding, may be, a finite set $T_1, T_2, \dots, T_m$ the interior part $\Gamma_s^T := \partial\Omega_s^T \setminus \Gamma$ of the boundary of the subdomain Ω_s^T consists of a finite number of piece wise smooth connected components.

Note that smoothness of Γ_s^T can be broken only at the so-called separation set (cut locus) ω_s of the domain Ω with respect to its boundary. This set determined by the metric $\mathrm{d}\tau_s^2$ (see e.g. [2]) is "small": ω_s is closed subset of zero volume, separated from Γ and determined by the metric $\mathrm{d}\tau_s^2$; the complement $\Omega \setminus \omega_s$ is connected. The main result of the paper is

Theorem 1 *Let $T > 0$ be fixed and (A) be fulfilled. The operator R^{2T} determines $\kappa \mid_{\Omega_p^T}$ and $\mu \mid_{\Omega_s^T}$ uniquely.*

1.5 Remarks

The proof of Theorem 1 is constructive: we propose a sampling algorithm based on the blow up effect which recovers κ and μ. More exactly, splitting the waves by $u = \nabla p + \mathrm{rot} s$, we select the acoustical and Maxwell subsystems α_p^T and α_s^T of the system α^T and reduce the inverse problem to the ones for the subsystems. The sampling algorithm for the system α_p^T governed by the scalar wave equation $p_{tt} = \kappa\Delta p$ has been proposed in [1] and described in [2] in detail; therefore we restrict ourselves by referring to the latter paper. The Maxwell part is governed by the equation $s_{tt} = -\mu \mathrm{rot} \mathrm{rot} s$ and the version of the algorithm recovering μ, in fact, is the main subject of our paper.

The procedures recovering κ and μ are local with respect to time: the longer is the interval of observation $[0, T]$, the bigger is the domain of determination. Moreover, such a determination is *time-optimal*: $R^{2T'}$ with $T' < T$ doesn't determine $\kappa \mid_{\Omega_p^T}$ and $\mu \mid_{\Omega_s^T}$ uniquely in principle. This feature is the main advantage of the BC method.

In view of evident importance of the inverse problem for theory and applications (acoustics, geophysics, nondestructive control etc) the literature devoted to the Lamé system is hardly observable. In the important case of the half space and coefficients depending on one coordinate (the layered media) the problem can be reduced to the 1d problems and treated by the local method of A.S.Blagoveschenskii in [7],[8] Our list of references is so short by the only reason: to the best of our knowledge, the *time–optimal* procedures for 3d case haven't been proposed yet.

2 Wave splitting

2.1 $p-$waves

Taking $f \in \mathcal{M}^T$ and applying $\kappa\mathrm{div}$ to (1.1) one gets

$$\kappa \mathrm{div} u^f \;=\; \kappa \Delta \kappa \mathrm{div} u^f$$

and arrives at the system α_p^T of the form

$$p_{tt} = c_p^2 \Delta p \qquad \text{in } Q^T \tag{2.1}$$

$$p \mid_{t=0} = p_t \mid_{t=0} = 0 \qquad \text{in } \Omega \tag{2.2}$$

$$p = g \qquad \text{on } \Sigma^T \tag{2.3}$$

($c_p := \sqrt{\mu}$) with the solution $p = p^g(x, t)$ connected with u^f through the relations

$$p^g = \kappa \mathrm{div} u^f, \quad g = \kappa \mathrm{div} u^f \mid_{\Sigma^T} = (R^T f)^\nu. \tag{2.4}$$

We call α_p^T the acoustical subsystem of the system α^T and associate with it the outer space of the (scalar) controls

$$\mathcal{F}_p^T := L_2(\Sigma^T) = \nu \cdot \mathcal{F}^T$$

and the response operator $R_p^T : \mathcal{F}_p^T \to \mathcal{F}_p^T$ defined on

$$\mathrm{Dom} R_p^T = \mathcal{M}_p^T := \left\{ g \in C^\infty(\Sigma^T) \mid g = 0 \quad \text{near } \Gamma \times \{0\} \right\} = \nu \cdot \mathcal{M}^T$$

by the rule

$$R_p^T g := \frac{\partial p^g}{\partial \nu} \qquad \text{on } \ \Sigma^T.$$

2.2 $s-$ waves

Applying μrot to (1.1) we get

$$(\mu \mathrm{rot} u^f)_{tt} = - \ \mu \mathrm{rot} \mathrm{rot} \mu \mathrm{rot} u^f$$

and arrive at the system α_s^T of the form

$$s_{tt} = -c_s^2 \mathrm{rot} \mathrm{rot} s \qquad \text{in } Q^T \tag{2.5}$$

$$s \mid_{t=0} = s_t \mid_{t=0} = 0 \qquad \text{in } \Omega \tag{2.6}$$

$$\Phi s_\theta = h \qquad \text{on } \Sigma^T \tag{2.7}$$

($c_s := \sqrt{\mu}$) with the solution $s = s^h(x,t)$ connected with u^f through the relations

$$s^h = \mu \mathrm{rot} u^f, \quad h = \mu \Phi (\mathrm{rot} u^f)_\theta \mid_{\Sigma^T} = (R^T f)_\theta \ . \tag{2.8}$$

We say α_s^T to be the Maxwell subsystem of the system α^T and associate with it the outer space of controls

$$\mathcal{F}_s^T \ := \ \{ L_2(\Sigma^T; \mathbf{R^3}) \mid h \cdot \nu = 0 \quad \text{on } \Gamma \} = (\mathcal{F}^T)_\theta$$

(so that

$$\mathcal{F}^T = (\mathcal{F}_p^T) \ \nu \oplus \mathcal{F}_s^T \tag{2.9}$$

in accordance with (1.4)) and the response operator $R_s^T : \mathcal{F}_s^T \to \mathcal{F}_s^T$ defined on

$$\mathrm{Dom} R_s^T \ = \ \mathcal{M}_s^T :=$$

$$\left\{ h \in C^\infty(\Sigma^T; \mathbf{R^3}) \mid h = 0 \ \text{ near } \Gamma \times \{0\}, \ h \cdot \nu = 0 \ \text{ on } \Gamma \right\} = (\mathcal{M}^T)_\theta$$

by the rule

$$R_s^T h := (\mathrm{rot} s^h)_\theta \qquad \text{on } \Sigma^T.$$

2.3 Splitting R^T

The following result plays the key role in solving the inverse problem: it shows that the response operator of the system α^T determines the ones of the subsystems α_p^T, α_s^T explicitly. In the formulation we use the auxiliary operators:

the tangent gradient $\nabla_\theta : \mathcal{F}_p^T \to \mathcal{F}_s^T$, $\ \mathrm{Dom}\nabla_\theta = \mathcal{M}_p^T$,

$$\nabla_\theta \varphi := (\nabla \tilde{\varphi})_\theta$$

where $\tilde{\varphi}$ is (any) smooth continuation of φ from Γ onto an $\mathbf{R^3}$- neighborhood of Γ;

the tangent divergence $\mathrm{div}_\theta : \mathcal{F}_s^T \to \mathcal{F}_p^T$, $\ \mathrm{div}_\theta := -(\nabla_\theta)^*$; as is known,

$$\nu \cdot \mathrm{rot}\psi = -\mathrm{div}_\theta\ \nu \times \psi_\theta = \mathrm{div}_\theta \Phi \psi_\theta \qquad \text{on } \Gamma \tag{2.10}$$

for a ψ smooth near Γ;

the integration $J^T : \mathcal{F}^T \to \mathcal{F}^T$,

$$(J^T f)(\cdot, t) := \int_0^t f(\cdot, \eta)\ \mathrm{d}\eta\ , \quad t \in [0, T]\ .$$

Also, we use the matrix notations coordinated with decompositions (1.4) and (2.9).

Lemma 1 *The relations*

$$\mathrm{Ker} R^T = \{0\}\ , \quad \mathrm{Ran} R^T = \mathcal{M}^T$$

and the representation

$$(R^T)^{-1} = (J^T)^2 \begin{pmatrix} R_p^T & -\mathrm{div}_\theta \\ \nabla_\theta & -R_s^T \end{pmatrix} \tag{2.11}$$

hold for any $T > 0$.

Proof For $f \in \mathrm{Ker} R^T$ one has

$$g := \kappa \mathrm{div} u^f = (R^T f)^\nu = 0, \quad h := \mu \Phi (\mathrm{rot} u^f)_\theta = (R^T f)_\theta = 0 \quad \text{on } \Sigma^T$$

that implies $p^g = \kappa \mathrm{div} u^f = 0$ in Q^T by virtue of uniqueness of solution of the problem (2.1)–(2.3), and $s^h = \mu \mathrm{rot} u^f = 0$ in Q^T by uniqueness for (2.5)–(2.7). Hence, $\mathrm{div} u^f(\cdot, t) = 0$ and $\mathrm{rot} u^f(\cdot, t) = 0$ for all $t \in [0, T]$ that easily leads to $u^f = 0$ and, then, to $f = 0$. Thus, $\mathrm{Ker} R^T = |0|$.

The embedding $\mathrm{Ran} R^T \subset \mathcal{M}^T$ follows from the smoothness of u^f. Let us verify that $\mathrm{Ran} R^T \supset \mathcal{M}^T$.

Take arbitrary $g \in \mathcal{M}_p^T$; let p^g be the solution of (2.1)–(2.3). Applying ∇ to (2,1) it is easy to see that ∇p^g satisfies the equation

$$(\nabla p^g)_{tt} = L \nabla p^g \qquad \text{in } Q^T$$

and the conditions

$$\nabla p^g \mid_{t=0} = (\nabla p^g)_t \mid_{t=0} = 0 \qquad \text{in } \Omega.$$

Hence, ∇p^g is the solution of (1.1)–(1.3): $\nabla p^g = u^{f'}$ corresponding to the control

$$f' = u^{f'} \mid_{\Sigma^T} = \begin{pmatrix} u^{f'} \cdot \nu \\ (u^{f'})_\theta \end{pmatrix} = \begin{pmatrix} \nabla p^g \cdot \nu \\ (\nabla p^g)_\theta \end{pmatrix} =$$
$$= \begin{pmatrix} \frac{\partial p^g}{\partial \nu} \\ \nabla_\theta p^g \end{pmatrix} = \begin{pmatrix} R_p^T g \\ \nabla_\theta g \end{pmatrix}.$$

At the same time, by the definition of the response operator one has

$$R^T f' = R^T \begin{pmatrix} R_p^T g \\ \nabla_\theta g \end{pmatrix} = \begin{pmatrix} \kappa \mathrm{div} \nabla p^g \\ \mu \Phi (\mathrm{rot} \nabla p^g)_\theta \end{pmatrix} =$$

$$= \begin{pmatrix} \kappa \Delta p^g \\ 0 \end{pmatrix} = \langle \text{see } (2.1) \rangle = \begin{pmatrix} p_{tt} \\ 0 \end{pmatrix} = \begin{pmatrix} g_{tt} \\ 0 \end{pmatrix}.$$

Integrating on time with regards to the commutation $R^T J^T = J^T R^T$ and the initial data (2.2) one has

$$R^T (J^T)^2 \begin{pmatrix} R_p^T g \\ \nabla_\theta g \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}. \tag{2.12}$$

Take arbitrary $h \in \mathcal{M}_s^T$, let s^h be the solution of (2.5)–(2.7). Applying rot to (2.5) it is easy to see that $\mathrm{rot} s^h$ satisfies the equation

$$(\mathrm{rot} s^h)_{tt} = L \mathrm{rot} s^h \qquad \text{in } Q^T$$

and the conditions

$$\mathrm{rot} s^h \mid_{t=0} = (\mathrm{rot} s^h)_t \mid_{t=0} = 0 \qquad \text{in } \Omega\ .$$

Hence, $\mathrm{rot}s^h$ is the solution of (1.1)–(1.3): $\mathrm{rot}s^h = u^{f''}$ corresponding to the control

$$f'' = u^{f''}\mid_{\Sigma^T} = \begin{pmatrix} u^{f''}\cdot\nu \\ (u^{f''})_\theta \end{pmatrix} = \begin{pmatrix} \nu\cdot\mathrm{rot}s^h \\ (\mathrm{rot}s^h)_\theta \end{pmatrix} =$$

$$\langle\text{see (2.10)}\rangle = \begin{pmatrix} \mathrm{div}_\theta\Phi s^h_\theta \\ (\mathrm{rot}s^h)_\theta \end{pmatrix} = \begin{pmatrix} \mathrm{div}_\theta h \\ R^T_s h \end{pmatrix}.$$

At the same time, by the definition of the response operator one has

$$R^T f'' = R^T \begin{pmatrix} \mathrm{div}_\theta h \\ R^T_s h \end{pmatrix} = \begin{pmatrix} \kappa\mathrm{div}\mathrm{rot}s^h \\ \mu\Phi(\mathrm{rot}\mathrm{rot}s^h)_\theta \end{pmatrix} =$$

$$\langle\text{see (2.5)}\rangle = \begin{pmatrix} 0 \\ -\Phi(s^h_{tt})_\theta \end{pmatrix} = -\begin{pmatrix} 0 \\ h_{tt} \end{pmatrix}.$$

Integrating on time with regards to (2.6) one has

$$R^T(J^T)^2 \begin{pmatrix} \mathrm{div}_\theta h \\ R^T_s h \end{pmatrix} = -\begin{pmatrix} 0 \\ h \end{pmatrix}. \tag{2.13}$$

Joining (2.12) with (2.13) we arrive at the relation

$$R^T(J^T)^2 \begin{pmatrix} R^T_p & -\mathrm{div}_\theta \\ \nabla_\theta & -R^T_s \end{pmatrix}\begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix}$$

that is equivalent to (2.13) by arbitrariness of g and h. The lemma is proved.

The straightforward consequence of (2.11) is the relations

$$R^T_p g = \frac{\partial^2}{\partial t^2}[R^T]^{-1}\begin{pmatrix} g \\ 0 \end{pmatrix}; \quad R^T_s h = -\left\{\frac{\partial^2}{\partial t^2}[R^T]^{-1}\begin{pmatrix} 0 \\ h \end{pmatrix}\right\}_\theta \tag{2.14}$$

representing the response operators of the subsystems through the response operator of the system. Sure, such a splitting of R^T occurs only due to $\rho = \mathrm{const}$.

One more remarkable fact easily follows from analysis of the proof: the wave u^f is of the form of $\mathrm{rot}s^h$ (the case $f = f''$) if and only if

$$f = \begin{pmatrix} \mathrm{div}_\theta h \\ R^T_s h \end{pmatrix}, \qquad h \in \mathcal{M}^T_s$$

i.e., if and only if the components of the control are linked as

$$f^\nu = \mathrm{div}_\theta[R^T_s]^{-1} f_\theta \qquad \text{on } \Sigma^T.$$

Such controls initiate the waves propagating with "slow" velocity $c_s = \sqrt{\mu}$:

$$\operatorname{supp} u^f(\cdot, t) \subset \Omega_s^t, \qquad t > 0. \tag{2.15}$$

It is worth of noting that the similar effect holds in the case of the proper Lamé system governed by the equation (1.0) with *variable* coefficients: there exists a class of controls initiating the waves propagating with the slow velocity, i.e., satisfying (2.15) for $c_s = \sqrt{\frac{\mu}{\rho}}$. Sure, in general case these waves aren't represented as $u^f = \operatorname{rot} s^h$, they are of more complicated structure. Nevertheless, they exist: this fact is derived from the results on boundary controllability of the Lamé system established in [6]. Moreover, the experience obtained in 1d problems (see [5], [7], [8]) allows to presume that it is the class of waves which can play the key role in time optimal determination in general case.

3 Inverse problem

3.1 Determination of κ

Given R^{2T} one can find R_p^{2T} by (2.14). The latter operator determines κ in Ω_p^T that was one of the first results of the BC method. One of the procedures recovering $\kappa\, |_{\Omega_p^T}$ based on the blow up effect has been proposed in [1]; its modernized version is presented in [2]. So, referring to the mentioned papers, we assume κ recovered. The rest of our paper is devoted to the blow up procedure recovering $\mu\, |_{\Omega_s^T}$.

3.2 Preliminaries

Here we present a short list of results of the BC method for the Maxwell system α_s^T. The reader is referred to [4], [3] for details.

(i) *Controllability* Introduce the space

$$\mathcal{J} := \{y \in L_2(\Omega; \mathbf{R^3}) \mid \operatorname{div}\frac{1}{\mu}y = 0 \quad \text{in } \Omega\}$$

of $\mu^{-1}-$ solenoidal vector fields with the inner product

$$(y, v)_{\mathcal{J}} = \int_\Omega \frac{\mathrm{d}x}{\mu}\; y \cdot v$$

which plays the role of the inner space of the system α_s^T. The solutions s^h of the problem (2.5)–(2.7) are considered as $\mathcal{J}$- valued functions of $t \in [0,T]$. The subspaces

$$\mathcal{J}^T := \{y \in \mathcal{J} \mid \text{supp } y \subset \overline{\Omega}_s^T\}, \quad T > 0$$

increase as T grows.

The operator $W^T : \mathcal{F}_s^T \to \mathcal{J}, \mathrm{Dom}W^T = \mathcal{M}_s^T$,

$$W^T h := s^h(\cdot, T)$$

realizes the "input $\mapsto$ state" map; the set

$$\mathcal{S}^T := \mathrm{Ran}W^T = \{s^h(\cdot, T) \mid h \in \mathcal{M}_s^T\}$$

is said to be reachable (at the moment $t = T$). By the property (2.15) (hyperbolicity of the problem (2.5)–(2.7)) the embedding $\mathcal{S}^T \subset \mathcal{J}^T$ holds; in [3] the character of this embedding is clarified as follows.

Assume the condition (A) of section 1.4 to be fulfilled; in this case the representation

$$\mathcal{J}^T = \mathrm{clos}\mathcal{S}^T \oplus \mathcal{D}^T \tag{3.1}$$

holds where $\mathcal{D}^T$ is a finite dimensional subspace consisting of elements d satisfying

$$\mathrm{rot}d = 0 \;\text{ in } \Omega_s^T, \quad d \times \nu = 0 \;\text{ on } \Gamma, \quad d \cdot \nu \;\text{ on } \Gamma_s^T \tag{3.2}$$

where

$$\Gamma_s^T := \{x \in \Omega \mid \tau_s(x) = T\} = \partial\Omega_s^T \setminus \Gamma$$

is the inner part (may be, unconnected) of the boundary of Ω_s^T which can be regarded as the forward front of $s-$ waves propagating from Γ into Ω. The value of $\dim\mathcal{D}^T$ is determined by topology of Ω_s^T, – see [3]. So, the system α_s^T is approximately controllable up to a defect subspace of finite dimension. Note that for small enough T one has $\mathcal{D}^T = \{0\}$. Note also that the relation (4.1) is derived from the vectorial version of the fundamental Holmgren–John–Tataru uniqueness theorem [9].

(ii) *Wave basis* The map $C^T : \mathcal{F}_s^T \to \mathcal{F}_s^T$, $\mathrm{Dom}C^T = \mathcal{M}_s^T$,

$$C^T \;:=\; (W^T)^* W^T$$

is called the connecting operator. The basic fact of the BC method is that this operator is expressed through the response operator (inverse data) in explicit form:

$$C^T = \frac{1}{2}\,[O^T]^* R_s^{2T} J^{2T} O^T \tag{3.3}$$

where $O^T : \mathcal{F}_s^T \to \mathcal{F}_s^{2T}$ extends the controls from $[0,T]$ onto $[0,2T]$ as odd (with respect to $t=T$) functions of t. So, possessing of R_s^{2T} one can find the product

$$(s^{h'}, s^{h''})_{\mathcal{J}} = (W^T h', W^T h'')_{\mathcal{J}} = (C^T h', h'')_{\mathcal{F}_s^T} \tag{3.4}$$

for $h', h'' \in \mathcal{M}_s^T$.

Let $\hat{h}^T = \{h_j\}_{j=1}^\infty \subset \mathcal{M}_s^T$ be a set of controls satisfying

$$(C^T h_i, h_j)_{\mathcal{F}_s^T} = \delta_{ij}, \quad \text{clos span}\hat{h}^T = \mathcal{F}_s^T; \tag{3.5}$$

such a system can be constructed from any complete system by C^T- orthogonalization. By virtue of (3.4) the system of waves $\hat{s}^T = \{s_j\}_{j=1}^\infty,\quad s_j := s^{h_j}(\cdot, T) = W^T h_j$ satisfies

$$(s_i, s_j)_{\mathcal{J}} = \delta_{ij}, \quad \text{clos span}\hat{s}^T = \text{clos}\mathcal{S}^T,$$

i.e., forms the orthonormalized basis in the reachable subspace clos$\mathcal{S}^T$. Denoting by P^T the orthogonal projection in $\mathcal{J}$ onto clos$\mathcal{S}^T$ one has

$$P^T = \sum_{j=1}^\infty (\cdot, s_j)_{\mathcal{J}}\ s_j\ . \tag{3.6}$$

Let $a \in \mathcal{J}^T$ be a smooth vector field satisfying

$$\text{rotrot}a = 0 \qquad \text{in } \Omega_s^T. \tag{3.7}$$

Expanding over the wave basis one has

$$P^T a = \sum_{j=1}^\infty a_j\ s_j; \quad a_j = (a, s_j)_{\mathcal{J}}$$

whereas the Fourier coefficients can be found as follows:

$$a_j = \int_\Omega \frac{\mathrm{d}x}{\mu}\ a \cdot s^{h_j}(\cdot, T) = \langle \text{see (2.6)} \rangle$$

$$= \int_\Omega \frac{\mathrm{d}x}{\mu}\; a \cdot \int_0^T \mathrm{d}t\; (T-t) s^{h_j}(\cdot,t) = \langle \text{see } (2.5) \rangle$$

$$= -\int_0^T \mathrm{d}t\; (T-t) \int_\Omega \mathrm{d}x\; a \cdot \mathrm{rot}\mathrm{rot} s^{h_j}(\cdot,t) = \langle \text{see } (3.7) \rangle$$

$$= -\int_0^T \mathrm{d}t\; (T-t) \int_\Gamma \mathrm{d}\gamma\; [\nu \times \mathrm{rot} s^{h_j}(\cdot,t) \cdot a - \nu \times s^{h_j}(\cdot,t) \cdot \mathrm{rot} a] =$$

$$= \int_{\Sigma^T} \mathrm{d}\gamma\; \mathrm{d}t\; \beta^T\; [\Phi R^T_s h_j \cdot a \;-\; h_j \cdot \mathrm{rot} a] \tag{3.8}$$

where $\beta^T(t) := T - t$. Note that, in course of integrating by parts, the integrals over the surface Γ^T_s vanish since$s^{h_j}(\cdot,t)$ vanishes near Γ^T_s. This follows from $h = 0$ near $\Gamma \times \{0\}$ (see the definition of the class $\mathcal{M}^T_s$) and the property (2.15). So, provided (3.7), the Fourier coefficients can be determined from the response operator R^T_s given in the inverse problem.

(iii) *Indicator* Fix an interior point $x_0 \in \Omega$ and a vector $l \in \mathbf{R^3}$, $|l| = 1$; denote $r_{x_0}(x) := |x - x_0|$; the vector field

$$m_{x_0}(x) \;:=\; \nabla r_{x_0}(x) \times l \;=\; \mathrm{rot}\frac{l}{r_{x_0}(x)}\;, \qquad x \in \Omega \setminus \{x_0\}$$

is said to be the indicator of x_0. The following properties make the indicator to be a tool for solving the inverse problem and can be checked by the straightforward calculation.

1) The relation

$$\mathrm{rot}\mathrm{rot} m_{x_0} = 0 \qquad \text{in } \Omega \setminus \{x_0\} \tag{3.9}$$

is valid.

2) Let $T < \tau_s(x_0)$ (so that $x_0 \notin \Omega^T_s$), denote $m^T_{x_0} := m_{x_0}\,|_{\Omega^T_s}$; then

$$||m^T_{x_0}||^2_{\mathcal{J}} \to \infty \qquad \text{as } T \to \tau_s(x_0), \tag{3.10}$$

i.e., the norm of the cut off function of the indicator onto Ω^T_s blows up as the forward front of $s-$ waves Γ^T_s approaches to x_0. Justifying this fact one uses the cone property of the subdomain Ω^T_s which it possesses of due to the condition (A). In result, the value of the $s-$ eikonal can be characterized as follows:

$$\tau_s(x_0) = \sup\{T > 0 \mid ||m^T_{x_0}||^2_{\mathcal{J}} < \infty\}. \tag{3.11}$$

3) If $x_0 \notin \omega_s$ (see the comments to the condition (A) in section 1.4) the norm of the projection of $m^T_{x_0} \in \mathcal{J}^T$ onto the defect subspace $\mathcal{D}^T$ remains bounded as $T \to \tau_s(x_0)$. Indeed, for $d \in \mathcal{D}^T$ one has

$$(m^T_{x_0}, d)_{\mathcal{J}} = \int_\Omega \frac{\mathrm{d}x}{\mu}\, \mathrm{rot}\frac{l}{r_{x_0}} \cdot d =$$

$$= \int_{\partial\Omega^T_s} \frac{\mathrm{d}\gamma}{\mu}\, \nu \times \frac{l}{r_{x_0}} \cdot d + \int_{\Omega^T_s} \mathrm{d}x\, \frac{l}{r_{x_0}} \cdot \mathrm{rot}\frac{d}{\mu} = \langle \text{see } (3.2) \rangle$$

$$= \int_{\Gamma^T_s} \mathrm{d}\gamma\, \nu \times \frac{d}{\mu} \cdot \frac{l}{r_{x_0}} + \int_{\Omega^T_s} \mathrm{d}x\, \frac{l}{r_{x_0}} \cdot \nabla\frac{1}{\mu} \times d.$$

The second of the last integrals is bounded uniformly with respect to T since the function $r^{-1}_{x_0}$ is square summable in Ω. The first one is finite for $T \in (0, \tau_s(x_0))$ whereas at $T = \tau_s(x_0)$ it becomes singular. Nevertheless, since x_0 lies at the smooth part of the surface Γ^T_s where the singularity is summable and defect elements d are smooth, it is easy to proof that the surface integral is bounded uniformly on $T \in (0, \tau_s(x_0)]$. So, the projection on $\mathcal{D}^T$ is bounded and, taking into account (3.10) we conclude that the projection of the indicator onto the reachable subspace blows up:

$$||P^T m^T_{x_0}||^2_{\mathcal{J}} = \langle \text{see } (3.6) \rangle = \sum_{j=1}^{\infty} (m^T_{x_0}, s_j)^2_{\mathcal{J}} \to \infty \quad \text{as } T \to \tau_s(x_0).$$

Therefore, (3.11) can be modified as

$$\tau_s(x_0) = \sup\{T > 0 \mid ||P^T m^T_{x_0}||^2_{\mathcal{J}} < \infty\} \tag{3.12}$$

whereas the norm entering in this criterion can be found from the inverse data: by virtue of (3.8) for $T < \tau_s(x_0)$ one has

$$||P^T m^T_{x_0}||^2_{\mathcal{J}} = \sum_{j=1}^{\infty} \left\{ \int_{\Sigma^T} \mathrm{d}\gamma\, \mathrm{d}t\, \beta^T [\Phi R^T_s h_j \cdot m_{x_0} - h_j \cdot \mathrm{rot} m_{x_0}] \right\}^2. \tag{3.13}$$

3.3 Determination of μ

We prove the Theorem 1 by summarizing the previous results in the following procedure. Just for simplicity we assume an upper bound $\mu^* \geq \mu(\cdot)$ known. So, assuming R^{2T} given,

(i) fix an $x_0 \in \Omega$ and take $T' \in (0,T)$ such that $\sqrt{\mu^*}\, T' < \text{dist}_{\,\mathbf{R}^3}(x_0, \Gamma)$, that ensures $x_0 \notin \Omega_s^{T'}$;

(ii) find $R_s^{2T'}$ by (2.14) and determine $C^{T'}$ by (3.3);

(iii) construct a system $\hat{h}^{T'}$ satisfying (3.5) and find the norm $||P^{T'} m_{x_0}^{T'}||^2_{\mathcal{J}}$ by (3.13);

(iv) growing T', check by the use of the criterion (3.12) whether the inclusion $x_0 \in \overline{\Omega}_s^T$ holds: if the norm found above remains bounded as $T' \to T$ then $x_0 \notin \Omega_s^T$; otherwise, if the norm blows up as $T' \to T_0 < T$ then $x_0 \in \Omega_s^T$ and $\tau_s(x_0) = T_0$ (so, in fact we detect the moment when the forward front $\Gamma_s^{T'}$ of $s-$ waves moving from Γ touches x_0);

(v) varying x_0 and repeating the same procedure find τ_s in Ω_s^T; then determine μ by the eikonal equation

$$|\nabla \tau_s|^2 = \frac{1}{\mu}$$

that holds in $\Omega \setminus \omega_s$, i.e., by the properties of the separation set, almost everywhere in Ω. Since μ is continuous it is determined everywhere in Ω_s^T. The inverse problem is solved.

3.4 Comments

- A simple modification of the procedure described in [2] (see sec. 7.6) enables to determine κ and μ from the data given on any open portion $\sigma \subset \Gamma$ in the subdomains $\Omega_p^{T,\sigma}$ and $\Omega_p^{T,\sigma}$ filled with $p-$ and $s-$ waves initiated by controls acting on σ.

- The above mentioned modification allows to deal with unbounded Ω. Smoothness of κ and μ is restricted only by applicability of the Holmgren–John–Tataru theorem: C^2- smoothness is available [9]. Real requirements on smoothness of Γ are also rather weak. Namely, a simple analysis shows that the procedure remains workable if the boundary satisfies the following condition: for any $\varepsilon > 0$ one can find a domain $\Omega_\varepsilon \supset \Omega$, $\partial\Omega_\varepsilon =: \Gamma_\varepsilon \in C^\infty$ such that Γ lies in $\varepsilon-$ neighborhood of Γ_ε (in $\mathbf{R}^3$). So, edges, cone points etc are admissible.

- In principle, there exists an analog of the formulas (3.8) and (3.13) for the system governed by the equation $\rho u_{tt} = \nabla\kappa \text{div} u - \text{rot}\mu\text{rot} u$; with variable ρ, so that one can find the norms of the cut off functions of $p-$ and $s-$ indicators from R^{2T}. Most probably, the same blow up effect holds in this case and allows to recover the velocities $c_p = \sqrt{\frac{\kappa}{\rho}}$ and $c_s = \sqrt{\frac{\mu}{\rho}}$ in the filled domains. However, this is only a hypothesis. Also, in the case of the complete Lamé equation (1.0) with variable ρ, λ, μ the trick using the indicators certainly doesn't work.

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