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ON STRATEGIC BACKGROUND OF RANDOM PRICE FLUCTUATIONS AT FINANCE MARKETS

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Abstract

Models of repeated bidding for risky assets (shares) are considered. Such model with arbitrary bids was introduced in De Meyer, Saley (2002) for strategic reasoning of Brownian Motion appearence in the price evolution at finance markets. We consider the models with discrete admissible bids. We show that the optimal strategy of informed agent generates random walks of posterior probabilities over admissible bids.

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1 Introduction

Wiener process and its discrete analog, the random walk, are often used to model the evolution of price system at finance markets. The random fluctuations of prices are usually motivated with the effect of multiple exogenous factors subjected to accidental variations.

A different strategic motivation for these phenomena is proposed in the recent work of De Meyer, Saley (2002). The authors assume that the Brownian component in the evolution of prices at stock market may originate from asymmetric information of stockbrokers on events determining market prices. "Insiders" are not interested in immediate revelation of their private information. This enforces them to randomize their actions and results in the appearance of the Brownian component in price evolution.

De Meyer Saley demonstrate this idea on a simplified model of multistage biddings between two agents for risky assets (shares). The final value of one share depends on a random "state of nature". Before the biddings start a chance move determines the "state of nature" and therefore the final value of one share once for all. This value is 1 with probability p and 0 with probability $1-p$. Player 1 is informed on the "state of nature", Player 2 is not. Both players know probabilities of a chance move. Player 2 knows that Player 1 is an insider.

At each subsequent step $t = 1, 2, ..., n$ both players propose simultaneously their prices for one share. The maximal bid wins and one share is transacted at this price. Each player aims to maximize the value of his final portfolio (money plus shares).

The authors reduce described model to zero-sum repeated game $G_n^c(p)$ with lack of information on one side, as introduced by Aumann, Maschler (1995). Unlike the Aumann, Maschler model, these games have a continuum of possible actions of players. Consequently, the existence of value for the games $G_n^c(p)$ has to be proved. They prove it by constructing the optimal strategies.

The values $V_n^c(p)$ are expressed through the probability densities of sums of n independent random variables uniformly distributed over the interval $[-1, 1]$. The values $V_n^c(p)$ infinitely grow up as $\sqrt{n}$. The term $V_n^c(p)/\sqrt{n}$ converges to a limit expressed through the density of normal distribution.

They represent the price process $p_1^n, \ldots, p_n^n$ with a continuous time process Π^n on $[0, 1]$ and show that, as $n \to \infty$, the process Π^n converges in law, in the sense of finite distributions, to the limit process Π expressed by means of Brownian Motion.

It is more realistic to assume that players may assign only discrete bids. In the preprint (2002) De Meyer and Marino considered the games modelling the biddings with admissible bids being multiples of $1/m$. The main result of the preprint is that, as n goes to ∞, the sequence of values divided by $\sqrt{n}$ converges uniformly to zero.

We consider the same games $G_n^m(p)$ and show that, as n tends to ∞, the sequence of values $V_n^m(p)$ of the games $G_n^m(p)$ with the summarized non-averaged payoffs is bounded from above. The result of De Meyer and Marino follows immediatelly from this fact. We prove that, as n tends to ∞, the sequence $V_n^m(p)$ converges. The limit $H^m(p)$ is a continuous concave piecewise linear function with m domains of linearity $[k/m, (k+1)/m], k = 0, \ldots, m-1$, and $H^m(k/m) = k(m-k)/2m$. The extended abstract of these results is published in Domansky, Kreps (2005b) (in Russian), Domansky (2005).

As the sequence $V_n^m(p)$ is bounded from above, it is reasonable to consider the games $G_\infty^m(p)$ with infinite number of steps and with total payoff. However, unlike the case of $n < \infty$, the existence of value for the games $G_\infty^m(p)$ has to be proved. We prove it by constructing explicitly the optimal strategies. We show that the value $V_\infty^m = H^m$ and that both Players have optimal strategies.

The optimal strategy of Player 1 in the infinite game $G_\infty^m(p)$ generates a symmetric random walk of posterior probabilities over the points $l/m, l = 0, \ldots, m$ with the absorbing extreme points 0 and 1. As $k(m-k)$ is its expected duration for the initial point k/m, and $1/2m$ is the gain per step, the result $V_\infty^m(k/m) = k(m-k)/2m$ turns to be rather intuitive.

The Markov time of absorption of posterior probabilities represents the time of revelation the "true" value of share by Player 2 and, generally speaking, the time of bidding termination. Thus, unlike the biddings with arbitrary bids, the biddings with the admissible bids being multiples of $1/m$ terminate almost surely in a finite number of steps, and the expected number of steps before termination is also finite.

The random sequence of Player 1's bids differs from the random sequence of posterior probabilities at most for $1/m$. As Player 2's bids "copy" Player 1's bids, the sequence of realizing market prices turns to be close to a symmetric random walk of posterior probabilities over the points $l/m, l = 0, \ldots, m$ with the absorbing extreme bids 0 and $(m-1)/m$.

The appearance of symmetric random walk over the admissible bids in the solutions for the market games $G_\infty^m(p)$ is a natural discrete analogue for strategic reasoning of Brownian Motion in the evolution of the price systems at finance markets with asymmetric information obtained in De Meyer and Saley (2002).

The extended abstract of results on the games $G_\infty^m(p)$ is published in Domansky, Kreps (2005a).

2 Repeated games with one-sided information and their recursive structure

Here we describe repeated zero-sum games with lack of information on one side $G_n(p)$, as introduced by Aumann, Maschler (1995). Such game is given with a payoff function $a^s(i,j)$ defined over a product of finite sets $S \times I \times J$. Here S is the state space, I and J are the respective action sets of Player 1 and Player 2.

At the initial stage 0 a chance move chooses the state $s \in S$ according to the probability distribution $p = (p^s) \in \Delta(S)$, known to both Players. Player 1 is informed of s and Player 2 is not.

At each subsequent stage $t = 1, \ldots, n$ (n may be infinite) both Players select simultaneously their actions in the respective action sets, $i_t \in I$ for Player 1 and $j_t \in J$ for Player 2. The pair (i_t, j_t) is then announced to both Players before proceeding to the next stage. At the end of the game Player 2 pays to Player 1 $\sum_{t=1}^n a^s(i_t, j_t)$. This description is common knowledge to both Players. (In Aumann, Maschler model the payoff is $1/n \sum a^s(i_t, j_t)$.)

At step t it is enough for both players to take into account the sequence $(i_1, \ldots, i_{t-1})$ of informed Player 1's actions only. Thus a behavior strategy σ for Player 1 is a sequence of moves

$$\sigma = (\sigma_1, \ldots, \sigma_t, \ldots),$$

where $\sigma_t : S \times I^{t-1} \to \Delta(I)$ is the probability distribution used by Player 1 to select his action at stage t, given the state s and previous observations. Analogously, a strategy τ for Player 2 is a sequence $\tau = (\tau_1, \ldots, \tau_t, \ldots)$ where $\tau_t : I^{t-1} \to \Delta(J)$.

A pair of strategies (σ, τ) induces a probability distribution $\Pi_{(\sigma,\tau)}$ over $(I \times J)^\infty$. The payoff function of the game $G_n(p)$ is

$$K_n(p, \sigma, \tau) = \sum_{s \in S} p^s h_n^s(\sigma, \tau),$$

where

$$h_n^s(\sigma, \tau) = \mathbf{E}_{(\sigma,\tau)}[\sum_{t=1}^{n} a^s(i_t, j_t)]$$

is the s-component of the n-step vector payoff $h_n(\sigma, \tau)$ for the pair of strategies (σ, τ). Here the expectation is taken with respect to the probability distribution $\Pi_{(\sigma,\tau)}$.

For the initial probability p, the strategy σ ensures the n-step payoff

$$w_n(p, \sigma) = \inf_\tau K_n(p, \sigma, \tau).$$

The strategy τ ensures the n-step vector payoff $z_n(\tau)$ with the components

$$z_n^s(\tau) = \sup_\sigma z_n^s(\sigma^n, \tau^n).$$

Now we describe the recursive structure of $G_{n+1}(p)$. A strategy σ may be regarded as a pair $(\sigma_1, (\sigma(i))_{i \in I})$, where $\sigma_1(i|s)$ is a probability on I depending on s, and $\sigma(i)$ is a strategy depending on the first move $i_1 = i$. Analogously, a strategy τ is regarded as $(\tau_1, (\tau(i))_{i \in I})$, where τ_1 is a probability on J.

A pair (p, σ_1) induces the probability π over $S \times I$, $\pi(s, i) = p(s)\sigma_1(i|s)$. Let

$$q \in \Delta(I), \quad q(i) = \sum_S p(s)\sigma_1(i|s),$$

be the marginal distribution of π on I (total probabilities of actions), and let

$$p(i) \in \Delta(S), p^s(i) = p(s)\sigma_1(i|s)/q(i),$$

be the conditional probability on S given $i_1 = i$ (posterior probability).

The following recursive representation for the payoff function corresponds to the recursive representation of strategies:

$$K_{n+1}(p, \sigma, \tau) = K_1(p, \sigma_1, \tau_1) + \sum_{i \in I} q(i) K_n(p(i), \sigma(i), \tau(i)).$$

Let, for all $i \in I$, the strategy $\sigma(i)$ ensure the payoff $w_n(p(i), \sigma(i))$ in the game $G_n(p^i)$. Then the strategy $\sigma = (\sigma_1, (\sigma(i))_{i \in I})$ ensures the payoff

$$w_{n+1}(p, \sigma) = \min_{j \in J} \sum_{s \in S} \sum_{i \in I} p^s \sigma_1(i|s) a(s, i, j) + q(i) w_n(p^i, \sigma(i)). \tag{2.1}$$

Let, for all $i \in I$, the strategy $\tau(i)$ ensure the vector payoff $h_n(\tau(i))$. Then the strategy $\tau = (\tau_1, (\tau^n(i))_{i \in I})$ ensures the vector payoff $h_{n+1}(\tau)$ with the components

$$h_{n+1}^s(\tau) = \max_{i \in I} \sum_{j \in J} \tau_1(j)(a(s, i, j) + h_n^s(\tau(i)) \quad \forall s \in S. \tag{2.2}$$

For finite n the game $G_n(p)$ has a value $V_n(p)$. The functions $V_n(p)$ are continuous and concave in p. Both players have optimal strategies σ^* and τ^*:

$$V_n(p) = \inf_\tau K_n(p, \sigma^*, \tau) = \sup_\sigma K_n(p, \sigma, \tau^*),$$

or, in above introduced notation, $V_n(p) = w_n(p, \sigma^*) = \sum_{s \in S} p^s h_n^s(\tau^*)$.

3 Games $G_n^m(p)$. Upper bounds for the values

We consider the games $G_n^m(p)$ modelling the auctions with the admissible bids being multiples of $1/m$, namely k/m, $k = 0, \ldots, m-1$. So the action spaces are $I = J = 0, \ldots, m-1$. The state space is $S = \{L, H\}$.

At the state L, i.e. if the final value of the share is equal to zero, the gains of Player 1 are given with the following matrix A^L:

$$1/m \begin{pmatrix} 0 & 1 & 2 & \ldots & k & \ldots & m-1 \\ -1 & 0 & 2 & \ldots & k & \ldots & m-1 \\ -2 & -2 & 0 & \ldots & k & \ldots & m-1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -k & -k & -k & \ldots & 0 & \ldots & m-1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ -m+1 & -m+1 & -m+1 & \ldots & -m+1 & \ldots & 0 \end{pmatrix}$$

At the state H, i.e. if the final value of the share is equal to one, then the matrix A^H of the gains of Player 1 takes the form

$$1/m \begin{pmatrix} 0 & -m+1 & -m+2 & \ldots & -m+k & \ldots & -1 \\ m-1 & 0 & -m+2 & \ldots & -m+k & \ldots & -1 \\ m-2 & m-2 & 0 & \ldots & -m+k & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ m-k & m-k & m-k & \ldots & 0 & \ldots & -1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & 1 & \ldots & 1 & \ldots & 0 \end{pmatrix}$$

We consider the n-step games $G_n^m(p)$ with total non-averaged payoffs. We show that as the number of steps n increases the sequence of values $V_n^m(p)$ of these games remains bounded.

For the games $G_n^m(p)$, we define recursively the set of strategies of Player 2 $\tau^{k,m}, k = 0, \ldots, m-1$.

Definition 3.1. *The first move $\tau_1^{k,m}$ is the action k. Further, if action l of Player 1 occurs, then*

$$\tau^{k,m}(l) = \begin{cases} \tau^{k-1,m}, & \text{for } l < k; \\ \tau^{k,m}, & \text{for } l = k; \\ \tau^{k+1,m}, & \text{for } l > k. \end{cases}$$

Proposition 3.2. *The strategies $\tau^{k,m}, k = 0, \ldots, m-1$, ensure the vector payoffs $h_n(\tau^{k,m})$ with components given by*

$$h_n^L(\tau^{k,m}) = 1/m \sum_{l=0}^{n-1} (k-l)^+, \tag{3.1}$$

$$h_n^H(\tau^{k,m}) = 1/m \sum_{l=0}^{n-1} (m-k-1-l)^+, \tag{3.2}$$

where $(a)^+ := \max\{0, a\}$.

Proof. The proof is by induction on n.

$n = 1$. For $k = 0, \ldots, m-1$,

$$h_1^L(\tau^{k,m}) = \max_l a_{l,k}^L = k/m.$$

For $k = 0, \ldots, m-1$,

$$h_1^H(\tau^{k,m}) = \max_l a_{l,k}^H = a_{k+1,k}^H = (m-k-1)/m.$$

Thus, for $k = 0, \ldots, m-1$, $h_1(\tau^{k,m}) = (k/m, (m-k-1)/m$.

$n \to n+1$. Assume that the vector payoffs $h_n(\tau^{k,m})$ are given with (3.1), (3.2).

For the state L, the first move of Player 1's best reply for the strategy $\tau^{0,m}$ is the action 0. According to (2.2) it results in $h_{n+1}^L(\tau^{0,m}) = 0$. For the strategy $\tau^{k,m}, k = 1, \ldots, m-1$, this first move is any action l with $l < k$. We have

$$h_{n+1}^L(\tau^{k,m}) = \max_l (a_{l,k}^L + h_n^L(\tau^{l,m})) = k/m + 1/m \sum_{l=0}^{n-1} (k-1-l)^+ = 1/m \sum_{l=0}^{n} (k-l)^+.$$

For the state H, the first move of Player 1's best reply for the strategy $\tau^{k,m}, k = 1, \ldots, m-2$, is the action $k+1$. We obtain

$$h_{n+1}^H(\tau^{k,m}) = \max_l (a_{l,k}^H + h_n^H(\tau^{l,m}))$$

$$= (m-k-1)/m + 1/m \sum_{l=0}^{n-1} (m-k-2-l)^+ = 1/m \sum_{l=0}^{n} (m-k-1-l)^+.$$

For the strategy $\tau^{m-1,m}$, this first move is the action $m-1$. We have $h_{n+1}^H(\tau^{m-1,m}) = 0$.

Thus, the strategy $\tau^{k,m}$, for $k = 0, \ldots, m-1$, ensures the vector payoff $(h_{n+1}^L(\tau^{k,m}), h_{n+1}^H(\tau^{k,m}))$.

□

It is easy to see that, for $n \geq m-1$, the vector payoffs $h_n(\tau^{k,m})$ stabilize, i.e.

$$h_n(\tau^{k,m}) = h_{n+1}(\tau^{k,m}), \quad k = 0, \ldots, m-1.$$

Corollary 3.3. *There is the following not depending on n upper bound for $V_n^m(p)$:*

$$V_n^m(p) < \min_k \{h^L(\tau^{k,m})(1-p) + h^H(\tau^{k,m})p\} = H^m(p), \quad k = 0, \ldots, m-1,$$

with the vector payoffs $(h^L(\tau^{k,m}), h^H(\tau^{k,m}))$ given by

$$h^L(\tau^{k,m}) = h_{m-1}^L(\tau^{k,m}) = 1/m \sum_{l=1}^{k} l = k(k+1)/2m,$$

$$h^H(\tau^{k,m}) = h_{m-1}^H(\tau^{k,m}) = 1/m \sum_{l=1}^{m-k-1} l = (m-k)(m-k-1)/2m.$$

Remark. The strategies $\tau^{k,m}, k = 0, \ldots, m-1$ guarantee the same upper bound H^m for the upper value of the game G_∞^m.

4 Lower bounds for the values of the games $G_n^m(p)$

Here we construct lower bounds for the values $V_n^m(p)$ of the n-step games $G_n^m(p)$. We define recursively the set of strategies of Player 1 $\sigma^{k,m} = (\sigma_1^{k,m}, (\sigma^{k,m}(l)_{l\in I}), k = 1, \dots, m-1$.

Definition 4,1. *The first move $\sigma_1^{k,m}$ makes use only of two actions $k-1$ and k. At state L it uses them with probabilities*

$$\sigma_1^{k,m}(k-1|L) = (m-k+1)/2(m-k); \quad \sigma_1^{k,m}(k|L) = (m-k-1)/2(m-k).$$

At state H it uses them with probabilities

$$\sigma_1^{k,m}(k-1|H) = (k-1)/2k; \quad \sigma_1^{k,m}(k|H) = (k+1)/2k.$$

Further, if action l of Player 1 occurs, then

$$\sigma^{k,m}(l) = \sigma^{k-1,m}, \quad \text{for} \quad l < k;$$

$$\sigma^{k,m}(l) = \sigma^{k+1,m}, \quad \text{for} \quad l \geq k.$$

Here $\sigma^{0,m} = \sigma^{m,m}$ are the completely revealing strategies: 0 for the state L and $m-1$ for the state H.

Consider the elementary symmetric random walk $(x_t)_{t=1}^n$ over the points $k = 0, \dots, m$ with the absorbing extreme points 0 and m, i.e. the Markov chain with the transition probabilities

$$P(k, k-1) = P(k, k+1) = 1/2, \quad k = 1, \dots, m-1, \quad P(0,0) = P(m,m) = 1.$$

Let $D_n^m(k)$ and $D_\infty^m(k)$ denote the expected duration until the absorption $x_t = 0, m$, for the chains $(x_t)_{t=1}^n$ and $(x_t)_{t=1}^\infty$ respectively, with the initial condition $x_1 = k$.

Values $D_n^m(k)$ are defined recursively. For $n = 1, 2, \dots$, $k = 1, 2, \dots, m-1$

$$D_n^m(k) = 1 + 1/2(D_{n-1}^m(k-1) + D_{n-1}^m(k+1)), \tag{4.1}$$

with the initial condition $D_0^m(k) = 0$, and the boundary conditions $D_n^m(0) = D_n^m(m) = 0$.

$D_\infty^m(k) = k(m-k)$. The following equalities hold:

$$\lim_{n\to\infty} D_n^m(k) = D_\infty^m(k) \quad k = 1, 2, \dots, m-1. \tag{4.2}$$

Proposition 4.2. *For the initial probability k/m, the strategy $\sigma^{k,m}$ ensures the payoff*

$$w_n(k/m, \sigma^{k,m}) = D_n^m(k/m)/2m, \quad k = 1, \dots, m-1.$$

Proof. The proof is by induction on n.
$n = 1$. In the one-step game $G_1^m(k/m)$ the strategy $\sigma^{k,m}$ ensures the payoff

$$w_1(k/m, \sigma^{k,m}) = D_1^m(k)/2m = 1/2m, \quad k = 0, \dots, m-1.$$

In fact, $for l < k-1$,

$$K_1(k/m, \sigma_1^{k,m}, l) = k/m - 1/2((k-1)/m + k/m) = 1/2m.$$

For $l = k-1$,
$$K_1(k/m, \sigma_1^{k,m}, k-1) = k/m(k+1)/2k - 1/2k/m = 1/2m.$$
For $l = k$,
$$K_1(k/m, \sigma_1^{k,m}, k) = -k/m(k-1)/2k + 1/2k/m) = 1/2m.$$
For $l > k$,
$$K_1(k/m, \sigma_1^{k,m}, l) = -k/m + l/m > 1/2m.$$
Thus, the best answer of Player 2 to this strategy is any action $l \leq k$. The resulting gain of Player 1 is equal to $1/2m$.
$n \to n+1$. Assume that the strategy $\sigma^{k,m}$ ensures the payoffs $D_n^m(l)/2m$ in the games $G_n^m(l/m), l = 1, \ldots, m-1$.

The best answer of Player 2 to the first move of the strategy $\sigma^{k,m}$ is any action l with $l \leq k$. The resulting immediate gain of Player 1 is equal to $1/2m$.

For this first move the actions $k-1$ and k occur with the same total probabilities $1/2$. In fact, for the action $k-1$,

$$q(k-1) = (1-p)\sigma_1^{k,m}(k-1|L) + p\sigma_1^{k,m}(k-1|H)$$
$$= (m-k)/m \cdot (m-k+1)/2(m-k) + k/m \cdot (k-1)/2k = 1/2,$$

and, consequently, $q(k) = 1/2$.

The corresponding conditional posterior probabilities of the state H are

$$p^H(k-1) = (k-1)/2k \cdot k/m \cdot 2 = (k-1)/m,$$

for the action $k-1$, and

$$p^H(k) = (k+1)/2k \cdot k/m \cdot 2 = (k+1)/m,$$

for the action k.

According to the induction assumption and formulas (2.1),(4.1), the resulting total gain of Player 1 is equal to

$$1/2m + 1/2(D_n^m(k-1)/2m + L_n^m(k+1)/2m) = D_{n+1}^m(k)/2m).$$

Thus, the strategy $\sigma^{k,m}$ ensures the payoff $D_{n+1}^m(k)/2m$ in the games $G_{n+1}^m(k/m), k = 1, \ldots, m-1$.
□

Remark. It follows from the proof that, for the initial probability k/m, the strategy $\sigma^{k,m}$ ensures the payoff
$$w_\infty(k/m, \sigma^{k,m}) = D_\infty^m(k/m)/2m, \quad k = 1, \ldots, m-1,$$
in the infinite game $G_\infty^m(p)$.

Consider the sequence $L_n^m(p)$ of continuous concave piecewise linear functions on the interval $[0,1]$ with m domains of linearity $[k/m, (k+1)/m]$, $k = 0, \ldots, m-1$. At the peak points k/m, the values $L_n^m(k/m) = D_n^m(k)/2m)$.

Corollary 4.3. *The following inequalities hold:*

$$L_n^m(p) \leq V_n^m(p) \quad \forall p \in [0,1].$$

Proof. For $p = k/m, k = 1, \dots, m-1$, it follows immediatelly from Proposition 4.2. Because of the continuity and concavity of the functions V_n^m it is true for all $p \in [0,1]$. This completes the proof of Corollary 4.3. $\square$

Remark. The strategy of Player 1 is completely defined for the initial probabilities $k/m, k = 1, 2, \dots, m-1$. For the interior points $p \in (k/m, (k+1)/m)$ Player 1 can use the convex combination of the one-shot strategies corresponding to extreme points of this interval.

5 Asymptotics for the values $V_n^m(p)$ and solutions for the games $G_\infty^m(p)$

In this section we show that as the number of steps n increases the sequence of lower bounds constructed in previous section converges to the upper bound H^m given in Corollary 3.3. Consequently, the sequence of values $V_n^m(p)$ of these games converges to H^m. We show also that the infinite game $G_\infty^m(p)$ has a value and this value is equal to H^m.

Theorem 5.1. *The following equalities hold:*

$$\lim_{n\to\infty} V_n^m(p) = H^m(p), \quad m = 2, 3, \dots.$$

Proof. According to Corollary 3.3. and Corollary 4.3. the following inequalities hold:

$$L_n^m(p) \le V_n^m(p) \le H^m(p), \quad \forall p \in [0,1].$$

The upper bound H^m given in Corollary 3.3. has the form

$$H^m(p) = \min_k\{(1-p)k(k+1)/2m + p(m-k)(m-k-1)/2m\}, \quad k = 0, \dots, m-1.$$

Observe that, for $p \in [k/m, (k+1)/m]$, the minimum in this formula is attained on the k-th vector payoff, and consequently,

$$H^m(p) = (1-p)k(k+1)/2m + p(m-k)(m-k-1)/2m, \quad p \in [k/m, (k+1)/m].$$

Thus, $H^m(p)$ is a continuous concave piecewise linear function with m domains of linearity $[k/m, (k+1)/m]$, $k = 0, \dots, m-1$. It is completely determined with its values at the peak points k/m, $k = 0, \dots, m$:

$$H^m(k/m) = (m-k)/m \cdot k(k+1)/2m + k/m \cdot (m-k)(m-k-1)/2m$$

$$= (m-k)/m \cdot k(k-1)/2m + k/m \cdot (m-k)(m-k+1)/2m = k(m-k)/2m.$$

So the functions L_n^m and H^m have the identical form. Because of the continuity and concavity of the functions L_n^m and H^m, to prove that the sequence L_n^m converges to H^m as n tends to ∞, it is enough to show that

$$\lim_{n\to\infty} L_n^m(k/m) = H^m(k/m). \tag{5.1}$$

But since $L_n^m(k/m) = D_n^m(k)/2m$ and $H^m(k/m) = D_\infty^m(k)/2m$, where $D_\infty^m(k)$ and $D_n^m(k)$ denote the expected duration until the absorption $x_t = 0, m$ for the chains $(x_t)_{t=1}^\infty$ and $(x_t)_{t=1}^n$ respectively with the initial condition $x_1 = k$. The equality (5.1) follows from (4.2) This proves Theorem 5.1.

As the values $V_n^m(p)$ are bounded from above on the number of steps n, the consideration of the games $G_\infty^m(p)$ with infinite number of steps becomes reasonable. The existence of values for these games does not follow from common considerations and has to be proved. We prove it by providing explicitly the optimal strategies.

Theorem 5.2. *The game $G_\infty^m(p)$ has a value $V_\infty^m(p)$ equal to $H^m(p)$. Both Players have optimal strategies.*

For $p = k/m, k = 1, \ldots, m-1$, the optimal strategy of Player 1 is the strategy $\sigma^{k,m}$, given by Definition 4.1. For the interior points $p \in (k/m, (k+1)/m)$, the optimal strategy of Player 1 is the convex combination of the strategies $\sigma^{k+1,m}$ and $\sigma^{k,m}$, corresponding to the extreme points of this interval.

For $p \in [k/m, (k+1)/m], k = 0, \ldots, m-1$, the optimal strategy of Player 2 is the strategy $\tau^{k,m}$, given by Definition 3.1. For the peak points $k/m, k = 1, \ldots, m-1$, any convex combination of the strategies $\tau^{k-1,m}$ and $\tau^{k,m}$ is optimal.

We give only an outline of proof. According to the remark to Proposition 4.2, the function H^m is the lower bound for the lower value of the game G_∞^m. The strategies $\sigma^{k,m}, k = 0, \ldots, m-1$, ensure this lower bound.

According to the remark to Corollary 3.3. the function H^m is the upper bound for the upper value of the game G_∞^m. The strategies $\tau^{k,m}, k = 0, \ldots, m-1$, ensure this upper bound. □

Remark. The Markov time of absorption of posterior probabilities represents the time of revelation the "true" value of share by Player 2 and, generally speaking, the time of bidding termination. Thus, unlike the biddings with arbitrary bids, the biddings with the admissible bids being multiples of $1/m$ terminate almost surely in a finite number of steps, and the expected number of steps before termination is also finite.

The random sequence of Player 1's bids differs from the random sequence of posterior probabilities at most for $1/m$. As Player 2's bids "copy" Player 1's bids, the sequence of realizing market prices turns to be close to a symmetric random walk of posterior probabilities over the points $l/m, l = 0, \ldots, m$ with the absorbing extreme bids 0 and $(m-1)/m$.

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