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# SCHAUDER ESTIMATES FOR THE EVOLUTION GENERALIZED STOKES PROBLEM

**V. A. Solonnikov**

Russian Academy of Sciences
Steklov Mathematical Institute
St.Petersburg Department
Fontanka 27, St. Petersburg 191011, Russia

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ABSTRACT

We prove coercive estimates in anisotropic Hölder spaces of the solution of the Cauchy–Dirichlet problem for the system of generalized Stokes equations arising in the linearization of equations of the motion of a non-Newtonian liquid.

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**Dedicated to the memory of Academician O. A. Ladyzhenskaya**

## 1. Introduction

Let $\Omega$ be a bounded domain in $R^n$, $n \geq 2$, whose boundary $S$ may consist of several connected components. In the present paper we consider the initial-boundary value problem

$$\frac{\partial \mathbf{v}(x,t)}{\partial t} + \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t),$$

$$\nabla \cdot \mathbf{v}(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T), \tag{1.1}$$

$$\mathbf{v}(x,0) = \mathbf{v}_0(x), \qquad \mathbf{v}(x,t)|_{x\in S} = \mathbf{a}(x,t), \tag{1.2}$$

where the unknowns are a vector field $\mathbf{v}(x,t) = (v_1(x,t), ..., v_n(x,t))$ (the velocity) and a function $p(x,t)$ (the pressure). As usual, $\nabla p \equiv \mathrm{grad} p = \left(\frac{\partial p}{\partial x_1}, ..., \frac{\partial p}{\partial x_n}\right)$, $\nabla \cdot \mathbf{v} \equiv \mathrm{div}\mathbf{v} = \sum_{j=1}^n \frac{\partial v_j}{\partial x_j}$, and $\nabla^2 = \Delta$ is the Laplacean. By $\mathcal{A}(x,t,\frac{\partial}{\partial x})$ we mean the matrix elliptic differential operator of the second order

$$\mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t) = \Big(\sum_{k=1}^n a_{jk}(x,t,\frac{\partial}{\partial x})v_k(x,t)\Big)_{j=1,...,n}$$

with real coefficients whose principal part, i.e., the sum of all terms containing only second derivatives, is denoted by $\mathcal{A}_0(x,t,\frac{\partial}{\partial x})$. Ellipticity means that for arbitrary $\xi, \eta \in R^n$ the inequality

$$C^{-1}|\xi|^2|\eta|^2 \leq \mathcal{A}_0(x,t,i\xi)\eta \cdot \eta \leq C|\xi|^2|\eta|^2 \tag{1.3}$$

holds with a constant $C$ independent of $\xi, \eta, x, t$. The Stokes system is a particular case of (1.1) corresponding to $\mathcal{A} = \mathcal{A}_0 = -\nu I\nabla^2$, $\nu = const > 0$.

The main result of the paper is an a-priori estimate of the solution of problem (1.1), (1.2) in anisotropic Hölder spaces (the definition is recalled in Sec. 2) and a respective existence theorem. Concerning the data of the problem, it is assumed that they satisfy the necessary compatibility conditions, in the first line,

$$\nabla \cdot \mathbf{v}_0(x) = 0, \quad \mathbf{v}_0(x)|_{x\in S} = \mathbf{a}(x,0), \quad \int_S \mathbf{a}(x,t)\cdot \mathbf{n}(x)dS = 0, \tag{1.4}$$

where $\mathbf{n}(x)$ is the exterior normal to $S$ and some further conditions of a non-local character involving $\mathbf{v}_t$. Let $p_0(x)$ be a solution of the Neumann problem

$$\nabla^2 p_0(x) = -\nabla \cdot \mathcal{A}(x,0,\frac{\partial}{\partial x})\mathbf{v}_0(x) + \nabla \cdot \mathbf{f}(x,0),$$

$$\frac{\partial p_0(x)}{\partial n}\Big|_{x\in S} = -\mathbf{n}\cdot\Big(\mathcal{A}(x,0,\frac{\partial}{\partial x})\mathbf{v}_0(x) - \mathbf{f}(x,0) + \mathbf{a}_t(x.0)\Big) \tag{1.5}$$

(if $\mathbf{v}$ does not possess third derivatives and $\mathbf{f}$ does not have first ones, the first equation in (1.5) should be understood in a weak sense). The compatibility condition reads

$$\mathbf{a}_t(x,0) + \mathcal{A}(x,0,\frac{\partial}{\partial x})\mathbf{v}_0(x) + \nabla p_0(x) = \mathbf{f}(x,0), \quad \forall x \in S \tag{1.6}$$

(the boundary condition in (1.5) is a normal projection of (1.6)). We often assume that

$$\nabla \cdot \mathbf{f}(x,t) = 0, \qquad \mathbf{f}(x,t) \cdot \mathbf{n}(x)|_{x \in S} = 0, \tag{1.7}$$

or, what is the same, $\int_\Omega \mathbf{f}(x,t) \cdot \nabla \varphi(x) dx = 0$ for arbitrary smooth $\varphi(x)$. In this case the terms with $\mathbf{f}$ in (1.5) drop out. If, in addition, $\mathbf{v}_0 = 0$ and $\mathbf{a} \cdot \mathbf{n} = 0$, then $\nabla p_0 = 0$, and (1.6) takes the form

$$\mathbf{a}_t(x,0) - \mathbf{f}(x,0) = 0, \quad x \in S. \tag{1.8}$$

As for the coefficients of the operator $\mathcal{A}$, we assume that they are bounded and satisfy the Hölder condition with the exponent $\alpha \in (0,1)$ with respect to $x$ and with the exponent $\alpha_1/2$, $\alpha_1 \in (\alpha, 1)$, with respect to $t$; moreover, the leading coefficients have bounded derivatives with respect to $x_k$. Under these conjectures, we prove the following theorem.

**Theorem 1.1.** *Let $S \in C^{2+\alpha}$, $\alpha \in (0,1)$, and let the operator $\mathcal{A}$ satisfy the above conjectures. Assume also that $\mathbf{f} \in C^{\alpha,\alpha/2}(Q_T)$, $\mathbf{v}_0 \in C^{2+\alpha}(\Omega)$, $\mathbf{a} \in C^{2+\alpha,1+\alpha/2}(\Sigma_T)$, where $Q_T = \Omega \times (0,T)$, $\Sigma_T = S \times (0,T)$, and that*

$$\sum_{k=1}^{n} [\mathcal{R}_k(\mathbf{a}_t \cdot \mathbf{n})]_{t,\Sigma_T}^{(\alpha/2)} < \infty,$$

*where*

$$\mathcal{R}_k(b) = -2\partial_k \int_S E(x-y) b(y) dS,$$

*$E$ is the fundamental solution of the Laplace equation, i.e.,*

$$E(x) = -\Gamma(n/2)|x|^{2-n}(2\pi^{n/2}(n-2))^{-1}, \ \ if \ \ n > 2,$$

$$E(x) = (2\pi)^{-1}\log|x|, \ \ if \ \ n = 2;$$

*$\partial_k = \frac{\partial}{\partial x_k} - n_k \frac{\partial}{\partial n}$ is the kth component of the surface gradient $\nabla_S$ on $S$. Finally, let conditions (1.4)-(1.7) be satisfied. Then problem (1.1), (1.2) has a unique solution $(\mathbf{v}, p)$, $\mathbf{v} \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$, and it satisfies the inequality*

$$|\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_T)} + |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}$$

$$+ |\mathbf{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n} [\mathcal{R}_k(\mathbf{a}_t \cdot \mathbf{n})]_{t,\Sigma_T}^{(\alpha/2)}\Big). \tag{1.9}$$

The operators $\mathcal{R}_k$, which may be considered as the Riesz operators on $S$ are continuous in the space $C^\alpha(S)$ but not in $C^{\alpha,\alpha/2}(\Sigma_T)$. However, estimate (1.9) is coercive in the sense that its second term can be majorized by the first term multiplied by a certain constant (see inequality (2.52) below) . If (1.7) does not hold or the equation $\nabla \cdot \mathbf{v} = 0$ in (1.1) is replaced with $\nabla \cdot \mathbf{v} = g \neq 0$, then the coercivity is generally lost (see Sec.5).

If (1.7) does not hold, then we can use the Weyl orthogonal decomposition

$$\mathbf{f} = \mathbf{f}_1 + \nabla \phi \equiv P_J \mathbf{f} + P_G \mathbf{f} \tag{1.10}$$

where $\mathbf{f}_1$ satisfies (1.7), and add $-\phi$ to the pressure. If $\mathbf{f}_1 = P_J\mathbf{f} \in C^{\alpha,\alpha/2}(Q_T)$ (which is not always the case), then a solution $\mathbf{v}, p_1 = p - \phi$ of the transformed problem exists and satisfies (1.9) with $P_J\mathbf{f}$ instead of $\mathbf{f}$. This implies the following estimate, which is often used below:

$$|\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_T)} + [P_G\mathbf{f}]^{(\alpha/2)}_{t,Q_T}$$

$$+|\mathbf{v}_0|_{C^{2+\alpha}(\Omega)} + |\mathbf{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n}[\mathcal{R}_k(\mathbf{a}_t\cdot\mathbf{n})]^{(\alpha/2)}_{t,\Sigma_T}\Big). \tag{1.11}$$

It is a consequence of (1.9) and of the boundedness of the projectors $P_G$ and $P_J$ in $C^\alpha(\Omega)$.

As in the case of the Stokes system, the pressure $p(x,t)$ is defined up to a summand $P(t)$, which can be fixed, for instance, by the requirement

$$\int_\Omega p(x,t)dx = 0$$

or by another normalization condition.

Inequality (1.9) is proved by the method of localization due to J.Schauder supplemented with the estimate of a certain mixed norm of the pressure (of the type of the norm in $C^{1/2+\alpha/2-\mu/2}(0,T;C^\mu(\Omega))$, which is denoted here by $|||p|||^{(\mu_1,\alpha_1/2)}_{(Q_T)}$ (see Sec. 2). For the Stokes problem (i.e., when $\mathcal{A} = -\nu I\nabla^2$), (1.9) was obtained by this method in [1] in the case $\mathbf{a}\cdot\mathbf{n} = 0$. This result was preceded by the construction and estimate of the solution of the half-space problem in [2,3]. For the generalized system with $\mathcal{A}(x,t,\frac{\partial}{\partial x}) = \mathcal{A}_0(\frac{\partial}{\partial x})$, the Cauchy problem was studied in [4–6] and the half-space Cauchy-Dirichlet problem was investigated in [7, 8]. Coercive estimates of the solution of both problems in the Sobolev spaces were proved in [6, 9].

It is natural to expect that a coercive estimate for the linearized problem enables one possible to prove the local solvability of a nonlinear problem. This is done at the end of the paper, in Sec. 6.

The results presented in [1-3] were obtained as early as in the 60s at the laboratory of mathematical physics of the Leningrad Branch of V.A.Steklov Mathematical Institute headed by Professor O.A.Ladyzhenskaya, and of course they were motivated by her activity. Professor Ladyzhenskaya was interested in coercive estimates for the Stokes problem until the last years of her life. It is also necessary to mention important contributions of K.K.Golovkin who also was one of her first students and collaborators (unfortunately, he died in 1969). His papers [10-12] have a direct relation to the problem studied here.

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## 2. Auxiliary propositions.

This section contains some auxiliary estimates in Hölder norms obtained by the methods of the potentials theory. We recall that $C^l(\Omega)$ with non-integral positive $l = [l] + \lambda$, $\lambda \in (0,1)$, is the Hölder space of functions (or vector fields) given in $\Omega$ and possessing finite norm

$$|u|_{C^l(\Omega)} = \sum_{0\le|j|\le[l]}\sup_\Omega|D^j u(x)| + [u]^{(l)}_\Omega,$$

where $j=(j_1,...,j_n)$, $|j|=j_1+...+j_n$, $D^ju(x)=\frac{\partial^{|j|}u(x)}{\partial x_1^{j_1}...\partial x_n^{j_n}}$,

$$[u]_\Omega^{(l)}=\sum_{|j|=[l]}[D^ju]_\Omega^{(\lambda)},\quad [v]_\Omega^{(\lambda)}=\sup_{x,y\in\Omega}\frac{|v(x)-v(y)|}{|x-y|^\lambda},$$

and $C^{l,l/2}(Q_T)$, $Q_T=\Omega\times(0,T)$, is the space of functions depending on $(x,t)\in Q_T$ with the norm

$$|u|_{C^{l,l/2}(Q_T)}=\sum_{0\le 2\,k+|j|\le[l]}\sup_{Q_T}|D_t^kD^ju(x,t)|+[u]_{Q_T}^{(l,l/2)},$$

where

$$[u]_{Q_T}^{(l,l/2)}=[u]_{x,Q_T}^{(l)}+[u]_{t,Q_T}^{(l/2)},$$

$$[u]_{x,Q_T}^{(l)}=\sup_{t<T}[u(\cdot,t)]_\Omega^{(l)},\quad [u]_{t,Q_T}^{(l/2)}=\sup_\Omega[u(x,\cdot)]_{(0,T)}^{(l/2)}.$$

In addition, we introduce the norm

$$|||u|||_{Q_T}^{(\mu,\beta)}=\langle u\rangle_{Q_T}^{(\mu,\beta)}+[u]_{t,Q_T}^{(\beta)}$$

with

$$\langle u\rangle_{Q_T}^{(\mu,\beta)}=\sup_{t,t'\in(0,T)}|t-t'|^{-\beta}[u(\cdot,t)-u(\cdot,t')]_\Omega^{(\mu)},$$

$\mu,\beta\in(0,1)$. We mainly work in the Hölder spaces with $l=k+\alpha$, $\alpha\in(0,1)$, $k=0,1,2$.

The following estimates are used below:

$$[\nabla u]_{t,Q_T}^{(\frac{1+\alpha}{2})}\le c\Big([u]_{x,Q_T}^{(2+\alpha)}+[u_t]_{x,Q_T}^{(\alpha)}\Big),$$

$$[D^ju]_{t,Q_T}^{(\frac{\alpha}{2})}\le c\Big([u]_{x,Q_T}^{(2+\alpha)}+[u_t]_{x,Q_T}^{(\alpha)}\Big),\quad |j|=2,$$

$$\langle u\rangle_{Q_T}^{(\mu,\beta/2)}\le c[u]_{Q_T}^{(\mu+\beta,\mu/2+\beta/2)}. \tag{2.1}$$

They follow from the interpolation inequalities

$$\sup_\Omega|\nabla w(x,t)|\le c\Big(h^{(1+\alpha)/2}[w(\cdot,t)]_\Omega^{(2+\alpha)}+h^{-1/2+\alpha/2}[w(\cdot,t)]_\Omega^{(\alpha)}\Big),$$

$$\sup_\Omega|D^jw(x,t)|\le c\Big(h^{\alpha/2}[w(\cdot,t)]_\Omega^{(2+\alpha)}+h^{-1+\alpha/2}[w(\cdot,t)]_\Omega^{(\alpha)}\Big),$$

$$[w(\cdot,t)]_\Omega^{(\mu)}\le c\Big(h^{\beta/2}[w(\cdot,t)]_\Omega^{(\mu+\beta)}+h^{-\mu/2}\sup_\Omega|w(x,t)|\Big)$$

applied to

$$w(x,t)=\Delta_t(h)u(x,t)\equiv u(x,t+h)-u(x,t),\quad h\in(0,T-t).$$

Indeed, for instance, the last inequality implies

$$[\Delta_t(h)u(\cdot,t)]_\Omega^{(\mu)}\le ch^{\beta/2}\Big([u]_{x,Q_T}^{(\mu+\beta)}+[u]_{t,Q_T}^{(\mu/2+\beta/2)}\Big)$$

which yields (2.1). Finally, we use the well-known estimate

$$[u_t]^{(\alpha)}_{x,Q_T} \le c|u|_{C^{2+\alpha,1+\alpha/2}(Q_T)}$$

(in many cases, in particular, when $\Omega = R^n$ or $\Omega = R^n_+$, the norm in the right-hand side can be replaced with $[u]^{(2+\alpha,1+\alpha/2)}_{Q_T}$). The proof of this estimate can be found, for instance, in [3], Lemma 2.

We shall make use of some classical estimates for the volume and surface potentials. At first we consider a volume potential in the space or in the half-space of the form

$$V(x) = \int_R K(x,t)u(y)dy = \lim_{\epsilon\to 0}\int_{R\setminus B_\epsilon(x)} K(x,y)u(y)dy, \quad x \in R, \tag{2.2}$$

where $R = R^n$ or $R = R^n_+$ and $B_r(x) = \{y \in R : |x-y| \le r\}$. We assume that $u$ decays at infinity sufficiently rapidly, so that the integral in (2.2) is convergent.

**Proposition 2.1.** *If the kernel $K(x,y)$ satisfies the inequalities*

$$|K(x,y)| \le c|x-y|^{-n}, \quad |\nabla_x K(x,y)| \le c|x-y|^{-n-1}, \tag{2.3}$$

*the integrals*

$$\int_{B_r(x)} K(x,y)dy = \lim_{\epsilon\to 0}\int_{B_r(x)\setminus B_\epsilon(x)} K(x,y)dy$$

*are uniformly bounded:*

$$\Big|\int_{B_r(x)} K(x,y)dy\Big| \le c, \quad \forall x \in R,\ r > 0, \tag{2.4}$$

*and, finally,*

$$\int_R (K(x,y) - K(z,y))dy = 0, \quad \forall x, z \in R, \tag{2.5}$$

*then*

$$[V]^{(\alpha)}_R \le c[u]^{(\alpha)}_R. \tag{2.6}$$

Inequality (2.6) in proved exactly as (2.9) in [13], Chap.3. The hypotheses of the proposition are satisfied by $D^jE(x-y)$, $|j| = 2$, in $R^n$, by $D^jE(x-y)$ and $D^jE(x-y^*)$, $y^* = (y_1, ..., y_{n-1}, -y_n)$, in $R^n_+$, and, as a consequence, by $D^jN(x,y)$, where

$$N(x,y) = E(x-y) + E(x-y^*)$$

is the Green function for the Neumann problem in the half-space. Finally, they are also satisfied by the first derivatives of the vector-valued kernel

$$\mathbf{L}(x,y) = \nabla_x(N(x,y) - 2x_n\frac{\partial}{\partial x_n}E(x-y^*)) + 2\nabla_y E(x-y^*), \quad x, y \in R^n_+.$$

The integral

$$\mathbf{v}(x) = \int_{R^n_+} \mathbf{L}(x,y)g(y)dy \tag{2.7}$$

solves the problem

$$-\nabla^2 \mathbf{v}(x) + \nabla p(x) = 0, \quad \nabla \cdot \mathbf{v}(x) = g(x), \quad \mathbf{v}|_{x_n=0} = 0$$

in $R^n_+$. The corresponding pressure equals

$$p(x) = g(x) - 4\frac{\partial^2}{\partial x_n^2} \int_{R^n_+} E(x - y^*) g(y) dy.$$

The kernel $\mathbf{L}$ can be represented in the form

$$\mathbf{L}(x, y) = \sum_{k=1}^{n} \frac{\partial \mathbf{L}_k(x, y)}{\partial x_k},$$

where $\mathbf{L}_k = (L_{jk})_{j=1,\dots,n}$ and

$$L_{jk}(x, y) = \delta_{jk}(N(x, y) - 2x_n \frac{\partial}{\partial x_n} E(x - y^*) - 2E(x - y^*)) + 4\delta_{jn}\delta_{kn} E(x - y^*). \quad (2.8)$$

Next, we consider the potential

$$v_1(x) = \int_R K_1(x, y) \mathbf{a} \cdot \nabla U(y) dy, \quad \mathbf{a} = constant. \quad (2.9)$$

**Proposition 2.2.** *If*

$$|K_1(x, y)| \le c|x - y|^{-n+1}$$

*and the derivatives of $K_1$ with respect to both arguments satisfy (2.3)-(2.5), then*

$$[v_1]^{(\mu)}_R \le c|\mathbf{a}|[U]^{(\mu)}_R, \quad \forall \mu \in (0, 1). \quad (2.10)$$

The proof is easily obtained from the representation formula

$$\begin{aligned} v_1(x) - v_1(z) = & \int_{B_r(x)} ((\mathbf{a} \cdot \nabla_y) K_1(x, y))(U(x) - U(y)) dy \\ & + \int_{\partial B_r(x)} (\mathbf{a} \cdot \mathbf{n}(y)) K_1(x, y)(U(y) - U(x)) dS \\ & - \int_{B_r(x)} ((\mathbf{a} \cdot \nabla_y) K_1(z, y))(U(z) - U(y)) dy \\ & - \int_{\partial B_r(x)} (\mathbf{a} \cdot \mathbf{n}(y)) K_1(z, y)(U(y) - U(z)) dS \\ & + \int_{R \setminus B_r(x))} ((\mathbf{a} \cdot \nabla_y)(K_1(x, y) - K_1(z, y)))(U(z) - U(y)) dy \end{aligned}$$

$$+\int_{\partial(R\setminus B_r(x))}(\mathbf{a}\cdot\mathbf{n}(y))(K_1(x,y)-K_1(z,y))(U(y)-U(z))dS, \tag{2.11}$$

where $r=2|x-z|$.

It follows from (2.6), (2.10) and from interpolation inequality that

$$\sup_R|\nabla v_1(x)|\le c|\mathbf{a}|\Big(r^{\alpha}[U]_R^{(1+\alpha)}+r^{-1+\mu}[U]_R^{(\mu)}\Big) \tag{2.12}$$

with $\alpha,\mu\in(0,1)$ and with arbitrary $r>0$.

The kernel $K_1$ can be equal to the first derivative of $E(x-y)$ in $R^n$ or in $R^n_+$, to the first derivative of $E(x-y^*)$ or of $L_{kj}$ in $R^n_+$.

Let us consider the Newtonian and single layer potentials

$$v(x)=\int_\Omega E(x-y)u(y)dy,\qquad w(x)=\int_S E(x-y)\varphi(y)dS. \tag{2.13}$$

We recall classical estimates of the theory of potentials:

$$|v|_{C^{2+\alpha}(\Omega)}\le c|u|_{C^{\alpha}(\Omega)}, \tag{2.14}$$

$$|w|_{C^{1+\alpha}(\Omega)}\le c|\varphi|_{C^{\alpha}(S)}. \tag{2.15}$$

In addition, the following proposition holds.

**Proposition 2.3.** *1. If*

$$u(x)=\mathbf{a}(x)\cdot\nabla U(x) \tag{2.16}$$

*with* $\mathbf{a}\in C^1(\Omega)$*, we have*

$$|\nabla v|_{C^{\mu}(\Omega)}\le c\Big(\sup_\Omega|\mathbf{a}(x)||U|_{C^{\mu}(\Omega)}+|\mathbf{a}|_{C^1(\Omega)}\sup_\Omega|U(x)|\Big),\quad\forall\mu\in(0,1). \tag{2.17}$$

*2. If* $\varphi(x)=\mathrm{div}_S\Phi(x)$*, where* $\Phi(x)$ *is a tangential vector field on* $S$*, then*

$$|w|_{C^{\mu}(\Omega)}\le c|\Phi|_{C^{\mu}(S)}. \tag{2.18}$$

**Proof.** Estimate (2.17) follows from the representation formula for $\nabla v(x)$ and for $\nabla v(x)-\nabla v(z)$ similar to (2.11), namely,

$$\begin{aligned}\nabla v(x)=-\sum_{k=1}^n\Big(&\int_\Omega\frac{\partial}{\partial y_k}(a_k(y)\nabla E(x-y))(U(y)-U(x))dy\\&-\int_{\partial\Omega}a_k(y)n_k(y)\nabla E(x-y)(U(y)-U(x))dS\Big)\\&=-\int_\Omega\nabla_y\cdot(\mathbf{a}(y)\otimes\mathcal{E}(x,y))(U(y)-U(x))dy\\&+\int_{\partial\Omega}(\mathbf{a}(y)\cdot\mathbf{n}(y)\mathcal{E}(x-y)(U(y)-U(x))dS,\end{aligned}$$

$$\nabla v(x) - \nabla v(z) = \int_{B_r(x)} (\nabla_y \cdot (\mathbf{a}(y) \otimes \mathcal{E}(x,y)))(U(x) - U(y))dy$$
$$+ \int_{\partial B_r(x)} (\mathbf{a}(y) \cdot \mathbf{n}(y))\mathcal{E}(x,y)(U(y) - U(x))dS$$
$$- \int_{B_r(x)} (\nabla_y \cdot (\mathbf{a}(y) \otimes \mathcal{E}(z,y)))(U(z) - U(y))dy$$
$$- \int_{\partial B_r(x)} (\mathbf{a}(y) \cdot \mathbf{n}(y))\mathcal{E}(z,y)(U(y) - U(z))dS$$
$$+ \int_{\Omega \setminus B_r(x))} (\nabla_y \cdot (\mathbf{a}(y) \otimes (\mathcal{E}(x,y) - \mathcal{E}(z,y))))(U(z) - U(y))dy$$
$$+ \int_{\partial(\Omega \setminus B_r(x))} (\mathbf{a}(y) \cdot \mathbf{n}(y))(\mathcal{E}(x,y) - \mathcal{E}(z,y))(U(y) - U(z))dS, \tag{2.19}$$

where $B_r(x) = \{y \in R : |x-y| \le r\}$, $r = 2|x-z|$, $\mathcal{E}(x,y) = \nabla E(x-y)$. The second statement of the proposition follows from (2.15), since

$$w(x) = -\int_S \nabla_S E(x-y) \cdot \Phi(y) dS.$$

**Remark.** Inequality (2.17) holds also for the potential (2.9) in the domain $\Omega = R$, if $\mathbf{a} = \mathbf{a}(y)$ or $U$ has a compact support. We have

$$|v_1|_{C^\mu(R)} \le c\Big( \sup_R |\mathbf{a}(x)| |U|_{C^\mu(R)} + |\mathbf{a}|_{C^1(R)} \sup_R |U(x)| \Big). \tag{2.20}$$

The proof is essentially the same. It is based on the representation formulas of the type (2.19) where $\Omega$ is a bounded domain containing $\operatorname{supp} u(x)$.

The next two propositions concern the Weyl orthogonal decomposition of an arbitrary vector field in the sum of solenoidal and potential vector fields,

$$\mathbf{u}(x) = \mathbf{v}(x) + \nabla\phi(x) \equiv P_J \mathbf{u} + P_G \mathbf{u}, \qquad x \in \Omega, \tag{2.21}$$

where $\phi(x)$ is a solution of the Neumann problem

$$\nabla^2 \phi(x) = \nabla \cdot \mathbf{u}(x), \quad x \in \Omega, \qquad \frac{\partial \phi(x)}{\partial n}\Big|_S = \mathbf{u} \cdot \mathbf{n}. \tag{2.22}$$

Hence, $\mathbf{v}(x)$ satisfies the conditions (1.7).

In the case of the half-space, $\phi(x)$ is given by

$$\phi(x) = -\int_{R^n_+} \nabla_y N(x,y) \cdot \mathbf{u}(y) dy, \quad x \in R^n_+,$$

and if $\Omega = R^n$,

$$\phi(x) = \int_{R^n} \nabla E(x-y) \cdot \mathbf{u}(y) dy.$$

It is well known that

$$|\mathbf{v}|_{C^\alpha(\Omega)} + |\nabla\phi|_{C^\alpha(\Omega)} \le c|\mathbf{u}|_{C^\alpha(\Omega)}, \tag{2.23}$$

(see, for instance, [14]), and in the case $\Omega = R^n$ or $\Omega = R^n_+$

$$[\mathbf{v}]^{(\alpha)}_\Omega + [\nabla\phi]^{(\alpha)}_\Omega \le c[\mathbf{u}]^{(\alpha)}_\Omega. \tag{2.24}$$

For the $C^{\alpha,\alpha/2}(Q_T)$- norms of vector fields depending on $(x,t) \in Q_T$ analogous estimates do not hold but it is possible to estimate the Hölder constant of $\mathbf{v}(x,t)$ with respect to $t$, if $\mathbf{u}$ has some extra regularity or a special structure. The proof is based on the following inequalities for the function $\phi$ in (2.22).

**Proposition 2.4.** *Let $S \in C^{2+\alpha}$. The solution of problem (2.22) satisfies the inequality*

$$|\phi|_{C^\mu(\Omega)} \le c \sup_\Omega |\mathbf{u}(x)|, \qquad \forall \mu \in (0,1), \tag{2.25}$$

*and if*

$$\mathbf{u}(x) = a(x)\nabla U(x) = \Big(\sum_{k=1}^n a_{jk}(x)\frac{\partial U(x)}{\partial x_k}\Big)_{j=1,\dots,n} \tag{2.26}$$

*with $a_{jk} \in C^1(\Omega)$, then*

$$|\phi|_{C^\mu(\Omega)} \le c\Big(\sup_\Omega |a(x)||U|_{C^\mu(\Omega)} + |a|_{C^1(\Omega)} \sup_\Omega |U(x)|\Big), \quad \forall \mu \in (0,1), \tag{2.27}$$

*where* $\sup_\Omega |a(x)| = \max_{k,j=1,\dots,n} \sup_\Omega |a_{kj}(x)|$, $|a|_{C^1(\Omega)} = \max_{k,j=1,\dots,n} |a_{kj}|_{C^1(\Omega)}$.

**Proof.** Let

$$\phi_1(x) = \int_\Omega \nabla E(x-y)\cdot\mathbf{u}(y)dy.$$

It is clear that $\nabla^2\phi_1(x) = \nabla\cdot\mathbf{u}(x)$ and

$$|\phi_1|_{C^\mu(\Omega)} \le c \sup_\Omega |\mathbf{u}(x)|, \qquad \forall \mu \in (0,1);$$

moreover, if (2.26) holds, then

$$|\phi_1|_{C^\mu(\Omega)} \le c\Big(\sup_\Omega |a(x)||U|_{C^\mu(\Omega)} + |a|_{C^1(\Omega)} \sup_\Omega |U(x)|\Big). \tag{2.28}$$

The difference $\phi_2 = \phi - \phi_1$ is a solution to the problem

$$\nabla^2\phi_2(x) = 0, \quad x \in \Omega, \quad \frac{\partial\phi_2}{\partial n}\Big|_S = \mathbf{u}\cdot\mathbf{n} - \frac{\partial\phi_1}{\partial n}\Big|_S \equiv \psi(x), \tag{2.29}$$

where

$$\frac{\partial\phi_1}{\partial n} = \sum_{m,k=1}^n n_m \frac{\partial^2 R_k(x)}{\partial x_m \partial x_k}, \qquad R_k(x) = \int_\Omega E(x-y)u_k(y)dy.$$

Since

$$\nabla^2 R_k = u_k,$$

we have

$$\psi(x) = \sum_{k,q=1}^{n} \Big( n_k \frac{\partial}{\partial x_q} - n_q \frac{\partial}{\partial x_k} \Big) \frac{\partial R_k}{\partial x_q} = \mathrm{div}_S \mathbf{M}(x) \tag{2.30}$$

with

$$\mathbf{M}(x) = \sum_{k,q=1}^{n} n_k \mathbf{e}_q \Big( \frac{\partial R_k}{\partial x_q} - \frac{\partial R_q}{\partial x_k} \Big), \quad \mathbf{e}_q = (\delta_{jq})_{j=1,\dots,n}.$$

By (2.28),

$$|\mathbf{M}|_{C^\mu(S)} \le c \Big( \sup_\Omega |a(x)| |U|_{C^\mu(\Omega)} + |a|_{C^1(\Omega)} \sup_\Omega |U(x)| \Big). \tag{2.31}$$

Neumann problem (2.29) whose boundary data have a structure similar to (2.30) was considered in [14]. By repeating the arguments in [14], Lemma 6.3 (they are essentially the same as in Proposition 2.7 below) it can be shown that

$$|\phi_2|_{C^\mu(\Omega)} \le c |\mathbf{M}|_{C^\mu(S)}.$$

Together with (2.28) and (2.31), this inequality yields (2.27). The proposition is proved.

Now, we assume that $\mathbf{u}$ depends on $x$ and $t$.

**Proposition 2.5.** *1. If* $\mathbf{u}$ *has a finite norm* $|||\mathbf{u}|||_{Q_T}^{(\rho,\beta)}$ *with some* $\rho, \beta \in (0,1)$, *then*

$$[\nabla \phi]_{t,Q_T}^{(\beta)} \le c |||\mathbf{u}|||_{Q_T}^{(\rho,\beta)}. \tag{2.32}$$

*2. Assume that* (2.26) *holds with* $a_{jk} = a_{jk}(x,t)$ *having finite norm*

$$\sup_{t<T} |a_{jk}(\cdot,t)|_{C^1[\Omega)} + [a_{jk}]_{t,Q_T}^{(\alpha_1/2)}, \quad \alpha_1 \in (\alpha, 1). \tag{2.33}$$

*Then*

$$|||\phi|||_{Q_T}^{(\mu,\beta)}$$

$$\le c \Big( \sup_{Q_T} |a(x,t)| |||U|||_{Q_T}^{(\mu,\beta)} + \sup_{t<T} |a(\cdot,t)|_{C^1(\Omega)} [U]_{t,Q_T}^{(\beta)} + [a]_{t,Q_T}^{(\beta)} \sup_{Q_T} |\nabla U(x,t)| \Big), \tag{2.34}$$

$\forall \mu \in (0,1), \ \beta \in (0, \alpha_1/2]$, *and*

$$[\nabla \phi]_{t,Q_T}^{(\alpha/2)} \le c \Big( \sup_{t<T} |\mathbf{u}(\cdot,t)|_{C^\alpha(\Omega)}$$

$$+ \sup_{Q_T} |a(x,t)| |||U|||_{Q_T}^{(\mu_1,\alpha_1/2)} + \sup_{t<T} |a(\cdot,t)|_{C^1(\Omega)} [U]_{t,Q_T}^{(\alpha_1/2)} + [a]_{t,Q_T}^{(\alpha_1/2)} \sup_{Q_T} |\nabla U(x,t)| \Big), \tag{2.35}$$

*where* $\alpha \in (0, \alpha_1)$, $\mu_1 = 1 + \alpha - \alpha_1 \in (0,1)$.

**Proof.** We apply Proposition 2.4 to $\Delta_t(h)\mathbf{u}(x,t) = \mathbf{u}(x,t+h) - \mathbf{u}(x,t)$. By (2.23),

$$\sup_\Omega |\Delta_t(h) \nabla \phi(x,t)| \le |\Delta_t(h) \nabla \phi(\cdot,t)|_{C^\rho(\Omega)} \le c |\Delta_t(h) \mathbf{u}(\cdot,t)|_{C^\rho(\Omega)}$$

which implies (2.32). If (2.26) holds, we have

$$\Delta_t(h)\mathbf{u}(x,t) = (\Delta_t(h)a(x,t))\nabla U(x,t+h) + a(x,t)\nabla\Delta_t(h)U(x,t)$$

$$\equiv \mathbf{u}'(x,t) + \mathbf{u}''(x,t).$$

Let $\nabla\phi' = P_G\mathbf{u}'$ and $\nabla\phi'' = P_G\mathbf{u}''$. By (2.25) and (2.27) ,

$$|\phi'(\cdot,t)|_{C^\mu(\Omega)} \le ch^\beta [a]^{(\beta)}_{t,Q_t} \sup_\Omega |\nabla U(x,t+h)|,$$

$$|\phi''(\cdot,t)|_{C^\mu(\Omega)}$$

$$\le c\Big(\sup_\Omega |a(x,t)||\Delta_t(h)U(\cdot,t)|_{C^\mu(\Omega)} + |a(\cdot,t)|_{C^1(\Omega)} \sup_\Omega |\Delta_t(h)U(x,t)|\Big).$$

These inequalities imply (2.34). Finally, we make use of the interpolation inequality

$$\sup_\Omega |\nabla f(x)| \le c\Big(h^{\alpha/2}[\nabla f]^{(\alpha)}_\Omega + h^{-(1-\mu_1)/2}[f]^{(\mu_1)}_\Omega\Big) \tag{2.36}$$

with $f = \Delta_t(h)\phi(x,t)$ and obtain

$$[\nabla\phi]^{(\alpha/2)}_{t,Q_T} \le c\Big([\nabla\phi]^{(\alpha)}_{x,Q_T} + |||\phi|||^{(\mu_1,\alpha_1/2)}_{Q_T}\Big)$$

which yields (2.35), in view of (2.23), (2.34). The proposition is proved.

**Corollary.** *For arbitrary* $\mathbf{u} \in C^{\alpha,\alpha/2}(Q_T)$

$$|P_G\mathbf{u}|_{C^{\alpha,\alpha/2}(Q_T)} + |P_J\mathbf{u}|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(|\mathbf{u}|_{C^{\alpha,\alpha/2}(Q_T)} + |||\mathbf{u}|||^{(\rho,\alpha/2)}_{Q_T}\Big), \quad \forall\rho\in(0,1),$$

*and if* (2.26) *holds, then*

$$|P_G\mathbf{u}|_{C^{\alpha,\alpha/2}(Q_T)} + |P_J\mathbf{u}|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(\sup_{t<T}|\mathbf{u}(\cdot,t)|_{C^\alpha(\Omega)}$$

$$+\sup_{Q_T}|a(x,t)||||U|||^{(\mu_1,\alpha_1/2)}_{Q_T} + \sup_{t<T}|a(\cdot,t)|_{C^1(\Omega)}[U]^{(\alpha_1/2)}_{t,Q_T} + [a]^{(\alpha_1/2)}_{t,Q_T}\sup_{Q_T}|\nabla U(x,t)|\Big).$$

Estimates similar to (2.34), (2.35) hold for the potential

$$v_1(x,t) = \int_R K_1(x,y)u(y,t)dy,$$

where $u(x,t) = \mathbf{a}(x,t)\cdot\nabla U(x,t)$ is a function with a compact support and $a_i(x,t)$ satisfy the hypotheses of Proposition 2.5. We have

$$|||v_1|||^{(\mu,\beta)}_{R_T}$$

$$\le c\Big(\sup_{R_T}|\mathbf{a}(x,t)||||U|||^{(\mu,\beta)}_{R_T} + \sup_{t<T}|\mathbf{a}(\cdot,t)|_{C^1(\Omega)}[U]^{(\beta)}_{t,R_T} + [\mathbf{a}]^{(\beta)}_{t,Q_T}\sup_{R_T}|\nabla U(x,t)|\Big), \tag{2.37}$$

$\forall \mu \in (0,1)$, $\beta \in (0, \alpha_1/2]$, and

$$[\nabla v_1]^{(\alpha/2)}_{t,R_T} \le c\Big([\mathbf{u}]^{(\alpha)}_{x,R_T}$$

$$+ \sup_{R_T} |\mathbf{a}(x,t)| |||U|||^{(\mu_1,\alpha_1/2)}_{R_T} + \sup_{t<T} |\mathbf{a}(\cdot,t)|_{C^1(R)} [U]^{(\alpha_1/2)}_{t,R_T} + [\mathbf{a}]^{(\alpha_1/2)}_{t,R_T} \sup_{R_T} |\nabla U(x,t)|\Big), \quad (2.38)$$

where $\alpha \in (0,\alpha_1)$, $\mu_1 = 1 + \alpha - \alpha_1 \in (0,1)$, $R_T = R \times (0,T)$. As above, these estimates can be deduced from (2.6), (2.20) and (2.36).

If $\mathbf{a} = const$, then

$$[\nabla v_1]^{(\alpha/2)}_{t,R_T} \le c|\mathbf{a}|\Big([\nabla U]^{(\alpha)}_{x,R_T} + \langle U \rangle^{(\mu_1,\alpha_1/2)}_{R_T}\Big) \le c|\mathbf{a}|[U]^{(1+\alpha,1/2+\alpha/2)}_{R_T}. \quad (2.39)$$

This estimate follows from (2.6), (2.10), (2.36). The function $U$ does not need to have a compact support.

Finally, for the surface potential $w$ in (2.13) with $\varphi(x,t) = \mathrm{div}_S \Phi(x,t)$ the inequality

$$[\nabla w]^{(\alpha/2)}_{t,Q_T} \le c\Big(\sup_{t<T} |\varphi(\cdot,t)|_{C^\alpha(S)} + |||\Phi|||^{(\mu_1,\alpha_1/2)}_{\Sigma_T}\Big) \quad (2.40)$$

holds. It is a consequence of (2.15) and (2.18).

Inequalities (2.34), (2.35) make it possible to obtain important estimates of the pressure in problem (1.1), (1.2). If $\mathbf{a} \cdot \mathbf{n} = 0$ and (1.7) holds, then $p(x,t)$ can be considered as a solution of the Neumann problem

$$\nabla^2 p(x,t) = -\nabla \cdot \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t), \quad x \in \Omega,$$

$$\frac{\partial p}{\partial n}\Big|_S = -\mathbf{n}(x) \cdot \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t),$$

in other words,

$$\nabla p = -P_G \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t). \quad (2.41)$$

**Proposition 2.6.** *The function $p(x,t)$ defined by (2.41) satisfies the inequalities*

$$|\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} + |||p|||^{(\mu_1,\alpha_1/2)}_{Q_T}$$

$$\le c\Big(\sup_{t<T} |\mathbf{v}_t(\cdot,t)|_{C^\alpha(\Omega)} + \sup_{t<T} |\mathbf{v}(\cdot,t)|_{C^{2+\alpha}(\Omega)}\Big) \quad (2.42)$$

*with $\mu_1 + \alpha_1 = 1 + \alpha$ and*

$$\sup_{Q_T} |p(x,\tau)| + [p]^{(\beta)}_{t,Q_T}$$

$$\le c\Big(\sum_{|j|\le 2} \sup_{Q_T} |D^j \mathbf{v}(x,\tau)| + |||\nabla \mathbf{v}|||^{(\rho,\beta)}_{Q_T} + |||\mathbf{v}|||^{(\rho,\beta)}_{Q_T}\Big), \quad (2.43)$$

*where $\beta \in (0,\alpha_1/2]$, $\rho > 0$.*

Indeed, $\nabla p = \nabla p_0 + \nabla p_1$, where

$$\nabla p_0 = -P_G \mathcal{A}_0(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t), \quad \nabla p_i = -P_G \mathcal{A}_1(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t),$$

$\mathcal{A}_1 = \mathcal{A} - \mathcal{A}_0$. Since $\mathcal{A}_0\mathbf{v}$ is a sum of the terms like (2.26) with $U = \frac{\partial v_j}{\partial x_k}$, we estimate $p_0$ by (2.34), (2.35), $p_1$ by (2.32), (2.34) and arrive at (2.42). Inequality (2.43) follows from

$$\sup_{Q_T} |p(x,\tau)| + [p]^{(\beta)}_{t,Q_T} \le \sup_{Q_T} |p(x,\tau)|_{C^\rho(\Omega)} + |||p|||^{(\rho,\beta)}_{Q_T},$$

where $\rho$ is a small positive number, and from (2.27), (2.34).

Let us estimate the solution of the problem

$$\nabla^2 q(x) = 0, \quad x \in \Omega, \quad \frac{\partial q}{\partial n}\Big|_S = b(x) \tag{2.44}$$

with $b(x)$ satisfying the necessary compatibility condition $\int_S b(x)dS = 0$. The theory of potentials furnishes the estimate

$$|q|_{C^{1+\alpha}(\Omega)} \le c|b|_{C^\alpha(S)}. \tag{2.45}$$

**Proposition 2.7.** *The solution of problem* (2.44) *satisfies the inequality*

$$\sup_\Omega |\nabla q(x)| \le c(\sup_S |b(x)| + \sum_{k=1}^n \sup_S |\mathcal{R}_k(b)|). \tag{2.46}$$

**Proof.** It is well known that $q$ can be represented in the form of the single layer potential

$$q(x) = -2\int_S E(x-y)\mu(y)dS$$

with $\mu(x)$ satisfying the integral equation

$$b(x) = -2\int_S \frac{\partial E(x-y)}{\partial n_x}\mu(y)dS + \mu(x) \tag{2.47}$$

and the orthogonality condition $\int_S \mu(x)dS = 0$. The function

$$\sigma(x) = -2\int_S \frac{\partial E(x-y)}{\partial n_x}\mu(y)dS$$

satisfies the same kind of equation

$$\sigma(x) - 2\int_S \frac{\partial E(x-y)}{\partial n_x}\sigma(y)dS = -2\int_S \frac{\partial E(x-y)}{\partial n_x}b(y)dS \equiv \tau(x)$$

and the same orthogonality condition. Since $\frac{\partial E}{\partial n}$ is a weakly singular kernel on $S$, we have

$$|\sigma|_{C^\rho(S)} \le c|\tau|_{C^\rho(S)} \le c\sup_S |b(y)|.$$

By virtue of (2.47),

$$q = q_1 + q_2, \tag{2.48}$$

where

$$q_1(x) = -2\int_S E(x-y)b(y)dS, \quad q_2(x) = 2\int_S E(x-y)\sigma(y)dS.$$

The second function satisfies the inequality

$$|\nabla q_2(x)| \le c|\sigma|_{C^\rho(S)} \le c\sup_S |b(x)| \tag{2.49}$$

and the derivatives of $q_1$ can be written in the form

$$\frac{\partial q_1(x)}{\partial x_k} = 2\int_S \frac{\partial E(x-y)}{\partial y_k} b(y)dS$$

$$= 2\int_S \frac{\partial E(x-y)}{\partial n_y} n_k(y)b(y)dS + 2\int_S \partial_{y_k} E(x-y)b(y)dS \equiv q_k'(x) + q_k''(x).$$

Since

$$\int_S \Big|\frac{\partial E(x-y)}{\partial n_y}\Big| dS \le c$$

with the constant independent of $x$, we have

$$|q_k'(x)| \le c\sup_S |b(y)|.$$

The function $q_k''$ is harmonic, so, due to the maximum principle,

$$|q_k''(x)| \le \sup_S |q_k''(x)| = \sup_S |\mathcal{R}_k(b)(x)| + c\sup_S |b(x)|.$$

Hence, (2.46) holds, and the proposition is proved.

**Corollary.** *If $b = b(x,t)$, then*

$$[\nabla q]^{(\alpha/2)}_{t,Q_T} \le c\Big([b]^{(\alpha/2)}_{t,\Sigma_T} + \sum_{k=1}^n [\mathcal{R}_k b]^{(\alpha/2)}_{t,\Sigma_T}\Big). \tag{2.50}$$

Let us go back to the estimates of the pressure in (1.1), (1.2). In the general case,

$$\nabla p = -P_G \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t) - \nabla q,$$

where $q$ is a solution of problem (2.44) with $b = \mathbf{a}_t \cdot \mathbf{n}$. By (2.45) and (2.50),

$$|\nabla p|_{C^{\alpha,\alpha/2}(Q_T)}$$

$$\le c\Big(\sup_{t<T} |\mathbf{v}_t(\cdot,t)|_{C^\alpha(\Omega)} + \sup_{t<T} |\mathbf{v}(\cdot,t)|_{C^{2+\alpha}(\Omega)} + \sum_{k=1}^n [\mathcal{R}_k(\mathbf{a}_t\cdot\mathbf{n})]^{(\alpha/2)}_{t,\Sigma_T}\Big). \tag{2.51}$$

Moreover, representing $q$ in the form (2.48) with

$$q_1 = -2\int_S E(x-y)\mathbf{a}_t(y,t)\cdot\mathbf{n}(y)dS$$

we find

$$\nabla_S q_1(x,t) = -\nabla_S p(x,t) - \nabla_S p'(x,t) - \nabla_S q_2(x,t), \quad x \in S,$$

where $\nabla p' = P_G\mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t)$. Since $\mathcal{R}_k(\mathbf{a}_t\cdot\mathbf{n})$ is the $k$th component of $\nabla_S q_1$, we obtain, by (2.42) and (2.49):

$$[\mathcal{R}_k(\mathbf{a}_t\cdot\mathbf{n})]^{(\alpha/2)}_{t,\Sigma_T} \le c\Big(\sup_{t<T}|\mathbf{v}_t(\cdot,t)|_{C^\alpha(\Omega)} + \sup_{t<T}|\mathbf{v}(\cdot,t)|_{C^{2+\alpha}(\Omega)} + [\nabla p]^{(\alpha/2)}_{t,Q_T}\Big). \tag{2.52}$$

This inequality proves the coercivity of the estimate (1.9).

Finally, we consider the stationary Stokes problem

$$-\nabla^2\mathbf{w}(x,t) + \nabla r(x,t) = 0, \quad \nabla\cdot\mathbf{w}(x,t) = 0, \quad x\in\Omega,$$

$$\mathbf{w}(x,t) = \mathbf{b}(x,t), \quad x\in S, \tag{2.53}$$

with the boundary data depending on the parameter $t\in(0,T)$. The solution satisfies the well-known coercive estimate

$$|\mathbf{w}(\cdot,t)|_{C^{2+\alpha}(\Omega)} \le c|\mathbf{b}(\cdot,t)|_{C^{2+\alpha}(S)}. \tag{2.54}$$

In addition, potentials theory due to F.Odquist (see [15-17]) furnishes

$$|\mathbf{w}(\cdot,t)|_{C^{\alpha}(\Omega)} \le c|\mathbf{b}(\cdot,t)|_{C^{\alpha}(S)} \tag{2.55}$$

and

$$\sup_{x\in\Omega}|\mathbf{w}(x,t)| \le c\sup_{x\in S}|\mathbf{b}(x,t)|. \tag{2.56}$$

These inequalities imply the following proposition.

**Proposition 2.8.** *If* $\mathbf{b}\in C^{2+\alpha,1+\alpha/2}(\Sigma_T)$, *then* $\mathbf{w}\in C^{2+\alpha,1+\alpha/2}(Q_T)$, *and*

$$|\mathbf{w}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \le c|\mathbf{b}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)}. \tag{2.57}$$

*Moreover,*

$$[P_G\mathbf{w}_t]^{(\alpha/2)}_{t,Q_T} \le c\Big([\mathbf{b}_t\cdot\mathbf{n}]^{(\alpha/2)}_{t,\Sigma_T} + \sum_{k=1}^n[\mathcal{R}_k(\mathbf{b}_t\cdot\mathbf{n})]^{(\alpha/2)}_{t,\Sigma_T}\Big). \tag{2.58}$$

It is clear that (2.57) follows from (2.54) and (2.56) applied to $\Delta_t(h)\mathbf{w}_t(x,t)$. Since $P_G\mathbf{w}_t = \nabla\Phi$, where $\Phi$ is a solution of problem (2.44) with $b = \mathbf{b}_t\cdot\mathbf{n}$, inequality (2.58) follows from (2.50).

## 3. Model problems in the whole space and in the half-space.

In this section we consider the Cauchy problem

$$\mathbf{v}_t + \mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{v} + \nabla p = \mathbf{f}(x,t), \qquad \nabla \cdot \mathbf{u} = 0, \qquad x \in R^n, \quad t \in (0,T), \tag{3.1}$$

$$\mathbf{v}(x,0) = 0, \tag{3.2}$$

and initial-boundary value problems

$$\mathbf{v}_t + \mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{v} + \nabla p = 0, \qquad \nabla \cdot \mathbf{v} = 0, \qquad x \in R^n_+ = \{x_n > 0\}, \quad t \in (0,T),$$

$$\mathbf{v}(x,0) = 0, \qquad \mathbf{v}(x,t)\Big|_{x_n=0} = \mathbf{b}(x',t), \qquad x' = (x_1,...,x_{n-1}) \in R^{n-1}, \tag{3.3}$$

$$\mathbf{v}_t + \mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{v} + \nabla p = \mathbf{f}(x,t), \qquad \nabla \cdot \mathbf{v} = 0, \qquad x \in R^n_+, \quad t \in (0,T),$$

$$\mathbf{v}(x,0) = 0, \qquad \mathbf{v}(x,t)\Big|_{x_n=0} = 0, \tag{3.4}$$

where $\mathcal{A}_0$ is a matrix second order differential elliptic operator with real constant coefficients containing only second order derivatives. Assume that $\mathcal{A}_0(i\xi) = \Big(a_{km}(i\xi)\Big)_{k,m=1,\dots,n}$ satisfies (1.3).

We recall the representation formulas for the solutions of problems (3.1)-(3.2), (3.3) obtained in [6-8]. We write system (3.1) in the form

$$\mathcal{L}(\frac{\partial}{\partial x}, \frac{\partial}{\partial t})(\mathbf{v}, p)^T = \mathbf{f}(x,t), \tag{3.5}$$

where

$$\mathcal{L}(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}) = \begin{pmatrix} I\frac{\partial}{\partial t} + \mathcal{A}_0(\frac{\partial}{\partial x}) & (\frac{\partial}{\partial x})^T \\ \frac{\partial}{\partial x} & 0 \end{pmatrix},$$

and we consider the matrix

$$\mathcal{L}(i\xi, s) = \begin{pmatrix} sI + \mathcal{A}_0(i\xi) & (i\xi)^T \\ i\xi & 0 \end{pmatrix}.$$

Let $L(i\xi, s) = \det \mathcal{L}(i\xi, s)$. It follows from the positivity of $\mathcal{A}_0(i\xi)$ that the roots of $L(i\xi, s), \forall \xi \in R^n \setminus \{0\}$, with respect to $s$ have negative real parts. Indeed, if $L(i\xi, s) = 0$, then the system $\mathcal{L}(i\xi, s)y = 0$ has a non-zero solution $y = (\eta, y_{n+1})$ with $\eta = (\eta_1, ..., \eta_n) \neq 0$ (in the case $\eta = 0$ the system reduces to $i\xi_j y_{n+1} = 0$, $j = 1, ..., n$ which implies $y_{n+1} = 0$, because $\xi \neq 0$). Hence, it can be assumed that $|\eta| = 1$. The system $\mathcal{L}(i\xi, s)y = 0$ yields

$$\xi \cdot \eta \equiv \sum_{j=1}^{n} \xi_j \eta_j = 0, \quad s\eta + \mathcal{A}_0(i\xi)\eta + i\xi y_{n+1} = 0,$$

and, as a consequence, $s|\eta|^2 + \bar{\eta} \cdot \mathcal{A}_0(i\xi)\eta = 0$ and

$$\mathrm{Re}\, s = -\mathrm{Re}(\bar{\eta} \cdot \mathcal{A}_0 \eta) \leq -\delta_0 |\xi|^2, \qquad \delta_0 > 0.$$

Hence,

$$|L(i\xi,s)| > c|\xi|^2(|s|+|\xi|^2)^{n-1}$$

for arbitrary $\xi \in R^n \setminus \{0\}$ and $s \in C$ with Re $s \geq -\delta_1|\xi|^2,\ \delta_1 \in (0,\delta_0)$.

The fundamental matrix of solutions of the system (3.5) is defined by a standard formula

$$\mathcal{T}(x,t) = (FL)^{-1}\mathcal{L}^{-1}(i\xi,s) = (T_{km})_{k,m=1,..,n+1},$$

where $FL$ means the Fourier transform with respect to $x$ and the Laplace transform with respect to $t$:

$$FLf(\xi,s) \equiv \widetilde{f}(\xi,s) = \int_0^\infty e^{-st}dt \int_{R^n} f(x,t)e^{-ix\cdot\xi}dx,$$

whereas $(FL)^{-1}$ denotes the inverse transformation. The solution of the Cauchy problem (3.1), (3.2) is given by

$$v_k(x,t) = \sum_{m=1}^n \int_0^t \int_{R^n} T_{km}(x-y,t-\tau)f_m(y,\tau)dyd\tau, \quad k=1,...,n, \tag{3.6}$$

$$p(x.t) = \sum_{m=1}^n \frac{\partial}{\partial x_m}\int_{R^n} E(x-y)f_m(y,t)dy$$

$$+ \sum_{m=1}^n \int_0^t \int_{R^n} T'_{n+1\,m}(x-y,t-\tau)f_m(y,\tau)dyd\tau. \tag{3.7}$$

Since $p(x,t)$ can be considered as a solution of the equation

$$\nabla^2 p(x,t) = \nabla\cdot\mathbf{f}(x,t) - \nabla\cdot\mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{v}(x,t), \qquad x\in R^n,$$

we have

$$\widetilde{p}(\xi,s) = -\sum_{m=1}^n \frac{i\xi_m}{|\xi|^2}\widetilde{f}_m(\xi,s) + \sum_{m=1}^n \widetilde{T}'_{n+1\,m}(\xi,s)\widetilde{f}(\xi,s),$$

where

$$\widetilde{T}'_{n+1\,m}(\xi,s) = \sum_{k,j=1}^n \frac{i\xi_j}{|\xi|^2}a_{jk}(i\xi)\frac{\widehat{L}_{km}(i\xi,s)}{L(i\xi,s)}$$

and $\widehat{L}_{km}$ are elements of the matrix $L(i\xi,s)\mathcal{L}^{-1}(i\xi,s)$. It is shown in [7] that

$$|D_t^k D_x^j T_{km}(x,t)| \leq \frac{c}{t^k(|x|^2+t)^{(n+|j|)/2}},$$

$$|D_t^k D_x^j T'_{n+1\,m}(x,t)| \leq \frac{c}{t^k(|x|^2+t)^{(n+1+|j|)/2}}$$

for arbitrary $t>0$; if $t<0$, then $T_{km}(x.t)=0$, $T'_{n+1\,m}(x,t)=0$. These inequalities make it possible to estimate functions (3.6) and (3.7) in Hölder norms (in [4,5] another method

was used). If $\mathbf{f}$ is divergence free, then the first integral in (3.7) drops out, and the solution of Cauchy problem (3.1), (3.2) satisfies the inequality

$$[\mathbf{v}]^{(2+\alpha,1+\alpha/2)}_{\Pi_T} + [\nabla p]^{(\alpha,\alpha/2)}_{\Pi_T} \le c[\mathbf{f}]^{(\alpha,\alpha/2)}_{\Pi_T}, \tag{3.8}$$

where $\Pi_T = R^n \times (0,T)$. In the general case, it is necessary to represent $\mathbf{f}$ in the form (1.10) (for $\Omega = R^n$) and to add $-\phi$ to the pressure. If $\mathbf{f}_1 = P_J\mathbf{f} \in C^{\alpha,\alpha/2}(R_T)$, instead of (3.8) we have

$$[\mathbf{v}]^{(2+\alpha,1+\alpha/2)}_{\Pi_T} + [\nabla p_1]^{(\alpha,\alpha/2)}_{\Pi_T} \le c[\mathbf{f}_1]^{(\alpha,\alpha/2)}_{\Pi_T},$$

$p_1 = p - \phi$, which implies

$$[\mathbf{v}]^{(2+\alpha,1+\alpha/2)}_{\Pi_T} + [\nabla p]^{(\alpha,\alpha/2)}_{\Pi_T} \le c\Big([\mathbf{f}]^{(\alpha,\alpha/2)}_{\Pi_T} + [P_G\mathbf{f}]^{(\alpha,\alpha/2)}_{\Pi_T}\Big)$$

$$\le c\Big([\mathbf{f}]^{(\alpha,\alpha/2)}_{\Pi_T} + [P_G\mathbf{f}]^{(\alpha/2)}_{t,\Pi_T}\Big). \tag{3.9}$$

Now, we consider initial-boundary value problem (3.3) under the assumption

$$\mathbf{b}(x',0) = \mathbf{b}_t(x',0) = 0.$$

The Fourier-Laplace transform with respect to $x'$ and $t$

$$\widehat{u}(\xi',s) \equiv F'Lu = \int_{R^{n-1}} e^{-i\xi'\cdot x'}dx' \int_0^\infty e^{-st}u(x',t)dt, \qquad \xi' \in R^{n-1},$$

reduces (3.3) to the boundary value problem for the system of ordinary differential equations on the half-line $\{x_n > 0\}$:

$$s\widehat{\mathbf{v}} + \mathcal{A}_0\Big(i\xi', \frac{\partial}{\partial x_n}\Big)\widehat{\mathbf{v}} + \widehat{\nabla}\widehat{p} = 0, \qquad \widehat{\nabla}\cdot\widehat{\mathbf{v}} = 0, \qquad x_n > 0,$$

$$\widehat{\mathbf{v}}(x_n) \to 0, \qquad \widehat{p}(x_n) \to 0, \qquad (x_n \to +\infty), \qquad \widehat{\mathbf{v}}|_{x_n=0} = \widehat{\mathbf{b}}, \tag{3.10}$$

where $\widehat{\nabla} = \Big(i\xi_1, ..., i\xi_{n-1}, \frac{d}{dx_n}\Big)$, $\widehat{\nabla}\cdot\widehat{\mathbf{v}} = i\xi_1\widehat{v}_1 + ... + i\xi_{n-1}\widehat{v}_{n-1} + \frac{dv_n}{dx_n}$, $\xi' = (\xi_1, ..., \xi_{n-1}) \in R^{n-1}\setminus\{0\}$, $s \in C$, $\mathrm{Re}\, s \ge -\delta_1|\xi'|^2$, $\delta_1 \in (0,\delta_0)$. This problem was analyzed in [7] in the case $n = 3$ and in [8] for $n \ge 2$, and it was shown that the solution of (3.3) can be represented in the form

$$v_k(x,t) = \sum_{m=1}^n \int_0^t \int_{R^{n-1}} T_{km}(x'-y', x_n, t-\tau)h_m(y',\tau)dy'd\tau, \qquad k = 1, ..., n, \tag{3.11}$$

$$p(x,t) = \int_{R^{n-1}} \nabla E(x'-y', x_n)\cdot\mathbf{h}(y',t)dy'$$

$$+ \sum_{m=1}^n \int_0^t \int_{R^{n-1}} T'_{n+1\,m}(x'-y', x_n, t-\tau)h_m(y',\tau)dy'd\tau \equiv p_1(x,t) + p_2(x,t),$$

where $h_m(x',t)$ satisfy the system of integral equations

$$\sum_{m=1}^{n}\int_0^t\int_{R^{n-1}} T_{km}(x'-y',0,t-\tau)h_m(y',\tau)dy'd\tau = b_k(x',t), \qquad k=1,...,n. \tag{3.12}$$

Moreover, it was proved that

$$[\mathbf{v}]_{R_T}^{(2+\alpha,1+\alpha/2)} + [\nabla p]_{R_T}^{(\alpha,\alpha/2)} \le c\Big([\mathbf{b}]_{R'_T}^{(2+\alpha,1+\alpha/2)} + \sum_{j=1}^{n-1}[R_j b_{nt}(x',\cdot)]_{t,R'_T}^{(\alpha/2)}\Big), \tag{3.13}$$

where $R'_T = R^{n-1}\times(0,T)$ and $R_j$ is the Riesz operator with the symbol $i\xi_j|\xi'|^{-1}$ :

$$R_j f = -2\frac{\partial}{\partial x_j}\int_{R^{n-1}} E(x'-y',0)f(y')dy'.$$

The pressure $p(x,t)$ can be considered as a solution of the Neumann problem

$$\nabla^2 p(x,t) = -\nabla\cdot\mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{v}(x,t), \qquad x_n>0,$$

$$\frac{\partial p(x,t)}{\partial x_n} = -\frac{\partial b_n(x',t)}{\partial t} - \sum_{m=1}^{n} a_{nm}(\frac{\partial}{\partial x})v_m(x,t), \qquad x_n=0,$$

that is given by the formula

$$p(x,t) = \int_{R^n_+}\nabla_y N(x,y)\cdot\mathcal{A}_0(\frac{\partial}{\partial y})\mathbf{v}(y,t)dy - 2\int_{R^{n-1}} E(x'-y',x_n)\frac{\partial b_n(y',t)}{\partial t}dy'.$$

Hence,

$$R_j\frac{\partial b_n(x',t)}{\partial t} = \frac{\partial p(x,t)}{\partial x_j}\Big|_{x_n=0} - \frac{\partial p'(x,t)}{\partial x_j}\Big|_{x_n=0},$$

where

$$p'(x,t) = \int_{R^n_+}\nabla_y N(x,y)\cdot\mathcal{A}_0(\frac{\partial}{\partial y})\mathbf{v}(y,t)dy,$$

and, by (2.39),

$$\Big[R_j\frac{\partial b_n(x',\cdot)}{\partial t}\Big]_{t,R'_T}^{(\alpha/2)} \le c\Big([\mathbf{v}]_{R_T}^{(2+\alpha,1+\alpha/2)} + [\nabla p]_{R_T}^{(\alpha,\alpha/2)}\Big).$$

This means that estimate (3.13) is coercive in the sense explained in Sec.1.

For the Stokes system (with $\nu=1$), the elements of the fundamental matrix are given by

$$T_{km}(x,t) = \delta_{km}\Gamma(x,t) - \frac{\partial^2}{\partial x_k\partial x_m}\int_{R^n}\Gamma(x-y,t)E(y)dy,$$

$$T_{n+1\,m}(x,t) = \delta(t)\frac{\partial E(x)}{\partial x_m}, \qquad T'_{n+1\,m}(x.t) = 0,$$

where $k, m = 1, ..., n$, $\delta(t)$ is the Dirac delta-function, and

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad t > 0,$$

is the fundamental solution of the heat equation. The solution of problem (3.10) has a simple explicit form:

$$\widehat{\mathbf{v}}(\xi', s, x_n) = \widehat{\mathbf{b}} e^{-r x_n} - \mathbf{k}(i\xi) \frac{\sum_{j=1}^{n-1} i\xi_j \widehat{b}_j - r\widehat{b}_n}{r - |\xi|} (e^{-r x_n} - e^{-|\xi| x_n}),$$

$$\widehat{p}(\xi', s, x_n) = -(\sum_{j=1}^{n-1} i\xi_j \widehat{b}_j - r\widehat{b}_n)(1 + |\xi'|^{-1} r) e^{-|\xi'| x_n},$$

where $r = \sqrt{s + |\xi'|^2}$, $\mathbf{k}(i\xi) = (\frac{i\xi_1}{|\xi'|}, ..., \frac{i\xi_{n-1}}{|\xi'|}, -1)$. The components of the vector $\widehat{\mathbf{h}} = F' L \mathbf{h}$ in (3.11), (3.12) are related to $\widehat{b}_k(\xi, s)$ as follows:

$$\widehat{h}_k = 2\Big(r\widehat{b}_k + \sum_{j=1}^{n-1} \frac{\xi_k \xi_j}{|\xi'|} \widehat{b}_j\Big), \quad k = 1, ..., n-1, \quad \widehat{h}_n = \frac{2r(r + |\xi'|)}{|\xi'|} \widehat{b}_n.$$

Finally, we consider problem (3.4) with $\mathbf{f}$ satisfying (1.7), i.e.,

$$\nabla \cdot \mathbf{f}(x,t) = 0, \quad f_n(x,t)|_{x_n = 0} = 0,$$

as well as the compatibility conditions $\mathbf{f}(x,0)|_{x_n=0} = 0$, and we reduce it to problem (3.3). Let $(\mathbf{u}, q)$ be a solution of the Cauchy problem

$$\mathbf{u}_t + \mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{u} + \nabla q = \mathbf{f}^*, \quad \nabla \cdot \mathbf{u} = 0, \qquad x \in R^n_+, \quad t \in (0,T),$$

$$\mathbf{u}(x,0) = 0,$$

where $\mathbf{f}^*$ is a divergence free extension of $\mathbf{f}$ from $R_T = R^n_+ \times (0,T)$ into $\Pi_T = R^n \times (0,T)$. It can be defined by

$$f^*_j(x', x_n, t) = f_j(x', -x_n, t), \quad j = 1, ..., n-1, \quad f^*_n(x', x_n, t) = -f_n(x', -x_n, t)$$

for $x_n < 0$; it is easily seen that

$$[\mathbf{f}^*]^{(\alpha,\alpha/2)}_{\Pi_T} \le c[\mathbf{f}]^{(\alpha,\alpha/2)}_{R_T}$$

and, by (3.8),

$$[\mathbf{u}]^{(2+\alpha,1+\alpha/2)}_{\Pi_T} + [\nabla q]^{(\alpha,\alpha/2)}_{\Pi_T} \le c[\mathbf{f}^*]^{(\alpha,\alpha/2)}_{\Pi_T} \le c[\mathbf{f}]^{(\alpha,\alpha/2)}_{R_T}. \tag{3.14}$$

Moreover, $q$ can be considered as a solution of the Neumann problem

$$\nabla^2 q = -\nabla \cdot \mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{u}, \qquad x_n > 0,$$

$$\frac{\partial q}{\partial x_n} = -\mathbf{e}_n \cdot \Big(\mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{u} + \mathbf{u}_t\Big), \qquad x_n = 0$$

that is given by the formula

$$q(x,t) = \int_{R^n_+} \nabla_y N(x,y) \cdot \mathcal{A}_0(\frac{\partial}{\partial y})\mathbf{u} dy - 2\int_{R^{n-1}} E(x'-y', x_n) u_{nt}(y',0,t) dy'.$$

Hence,

$$R_j \frac{\partial u_n(x',0,t)}{\partial t} = \frac{\partial q}{\partial x_j}\Big|_{x_n=0} - \frac{\partial q'}{\partial x_j}\Big|_{x_n=0}, \quad j = 1, ..., n-1, \tag{3.15}$$

where

$$q'(x,t) = \int_{R^n_+} \nabla_y N(x,y) \cdot \mathcal{A}_0(\frac{\partial}{\partial y})\mathbf{u} dy.$$

From (3.15) and (2.39) it follows that

$$[R_j u_{nt}]^{(\alpha/2)}_{t,R'_T} \le \Big[\frac{\partial q}{\partial x_j}\Big]^{(\alpha/2)}_{t,R'_T} + \Big[\frac{\partial q'}{\partial x_j}\Big]^{(\alpha/2)}_{t,R'_T}$$

$$\le [\nabla q]^{(\alpha/2)}_{t,R_T} + c[\mathbf{u}]^{(2+\alpha,1+\alpha/2)}_{R_T} \le c[\mathbf{f}]^{(\alpha,\alpha/2)}_{R_T}. \tag{3.16}$$

The functions $\mathbf{w} = \mathbf{v} - \mathbf{u}$, $r = p - q$ satisfy

$$\mathbf{w}_t + \mathcal{A}_0(\frac{\partial}{\partial x})\mathbf{w} + \nabla r = 0, \quad \nabla \cdot \mathbf{w} = 0, \qquad x \in R^n_+, \quad t \in (0,T),$$

$$\mathbf{w}(x,0) = 0, \qquad \mathbf{w}|_{x_n=0} = -\mathbf{u}|_{x_n=0} \equiv \mathbf{b}(x',t).$$

It is easily seen that $\mathbf{b}(x',0) = \mathbf{b}_t(x',0) = 0$, hence, by (3.13),

$$[\mathbf{w}]^{(2+\alpha,1+\alpha/2)}_{R_T} + [\nabla r]^{(\alpha,\alpha/2)}_{R_T} \le c\Big([\mathbf{b}]^{(2+\alpha,1+\alpha/2)}_{R'_T} + \sum_{j=1}^{n-1} [R_j b_{nt}]^{(\alpha/2)}_{t,R'_T}\Big).$$

Together with (3.14), and (3.16), this yields

$$[\mathbf{v}]^{(2+\alpha,1+\alpha/2)}_{R_T} + [\nabla p]^{(\alpha,\alpha/2)}_{R_T} \le c[\mathbf{f}]^{(\alpha,\alpha/2)}_{R_T}. \tag{3.17}$$

If the conditions $\nabla \cdot \mathbf{f} = 0$, $f_n|_{x_n=0} = 0$ are not satisfied but the norms $[\mathbf{f}]^{(\alpha,\alpha/2)}_{R_T}$ and $[P_G\mathbf{f}]^{(\alpha,\alpha/2)}_{R_T}$ are finite, we have

$$[\mathbf{v}]^{(2+\alpha,1+\alpha/2)}_{R_T} + [\nabla p]^{(\alpha,\alpha/2)}_{R_T} \le c\Big([\mathbf{f}]^{(\alpha,\alpha/2)}_{R_T} + [P_G\mathbf{f}]^{(\alpha/2)}_{t,R_T}\Big). \tag{3.18}$$

## 4. The proof of estimate (1.9).

We assume at first that $\mathbf{a} = 0$, $\mathbf{v}_0 = 0$, and we estimate the solution in the neighborhood of an arbitrary point in the interior of $\Omega$ or near the boundary, following Schauder's method.

We consider the second, more complicated case. Let $t_0 \in (0,T]$ and $x_0 \in S$. Without loss of generality we can assume that $x_0$ coincides with the origin and that the $x_n$-axis is directed along the interior normal $-\mathbf{n}(x_0)$. Let

$$x_n = F(x'), \qquad x' = (x_1, ..., x_{n-1}) \tag{4.1}$$

be the equation of the boundary near $x_0 = 0$. By our hypothesis concerning $S$, $F \in C^{2+\alpha}(B_0)$, $B_0 = \{|x'| \le d_0\}$, moreover, $F(0) = 0$, $\nabla F(0) = 0$. The change of variables

$$y' = x', \quad y_n = x_n - F(x')$$

transforms system (1.1) into

$$\mathbf{v}_t + \widetilde{\mathcal{A}}(y,t,\frac{\partial}{\partial y})\mathbf{v} + \widetilde{\nabla} p = \mathbf{f}, \qquad \widetilde{\nabla}\cdot\mathbf{v} = 0, \tag{4.2}$$

where $\widetilde{\mathcal{A}}$ is an elliptic operator with the same properties as $\mathcal{A}$ and

$$\widetilde{\nabla} = \nabla_y - \nabla F \frac{\partial}{\partial y_n} = \Big(\frac{\partial}{\partial y_1} - \frac{\partial F}{\partial y_1}\frac{\partial}{\partial y_n}, ..., \frac{\partial}{\partial y_{n-1}} - \frac{\partial F}{\partial y_{n-1}}\frac{\partial}{\partial y_n}, \frac{\partial}{\partial y_n}\Big).$$

The part of the boundary defined by (4.1) is transformed into the ball $B_0 \in R^{n-1}$. For the functions $\mathbf{v}$, $p$ etc. written in new variables, the same notations are conserved.

Let $\zeta(y,t)$ be a cut-off function equal to one in the "parabolic cylinder" $C_{d/2}(0,t_0) \equiv \{y,t : |y| \le d/2, 0 \le t_0 - t \le d^2/4\}$ and to zero for $t \le t_0$ outside $C_d$. The functions $\mathbf{u}(y,t) = \mathbf{v}(y,t)\zeta(y,t)$, $q(y,t) = p(y,t)\zeta(y,t)$ extended by zero into the domain $\{t < t_0,\ y_n > 0\}$ outside $C_d$ satisfy the relations

$$\frac{\partial \mathbf{u}(y,t)}{\partial t} + \widetilde{\mathcal{A}}(y,t,\frac{\partial}{\partial y})\mathbf{u}(y,t) + \widetilde{\nabla} q(y,t) = \mathbf{f}(y,t)\zeta(y,t) + \mathbf{f}'(y,t), \tag{4.3}$$

$$\widetilde{\nabla}\cdot\mathbf{u} = \mathbf{v}\cdot\widetilde{\nabla}\zeta, \qquad y_n > 0,\ t \le t_0,$$

$$\mathbf{u}(y,0) = 0, \qquad \mathbf{u}(y,t)|_{y_n=0} = 0$$

that can be also written as

$$\frac{\partial \mathbf{u}(y,t)}{\partial t} + \widetilde{\mathcal{A}}_{00}(\frac{\partial}{\partial y})\mathbf{u}(y,t) + \nabla q(y,t) = \mathbf{f}(y,t)\zeta(y,t) + \mathbf{f}_1(y,t),$$

$$\nabla\cdot\mathbf{u} = \mathbf{v}\cdot\widetilde{\nabla}\zeta + (\nabla - \widetilde{\nabla})\cdot\mathbf{u} \equiv g(y,t), \qquad t \in (0,t_0), \quad y_n > 0,$$

$$\mathbf{u}(y,0) = 0, \qquad \mathbf{u}(y,t)|_{y_n=0} = 0,$$

where

$$\mathbf{f}'(y,t) = [\widetilde{\mathcal{A}},\zeta]\mathbf{v} + p\widetilde{\nabla}\zeta + \mathbf{v}\zeta_t, \quad [\widetilde{\mathcal{A}},\zeta]\mathbf{v} = \widetilde{\mathcal{A}}(\zeta\mathbf{v}) - \zeta\widetilde{\mathcal{A}}\mathbf{v},$$

$$\mathbf{f}_1(y,t) = -\widetilde{\mathcal{A}}'(y,t,\frac{\partial}{\partial y})\mathbf{u} + (\nabla - \widetilde{\nabla})q + \mathbf{f}'(y,t),$$

$$\widetilde{\mathcal{A}}_{00}(\frac{\partial}{\partial y}) = \widetilde{\mathcal{A}}_0(0, t_0, \frac{\partial}{\partial y}), \quad \widetilde{\mathcal{A}}' = \widetilde{\mathcal{A}} - \widetilde{\mathcal{A}}_{00}.$$

Following [14], Sec.8, we represent $\mathbf{u}$ in the form $\mathbf{u} = \mathbf{u}^{(1)} + \mathbf{u}^{(2)} + \mathbf{u}^{(3)}$, where $\mathbf{u}^{(i)}$ are solutions to the problems

$$\frac{\partial \mathbf{u}^{(1)}(y,t)}{\partial t} + \widetilde{\mathcal{A}}_{00}(\frac{\partial}{\partial y})\mathbf{u}^{(1)}(y,t) = \mathbf{f}(y,t)\zeta(y,t), \quad y_n > 0,\ t \le t_0,$$

$$\mathbf{u}^{(1)}(y,0) = 0, \qquad \mathbf{u}^{(1)}(y,t)|_{y_n=0} = 0, \tag{4.4}$$

$$-\nabla^2 \mathbf{u}^{(2)}(y,t) + \nabla q^{(2)} = 0, \quad \nabla \cdot \mathbf{u}^{(2)}(y,t) = g(y,t) - \nabla \cdot \mathbf{u}^{(1)}(y,t), \quad y_n > 0,$$

$$\mathbf{u}^{(2)}(y,t)|_{y_n=0} = 0,$$

$$\frac{\partial \mathbf{u}^{(3)}(y,t)}{\partial t} + \widetilde{\mathcal{A}}_{00}(\frac{\partial}{\partial y})\mathbf{u}^{(3)}(y,t) + \nabla q(y,t) = \mathbf{f}_2(y,t), \quad \nabla \cdot \mathbf{u}^{(3)} = 0,$$

$$\mathbf{u}^{(3)}(y,0) = 0, \quad \mathbf{u}^{(3)}(y,t)|_{y_n=0} = 0$$

with

$$\mathbf{f}_2 = \mathbf{f}_1 - \frac{\partial \mathbf{u}^{(2)}}{\partial t} - \widetilde{\mathcal{A}}_{00}(\frac{\partial}{\partial y})\mathbf{u}^{(2)}.$$

Let us estimate $[\mathbf{u}^{(i)}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}$, $R_{t_0} = R^n_+ \times (0,t_0)$, $i = 1,2,3$. Since $\mathbf{f}(x,0) = 0$ for $x_n = 0$, the necessary compatibility conditions in parabolic problem (4.4) are satisfied, it has a unique solution $\mathbf{u}^{(1)} \in C^{2+\alpha,1+\alpha/2}(R_{t_0})$, and

$$[\mathbf{u}^{(1)}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}} \le c[\mathbf{f}\zeta]^{(\alpha,\alpha/2)}_{R_{t_0}}$$

(see [18]). Further, according to well-known estimates for the stationary Stokes problem in the half-space,

$$[\mathbf{u}^{(2)}(\cdot,t)]^{(2+\alpha)}_{R^n_+} \le c[g - \nabla \cdot \mathbf{u}^{(1)}]^{(1+\alpha)}_{R^n_+}.$$

To estimate the Hölder constant of $\mathbf{u}^{(2)}_t$ with respect to $t$, we use explicit formula (2.7) for $\mathbf{u}^{(2)}$:

$$\mathbf{u}^{(2)}(y,t) = \int_{R^n_+} \mathbf{L}(y,z)\Big(\mathbf{v}(z,t)\cdot\widetilde{\nabla}\zeta(z,t) + (\nabla - \widetilde{\nabla})\cdot\mathbf{u}(z,t) - \nabla\cdot\mathbf{u}^{(1)}(z,t)\Big)dz.$$

We integrate by parts, differentiate this formula with respect to $t$ and make use of equations (4.2)-(4.4) which leads to

$$\mathbf{u}^{(2)}_t(y,t) = \int_{R^n_+} \mathbf{L}(y,z)(-\widetilde{\mathcal{A}}\mathbf{v} - \widetilde{\nabla}p)\cdot\widetilde{\nabla}\zeta(z,t)dz$$

$$+\sum_{k=1}^{n}\frac{\partial}{\partial y_k}\Big(\int_{R^n_+}\frac{\partial \mathbf{L}_k(y,z)}{\partial z_n}\nabla F\cdot(\widetilde{\mathcal{A}}\mathbf{u}+\widetilde{\nabla}q-\mathbf{f}')dz-\int_{R^n_+}\nabla_z\mathbf{L}_k(y,z)\widetilde{\mathcal{A}}_{00}\,\mathbf{u}^{(1)}(z,t)dz\Big).$$

It is easily verified that the terms with $\mathbf{f}$ are cancelled, because $\mathbf{f}$ is divergence free as a function of $x$ and $\mathbf{f}\cdot\mathbf{n}|_S=0$. We observe that $|\nabla F|\le cd$ in $C_d$, and we estimate the first term by (2.37) with $\beta=\alpha/2$ and with a small $\mu=\rho>0$, the second by (2.38) and the third by (2.39). This leads to

$$[\mathbf{u}_t^{(2)}]^{(\alpha/2)}_{t,\Pi_{t_0}}\le c\Big([\mathbf{u}^{(1)}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}+\|\mathbf{f}'\cdot\nabla F\|^{(\rho,\alpha/2)}_{R_{t_0}}$$

$$+d[\mathbf{u}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}+d\|q\|^{(\mu_1,\alpha_1/2)}_{R_{t_0}}\Big)+c(d)\Big(\sup_{C_d}|\nabla p(x,t)|$$

$$+\sum_{|j|\le 2}\sup_{C_d}|D^j\mathbf{v}(x,t)|+\|\nabla\mathbf{v}\|^{(\rho,\alpha/2)}_{C_d}+\|\mathbf{v}\|^{(\rho,\alpha/2)}_{C_d}+\|p\|^{(\rho,\alpha/2)}_{C_d}\Big).$$

We also need to estimate $[P_G\frac{\partial\mathbf{u}^{(2)}}{\partial t}]^{(\alpha/2)}_{t,R_{t_0}}$. It is clear that $P_G\frac{\partial\mathbf{u}^{(2)}}{\partial t}=\nabla\psi$, where $\psi$ is a solution of the problem

$$\nabla^2\psi(y,t)=g_t(y,t)-\nabla\cdot\mathbf{u}_t^{(1)},\quad y_n>0,\quad \frac{\partial\psi}{\partial y_n}\Big|_{y_n=0}=0,$$

hence,

$$\psi(y,t)=\int_{R^n_+}N(y,z)(-\widetilde{\mathcal{A}}\mathbf{v}-\widetilde{\nabla}p)\cdot\widetilde{\nabla}\zeta(z,t)dz$$

$$+\int_{R^n_+}\frac{\partial N(y,z)}{\partial z_n}\nabla F\cdot(\widetilde{\mathcal{A}}\mathbf{u}+\widetilde{\nabla}q-\mathbf{f}')dz-\int_{R^n_+}\nabla_z N(y,z)\cdot\widetilde{\mathcal{A}}_{00}\,\mathbf{u}^{(1)}(z,t)dz,$$

and $\nabla\psi$ is estimated exactly as $\mathbf{u}_t^{(2)}$:

$$[P_G\mathbf{u}_t^{(2)}]^{(\alpha/2)}_{t,\Pi_{t_0}}\le c\Big([\mathbf{u}^{(1)}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}+\|\mathbf{f}'\|^{(\rho,\alpha/2)}_{R_{t_0}}$$

$$+d[\mathbf{u}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}+d|||q|||^{(\mu_1,\alpha_1/2)}_{R_{t_0}}\Big)+c(d)\Big(\sup_{C_d}|\nabla p(x,t)|$$

$$+\sum_{|j|\le 2}\sup_{C_d}|D^j\mathbf{v}(x,t)|+|||\nabla\mathbf{v}|||^{(\rho,\alpha/2)}_{C_d}+|||\mathbf{v}|||^{(\rho,\alpha/2)}_{C_d}+|||p|||^{(\rho,\alpha/2)}_{C_d}\Big).$$

Finally, $\mathbf{u}^{(3)}$ is estimated with the help of (3.18), (2.39):

$$[\mathbf{u}^{(3)}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}+[\nabla q]^{(\alpha,\alpha/2)}_{R_{t_0}}\le c\Big([\mathbf{f}_2]^{(\alpha,\alpha/2)}_{R_{t_0}}+[P_G\mathbf{f}_2]^{(\alpha/2)}_{t,R_{t_0}}\Big)$$

$$\le c\Big([\mathbf{f}_1]^{(\alpha,\alpha/2)}_{R_{t_0}}+[P_G\mathbf{f}_1]^{(\alpha/2)}_{t,R_{t_0}}+[\mathbf{u}^{(2)}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}+[P_G\mathbf{u}_t^{(2)}]^{(\alpha/2)}_{t,R_{t_0}}\Big);$$

in its turn,

$$[\mathbf{f}_1]^{(\alpha,\alpha/2)}_{R_{t_0}}+[P_G\mathbf{f}_1]^{(\alpha/2)}_{t,R_{t_0}}\le[\mathbf{f}']^{(\alpha,\alpha/2)}_{R_{t_0}}+|||\mathbf{f}'|||^{(\rho,\alpha/2)}_{R_{t_0}}+cd\Big(|||q|||^{(\mu_1,\alpha_1/2)}_{R_{t_0}}$$

$$+[\nabla q]^{(\alpha)}_{x,R_{t_0}}) + [\mathbf{u}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}}\Big)$$

$$+c\Big(\sum_{|j|\le 2}\sup_{R_{t_0}}|D^j\mathbf{u}(y,t)| + |||\mathcal{A}_1\mathbf{u}|||^{(\rho,\alpha/2)}_{R_{t_0}} + \sup_{R_{t_0}}|\nabla q(x,t)|\Big),$$

because the leading coefficients of $\mathcal{A}'$ and $\nabla F$ do not exceed $cd$. Collecting all the above estimates and taking $d$ sufficiently small, we obtain

$$[\mathbf{u}]^{(2+\alpha,1+\alpha/2)}_{R_{t_0}} + [\nabla q]^{(\alpha,\alpha/2)}_{R_{t_0}} \le c\Big(d|||q|||^{(\mu_1,\alpha_1/2)}_{R_{t_0}} + [\zeta\mathbf{f}]^{(\alpha,\alpha/2)}_{R_{t_0}} + M\Big),$$

where $M$ is the sum of some lower order norms of $p$, $\mathbf{v}$, $q=\zeta p$ and $\mathbf{u}=\zeta\mathbf{v}$ that can be estimated by interpolation inequalities as follows:

$$M \le \epsilon\Big([\mathbf{v}]^{(2+\alpha,1+\alpha/2)}_{C_d} + [\nabla p]^{(\alpha,\alpha/2)}_{C_d}\Big)$$

$$+c(\epsilon)\Big(\sup_{C_d}|\mathbf{v}(x,t)| + \sup_{C_d}|p(x,t)| + [p]^{(\alpha/2)}_{t,C_d}\Big),$$

$\forall\epsilon\in(0,1)$. A similar estimate holds for $(\mathbf{v},p)$ in the neighborhood of arbitrary point $x_0\in\Omega$ with $\mathrm{dist}(x_0,\Omega)\le d/2$. From these estimates we can conclude, by standard arguments of Schauder's method, that

$$|\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_{t'})} \le c\Big(|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_{t'})}$$

$$+\sup_{Q_{t'}}|\mathbf{v}(y,t)| + \sup_{t<t'}|p(\cdot,t)|_{C^{\mu_1}(\Omega)} + [p]^{(\alpha_1/2)}_{t,Q_{t'}} + d|||p|||^{(\mu_1,\alpha_1/2)}_{Q_{t'}}\Big).$$

for arbitrary $t'\le T$. Due to (2.42), we can eliminate the last term in the right-hand side choosing $d$ small. The remaining two norms of $p$ can be estimated with the help of (2.43). After this, using again interpolation inequalities with a small parameter, we obtain

$$|\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_{t'})} \le c\Big(\sup_{Q_{t'}}|\mathbf{v}(x,\tau)| + |\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_{t'})}\Big).$$

Finally, applying the inequality

$$|\mathbf{v}(x,\tau)| \le \int_0^\tau |\mathbf{v}_s(x,s)|ds,$$

and Gronwall's lemma we arrive at

$$|\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_{t'})} \le c|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_{t'})}, \quad \forall t'\le T. \tag{4.5}$$

Now, we turn to the general case. We introduce a divergence free extension $\mathbf{v}_0^*$ of $\mathbf{v}_0$ into $R^n$ with compact support such that

$$|\mathbf{v}_0^*|_{C^{2+\alpha}(R^n)} \le |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}.$$

As pointed out in [9], this extension can be made in the same way as it was done for the S.L.Sobolev spaces [19]. Let $\mathbf{v}_1(x,t)$ be a solution of the Cauchy problem

$$\mathbf{v}_{1t}(x,t) - \nabla^2 \mathbf{v}_1(x,t) = 0, \quad (x,t) \in \Pi_T, \quad \mathbf{v}_1(x,0) = \mathbf{v}_0^*(x,0).$$

It is clear that $\nabla \cdot \mathbf{v}_1 = 0$ and

$$|\mathbf{v}_1|_{C^{2+\alpha,1+\alpha/2}(\Pi_T)} \le c|\mathbf{v}_0^*|_{C^{2+\alpha}(R^n)} \le c|\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}, \tag{4.6}$$

$$[P_G \mathbf{v}_{1t}]_{t,\Pi_T}^{(\alpha/2)} = [P_G \nabla^2 \mathbf{v}_1]_{t,\Pi_T}^{(\alpha/2)} \le c|\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}. \tag{4.7}$$

Next, we define $\mathbf{v}_2(x,t)$ as a solution of the stationary Stokes problem

$$-\nabla^2 \mathbf{v}_2(x,t) + \nabla p_2(x,t) = 0, \quad \nabla \cdot \mathbf{v}_2 = 0, \quad x \in \Omega,$$

$$\mathbf{v}_2(x,t) = \mathbf{a}(x,t) - \mathbf{v}_1(x,t), \quad x \in S.$$

By (2.57), (2.58), (4.6), (4.7), we have

$$|\mathbf{v}_2|_{C^{2+\alpha,1+\alpha/2}(Q_T)} \le c\Big(|\mathbf{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}\Big),$$

$$[P_G \mathbf{v}_{2t}]_{t,Q_T}^{(\alpha/2)} \le c\Big([\mathbf{a}_t \cdot \mathbf{n}]_{t,\Sigma_T}^{(\alpha/2)} + \sum_{k=1}^{n} [\mathcal{R}_k(\mathbf{a}_t \cdot \mathbf{n})]_{t,\Sigma_T}^{(\alpha/2)}| + |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}\Big). \tag{4.8}$$

The difference $\mathbf{w}(x,t) = \mathbf{v}(x,t) - \mathbf{v}_1(x,t) - \mathbf{v}_2(x,t)$, is a solution of the problem

$$\mathbf{w}_t + \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{w} + \nabla p = \mathbf{h}, \qquad \nabla \cdot \mathbf{w} = 0,$$

$$\mathbf{w}(x,0) = 0, \qquad \mathbf{w}|_{x \in S} = 0 \tag{4.9}$$

with

$$\mathbf{h}(x,t) = \mathbf{f}(x,t) - \mathbf{v}_{1t}(x,t) - \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}_1(x,t) - \mathbf{v}_{2t}(x,t) - \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}_2(x,t)$$

satisfying the inequality

$$|\mathbf{h}|_{C^{\alpha,\alpha/2}(Q_T)} + [P_G \mathbf{h}]_{t,Q_T}^{(\alpha/2)} \le c\Big(|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_T)} + |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}$$

$$+|\mathbf{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n} [\mathcal{R}_k(\mathbf{a}_t \cdot \mathbf{n})]_{t,\Sigma_T}^{(\alpha/2)}\Big).$$

In addition, for $x \in S$ we have $\mathbf{v}_{1t}(x,0) + \mathbf{v}_{2t}(x,0) = \mathbf{a}_t(x,0)$, and, as a consequence,

$$\mathbf{h}(x,0) = \mathbf{f}(x,0) - \mathbf{a}_t(x,0) - \mathcal{A}(x,0,\frac{\partial}{\partial x})\mathbf{v}_0(x),$$

from which it follows that compatibility conditions (1.6) are satisfied in problem (4.9). Hence, as has already been shown,

$$|\mathbf{w}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(|\mathbf{h}|_{C^{\alpha,\alpha/2}(Q_T)} + [P_G\mathbf{h}]^{(\alpha/2)}_{t,Q_T}\Big)$$

$$\le c\Big(|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_T)} + |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)} + |\mathbf{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n}[\mathcal{R}_k(\mathbf{a}_t\cdot\mathbf{n})]^{(\alpha/2)}_{t,\Sigma_T}\Big).$$

Together with (4.6)-(4.8), this estimate implies (1.9).

## 5. On the solvability of problem (1.1), (1.2).

We pass to the proof of the solvability of problem (1.1), (1.2). It is enough to consider the case $\mathbf{a} = 0,\ \mathbf{v}_0 = 0$, moreover, we assume at first that

$$\mathbf{f}(x,0) = 0, \quad x \in \Omega, \tag{5.1}$$

$\mathcal{A}_1 = 0$ and that the coefficients of the operator $\mathcal{A}_0$ are independent of $t$. Thus, we consider the problem

$$\mathbf{v}_t(x,t) + \mathcal{A}_0(x,\frac{\partial}{\partial x})\mathbf{v}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t),$$

$$\nabla\cdot\mathbf{v} = 0, \quad x\in\Omega, \quad t\in(0,T), \tag{5.2}$$

$$\mathbf{v}(x,0) = 0, \qquad \mathbf{v}(x,t)|_{x\in S} = 0 \tag{5.3}$$

with $\mathbf{f}$ satisfying (1.7), (5.1).

We apply the device used in [20] in the proof of the solvability of the second initial-boundary value problem for the Stokes system. For arbitrary $\mathbf{f}\in L_2(Q_T)$ problem (5.2), (5.3) has a unique solution $(\mathbf{v},p)$, $\mathbf{v}\in W_2^{2,1}(Q_T)$, $\nabla p\in L_2(Q_T)$ (i.e., $D^j\mathbf{v}, \mathbf{v}_t \in L_2(Q_T),\ |j| = 0,1,2$), that can be constructed by the Galerkin method. Let us write $\mathcal{A}_0(x,\frac{\partial}{\partial x})\mathbf{v}$ in the form

$$\mathcal{A}_0(x,\frac{\partial}{\partial x})\mathbf{v} = \Big(\sum_{m,j,q=1}^{n} a_{km,jq}(x)\frac{\partial^2 v_m}{\partial x_j\partial x_q}\Big)_{k=1,\dots,n}$$

and define a weak solution of problem (5.2), (5.3) as a solenoidal vector field $\mathbf{v}\in L_2(Q_T)$ with $\mathbf{v}_{x_j}, \mathbf{v}_t \in L_2(Q_T)$ satisfying conditions (5.3) and the integral identity

$$\int_0^T\int_\Omega \mathbf{v}_t(x,t)\cdot\eta(x,t)dxdt + \int_0^T \mathcal{A}_0[\mathbf{v},\eta]dt = \int_0^T\int_\Omega \mathbf{f}(x,t)\cdot\eta(x,t)dxdt$$

for arbitrary solenoidal $\eta\in L_2(Q_T)$ with $\eta_{x_j}\in L_2(Q_T)$ vanishing for $x\in S$. By $\mathcal{A}_0[\mathbf{v},\eta]$ we mean the bilinear form

$$\mathcal{A}_0[\mathbf{v},\eta] = -\int_\Omega \sum_{k,m,j,q=1}^{n}\Big(a_{km,jq}(x)\frac{\partial v_m}{\partial x_q}\frac{\partial\eta_k}{\partial x_j} + \frac{\partial a_{km,jq}(x)}{\partial x_j}\frac{\partial v_m}{\partial x_q}\eta_k\Big)dx.$$

The existence of the unique weak solution $\mathbf{v}$ and the estimate

$$\int_0^T \int_\Omega (|\mathbf{v}_t(x,t)|^2 + |\nabla \mathbf{v}(x,t)|^2 + |\mathbf{v}(x,t)|^2) dx dt \le c \int_0^T \int_\Omega |\mathbf{f}|^2 dx dt \tag{5.4}$$

are proved by Galerkin's method in a quite standard way on the basis of the Gårding inequality

$$\mathcal{A}_0[\mathbf{v},\mathbf{v}] \ge c_1 \int_\Omega |\nabla \mathbf{v}(x,t)|^2 dx - c_2 \int_\Omega |\mathbf{v}(x,t)|^2 dx$$

(see, for example, [21], we omit the details). Now, we can consider $\mathbf{v}$ as a weak solution of the elliptic problem

$$\mathcal{A}_0(x, \frac{\partial}{\partial x})\mathbf{v} + \nabla p = \mathbf{f} - \mathbf{v}_t, \quad \nabla \cdot \mathbf{v} = 0, \quad x \in \Omega, \quad \mathbf{v}|_S = 0$$

depending on the parameter $t \in (0,T)$. By the regularity theorem for such problems, the second derivatives $\mathbf{v}_{x_j x_k} \in L_2(Q_T)$ exist, and their norms can be estimated by the $L_2$-norms of $\mathbf{f}$, $\mathbf{v}$ and of the first derivatives of $\mathbf{v}$. Moreover, there exists a pressure function $p \in L_2(Q_T)$ with $\nabla p \in L_2(Q_T)$ such that (5.2) holds in a strong sense, and, along with (5.4), the inequality

$$\sum_{|j|=2} \|D^j \mathbf{v}\|^2_{L_2(Q_T)} + \|\nabla p\|^2_{L_2(Q_T)} \le c \|\mathbf{f}\|^2_{L_2(Q_T)}$$

is satisfied.

Next, we show that $\mathbf{v}$ and $p$ belong to appropriate Hölder spaces. To this end, we approximate $\mathbf{f}$ with the vector fields

$$\mathbf{f}_\epsilon(x,t) = \omega_\epsilon(t) \star \mathbf{f}(x, t-2\epsilon) = \int_{|\tau| \le \epsilon} \omega_\epsilon(\tau) \mathbf{f}(x, t-\tau-2\epsilon) d\tau,$$

where $\omega_\epsilon$ is a standard mollifying kernel (see for instance [21]) and $\mathbf{f}(x,t) = 0$ for $t < 0$. The corresponding solutions of problem (5.2), (5.3) have the same structure:

$$\mathbf{v}_\epsilon(x,t) = \omega_\epsilon(t) \star \mathbf{v}(x, t-2\epsilon), \quad p_\epsilon(x,t) = \omega_\epsilon(t) \star p(x, t-2\epsilon),$$

$\mathbf{v}(x,t) = 0$, $p(x,t) = 0$ for $t < 0$. It is clear that $\mathbf{f}_\epsilon$, $\mathbf{v}_\epsilon$, $p_\epsilon$ are infinitely differentiable with respect to $t$, in particular, $\mathbf{v}_{\epsilon t} \in W_2^2(\Omega)$ for arbitrary fixed $t \in (0,T)$. By the embedding theorem, $\mathbf{v}_{\epsilon t} \in L_q(\Omega)$ with $1/q \le 1/2 - 2/n$. By the regularity theorem for elliptic problems, $\mathbf{v}_\epsilon \in W_q^2(\Omega)$, $\nabla p_\epsilon \in L_q(\Omega)$. Since $\mathbf{v}_\epsilon$ is infinitely differentiable with respect to $t$, it follows that $\mathbf{v}_{\epsilon t} \in L_{q_1}(\Omega)$ with $1/q_1 < 1/q - 2/n \le 1/2 - 4/n$, hence, $\mathbf{v}_\epsilon \in W^2_{q_1}(\Omega$, $\nabla p_\epsilon \in L_{q_1}(\Omega)$. Repeating these arguments, we increase the exponent $q$ indefinitely, until we can conclude that $\mathbf{v}_{\epsilon t} \in C^\alpha(\Omega)$ and, as a consequence, $\mathbf{v}_\epsilon \in C^{2+\alpha}(\Omega)$, $\nabla p_\epsilon \in C^\alpha(\Omega)$. By (1.9),

$$|\mathbf{v}_\epsilon|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p_\epsilon|_{C^{\alpha,\alpha/2}(Q_T)} \le c |\mathbf{f}_\epsilon|_{C^{\alpha,\alpha/2}(Q_T)}.$$

Now, let $\epsilon \to 0$. Since $\mathbf{f}(x,0) = 0$, the zero extension of $\mathbf{f}$ into the domain $\{t < 0\}$ belongs to the same class $C^{\alpha,\alpha/2}$. As $\epsilon \to 0$, the vector fields $\mathbf{f}_\epsilon$ approximate $\mathbf{f}$ in $C^{\alpha',\alpha'/2}(Q_T)$ with

arbitrary $\alpha' < \alpha$ and the norms $|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_T)}$ remain uniformly bounded. Hence, $\mathbf{v}_\epsilon \to \mathbf{v}$ in $C^{2+\alpha',1+\alpha'/2}(Q_T)$, $\nabla p_\epsilon \to \nabla p$ in $C^{\alpha',\alpha'/2}(Q_T)$, and, moreover, $(\mathbf{v}, p)$, $\mathbf{v} \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$, is the required solution of (5.2), (5.3).

Let us show that problem (5.2), (5.3) with the operator $\mathcal{A}$ of a general form instead of $\mathcal{A}_0(x, \frac{\partial}{\partial x})$ and with $\mathbf{f}$ satisfying (1.7) and (5.1) is solvable in a small time interval. It follows from (5.1) that

$$\mathbf{v}(x,0) = \mathbf{v}_t(x,0) = 0, \quad x \in \Omega.$$

For arbitrary $\mathbf{v} \in C^{\alpha,\alpha/2}(Q_T)$ satisfying these conditions we have

$$|\mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v} - \mathcal{A}_0(x,0,\frac{\partial}{\partial x})\mathbf{v}|_{C^{\alpha,\alpha/2}(Q_{t'})}$$

$$\le c\Big( \max_{km,jq} \sup_{Q_{t'}} |a_{km,jq}(x,t) - a_{km,jq}(x,0)| |\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})}$$

$$+ \max_{k,m,j,q} |a_{km,jq}|_{C^{\alpha,\alpha/2}(Q_{t'})} \sum_{|j|=2} \sup_{Q_{t'}} |D^j \mathbf{v}(x,t)|$$

$$+ |\mathbf{v}|_{C^{1+\alpha,1/2+\alpha/2}(Q_{t'})} \Big) \le c(t')^{\alpha/2} |\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})},$$

$$[P_G(\mathcal{A}\mathbf{v} - \mathcal{A}_0(x,0,\frac{\partial}{\partial x})\mathbf{v})]^{(\alpha/2)}_{t,Q_{t'}} \le [P_G(\mathcal{A}_0(x,t,\frac{\partial}{\partial x})\mathbf{v} - \mathcal{A}_0(x,0,\frac{\partial}{\partial x})\mathbf{v})]^{(\alpha/2)}_{t,Q_{t'}}$$

$$+ c|||\mathcal{A}_1\mathbf{v}|||^{(\rho,\alpha/2)}_{Q_{t'}} \le c(t')^{\alpha/2} |\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})}$$

(the last inequality follows from Proposition 2.5). Hence,

$$|P_J(\mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v} - \mathcal{A}_0(x,0,\frac{\partial}{\partial x})\mathbf{v})|_{C^{\alpha,\alpha/2}(Q_{t'})} \le c(t')^{\alpha/2} |\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})},$$

and the solvability of the problem under consideration in a small time interval $(0, t')$ can be deduced from the solvability of problem (5.2), (5.3) with $\mathcal{A}_0(x, \frac{\partial}{\partial x}) = \mathcal{A}_0(x, 0, \frac{\partial}{\partial x})$ and from the contraction mapping principle or proved by successive approximations. From this it follows that the solution exists for $t \in (0, T_0)$.

It remains to remove restriction (5.1). Let us consider problem (1.1), (1.2) with $\mathbf{v}_0 = 0$, $\mathbf{a} = 0$ and with arbitrary $\mathbf{f} \in C^{\alpha,\alpha/2}(Q_T)$ satisfying (1.7), (1.8). We construct a divergence free extension of $\mathbf{f}$, $\mathbf{f}^*$, into the whole space $R^n$ with a compact support, such that

$$|\mathbf{f}^*(\cdot,t)|_{C^\alpha(R^n)} \le c|\mathbf{f}(\cdot,t)|_{C^\alpha(\Omega)}, \quad \sup_{x \in R^n} |\mathbf{f}^*(x,t)| \le c \sup_{x \in \Omega} |\mathbf{f}(x,t)|, \tag{5.5}$$

and, as a consequence,

$$|\mathbf{f}^*|_{C^{\alpha,\alpha/2}(\Pi_T)} \le c|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_T)}.$$

Let $\mathbf{v}^*(x,t)$ be a solution of the Cauchy problem

$$\mathbf{v}^*_t(x,t) - \nabla^2 \mathbf{v}^*(x,t) + \nabla p^*(x,t) = \mathbf{f}^*(x,t),$$

$$\nabla \cdot \mathbf{v}^* = 0, \quad x \in R^n, \quad t \in (0,T),$$

$$\mathbf{v}^*(x,0)=0,$$

i.e.,

$$\mathbf{v}^*(x,t)=\int_0^t\int_{R^n}\Gamma(x-y,t-\tau)\mathbf{f}^*(y,\tau)dyd\tau,$$

where $\Gamma$ is the fundamental solution of the heat equation, and $p^*=0$. Further, we define $\mathbf{w}$ as a solution of the Stokes problem in $\Omega$:

$$-\nabla^2\mathbf{w}+\nabla r=0,\quad \nabla\cdot\mathbf{w}=0,\quad \mathbf{w}|_S=-\mathbf{v}^*|_S.$$

Since $\mathbf{v}_t^*(x,0)=0$ for $x\in S$, we have $\mathbf{w}(x,0)=\mathbf{w}_t(x,0)=0$, $x\in\Omega$.

Both $\mathbf{v}^*$ and $\mathbf{w}$ belong to $C^{2+\alpha,1+\alpha/2}(Q_T)$, moreover, $P_G\mathbf{v}_t^*=P_G\nabla^2\mathbf{v}^*\in C^{\alpha,\alpha/2}(Q_T)$, and, as we have seen above,

$$[P_G\mathbf{w}_t]_{t,Q_T}^{(\alpha/2)}\le c\Big([\mathbf{v}_t^*\cdot\mathbf{n}]_{t,Q_T}^{(\alpha/2)}+\sum_{k=1}^n[\mathcal{R}_k(\mathbf{v}_t^*\cdot\mathbf{n})]_{t,\Sigma_T}^{(\alpha/2)}\Big).$$

Since $\mathbf{f}\cdot\mathbf{n}|_{x\in S}=0$, we have $\mathcal{R}_k(\mathbf{v}_t^*\cdot\mathbf{n})=\mathcal{R}_k(\nabla^2\mathbf{v}^*\cdot\mathbf{n})$. The function $\nabla^2\mathbf{v}^*\cdot\mathbf{n}$ can be written in the form

$$\nabla^2\mathbf{v}^*\cdot\mathbf{n}=\sum_{j,k=1}^n\Big(n_j\frac{\partial}{\partial x_k}-n_k\frac{\partial}{\partial x_j}\Big)\frac{\partial v_j^*}{\partial x_k}=\mathrm{div}_S\Phi$$

with

$$\Phi=\sum_{j,k=1}^n(n_j\mathbf{e}_k-n_k\mathbf{e}_j)\frac{\partial v_j^*}{\partial x_k},$$

hence, by (2.40),

$$[\mathcal{R}_k(\mathbf{v}_t^*\cdot\mathbf{n})]_{t,\Sigma_T}^{(\alpha/2)}\le c|\mathbf{v}^*|_{C^{2+\alpha,1+\alpha/2}(Q_T)}.$$

For the difference $\mathbf{u}=\mathbf{v}-\mathbf{v}^*-\mathbf{w}$ we obtain the problem

$$\mathbf{u}_t+\mathcal{A}\mathbf{u}+\nabla p=\mathbf{h},\quad \nabla\cdot\mathbf{u}=0,\qquad \mathbf{u}(x,0)=0,\quad \mathbf{u}|_{x\in S}=0,$$

where

$$\mathbf{h}=-\mathbf{w}_t-\mathcal{A}\mathbf{w}-\mathcal{A}\mathbf{v}^*-\nabla^2\mathbf{v}^*\in C^{\alpha,\alpha/2}(Q_T),\quad P_G\mathbf{h}\in C^{\alpha,\alpha/2}(Q_T).$$

In addition, $\mathbf{h}(x,0)=0$. As was shown above, this problem has a solution $(\mathbf{u},p)$, $\mathbf{u}\in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p\in C^{\alpha,\alpha/2}(Q_T)$. The proof of Theorem 1.1 is now complete.

Now, we say a few words about the construction of $\mathbf{f}^*$ satisfying (5.5) (it is different from the extension used at the end of Sec. 4). Let $\{\zeta_k(x)\}_{k=0,\dots,M}$ be a smooth partition of unity in $\Omega$ possessing the following properties: $\mathrm{supp}\zeta_0\subset\Omega_0$, where $\Omega_0$ is a strictly interior subdomain of $\Omega$: $\bar\Omega_0\subset\Omega$, and $\mathrm{supp}\zeta_k|_{\bar\Omega}\subset\omega_k$, $k=1,...,M$, where $\omega_k$ is contained in a certain neighborhood of the point $x_k\in S$. Let $\{z_j\}_{j=1,\dots,n}$ be Cartesian coordinates with the origin $x_k$ and with the $z_n$-axis directed along $-\mathbf{n}(x_k)$, and let $z_n=F_k(z')$ be the equation of $S$ in the neighborhood of $x_k$. We assume that

$$\omega_k=\{|z'|\le\lambda,\ 0\le z_n-F_k(z')\le\lambda\},$$

and, moreover, that $\zeta_k$ is defined also outside $\Omega$ in the domain

$$\omega_k' = \{|z'| \le \lambda,\ 0 \le F_k(z') - z_n \le \lambda\}$$

and $\mathrm{supp}\zeta_k \subset \omega_k \cup \omega_k'$.

Assuming that the coordinates $\{x\}$ and $\{z\}$ are related to each other by an orthogonal transformation

$$z = V^{(k)}(x - x_k),$$

we introduce the vector fields

$$\mathbf{f}'(z,t) = V^{(k)}\mathbf{f}(x,t)$$

and make the change of variables

$$\xi' = z', \quad \xi_n = z_n - F_k(z').$$

Then we extend $f_j'(\xi', \xi_n + F_k(\xi'), t) \equiv g_j(\xi', \xi_n, t)$, $j = 1, ..., n-1$, into the domain $\xi_n < 0$ as even functions of $\xi_n$ and

$$f_n'(\xi', \xi_n + F_k(\xi'), t) - \sum_{j=1}^{n-1} \frac{\partial F_k(\xi')}{\partial \xi_j} f_j'(\xi', \xi_n + F_k(\xi'), t) \equiv g_n(\xi', \xi_n, t)$$

as an odd function, and we return to original coordinates. As a result, we obtain a divergence free extension $\mathbf{f}_k = (V^{(k)})^{-1}\mathbf{f}'$ of $\mathbf{f}$ from $\omega_k$ into $\omega_k \cup \omega_k' \equiv \Omega_k$, and this extension satisfies the inequalities

$$|\mathbf{f}_k(\cdot,t)|_{C^\alpha(\Omega_k)} \le c|\mathbf{f}(\cdot,t)|_{C^\alpha(\omega_k)}, \quad \sup_{x\in\Omega_k} |\mathbf{f}_k(x,t)| \le c \sup_{x\in\omega_k} |\mathbf{f}(x,t)|.$$

The vector field $\mathbf{f}^{(1)}(x,t) = \sum_{k=0}^{M} \zeta_k(x,t)\mathbf{f}_k(x,t)$ is an extension of $\mathbf{f}(x,t)$ with compact support contained in $\Omega \cup \Omega'$, $\Omega' \subset R^n \setminus \Omega$, and

$$\nabla \cdot \mathbf{f}^{(1)} = \sum_{k=1}^{M} \nabla\zeta_k \cdot \mathbf{f}_k \equiv h(x,t)$$

has a support contained in $\Omega'$; moreover,

$$\int_{\Omega'} h(x,t)dx = -\sum_{k=1}^{M} \int_S \zeta_k(x,t)\mathbf{n}(x) \cdot \mathbf{f}(x,t)dS = 0.$$

Hence, there exists a vector field $\mathbf{f}^{(2)}(x,t)$ with a support in $\Omega'$ such that

$$\nabla \cdot \mathbf{f}^{(2)}(x,t) = h(x,t)$$

and

$$|\mathbf{f}^{(2)}(\cdot,t)|_{C^\alpha(\Omega')} \le c \sup_{\Omega'} |h(x,t)|$$

(see [19,9]). It is clear that $\mathbf{f}^* = \mathbf{f}^{(1)} - \mathbf{f}^{(2)}$ is the required extension of $\mathbf{f}$.

In conclusion, we consider briefly the problem with nonzero divergence:

$$\frac{\partial \mathbf{v}(x,t)}{\partial t} + \mathcal{A}(x,t,\frac{\partial}{\partial x})\mathbf{v}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), \quad \nabla \cdot \mathbf{v} = g(x,t), \quad (x,t) \in Q_T,$$

$$\mathbf{v}(x,0) = \mathbf{v}_0(x), \qquad \mathbf{v}(x,t)|_{x\in S} = \mathbf{a}(x,t). \tag{5.6}$$

The compatibility conditions take now the form

$$\nabla \cdot \mathbf{v}_0(x) = g(x,0), \quad \mathbf{v}_0(x)|_{x\in S} = \mathbf{a}(x,0), \quad \int_S \mathbf{a}(x,t)\cdot\mathbf{n}(x)dS = \int_\Omega g(x,t)dx, \tag{5.7}$$

$$\mathbf{a}_t(x,0) + \mathcal{A}(x,0,\frac{\partial}{\partial x})\mathbf{v}_0(x) + \nabla p_0(x) = \mathbf{f}(x,0), \quad \forall x \in S, \tag{5.8}$$

with $p_0(x)$ satisfying

$$\nabla^2 p_0(x) = -g_t(x,0) - \nabla \cdot \mathcal{A}(x,0,\frac{\partial}{\partial x})\mathbf{v}_0(x),$$

$$\frac{\partial p_0(x)}{\partial n}\Big|_{x\in S} = -\mathbf{n}\cdot\Big(\mathcal{A}(x,0,\frac{\partial}{\partial x})\mathbf{v}_0(x) + \mathbf{a}_t(x,0)\Big) \tag{5.9}$$

(we assume that $\mathbf{f}$ satisfies (1.7)). The following theorem holds.

**Theorem 5.1.** *Assume that* $\mathbf{f}$, $\mathbf{v}_0$, $\mathbf{a}$ *satisfy the hypotheses of Theorem 1.1, conditions* (1.7), (5.7), (5.8) *hold and*

$$g_t(x,t) = \nabla \cdot \mathbf{h}(x,t) + h_0(x,t)$$

*with* $\mathbf{h}$ *and* $h_0$ *having finite norm*

$$H = \sup_{t<T}|\mathbf{h}(\cdot,t)|_{C^\alpha(\Omega)} + |||\mathbf{h}|||_{Q_T}^{(\rho,\alpha/2)} + \sup_{Q_T}|h_0(x,t)| + [h_0]_{t,Q_T}^{(\alpha/2)},$$

$\rho > 0$. *Then problem* (5.6) *has a unique solution* $(\mathbf{v},p)$, $\mathbf{v} \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$, *and*

$$|\mathbf{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(|\mathbf{f}|_{C^{\alpha,\alpha/2}(Q_T)} + |\mathbf{v}_0|_{C^{2+\alpha}(\Omega)}$$

$$+|\mathbf{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^n [\mathcal{R}_k(\mathbf{a}_t\cdot\mathbf{n})]_{t,Q_T}^{(\alpha/2)} + H\Big). \tag{5.10}$$

**Proof.** First we observe that the function $p_0(x)$ that solves problem (5.9) with $g_t(x,0) = \nabla\cdot\mathbf{h}(x,0) + h_0(x,0)$ belongs to $C^{1+\alpha}(\Omega)$. Now, let $\mathbf{w}(x,t)$ be a solution of the stationary problem

$$-\nabla^2\mathbf{w}(x,t) + \nabla r(x,t) = 0, \quad \nabla\cdot\mathbf{w} = g(x,t), \quad x\in\Omega,$$

$$\mathbf{w}|_{x\in S} = k(t)\mathbf{N}(x), \tag{5.11}$$

where $\mathbf{N}(x)$ is a smooth vector field on $S$ such that

$$\int_S \mathbf{N}(x)\cdot\mathbf{n}(x)dS \equiv k_0 > 0$$

and

$$k(t) = k_0^{-1}\int_\Omega g(x,t)dx = k_0^{-1}\int_S \mathbf{a}_t\cdot\mathbf{n}(x)dS$$

(hence, the necessary compatibility conditions in (5.11) are satisfied). For arbitrary $t \in (0,T)$ we have

$$|\mathbf{w}(\cdot,t)|_{C^{2+\alpha}(\Omega)} \le c|g(\cdot,t)|_{C^{1+\alpha}(\Omega)}. \tag{5.12}$$

The time derivative $\mathbf{w}_t$ solves the problem

$$-\nabla^2\mathbf{w}_t(x,t) + \nabla r_t(x,t) = 0, \quad \nabla\cdot\mathbf{w}_t = \nabla\cdot\mathbf{h}(x,t) + h_0(x,t), \quad x\in\Omega,$$

$$\mathbf{w}_t|_{x\in S} = k'(t)\mathbf{N}(x);$$

it can be represented in the form $\mathbf{w}_t = \mathbf{w}' + \mathbf{w}''$, where

$$\mathbf{w}'(x,t) = \int_\Omega \nabla E(x-y)(\nabla\cdot\mathbf{h}(y,t) + h_0(y,t))dy,$$

and $\mathbf{w}''$ is a solution of (2.53) with $\mathbf{b}(x,t) = k'(t)\mathbf{N}(x) - \mathbf{w}'(x,t)|_{x\in S}$,

$$k'(t) = k_0^{-1}\Big(\int_S \mathbf{h}(y,t)\cdot\mathbf{n}(y)dS + \int_\Omega h_0(y,t)dy\Big).$$

We have

$$|\mathbf{w}'|_{C^{\alpha,\alpha/2}(Q_T)} \le \sup_{t<T}|\mathbf{w}'(\cdot,t)|_{C^\alpha(\Omega)} + |||\mathbf{w}'|||_{Q_T}^{(\rho,\alpha/2)} \le cH \tag{5.13}$$

and, by (2.55), (2.56),

$$|\mathbf{w}''|_{C^{\alpha,\alpha/2}(Q_T)} \le c|\mathbf{b}|_{C^{\alpha,\alpha/2}(\Sigma_T)} \le cH; \tag{5.14}$$

finally, $P_G\mathbf{w}_t = \nabla\psi$, where $\psi$ is a solution of the problem

$$\nabla^2\psi(x,t) = \nabla\cdot\mathbf{h} + h_0, \quad x\in\Omega, \quad \frac{\partial\psi(x,t)}{\partial n}\Big|_{x\in S} = k'(t)\mathbf{N}(x)\cdot\mathbf{n}(x),$$

hence,

$$[P_G\mathbf{w}_t]_{t,Q_T}^{(\alpha/2)} \le |||P_G\mathbf{w}_t|||_{Q_T}^{(\rho,\alpha/2)} \le cH. \tag{5.15}$$

Problem (5.6) is transformed into problem (1.1), (1.2) for $\mathbf{u} = \mathbf{v} - \mathbf{w}$ and $p$ with $\mathbf{f}$, $\mathbf{a}$, $\mathbf{v}_0$ replaced by

$$\mathbf{f}' = \mathbf{f} - \mathbf{w}_t - \mathcal{A}\mathbf{w} \in C^{\alpha,\alpha/2}(Q_T),$$

$\mathbf{a} - \mathbf{w}|_{x\in S} \in C^{2+\alpha,1+\alpha/2}(\Sigma_T)$, $\mathbf{v}_0 - \mathbf{w}(x,0) \in C^{2+\alpha}(\Omega)$, respectively. Since $P_J\mathbf{f}' \in C^{\alpha,\alpha/2}(Q_T)$, it has a solution $(\mathbf{u},p)$, $\mathbf{u}\in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$ and, by (1.11),

$$|\mathbf{u}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \le c\Big(|\mathbf{f}'|_{C^{\alpha,\alpha/2}(Q_T)} + |\mathbf{v}_0 - \mathbf{w}(\cdot,0)|_{C^{2+\alpha}(\Omega)}$$

$$+[P_G\mathbf{f}']^{(\alpha/2)}_{t,Q_T} + |\mathbf{a}-\mathbf{w}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n}[\mathcal{R}_k((\mathbf{a}_t-\mathbf{w}_t)\cdot\mathbf{n})]^{(\alpha/2)}_{t,\Sigma_T}\Big). \tag{5.16}$$

Since

$$[\mathcal{R}_k(\mathbf{w}_t\cdot\mathbf{n})]^{(\alpha/2)}_{t,\Sigma_T} \le |||\mathcal{R}_k(\mathbf{w}_t\cdot\mathbf{n})|||^{(\rho,\alpha/2)}_{\Sigma_T} \le cH,$$

inequality (5.10) follows from (5.12)-(5.16). Theorem 5.1 is proved.

## 6. Local solvability of a nonlinear problem.

Equations (1.1) arise in the linearization of equations of motions of a certain class of non-Newtonian liquids having the form (see [4])

$$\mathbf{v}_t - \nabla\cdot D_\sigma(\sigma(\mathbf{v})) + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}(x,t),$$

$$\nabla\cdot\mathbf{v}(x,t) = 0, \quad (x,t)\in Q_T. \tag{6.1}$$

Here $D(\sigma)$ is a smooth convex function of symmetric real matrices $\sigma = (\sigma_{jk})_{j,k=1,\dots,n}$,

$$\sigma(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T) = \frac{1}{2}\Big(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_j}{\partial x_k}\Big)_{j,k=1,\dots,n},$$

$$D_\sigma = (\frac{\partial D}{\partial\sigma_{jk}})_{j,k=1,\dots,n},$$

hence,

$$\nabla\cdot D_\sigma(\sigma(\mathbf{v})) = \Big(\sum_{k,q,r=1}^{n} D_{jk,qr}(\sigma(\mathbf{v}))\frac{\partial^2 v_q}{\partial x_k\partial x_r}\Big)_{j=1,\dots,n} \equiv -A(\sigma(\mathbf{v}),\frac{\partial}{\partial x})\mathbf{v}.$$

The functions $D_{jk,qr}(\sigma) = \frac{\partial^2 D(\sigma)}{\partial\sigma_{jk}\partial\sigma_{qr}}$ possess the symmetry properties

$$D_{jk,qr} = D_{kj,qr} = D_{kj,rq} = D_{qr,jk},$$

and the convexity means that

$$C^{-1}\sum_{i,j=1}^{n}\kappa_{ij}^2 \le \sum_{j,k,q,r=1}^{n} D_{jk,qr}(\sigma)\kappa_{jk}\kappa_{qr} \le C\sum_{i,j=1}^{n}\kappa_{ij}^2$$

for arbitrary symmetric matrix $\kappa = (\kappa_{jk})_{j,k=1,\dots,n}$. Taking $\kappa_{jk} = \frac{1}{2}(\xi_j\eta_k + \xi_k\eta_j)$ we obtain

$$\frac{1}{2C}(|\xi|^2|\eta|^2 + (\xi\cdot\eta)^2) \le A(\sigma,i\xi)\eta\cdot\eta \le \frac{C}{2}(|\xi|^2|\eta|^2 + (\xi\cdot\eta)^2) \tag{6.2}$$

which is equivalent to ellipticity condition(1.3) for the operator $A(\sigma,\frac{\partial}{\partial x})$.

Let us consider the initial-boundary value problem for equations (6.1) with the initial and boundary conditions

$$\mathbf{v}(x,0) = \mathbf{v}_0(x), \quad \mathbf{v}(x,t)|_{x\in S} = 0 \tag{6.3}$$

assuming that $\mathbf{f}$ and $\mathbf{v}_0$ satisfy (1.7) and the compatibility conditions

$$\nabla \cdot \mathbf{v}_0(x) = 0, \quad \mathbf{v}_0(x)|_S = 0,$$

$$A(\sigma(\mathbf{v}_0), \frac{\partial}{\partial x})\mathbf{v}_0 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 + \nabla p_0(x) = \mathbf{f}(x, 0), \tag{6.4}$$

where $p_0$ is a solution of the Neumann problem

$$\nabla^2 p_0(x) = -\nabla \cdot \Big((\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 + A\big(\sigma(\mathbf{v}_0), \frac{\partial}{\partial x}\big)\mathbf{v}_0\Big), \quad x \in \Omega,$$

$$\frac{\partial p_0(x)}{\partial n}\Big|_S = -\mathbf{n} \cdot \Big((\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 + A\big(\sigma(\mathbf{v}_0), \frac{\partial}{\partial x}\big)\mathbf{v}_0\Big),$$

in other words,

$$\nabla p_0 = -P_G\Big((\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 + A\big(\sigma(\mathbf{v}_0), \frac{\partial}{\partial x}\big)\mathbf{v}_0\Big).$$

If $\mathbf{v}_0 \in C^{2+\alpha}(\Omega)$ and the coefficients of the operator $A\big(\sigma(\mathbf{v}_0), \frac{\partial}{\partial x}\big)$ are continuously differentiable, then $\nabla p_0 \in C^{\alpha}(\Omega)$.

We prove local solvability of problem (6.1), (6.3).

**Theorem 6.1.** *Assume that $S \in C^{2+\alpha}$, $\alpha \in (0,1)$, $D(\sigma)$ is sufficiently smooth, and condition (6.2) is satisfied when $\sigma$ belongs to the $\lambda$-neighborhood of the image of $\sigma(\mathbf{v}_0)$, i.e.,*

$$|\sigma - \sigma(\mathbf{v}_0(x))| \le \lambda$$

*for some $x \in \Omega$. Then for arbitrary $\mathbf{f} \in C^{\alpha,\alpha/2}(Q_T)$, $\mathbf{v}_0 \in C^{2+\alpha}(\Omega)$ satisfying (1.7),(6.4) problem (6.1), (6.3) has a unique solution $(\mathbf{v}, p)$, $\mathbf{v} \in C^{2+\alpha,1+\alpha/2}(Q_\tau)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_\tau)$, in a certain (small) time interval $(0, \tau)$.*

**Proof.** We introduce the functions $(\mathbf{v}_1(x.t), p_1(x,t))$ as a solution of the linear problem

$$\mathbf{v}_{1t} + A\big(\sigma(\mathbf{v}_0), \frac{\partial}{\partial x}\big)\mathbf{v}_1 + \nabla p_1 = \mathbf{f} - (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0,$$

$$\nabla \cdot \mathbf{v}_1(x,t) = 0, \quad (x,t) \in Q_T,$$

$$\mathbf{v}_1(x,0) = \mathbf{v}_0, \quad \mathbf{v}_1|_{x \in S} = 0.$$

It is clear that compatibility conditions (1.6) are satisfied, hence, $\mathbf{v}_1 \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p_1 \in C^{\alpha,\alpha/2}(Q_T)$. The differences $\mathbf{w} = \mathbf{v} - \mathbf{v}_1$, $r = p - p_1$ satisfy the equations

$$\mathbf{w}_t - \nabla \cdot \Big(D_\sigma\big(\sigma(\mathbf{v}_1 + \mathbf{w})\big) - D_\sigma\big(\sigma(\mathbf{v}_1)\big)\Big) + \big((\mathbf{v}_1 + \mathbf{w}) \cdot \nabla\big)(\mathbf{v}_1 + \mathbf{w}) - (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1 + \nabla r$$

$$= A\Big(\sigma(\mathbf{v}_0), \frac{\partial}{\partial x}\Big)\mathbf{v}_1 - A\Big(\sigma(\mathbf{v}_1), \frac{\partial}{\partial x}\Big)\mathbf{v}_1 + (\mathbf{v}_0 \cdot \nabla)\mathbf{v}_0 - (\mathbf{v}_1 \cdot \nabla)\mathbf{v}_1, \tag{6.5}$$

$$\nabla \cdot \mathbf{w}(x,t) = 0, \quad (x,t) \in Q_T,$$

$$\mathbf{w}(x,0) = 0, \quad \mathbf{w}(x,t)|_{x \in S} = 0.$$

We separate out the linear part with respect to $\mathbf{w}$ of the operator in the left hand side of (6.5) and write the same equations in the form

$$\mathbf{w}_t + \mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\mathbf{w} + \nabla r = \mathbf{h}(x,t) + Q[\mathbf{w}],$$

$$\nabla\cdot\mathbf{w}(x,t) = 0, \quad (x,t)\in Q_T,$$

$$\mathbf{w}(x,0) = 0, \quad \mathbf{w}(x,t)|_{x\in S} = 0, \tag{6.6}$$

where

$$\mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\mathbf{w} = -\nabla\cdot\Big(D_{\sigma\sigma}\big(\sigma(\mathbf{v}_1)\big)\sigma(\mathbf{w})\Big) + (\mathbf{w}\cdot\nabla)\mathbf{v}_1 + (\mathbf{v}_1\cdot\nabla)\mathbf{w}, \tag{6.7}$$

$$D_{\sigma\sigma}\big(\sigma(\mathbf{v}_1)\big)\sigma(\mathbf{w}) = \Big(\sum_{q,r=1}^{n} D_{jk,qr}(\sigma(\mathbf{v}_1))\sigma_{qr}(\mathbf{w})\Big)_{j,k=1,\dots,n},$$

$$\mathbf{h}(x,t) = A\Big(\sigma(\mathbf{v}_0),\frac{\partial}{\partial x}\Big)\mathbf{v}_1 - A\Big(\sigma(\mathbf{v}_1),\frac{\partial}{\partial x}\Big)\mathbf{v}_1 - (\mathbf{v}_1\cdot\nabla)\mathbf{v}_1 + (\mathbf{v}_0\cdot\nabla)\mathbf{v}_0,$$

$$Q[\mathbf{w}] = \nabla\cdot\Big(D_\sigma\big(\sigma(\mathbf{v}_1+\mathbf{w})\big) - D_\sigma\big(\sigma(\mathbf{v}_1)\big) - D_{\sigma\sigma}\big(\sigma(\mathbf{v}_1)\big)\sigma(\mathbf{w})\Big) + (\mathbf{w}\cdot\nabla)\mathbf{w}$$

$$= \nabla\cdot\int_0^1 \Big(D_{\sigma\sigma}\big(\sigma(\mathbf{v}_1+s\mathbf{w})\big) - D_{\sigma\sigma}\big(\sigma(\mathbf{v}_1)\big)\Big)\sigma(\mathbf{w})ds + (\mathbf{w}\cdot\nabla)\mathbf{w}$$

is a sum of quadratic terms with respect to $\mathbf{w}$. The principal part of operator (6.7) equals $A\Big(\sigma(\mathbf{v}_1),\frac{\partial}{\partial x}\Big)$. Since $\mathbf{v}_1(x,0) = \mathbf{v}_0(x)$, this operator satisfies (1.3) for $t\in(0,T_1)$ if

$$[\sigma(\mathbf{v}_1)]_{t,Q_T}^{(1/2+\alpha/2)} T_1^{1/2+\alpha/2} \le \lambda.$$

We introduce the following spaces:
$X_\tau$ is the space of vector fields $\mathbf{f}'\in C^{\alpha,\alpha/2}(Q_\tau)$ satisfying (1.7), (5.1);
$Y_\tau$ is the space of divergence free vector fields $\mathbf{v}\in C^{2+\alpha,1+\alpha/2}(Q_\tau)$ satisfying

$$\mathbf{v}|_{x\in S} = 0, \quad \mathbf{v}(x,0) = \mathbf{v}_t(x,0) = 0,$$

and, assuming that $\tau\le T_1$, we define the linear continuous operator $\mathcal{L}: X_\tau\to Y_\tau$ that makes correspond to $\mathbf{f}'\in X_\tau$ the solution $\mathbf{v}\in Y_\tau$ of the linear problem

$$\mathbf{v}_t + \mathcal{A}\Big(x,t,\frac{\partial}{\partial x}\Big)\mathbf{v} + \nabla r = \mathbf{f}'(x,t),$$

$$\nabla\cdot\mathbf{v}(x,t) = 0, \quad (x,t)\in Q_\tau,$$

$$\mathbf{v}(x,0) = 0, \quad \mathbf{v}(x,t)|_{x\in S} = 0.$$

Then problem (6.6) in $Q_\tau$ is equivalent to the equation

$$\mathbf{w} = \mathcal{L}P_J\mathbf{h} + \mathcal{L}P_J Q[\mathbf{w}] \equiv \mathcal{P}[\mathbf{w}].$$

The difference $A\big(\sigma(\mathbf{v}_0),\frac{\partial}{\partial x}\big)\mathbf{v}_1 - A\big(\sigma(\mathbf{v}_1),\frac{\partial}{\partial x}\big)\mathbf{v}_1$ is a linear combination of the expressions of form (2.26) where the role of the coefficients $a_{km}$ is played by the components of $D_{\sigma\sigma}\big(\sigma(\mathbf{v}_0)\big) - D_{\sigma\sigma}\big(\sigma(\mathbf{v}_1)\big)$, and $U_j$ are the first derivatives of the components of $\mathbf{v}_1$. Application of inequality (2.35) leads to

$$|P_J\mathbf{h}|_{C^{\alpha,\alpha/2}(Q_\tau)} \le c\tau^\beta |\mathbf{v}_1|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)}$$

with $\beta < 1/2$ (we omit detailed calculations); hence,

$$|\mathcal{L}P_J\mathbf{h}|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)} \le c_1\tau^\beta |\mathbf{v}_1|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)}.$$

The same arguments prove that for arbitrary $\mathbf{w} \in Y_\tau$

$$|\mathcal{L}P_J Q[\mathbf{w}]|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)} \le c_2\tau^\beta |\mathbf{w}|^2_{C^{2+\alpha,1+\alpha/2}(Q_\tau)},$$

$$|\mathcal{L}P_J\big(Q[\mathbf{w}] - Q[\mathbf{w}']\big)|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)}$$

$$\le c_3\tau^\beta\Big(|\mathbf{w})|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)} + |\mathbf{w}'|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)}\Big)|\mathbf{w}-\mathbf{w}'|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)}.$$

Hence, $\mathcal{P}$ is a contraction operator in the ball

$$|\mathbf{w}|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)} \le \rho$$

of the space $Y_\tau$, if

$$c_1|\mathbf{v}_1|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)}\tau^\beta + c_2\tau^\beta\rho^2 \le \rho, \quad 2c_3\tau^\beta\rho < 1,$$

i.e., $2\max(c_2,c_3)\tau^\beta\rho < 1$,

$$c_1|\mathbf{v}_1|_{C^{2+\alpha,1+\alpha/2}(Q_\tau)}\tau^\beta \le \rho/2.$$

These conditions are satisfied if $\tau$ is sufficiently small, and in this case problem (6.6) has a solution $(\mathbf{w}, r)$, $\mathbf{w} \in C^{2+\alpha,1+\alpha/2}(Q_\tau)$, $\nabla r \in C^{\alpha,\alpha/2}(Q_\tau)$. The solution is unique, because the difference $\mathbf{u} = \mathbf{w} - \mathbf{w}'$ of two solutions from $Y_\tau$ satisfies the equation

$$\mathbf{u} = \mathcal{L}P_J Q[\mathbf{w}] - \mathcal{L}P_J Q[\mathbf{w}'].$$

Thus applying estimate (2.35) once again, we arrive at

$$|\mathbf{u}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})}$$

$$\le c_3 t'^\beta\Big(|\mathbf{w}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})} + |\mathbf{w}'|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})}\Big)|\mathbf{u}|_{C^{2+\alpha,1+\alpha/2}(Q_{t'})}$$

for arbitrary $t' \le \tau$. This proves that $\mathbf{u} = 0$ in a certain small time interval $(0, t')$; repeating this argument we show that the same is true in $(0, \tau)$. Theorem 6.1 is proved.

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