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THE BELLMAN FUNCTIONS

FOR A CERTAIN TWO WEIGHT INEQUALITY: THE CASE STUDY

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Abstract

We are going to give a formula for the exact Bellman function of a certain "toy" two weight problem. This adds one more function to a short list of other Bellman functions for which the precise formulae were found recently. Our case study reveals essential features of finding Bellman functions in general. It gives the extremal sequences for the problem. We also ask some open questions.

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0. Martingale transform: two weight estimate

Let T_0 be an operator defined on some class of "sufficiently good" functions $f : \mathbb{R} \to \mathbb{R}$. Let w, v be two (almost) everywhere positive locally integrable functions (from now on, we shall call such functions "weights"). The question we are going to discuss is

When is the operator T_0 bounded from $L^2(w)$ to $L^2(v)$?

or, in other words,

When does the inequality

$$\int_{\mathbb{R}} |T_0 f|^2 v \leqslant C \int_{\mathbb{R}} |f|^2 w$$

hold with some constant C independent of f?

(unless otherwise specified, all integrals are taken with respect to the standard Lebesgue measure on $\mathbb{R}$).

Denoting $u \stackrel{\text{def}}{=} w^{-1}$, we can reformulate the above question as follows:

When is the operator $T \stackrel{\text{def}}{=} M_{\sqrt{v}} T_0 M_{\sqrt{u}}$ bounded in L^2?

(here M_φ stands for the operator of multiplication by φ).

Such weighted norm inequalities arise naturally in many areas of analysis, operator theory (including the perturbation of self-adjoint operators), and probability theory.

The case when $w = v$ is now pretty well understood for many interesting operators T_0. For the Hilbert transform $Hf(t) = \dfrac{1}{\pi} \displaystyle\int_{\mathbb{R}} \frac{f(s)}{t-s}\, ds$ (defined as the principal value $\lim\limits_{\varepsilon \to 0} \dfrac{1}{\pi} \displaystyle\int_{\mathbb{R} \setminus (t-\varepsilon, t+\varepsilon)} \frac{f(s)}{t-s}\, ds$) the answer is given by the famous theorems of Helson–Szegö and Hunt–Muckenhoupt–Wheeden.

The Helson–Szegö theorem states that the Hilbert transform is bounded in $L^2(v)$ if and only if the weight v can be represented as $v = \exp\{\varphi + H\psi\}$, where $\varphi, \psi \in L^\infty$, $\|\psi\|_{L^\infty} < \pi/2$.

The Hunt–Muckenhoupt–Wheeden theorem states that the Hilbert transform is bounded in $L^2(v)$ if and only if the weight v satisfies the so-called A_2 condition

$$(A_2) \qquad \sup_I \Big(\frac{1}{|I|} \int_I v \Big) \cdot \Big(\frac{1}{|I|} \int_I v^{-1} \Big) < \infty ,$$

where the supremum is taken over all intervals $I \subset \mathbb{R}$.

It is worth mentioning that no direct proof of the equivalence of the Helson–Szegö condition and the A_2 condition is known.

The same A_2 condition is necessary and sufficient for weighted boundedness of the maximal function operator

$$\mathcal{M} f(x) \stackrel{\text{def}}{=} \sup_{I \ni x} \frac{1}{|I|} \int_I |f| ,$$

when weights are equal: $w = v$ (here the supremum is taken over all intervals I containing x).

Things look much more complicated in the two-weight case. Though there exists a beautiful two-weight analogue of the Helson–Szegö theorem due to Cotlar and Sadosky (see [CS]), the consensus now is that there can be no explicit (Muckenhoupt type) necessary and sufficient condition for the boundedness of the Hilbert Transform.

It was a big surprise when Eric Sawyer [S1] found a necessary and sufficient condition for boundedness of the maximal function operator $\mathcal{M}$ from $L^2(w)$ to $L^2(v)$. His theorem states that it is enough to test the boundedness on a very special class of "simple" test functions, namely on the functions

$$w^{-1} \chi_I$$

(here and below χ_I stands for the characteristic function of an interval $I \subset \mathbb{R}$). Later (see [S2]) Sawyer presented a necessary and sufficient condition of similar type for boundedness from $L^2(w)$

to $L^2(v)$ of certain integral operators T_0 with positive kernels. The main difference from the case of the maximal function operator was that now one should also check boundedness of the adjoint operator $T_0^* : L^2(v^{-1}) \to L^2(w^{-1})$ on an appropriate class of simple test-functions, namely on the functions

$$v\chi_I \,.$$

The reader should not be scared by the inverses of w and v. In most interesting cases one can assume that they are summable.

We can reformulate the Sawyer's result in a way that this gives the motivation for what is to follow. The following holds on a regular basis for operators T_0 with positive kernel.

Sawyer type conditions.

Operator $T = M_{\sqrt{v}} T_0 M_{\sqrt{u}}$ is bounded in L^2 if and only if it is bounded on the test-functions $\sqrt{u}\,\chi_I$ and the adjoint operator $T^* = M_{\sqrt{u}} T_0^* M_{\sqrt{v}}$ is bounded on the test-functions $\sqrt{v}\,\chi_I$.

Everywhere below we shall call conditions of such kind *Sawyer type* conditions. The main aim of the paper [NTV1] was to present a uniform approach that allows to recover many of Sawyer's results (at least in L^2) and to obtain at least one new theorem of Sawyer's type: the necessary and sufficient condition for boundedness of Haar multipliers. As far as we know, it is the first theorem of Sawyer's type for *singular* integral operators: all previous results concerned operators with positive kernels only. The method of the Bellman function was used in [NTV1], actually the 1995 preprint version of this paper was the place where harmonic analysis Bellman function appeared—except, of course, the famous Burkholder paper [B] that was implicitly related to Bellman function technique, but where he used another language.

Everywhere below we will use the notation $\langle f \rangle_I$ for the average $\frac{1}{|I|}\int_I f$ of the function $f : \mathbb{R} \to \mathbb{R}$ over the interval $I \subset \mathbb{R}$.

Let $\mathcal{D}$ be some dyadic lattice and let h_J $(J \in \mathcal{D})$ be the corresponding family of Haar functions normalized in L^2, namely $h_J = (\chi_{J+} - \chi_{J-})/\sqrt{|J|}$, where $J_\pm$ denote the right and the left halves of J. Let $H_J \stackrel{\text{def}}{=} \sqrt{|J|}h_J, \Delta_J f \stackrel{\text{def}}{=} \langle f \rangle_{J+} - \langle f \rangle_{J-}$.

Let $\mathbf{a}$ denote a bounded sequence a_J enumerated by dyadic intervals. Define the operator $T_0(\mathbf{a})$ by

$$T_0(\mathbf{a})f \stackrel{\text{def}}{=} \sum_{J\in\mathcal{D}} a_J \big(f, h_J\big)_{L^2} h_J = \sum_{J\in\mathcal{D}} a_J \,[\Delta_J f]\, H_J \,.$$

Sawyer's type question is whether the "weighted operator"

$$T(\mathbf{a}) \stackrel{\text{def}}{=} M_{\sqrt{v}} T_0(\mathbf{a}) M_{\sqrt{u}}$$

is bounded in L^2 if and only if it is bounded on the test-functions $\sqrt{u}\,\chi_I$ and the adjoint operator $T(\mathbf{a})^* = M_{\sqrt{u}} T_0(\mathbf{a}) M_{\sqrt{v}}$ is bounded on the test-functions $\sqrt{v}\,\chi_I$ $(I \in \mathcal{D})$.

The answer is "Yes", and this is proved in [NTV1] for the family $T(\mathbf{a})$ as a whole (meaning that Sawyer type conditions are fulfilled uniformly in $\mathbf{a}$ if and only if all operators $T(\mathbf{a})$ are uniformly bounded).

Theorem (Sawyer type theorem for Haar multipliers)**.** *The operators $T(\boldsymbol{\sigma}\mathbf{a})$ are uniformly bounded in L^2, i.e., $\sup_{\boldsymbol{\sigma}} \|T(\boldsymbol{\sigma}\mathbf{a})\|_{L^2\to L^2} < +\infty$, if and only if they are uniformly bounded on the test-functions $\sqrt{u}\,\chi_I$ and the adjoint operators $T(\boldsymbol{\sigma}\mathbf{a})^*$ are uniformly bounded on the test-functions $\sqrt{v}\,\chi_I$, that is, if and only if there exists a constant $C > 0$ such that*

$$(*) \qquad \int_{\mathbb{R}} \big[T_0(\boldsymbol{\sigma}\mathbf{a})(w\chi_I)\big]^2 v \leqslant C \int_I w \quad \textit{and} \quad \int_{\mathbb{R}} \big[T_0(\boldsymbol{\sigma}\mathbf{a})(v\chi_I)\big]^2 w \leqslant C \int_I v$$

for all dyadic intervals $I \in \mathcal{D}$ and sign sequences $\boldsymbol{\sigma}$.

One can ask, of course, whether this result is true for the individual $T_0(\mathbf{a})$. The answer is positive, see [NTV2].

Another natural question arises: whether a simpler conditions can be found? Here are some necessary conditions.

(1) $\langle u\rangle_I\langle v\rangle_I \leqslant \frac{1}{2}Ca_I^{-2}$ (Muckenhoupt type condition);

(2_u) $\displaystyle \frac{1}{|I|}\sum_{J\subset I} a_J^2\big[\Delta_J u\big]^2\langle v\rangle_J|J| \leqslant C\langle u\rangle_I\,;$

(2_v) $\displaystyle \frac{1}{|I|}\sum_{J\subset I} a_J^2\big[\Delta_J v\big]^2\langle u\rangle_J|J| \leqslant C\langle v\rangle_I\,.$

Indeed, fixing some $I\in\mathcal{D}$ and taking the average of $\int_{\mathbb{R}}\big[T_0(\boldsymbol{\sigma}\mathbf{a})(u\,\chi_I)\big]^2v$ over all possible sign sequences $\boldsymbol{\sigma}$, we get

$$\frac{1}{|I|}\sum_{J\in\mathcal{D}} a_J^2\big[\Delta_J(u\,\chi_I)\big]^2\langle v\rangle_J\,|J| = \mathrm{Avrg}_{\boldsymbol{\sigma}}\,\frac{1}{|I|}\int_{\mathbb{R}}\big[T_0(\boldsymbol{\sigma}\mathbf{a})(u\chi_I)\big]^2v \leqslant C\langle u\rangle_I\,.$$

Observing that $\Delta_J(u\,\chi_I)=\Delta_J u$ for all $J\subset I$ and that every square is non-negative, we immediately arrive at (2_u).

To prove (1), let us apply the above inequality to $u\,\chi_{I\pm}$ instead of u and take only one summand corresponding to $J=I$. Taking into account that $|\Delta_I(u\,\chi_{I\pm})|=\langle u\rangle_{I\pm}$ and $\langle u\,\chi_{I\pm}\rangle_I=\frac{1}{2}\langle u\rangle_{I\pm}$, we find

$$a_I^2\langle u\rangle_{I\pm}^2\,\langle v\rangle_I \leqslant \frac{1}{2}C\langle u\rangle_{I\pm}\,,$$

i.e.,

$$\langle u\rangle_{I\pm}\,\langle v\rangle_I \leqslant \frac{1}{2}Ca_I^{-2}\,.$$

Recalling that $\langle u\rangle_I \leqslant \max\{\langle u\rangle_{I-},\langle u\rangle_{I+}\}$, we get (1). The proof of (2_v) is similar to that of (2_u).

The condition (1) becomes the famous Hunt–Muckenhoupt–Wheeden condition when $v=w=u^{-1}$ (we used the relation between u and w introduced above). So it is natural to ask whether this assumption (1) alone is sufficient for the two-weight boundedness. This is not the case, and the proof can be found in [N]. It is worthwhile to mention here that it is also a Bellman function proof. We also mention that (1) and (2_u) together are not sufficient. Another Bellman function (actually its blowing to infinity) shows that in [NV].

But we are going to show that (1) alone is sufficient for "bilinear test of Sawyer".

1. Bilinear test

Let $\mathbf{a}$ denote now a bounded sequence a_J enumerated by dyadic intervals and such that each a_J is ±1. As we already told, one is brought to consider Sawyer tests with C independent of J:

$$\sup_a\|M_{\sqrt{v}}T_0(\mathbf{a})M_{\sqrt{u}}\sqrt{u}\chi_J\|_2 \leqslant C\|\sqrt{u}\chi_J\|_2 = C\langle u\rangle_J^{1/2}|J|^{1/2}\,,$$

$$\sup_a\|M_{\sqrt{u}}T_0(\mathbf{a})M_{\sqrt{v}}\sqrt{v}\chi_J\|_2 \leqslant C\|\sqrt{v}\chi_J\|_2 = C\langle v\rangle_J^{1/2}|J|^{1/2}\,.$$

We have told that a simple condition $\langle u\rangle_I\langle v\rangle_I \leqslant Q<\infty$ for all dyadic I *is not enough to ensure these tests.*

But let us consider a simpler test, where we apply our operator simultaneously to both of these test functions $\sqrt{u}\chi_J$, $\sqrt{v}\chi_J$. In other words we want now

$$\sup_{\mathbf{a}}|(M_{\sqrt{v}}T_0(\mathbf{a})M_{\sqrt{u}}\sqrt{u}\chi_J,\sqrt{v}\chi_J)_{L^2}| \leqslant C\langle u\rangle_J^{1/2}\langle v\rangle_J^{1/2}|J|\,. \tag{1.1}$$

Theorem 1. *The assumption* $\sup_{I\in\mathcal{D}}\langle u\rangle_I\langle v\rangle_I \leqslant M^2$ *implies* (1.1).

One can look closer to the sum appearing in the left hand side of (1.1). The most essential contribution is given by the sum over all the dyadic subintervals of J. Namely we are forced to consider the inequality

$$\frac{1}{|J|}\sum_{I\in\mathcal{D},I\subset J}|\langle u\rangle_{I^+}-\langle u\rangle_{I^-}|\,|\langle v\rangle_{I^+}-\langle v\rangle_{I^-}|\,|I|\leqslant C\langle u\rangle_J^{1/2}\langle v\rangle_J^{1/2}\,. \tag{1.2}$$

Obviously the next theorem proves Theorem 1.

Theorem 2. *The assumption* $\sup_{I\in\mathcal{D}}\langle u\rangle_I\langle v\rangle_I\leqslant M^2$ *implies* (1.2) *with* $C=16M$.

Actually our main goal is to fully investigate the inequality (1.2), to find its Bellman function precisely and to find the "extremal" u, v (there will be no extremal pairs, but we will find the extremal sequences of pairs). We believe that this goal is somewhat instructive.

2. Notation and definitions

From the beginning we would like to underline that we shall consider rather arbitrary partition of an interval than only a dyadic one. Moreover we shall compare the "dyadic" and "non-dyadic" cases, because we know that sometimes this technique gives drastically different results for these cases, but in the question under consideration it appears that the estimates in both cases are the same.

We shall say that an interval I and a pair of positive numbers $\alpha^\pm$ ($\alpha^++\alpha^-=1$) generate a pair of subintervals I^+ and I^- if $|I^\pm|=\alpha^\pm|I|$ and $I=I^-\cup I^+$. To every interval J and every sequence

$$\{\alpha_{n,m}\colon 0<\alpha_{n,m}<1,\quad 0\leqslant m<2^n,\quad 0<n<\infty\}$$

such that $\alpha_{n,2k}+\alpha_{n,2k+1}=1$ we put in correspondence a family $\mathcal{I}$ of intervals $\{I_{n,m}\}$ such that any interval $I_{n,m}$ generates the pair $I_{n+1,2m}=I^-_{n,m}$, $I_{n+1,2m+1}=I^+_{n,m}$ with $\alpha^-=\alpha_{n+1,2m}$, $\alpha^+=\alpha_{n+1,2m+1}$, starting with $I_{0,0}=J$. The symbol $\mathcal{I}=\mathcal{I}(\alpha)$ will denote the families of subintervals of J corresponding to a fixed choice of the numbers $\alpha_{n,m}$. For a special choice $\alpha_{n,m}=\frac12$ we get the dyadic family $\mathcal{I}=\mathcal{D}$. Every family $\mathcal{I}$ has its own set of Haar functions:

$$\forall I\in\mathcal{I}\qquad h_I(t)=\begin{cases}-\sqrt{\frac{\alpha^+}{\alpha^-|I|}} & \text{if } t\in I^-,\\ \sqrt{\frac{\alpha^-}{\alpha^+|I|}} & \text{if } t\in I^+.\end{cases}$$

If the family $\mathcal{I}$ is such that that the maximal length of the interval of n-th generation (i.e., $\max\{|I_{n,m}|\colon 0\leqslant m<2^n\}$) tends to 0 as $n\to\infty$, the Haar family forms an orthonormal basis in the space $L^2(J)\ominus\{\mathrm{const}\}$.

Definition. Define the following domain in $\mathbb{R}^2$

$$\Omega=\Omega_{m,M}=\{x=(x_1,x_2):\ x_i\geqslant 0,\ m^2\leqslant x_1x_2\leqslant M^2\}$$

and a function on this domain

$$\mathbf{B}(x)=\mathbf{B}(x;m,M)=\sup_{\mathcal{I},u,v}\Big\{\frac{1}{|J|}\sum_{I\in\mathcal{I}}|(u,h_I)|\,|(v,h_I)|\Big\},$$

where the supremum is taken over all possible partitions $\mathcal{I}$ of the interval J, over all nonnegative functions u, v from $L^1(J)$ such that $(\langle u\rangle_J,\langle v\rangle_J)=(x_1,x_2)=x$ and the inequalities

$$m^2\leqslant\langle u\rangle_I\langle v\rangle_I\leqslant M^2 \tag{2.1}$$

are fulfilled for all subintervals I of J.

The function $\mathbf{B}$ we call a Bellman function.

If we fix a partition $\mathcal{I}=\mathcal{D}$ and consider the supremum of the same sum over all pair u, v with restriction (2.1) only for the dyadic intervals I we get dyadic Bellman function $\mathbf{B}_{\mathcal{D}}$.

Remark 1. Recall that

$$\begin{aligned} |(u,h_{_I})|\,|(v,h_{_I})| &= \alpha^+\alpha^-|I|\,|\langle u\rangle_{_{I^+}}-\langle u\rangle_{_{I^-}}|\,|\langle v\rangle_{_{I^+}}-\langle v\rangle_{_{I^-}}| \\ &= \alpha^+\alpha^-|I|\,|\Delta_{_I}u|\,|\Delta_{_I}v|. \end{aligned}$$

Remark 2. The function $\mathbf{B}$ does not depend on the initial interval J. Indeed, consider two different intervals J and J'. The linear mapping of I onto I' maps $L^1(J)$ onto $L^1(J')$ and the family of intervals $\mathcal{I}$ onto the family $\mathcal{I}'$ not changing the averages $\langle u\rangle_{_I}=\langle u'\rangle_{_{I'}}$ and the splitting coefficients $\alpha_I^\pm=\alpha_{I'}^\pm$. Taking into account the relation from the preceding remark, we see that the supremum in definition of the function $\mathbf{B}$ is taken over the same set of numbers not depending on the choice of J.

Remark 3. Since in the case $m=M$ only constant functions u and v can satisfy (2.1), we have $\mathbf{B}(x;m,m)=\mathbf{B}_{\mathcal{D}}(x;m,m)=0$. And in all our calculation we may assume $m<M$.

Remark 4. For arbitrary point $x\in\Omega$ there exists a pair of functions u, v satisfying condition (2.1) and such that $(\langle u\rangle_{_J},\langle v\rangle_{_J})=(x_1,x_2)=x$, i.e., the supremum in the definition of $\mathbf{B}$ is taken over a nonempty set. Choosing the partition $J=J^+\cup J^-$ with the following splitting coefficients

$$\alpha^-=\frac{M-\sqrt{x_1x_2}}{M-m}\,,\qquad \alpha^+=\frac{\sqrt{x_1x_2}-m}{M-m}$$

we take a pair of step functions being constant on $J^\pm$:

$$u(t)=\begin{cases} M\sqrt{\frac{x_1}{x_2}} & t\in J^+\\ m\sqrt{\frac{x_1}{x_2}} & t\in J^-\end{cases}\qquad\text{and}\qquad v(t)=\begin{cases} M\sqrt{\frac{x_2}{x_1}} & t\in J^+\\ m\sqrt{\frac{x_2}{x_1}} & t\in J^-\end{cases}.$$

It is easy to check that $(\langle u\rangle_{_J},\langle v\rangle_{_J})=x$. The geometrical picture make the situation absolutely transparent: our initial point x belongs to the segment of the straight line with the endpoints $(M\sqrt{\frac{x_1}{x_2}},M\sqrt{\frac{x_2}{x_1}})$, $(m\sqrt{\frac{x_1}{x_2}},m\sqrt{\frac{x_2}{x_1}})$, which are on the boundary of the strip (2.1). The whole this segment is inside of Ω and since the averages $(\langle u\rangle_{_I},\langle v\rangle_{_I})$ over any subinterval $I\subset J$ are, clearly, on the same segment, condition (2.1) is fulfilled for all $I\subset J$.

3. Main result

Now we are ready to state the main result of the paper.

Theorem 3.

$$\mathbf{B}(x;m,M)=\mathbf{B}_{\mathcal{D}}(x;m,M)=4M\sqrt{x_1x_2}-x_1x_2+m^2-4mM. \tag{3.1}$$

Remark 5. To make the statement simpler we did not include an important detail in it, namely, equality (3.1) will be proved for all points $x\in\Omega$ in the dyadic case, i.e., for the function $\mathbf{B}_{\mathcal{D}}$, but for the function $\mathbf{B}$ it will be proved for all points except the points on the upper boundary, i.e., for $x=(x_1,x_2)$ with $m^2\leqslant x_1x_2<M^2$. For the points on the upper boundary we are able to prove only an inequality

$$\mathbf{B}(x;m,M)\leqslant 3M^2-m^2-4mM\,.$$

We cannot assert that the equality is here and the function is continuous on the upper boundary of the domain of definition. We discuss this problem later, after the proof.

We get immediately Theorem 2 if put here $m=0$ and take into account the relation mentioned in Remark 1 with $\alpha^\pm=\frac12$. Moreover, if the pair u, v satisfies condition (2.1) not only for dyadic but for arbitrary subinterval of the initial interval J, then the same inequality is true for a sum taken over not obligatory a dyadic but over an arbitrary partition of the initial interval:

Corollary 1. *The assumption* $\sup_{I\subset J}\langle u\rangle_{I}\langle v\rangle_{I}\leqslant M^2$ *implies*

$$\frac{1}{|J|}\sum_{I\in\mathcal{I}}|(u,h_{I})|\,|(v,h_{I})|\leqslant 4M\langle u\rangle_{J}^{1/2}\langle v\rangle_{J}^{1/2}$$

for an arbitrary partition $\mathcal{I}$.

4. How to find Bellman function

This section can be skipped by the reader who is interested in the result rather than in the method, because the proof of the theorem that will be presented in the next section does not required these arguments.

First we state some property of the Bellman function that help us to find it.

Lemma 1. *There exists a function* φ *of one variable defined on the interval* $[m^2,M^2]$ *satisfying the boundary condition* $\varphi(m^2;m,M)=0$ *and such that* $\mathbf{B}(x;m,M)=\varphi(x_1x_2;m,M)$.

Proof. For any admissible pair of functions u, v (i.e., satisfying condition (2.1)) and an arbitrary positive number t introduce another pair $\tilde u=tu$ and $\tilde v=t^{-1}v$. Since $\langle u\rangle_{I}\langle v\rangle_{I}=\langle\tilde u\rangle_{I}\langle\tilde v\rangle_{I}$, the pairs u, v and $\tilde u$, $\tilde v$ run over the set of all admissible pairs simultaneously. Summing identities $(u,h_{I})(v,h_{I})=(\tilde u,h_{I})(\tilde v,h_{I})$ and taking supremum over the set of all admissible pairs we come to the equality

$$\mathbf{B}(x)=\mathbf{B}(\tilde x)=\mathbf{B}(tx_1,\frac{1}{t}x_2)\,.$$

If we chose here $t=x_2$, we get

$$\mathbf{B}(x_1,x_2)=\mathbf{B}(x_1x_2,1)\stackrel{\text{def}}{=}\varphi(x_1x_2)\,,$$

where we put

$$\varphi(s;m,M)\stackrel{\text{def}}{=}\mathbf{B}(s,1;m,M)\,.$$

Furthermore, for an admissible pair u, v the equality $x_1x_2=m^2$ occurs if and only if $u=\text{const}$, $v=\text{const}$, and $uv=m^2$. Indeed, this is the consequence of the convexity of the domain $\{(x_1,x_2):x_i>0,\,x_1x_2\geqslant m^2\}$: inequalities $x_1^{\pm}x_2^{\pm}\geqslant m^2$ together with the equalities $x_i=\alpha^+x_i^++\alpha^-x_i^-$, $x_1x_2=m^2$ imply that all three points coincide: $x=x^+=x^-$. This means that the average of u and v is the same for all intervals $I\subset J$, therefore they are constant functions and

$$\varphi(m^2;m,M)=\mathbf{B}(uv,1;m,M)=\mathbf{B}(u,v;m,M)=0\,,$$

since for constant functions u, v we have $\Delta_{I}u=\Delta_{I}v=0$ for all intervals I. □

Now we deduce a basic inequality for the dyadic Bellman function $\mathbf{B}_{\mathcal{D}}$.

Lemma 2. *Let* $x^{\pm}$ *are two arbitrary points from the strip* Ω *such that their arithmetic mean, i.e., the point* $x=\frac12(x^++x^-)$ *belongs to* Ω *as well. Then*

$$\mathbf{B}_{\mathcal{D}}(x)\geqslant\frac12\big(\mathbf{B}_{\mathcal{D}}(x^+)+\mathbf{B}_{D}(x^-)\big)+\frac14|x_1^+-x_1^-|\,|x_2^+-x_2^-|\,. \tag{4.1}$$

Proof. Take two pairs of functions $u^{\pm}$, $v^{\pm}$ defined on $J^{\pm}$ respectively and satisfying (2.1) for all dyadic subintervals of $J^{\pm}$ and "glue" them into a pair u, v defined on J:

$$u(t)=\begin{cases}u^+(t) & \text{if } t\in J^+\\ u^-(t) & \text{if } t\in J^-\end{cases}\qquad\text{and}\qquad v(t)=\begin{cases}v^+(t) & \text{if } t\in J^+\\ v^-(t) & \text{if } t\in J^-\end{cases}.$$

Now we split the sum from the definition of the function $\mathbf{B}_{\mathcal{D}}$ in three parts

$$\frac{1}{|J|}\sum_{I\in\mathcal{D}}|(u,h_{_I})|\,|(v,h_{_I})| = \frac{1}{2|J^+|}\sum_{I\in\mathcal{D}^+}|(u,h_{_I})|\,|(v,h_{_I})|+ \tag{4.2}$$
$$\frac{1}{2|J^-|}\sum_{I\in\mathcal{D}^-}|(u,h_{_I})|\,|(v,h_{_I})| + \frac{1}{|J|}|(u,h_{_J})|\,|(v,h_{_J})|\,,$$

where $\mathcal{D}^\pm$ are dyadic subintervals of $J^\pm$. Note that $\frac{1}{|J|}|(u,h_{_J})|\,|(v,h_{_J})| = \frac{1}{4}|x_1^+-x_1^-|\,|x_2^+-x_2^-|$ and the pair u, v runs over a subset of admissible pairs for $\mathbf{B}_{\mathcal{D}}(x)$ when $u^\pm$, $v^\pm$ run over the set of *all* admissible pairs for $\mathbf{B}_{\mathcal{D}}(x^\pm)$. Passing to the supremum over this set in (4.2) we come to the desired inequality. $\square$

Remark 6. Underline that in the non-dyadic case the constructed pair u, v should not be admissible, i.e., should not satisfy (2.1) for *every* $I\subset J$ in spite of the fact that (2.1) is true for all $I\subset J^+$ or $I\subset J^-$. Nevertheless as an heuristic assumption we can look for a function B satisfying analogous inequality for arbitrary splitting coefficients:

$$B(x)\geqslant \alpha^+B(x^+)+\alpha^-B(x^-)+\alpha^+\alpha^-|x_1^+-x_1^-|\,|x_2^+-x_2^-|\,, \tag{4.3}$$

where $x=\alpha^+x^++\alpha^-x^-$, and after such a function B is found we shall be able to prove that $B=\mathbf{B}$.

Remark 7. Inequalities (4.1) and (4.3) have the following infinitesimal form:

$$-\Big(\frac{\partial^2B}{\partial x_1^2}\Delta_{_1}^2+2\frac{\partial^2B}{\partial x_1\partial x_2}\Delta_{_1}\Delta_{_2}+\frac{\partial^2B}{\partial x_2^2}\Delta_{_2}^2\Big)\geqslant 2|\Delta_{_1}\Delta_{_2}|\,, \tag{4.4}$$

where slightly abusing notation we use $\Delta_{_i}$ for $x_i^+-x_i^-$. But in fact we always assume that x_1 is an average of a certain summable function u over some interval I, i.e., $\Delta_{_1}=\Delta_{_I}u$, and similarly $\Delta_{_2}=\Delta_{_I}v$.

To deduce (4.4) from (4.3) we use the relations

$$x_i^+=x_i+\alpha^-\Delta_{_i}\qquad\text{and}\qquad x_i^-=x_i-\alpha^+\Delta_{_i}$$

and assuming function B smooth enough take three terms of the Taylor decomposition:

$$B(x^\pm)\approx B(x)\pm\alpha^\mp\sum_{i=1}^2\frac{\partial B}{\partial x_i}\Delta_{_i}+\frac12(\alpha^\mp)^2\sum_{i,j=1}^2\frac{\partial^2B}{\partial x_i\partial x_j}\Delta_{_i}\Delta_{_j},$$

whence

$$\alpha^+B(x^+)+\alpha^-B(x^-)\approx B(x)+\frac12\alpha^+\alpha^-\sum_{i,j=1}^2\frac{\partial^2B}{\partial x_i\partial x_j}\Delta_{_i}\Delta_{_j},$$

what immediately yields (4.4).

Remark 8. Condition (4.4) contains two following matrix inequalities:

$$\begin{pmatrix}\dfrac{\partial^2B}{\partial x_1^2} & \dfrac{\partial^2B}{\partial x_1\partial x_2}-1\\ \dfrac{\partial^2B}{\partial x_1\partial x_2}-1 & \dfrac{\partial^2B}{\partial x_2^2}\end{pmatrix}\leqslant 0\,, \tag{4.5}$$
$$\begin{pmatrix}\dfrac{\partial^2B}{\partial x_1^2} & \dfrac{\partial^2B}{\partial x_1\partial x_2}+1\\ \dfrac{\partial^2B}{\partial x_1\partial x_2}+1 & \dfrac{\partial^2B}{\partial x_2^2}\end{pmatrix}\leqslant 0\,.$$

In fact, (4.4) is equivalent to (4.5).

Remark 9. If there is an extremal pair u, v realizing supremum in the definition of the Bellman function, then one of the matrices (4.5) has to be degenerated. Indeed, assume that for some u, v with the averages $\langle u\rangle_{J}=x_1$, $\langle v\rangle_{J}=x_2$ the following equality is fulfilled:

$$\mathbf{B}(x)=\frac{1}{|J|}\sum_{I\in\mathcal{I}}\alpha^+\alpha^-|I|\,|\Delta_{I}u|\,|\Delta_{I}v|\,.$$

Then we have

$$\begin{gathered}\alpha^+\mathbf{B}(x^+)+\alpha^-\mathbf{B}(x^-)+\alpha^+\alpha^-|\Delta_{J}u|\,|\Delta_{J}v|\geqslant\\ \frac{\alpha^+}{|J^+|}\sum_{I\in\mathcal{I}^+}\alpha^+\alpha^-|I|\,|\Delta_{I}u|\,|\Delta_{I}v|+\\ \frac{\alpha^-}{|J^-|}\sum_{I\in\mathcal{I}^-}\alpha^+\alpha^-|I|\,|\Delta_{I}u|\,|\Delta_{I}v|+\alpha^+\alpha^-|\Delta_{J}u|\,|\Delta_{J}v|=\\ \frac{1}{|J|}\sum_{I\in\mathcal{I}}\alpha^+\alpha^-|I|\,|\Delta_{I}u|\,|\Delta_{I}v|=\mathbf{B}(x)\,.\end{gathered}$$

Together with the assumption that the function $\mathbf{B}$ satisfies (4.3) this gives the equality for some splitting of the initial interval J. If to assume that the infinitesimal *equality* in (4.4) is true as well, then one of the matrices (4.5) must be degenerated. Putting the determinants of these matrices equal to zero we get two variants of the Bellman equation:

$$\frac{\partial^2 B}{\partial x_1^2}\cdot\frac{\partial^2 B}{\partial x_1^2}=\left(\frac{\partial^2 B}{\partial x_1\partial x_2}\pm 1\right)^2. \tag{4.6}$$

Let us show that our assumption about the infinitesimal *equality* in (4.4) can be proved in some situation. Fix the splitting, for which we have equality in (4.3), and let the points x^+ and x^- generated by this splitting are such that the whole straight line segment L connecting $x^\pm$ is in Ω. Of course $x\in L$. Take $\sigma=\operatorname{sign}\Delta_1\Delta_2$ and consider the following function

$$\phi(x)=\mathbf{B}(x)+\sigma x_1x_2\,.$$

Assumption (4.3) exactly means that ϕ is a concave function, because

$$\alpha^+x_1^+x_2^++\alpha^-x_1^-x_2^-=x_1x_2+\alpha^+\alpha^-\Delta_1\Delta_2\,,$$

what will be checked in the beginning of the proof of Lemma 4. In particular it is concave on L. But we have just seen that

$$\alpha^+\phi(x^+)+\alpha^-\phi(x^-)\geq\phi(x)\,.$$

Concave function on an interval can satisfy the latter inequality only if it is linear on this interval. This means that one of the matrices (4.5) must be degenerated.

Reducing the Bellman equation to an ordinary differential equation can be done by using function φ introduced in Lemma 1. We shall look for a function B of the form $B(x_1,x_2)=\varphi(x_1x_2)$. Then

$$\frac{\partial B}{\partial x_1}=x_2\varphi',\qquad\frac{\partial B}{\partial x_2}=x_1\varphi',$$

$$\frac{\partial^2 B}{\partial x_1^2}=x_2^2\varphi'',\qquad\frac{\partial^2 B}{\partial x_1\partial x_2}=\varphi'+x_1x_2\varphi'',\qquad\frac{\partial^2 B}{\partial x_2^2}=x_1^2\varphi'',$$

and therefore the matrix

$$\begin{pmatrix} \dfrac{\partial^2 B}{\partial x_1^2} & \dfrac{\partial^2 B}{\partial x_1 \partial x_2} \pm 1 \\ \dfrac{\partial^2 B}{\partial x_1 \partial x_2} \pm 1 & \dfrac{\partial^2 B}{\partial x_2^2} \end{pmatrix} = \begin{pmatrix} x_2^2\varphi'' & \varphi' + x_1x_2\varphi'' \pm 1 \\ \varphi' + x_1x_2\varphi'' \pm 1 & x_1^2\varphi'' \end{pmatrix}$$

is degenerated if and only if

$$\begin{gathered} (x_2^2\varphi'')(x_1^2\varphi'') - (\varphi' + x_1x_2\varphi'' \pm 1)^2 = 0\,, \\ s^2(\varphi'')^2 - (\varphi' + s\varphi'' \pm 1)^2 = 0\,, \\ (\varphi' \pm 1)(\varphi' + 2s\varphi'' \pm 1) = 0\,. \end{gathered}$$

We immediately see that any solution of the equation $\varphi' \pm 1 = 0$ do not satisfy us because then one of the matrices in (4.5) is identically zero and the second one turns into $\begin{pmatrix} 0 & \pm 2 \\ \pm 2 & 0 \end{pmatrix}$ that is not negatively defined.

Solution of the ordinary differential equation. So, we need to find a solution of one of the differential equations

$$2s\varphi'' + \varphi' \pm 1 = 0 \tag{4.7}$$

such that conditions (4.5) or (in terms of φ)

$$\begin{pmatrix} x_2^2\varphi'' & \varphi' + x_1x_2\varphi'' \pm 1 \\ \varphi' + x_1x_2\varphi'' \pm 1 & x_1^2\varphi'' \end{pmatrix} \leqslant 0 \tag{4.8}$$

are fulfilled. Note that these conditions have to be fulfilled for both signs not depending on what sign was taken in equation (4.7). Dividing this equation by $2\sqrt{s}$ we get

$$\begin{gathered} \sqrt{s}\varphi'' + \frac{1}{2\sqrt{s}}(\varphi' \pm 1) = 0\,, \\ \big[\sqrt{s}(\varphi' \pm 1)\big]' = 0\,, \\ \varphi' \pm 1 = \frac{C_\pm}{2\sqrt{s}}\,, \\ \varphi \pm s = C_\pm\sqrt{s} + \mathrm{const}\,. \end{gathered}$$

The value of the latter constant of integration we can find from the boundary condition $\varphi(m^2) = 0$:

$$\varphi(s) = \mp(s - m^2) + C_\pm(\sqrt{s} - m)\,.$$

Rewrite conditions (4.8) in a scalar form

$$\varphi'' \leqslant 0, \qquad (s\varphi'')^2 \geqslant (s\varphi'' + \varphi' \pm 1)^2\,.$$

For our solutions the inequality $\varphi'' \leqslant 0$ is equivalent to $C_\pm \geqslant 0$. Let us check that the second condition does not fulfilled for the solution $\varphi = s - m^2 + C_-(\sqrt{s} - m)$:

$$s\varphi'' + \varphi' + 1 = s\big(-\frac{C_-}{4s^{3/2}}\big) + \big(1 + \frac{C_-}{2s^{1/2}}\big) + 1 = \frac{C_-}{4\sqrt{s}} + 2 > \frac{C_-}{4\sqrt{s}} = |s\varphi''|.$$

It remains to consider the solution

$$\varphi(s) = -s + m^2 + C_+(\sqrt{s} - m).$$

We have

$$\varphi'(s) = -1 + \frac{C_+}{2s^{1/2}}, \qquad \varphi''(s) = -\frac{C_+}{4s^{3/2}}\,,$$
$$s\varphi'' + \varphi' + 1 = \frac{C_+}{4s^{1/2}} = |s\varphi''|\,,$$
$$s\varphi'' + \varphi' - 1 = \frac{C_+}{4s^{1/2}} - 2\,,$$
$$(s\varphi'')^2 \geqslant (s\varphi'' + \varphi' \pm 1)^2 \iff \frac{C_+^2}{16s} \geqslant \Big(\frac{C_+}{4s^{1/2}} - 2\Big)^2 \iff C_+ \geqslant 4\sqrt{s}\,.$$

Since the latter inequality has to be fulfilled for all $s \in [m^2, M^2]$, we get the restriction

$$C_+ \geqslant 4M\,.$$

We shall check that the minimal possible value of the constant $C_+ = 4M$ gives us the true Bellman function.

5. The proof of Theorem 2 and Corollary to Theorem 3

For a while we forget about the found candidate in the Bellman function and prove the following general assertion.

Lemma 3. *Let $x^\pm$ be a pair of arbitrary points from Ω and let $\alpha^\pm$ be any pair of splitting numbers ($\alpha^\pm \geqslant 0$, $\alpha^+ + \alpha^- = 1$) such that the point $x \stackrel{\text{def}}{=} \alpha^+x^+ + \alpha^-x^-$ is in Ω as well. If a function B defined on Ω satisfies inequality* (4.3), *for all such triples (x^+, x^-, x) then*

$$\mathbf{B}(x; m, M) \leqslant B(x; m, M) \qquad \text{and} \qquad \mathbf{B}_{\mathcal{D}}(x; m, M) \leqslant B(x; m, M)\,.$$

Proof. Fix an arbitrary partition $\mathcal{I}$ of J (in dyadic case $\mathcal{I} = \mathcal{D}$) and any pair u, v of admissible function (i.e., satisfying condition (2.1) for all $I \subset J$ in non-dyadic case and for all $I \in \mathcal{D}$ in dyadic one).

Rewrite inequality (4.3) for $I = I_{n,m}$ in the following form

$$\begin{aligned}
&\alpha_I^+\alpha_I^- |\langle u\rangle_{I^+} - \langle u\rangle_{I^-}|\,|\langle v\rangle_{I^+} - \langle v\rangle_{I^-}| \\
&\quad = \alpha_{n+1,2m}\alpha_{n+1,2m+1}|x_1^{n+1,2m} - x_1^{n+1,2m+1}|\,|x_2^{n+1,2m} - x_2^{n+1,2m+1}| \\
&\quad \leqslant B(x^{n,m}) - \alpha_{n+1,2m}B(x^{n+1,2m}) - \alpha_{n+1,2m+1}B(x^{n+1,2m+1})\,,
\end{aligned}$$

where $x_1^{m,n} = \langle u\rangle_{I^{m,n}}$, $x_2^{m,n} = \langle v\rangle_{I^{m,n}}$. Multiply it by $|I_{n,m}|$ and sum over all intervals of the n-th generation $\mathcal{I}^{(n)}$

$$\begin{aligned}
&\sum_{m=0}^{2^n-1} \alpha_{n+1,2m}\alpha_{n+1,2m+1}|x_1^{n+1,2m} - x_1^{n+1,2m+1}|\,|x_2^{n+1,2m} - x_2^{n+1,2m+1}|\,|I_{n,m}| \\
&\leqslant \sum_{m=0}^{2^n-1}\Big[B(x^{n,m})\,|I_{n,m}| - B(x^{n+1,2m})\,|I_{n+1,2m}| - B(x^{n+1,2m+1})\,|I_{n+1,2m+1}|\Big] \\
&= \sum_{m=0}^{2^n-1} B(x^{n,m})\,|I_{n,m}| - \sum_{m=0}^{2^{n+1}-1} B(x^{n+1,m})\,|I_{n+1,m}| \\
&= \sum_{I\in\mathcal{I}^{(n)}} B(\langle u\rangle_I, \langle v\rangle_I)\,|I| - \sum_{I\in\mathcal{I}^{(n+1)}} B(\langle u\rangle_I, \langle v\rangle_I)\,|I|.
\end{aligned} \tag{5.1}$$

Now, taking sum over n we get

$$
\begin{aligned}
&\sum_{n=0}^{N}\sum_{I\in\mathcal{I}^{(n)}}\alpha_I^+\alpha_I^-\,|\langle u\rangle_{_{I+}}-\langle u\rangle_{_{I-}}|\,|\langle v\rangle_{_{I+}}-\langle v\rangle_{_{I-}}|\,|I|\\
&\qquad\leqslant B(\langle u\rangle_{_J},\langle v\rangle_{_J})\,|J|-\sum_{I\in\mathcal{I}^{(n+1)}}B(\langle u\rangle_{_I},\langle v\rangle_{_I})\,|I|\\
&\qquad\leqslant B(\langle u\rangle_{_J},\langle v\rangle_{_J})\,|J|.
\end{aligned}
\tag{5.2}
$$

Passing to the limit as $N\to\infty$ and taking supremum over all admissible pair of functions u, v with $\langle u\rangle_{_J}=x_1$, $\langle v\rangle_{_J}=x_2$ we obtain the assertion of the lemma for $\mathbf{B}_{\mathcal{D}}(x)$. Do get the desired inequality in non-dyadic case we need to take one more supremum over all possible partitions $\mathcal{I}$. $\square$

From now on we shall denote by the symbol $\mathcal{B}$ the candidate in the Bellman function we have found, i.e.,

$$\mathcal{B}(x;m,M)=\varphi(x_1x_2;m,M)=4M\sqrt{x_1x_2}-x_1x_2+m^2-4mM.$$

We are going to check that this function satisfies the assumptions of Lemma 3.

Lemma 4. *Let $x^\pm$ be a pair of arbitrary points from Ω and let $\alpha^\pm$ be any pair of splitting numbers ($\alpha^\pm\geqslant 0$, $\alpha^++\alpha^-=1$) such that the point $x\stackrel{\text{def}}{=}\alpha^+x^++\alpha^-x^-$ is in Ω as well. Then the function $\mathcal{B}$ satisfies inequality* (4.3).

Proof. As before, we put $\Delta_{_i}=x_i^+-x_i^-$ and check that $\alpha^+x_1^+x_2^++\alpha^-x_1^-x_2^-=x_1x_2+\alpha^+\alpha^-\Delta_{_1}\Delta_{_2}$.

$$
\begin{aligned}
&\alpha^+x_1^+x_2^++\alpha^-x_1^-x_2^-\\
&\qquad=\alpha^+(x_1+\alpha^-\Delta_{_1})(x_2+\alpha^-\Delta_{_2})+\alpha^-(x_1-\alpha^+\Delta_{_1})(x_2-\alpha^+\Delta_{_2})\\
&\qquad=(\alpha^++\alpha^-)x_1x_2+(\alpha^+(\alpha^-)^2+(\alpha^+)^2\alpha^-)\Delta_{_1}\Delta_{_2}\\
&\qquad=x_1x_2+\alpha^+\alpha^-\Delta_{_1}\Delta_{_2}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\mathcal{B}(x)-\alpha^+\mathcal{B}(x^+)-\alpha^-\mathcal{B}(x^-)\\
&=4M\Big[\sqrt{x_1x_2}-\alpha^+\sqrt{x_1^+x_2^+}-\alpha^-\sqrt{x_1^-x_2^-}\Big]-\big[x_1x_2-\alpha^+x_1^+x_2^+-\alpha^-x_1^-x_2^-\big]\\
&=4M\Big[\sqrt{x_1x_2}-\alpha^+\sqrt{x_1^+x_2^+}-\alpha^-\sqrt{x_1^-x_2^-}\Big]+\alpha^+\alpha^-\Delta_{_1}\Delta_{_2}.
\end{aligned}
$$

If $\Delta_{_1}\Delta_{_2}\geqslant 0$, then it is sufficient to refer to the concavity of the function $x\mapsto\sqrt{x_1x_2}$ in the first quadrant:

$$
d^2(\sqrt{x_1x_2})=\begin{pmatrix}-\dfrac14\sqrt{\dfrac{x_2}{x_1^3}} & \dfrac{1}{4\sqrt{x_1x_2}}\\[2ex] \dfrac{1}{4\sqrt{x_1x_2}} & -\dfrac14\sqrt{\dfrac{x_1}{x_2^3}}\end{pmatrix}\leqslant 0\,,
$$

whence

$$\sqrt{x_1x_2}-\alpha^+\sqrt{x_1^+x_2^+}-\alpha^-\sqrt{x_1^-x_2^-}\geqslant 0\,,$$

and

$$\mathcal{B}(x)-\alpha^+\mathcal{B}(x^+)-\alpha^-\mathcal{B}(x^-)\geqslant\alpha^+\alpha^-\Delta_{_1}\Delta_{_2}\,.$$

In the case $\Delta_1\Delta_2 < 0$ we use the concavity of the function $t \mapsto \sqrt{t}$ (indeed, $4(\sqrt{t})'' = -t^{-3/2} < 0$):

$$\begin{aligned}
&\sqrt{x_1x_2} - \alpha^+\sqrt{x_1^+x_2^+} - \alpha^-\sqrt{x_1^-x_2^-}\\
&\quad\geqslant \sqrt{x_1x_2} - \sqrt{\alpha^+x_1^+x_2^+ + \alpha^-x_1^-x_2^-}\\
&\quad= \sqrt{x_1x_2} - \sqrt{x_1x_2 + \alpha^+\alpha^-\Delta_1\Delta_2}\\
&\quad= -\frac{\alpha^+\alpha^-\Delta_1\Delta_2}{\sqrt{x_1x_2} + \sqrt{x_1x_2 + \alpha^+\alpha^-\Delta_1\Delta_2}}\\
&\quad\geqslant -\frac{\alpha^+\alpha^-\Delta_1\Delta_2}{2\sqrt{x_1x_2}}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\mathcal{B}(x) - \alpha^+\mathcal{B}(x^+) - \alpha^-\mathcal{B}(x^-) &\geqslant 4M\Big(-\frac{\alpha^+\alpha^-\Delta_1\Delta_2}{2\sqrt{x_1x_2}}\Big) + \alpha^+\alpha^-\Delta_1\Delta_2\\
&= \alpha^+\alpha^-|\Delta_1\Delta_2|\Big(\frac{2M}{\sqrt{x_1x_2}} - 1\Big) \geqslant \alpha^+\alpha^-|\Delta_1\Delta_2|.
\end{aligned}$$

□

This Lemma together with the preceding one prove Theorem 2 and the Corollary to Theorem 3. As you see this proof is very short and does not require the consideration of the preceding section if somehow we know the Bellman function a priori.

The rest of the paper is devoted to the proof of Theorem 3, what means constructing an extremal pair of function (in fact, approximating sequence) that gives us the opposite inequalities:

$$\mathbf{B}(x;m,M) \geqslant \mathcal{B}(x;m,M) \qquad \text{and} \qquad \mathbf{B}_{\mathcal{D}}(x;m,M) \geqslant \mathcal{B}(x;m,M)\,.$$

6. How to find an extremal function

The reader who is interested in the formal proof rather than in underlying ideas can skip this section as it was with Section 4. Because here we shall try to explain how we come to the construction that will be formally presented in the next section.

If we look for a pair u, v that realize the supremum in the definition of $\mathbf{B}$ we need to have an equality in the limit in (5.2) and therefore we need to have an equality in each term of the sum (5.1). So, in each step we have to do such a splitting that the corresponding differences $\Delta = x^+ - x^-$ were in the kernel of one of the matrices (4.5). By this reason we shall investigate the vector field of the kernel vectors of these matrices for our candidate $\mathcal{B}$ in the Bellman function. If we substitute $\varphi(x_1x_2) = -x_1x_2 + m^2 + 4M(\sqrt{x_1x_2} - m)$ into (4.8) that represent matrices (4.5) in terms of φ, then we get the following expressions

$$\begin{pmatrix} -M\sqrt{\dfrac{x_2}{x_1^3}} & \dfrac{M}{\sqrt{x_1x_2}} - 1 \pm 1\\ \dfrac{M}{\sqrt{x_1x_2}} - 1 \pm 1 & -M\sqrt{\dfrac{x_1}{x_2^3}} \end{pmatrix}. \tag{6.1}$$

For the "+" sign we get a degenerate matrix at every point x

$$\begin{pmatrix} -M\sqrt{\dfrac{x_2}{x_1}} & \dfrac{M}{\sqrt{x_1x_2}}\\ \dfrac{M}{\sqrt{x_1x_2}} & -M\sqrt{\dfrac{x_1}{x_2}} \end{pmatrix}$$

with the kernel vector $\Delta = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ directed along the ray started at the origin. This means that our basic inequality (4.3) turns into equality up to the terms of the second order with respect to the length of Δ when the points x, $x^\pm$ lay on that ray. Let us check that in fact the equality occurs not only infinitesimally but it is strict identity for all such triple of points. Take an arbitrary point $x = (x_1, x_2)$, $\Delta = (\Delta_1, \Delta_2) = (x_1, x_2)t$, $x^+ = x + \alpha^- \Delta = (1+\alpha^- t)x$, and $x^- = x - \alpha^+ \Delta = (1-\alpha^+ t)x$. Then we get

$$\mathcal{B}(x^\pm) = 4M(1 \pm \alpha^\mp t)\sqrt{x_1 x_2} - (1 \pm \alpha^\mp t)^2 x_1 x_2 + m^2 - 4mM\,,$$

whence

$$\begin{aligned} \alpha^+ \mathcal{B}(x_+) + \alpha^- \mathcal{B}(x_-) &= 4M\sqrt{x_1 x_2} - (1 + \alpha^+ \alpha^- t^2) x_1 x_2 + m^2 - 4mM \\ &= \mathcal{B}(x) - \alpha^+ \alpha^- t^2 x_1 x_2 = \mathcal{B}(x) - \alpha^+ \alpha^- |\Delta_1 \Delta_2|\,. \end{aligned}$$

Remark 10. So, we know, how to find an extremal pair u, v for arbitrary point x in the interior of Ω: we have to take points $x^\pm$, where the ray starting at the origin and passing through x intersect the boundary of Ω, we take, say x^-, on the lower boundary, i.e., $x_1^- = m\sqrt{\frac{x_1}{x_2}}$, $x_2^- = m\sqrt{\frac{x_2}{x_1}}$, and x^+ on the upper boundary, i.e., $x_1^+ = M\sqrt{\frac{x_1}{x_2}}$, $x_2^+ = M\sqrt{\frac{x_2}{x_1}}$. Then we put on J^- the extremal pair corresponding the point of the lower boundary, i.e., the constant functions $u = x_1^-$, $v = x_2^-$, and on J^+ the extremal pair corresponding the point of the upper boundary, which is unknown. That is, the problem is to construct the extremal pair for the points of the upper boundary.

The procedure has to be changed a little in the dyadic case, when we cannot choose the splitting coefficient and at every step we split our interval in two equal parts. In this situation we can take a rearrangement of the constructed pair. On J^- we leave the constant function as it was, but the interval J^+ we represent as a union of the dyadic intervals and on every such interval we define the pair u, v being the suitably scaled pair corresponding to the point x^+ of the upper boundary.

If in non-dyadic case we still need to check that this procedure supply us with an admissible pair (i.e., condition (2.1) is fulfilled for every subinterval $I \subset J$), in dyadic one the obtained pair is admissible automatically (i.e., condition (2.1) is fulfilled for every dyadic subinterval $I \subset J$). Indeed, average over any dyadic interval I is a convex combination of the averages over $I \cap J^-$ (what are the coordinates of the point x^-) and the averages over $I \cap J^+$ (what are the coordinates of the point x^+, because $I \cap J^+$ is a union of the dyadic intervals and averages over each of them are the same, namely, the coordinates of the point x^+). Therefore these averages give us a point on the straight line segment with the endpoints $x^\pm$, which is inside the domain Ω.

Returning to found extremal trajectories we see that they cannot help us to construct the extremal pair for the points of the upper boundary, because we cannot represent a point of the upper boundary as a convex combination of two points $x^\pm$ from the ray passing through the origin and different from x itself. But here the second matrix in (6.1) with the "$-$" sign can help us.

Indeed, this matrix is nondegenerate everywhere in Ω except the upper boundary. For $x_1 x_2 = M^2$ this matrix is

$$\begin{pmatrix} -\dfrac{x_2}{x_1} & -1 \\ -1 & -\dfrac{x_1}{x_2} \end{pmatrix}$$

and its kernel vector $\Delta = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix}$ is tangent to the upper boundary. Therefore, the upper boundary itself is an "extremal trajectory" and the points on it we can "split" in the tangent directory. However, for any finite Δ we have strict inequality in (4.3), but the difference between left and right parts is of the forth order in length of Δ. By this reason we cannot get an extremal pair u, v, but if we split the intervals corresponding to the points on the upper boundary so that the averages on the left and right subintervals would be closer and closer, we get a sequence of functions realizing supremum in the definition of the Bellman function.

7. Construction of an extremal sequence

To prove the reverse inequality we need to construct a sequence of test functions $u_n, v_n \in L^1(J)$ satisfying the conditions $m^2 \leqslant \langle u_n \rangle_I \langle v_n \rangle_I \leqslant M^2$, $\langle u_n \rangle_J = x_1$, $\langle v_n \rangle_J = x_2$, and such that the corresponding sums in the definition of $\mathbf{B}$ tend to $\mathbf{B}(x_1, x_2)$.

To this aim fix a constant C, $m < C \leqslant M$, an integer N, and the larger root a ($a > 1$) of the equation

$$2^N = \frac{C-m}{C} \cdot \frac{a^2+1}{(a-1)^2}\,, \tag{7.1}$$

i.e.,

$$a = \frac{2^N C + \sqrt{(C-m)(2^{N+1}C - C + m)}}{2^N C - C + m}\,.$$

Take $J = [0,1]$ and define the following sequence of step functions $u_n(t) = u_n(t; N, m, C)$, $v_n(t) = v_n(t; N, m, C)$ on $[0,1]$: $u_0(t) = v_0(t) = C$,

$$u_{n+1}(t) = \begin{cases} \dfrac{m}{a}, & 0 \leqslant t < 2^{-N-1}, \\[2ex] \dfrac{1}{a} u_n(2^{k+1}t - 1), & 2^{-k-1} \leqslant t < 2^{-k}, \\[2ex] a u_n(2 - 2^{k+1}(1-t)), & 1 - 2^{-k} < t \leqslant 1 - 2^{-k-1}, \\[2ex] am, & 1 - 2^{-N-1} < t \leqslant 1, \end{cases} \tag{7.2}$$

and

$$v_{n+1}(t) = \begin{cases} am, & 0 \leqslant t < 2^{-N-1}, \\[2ex] a v_n(2^{k+1}t - 1), & 2^{-k-1} \leqslant t < 2^{-k}, \\[2ex] \dfrac{1}{a} v_n(2 - 2^{k+1}(1-t)), & 1 - 2^{-k} < t \leqslant 1 - 2^{-k-1}, \\[2ex] \dfrac{m}{a}, & 1 - 2^{-N-1} < t \leqslant 1. \end{cases} \tag{7.3}$$

In these formulas k runs from 1 to N.

Lemma 5. *For all $n \geqslant 0$ the following equalities are true*

$$v_n(t) = u_n(1-t), \tag{7.4}$$

$$\langle u_n \rangle_{[0,1]} = \langle v_n \rangle_{[0,1]} = C, \tag{7.5}$$

$$\langle u_{n+1} \rangle_{[2^{-k-1}, 2^{-k}]} = \frac{C}{a}, \quad \langle v_{n+1} \rangle_{[2^{-k-1}, 2^{-k}]} = aC, \quad 1 \leqslant k \leqslant N, \tag{7.6}$$

$$
\begin{aligned}
&\langle u_{n+1}\rangle_{[0,\frac12]} = \langle v_{n+1}\rangle_{[\frac12,1]} = \frac{2C}{1+a^2},\\
&\langle u_{n+1}\rangle_{[\frac12,1]} = \langle v_{n+1}\rangle_{[0,\frac12]} = \frac{2a^2C}{1+a^2}.
\end{aligned}
\tag{7.7}
$$

Proof. Relation (7.4) we check by induction. For $n = 0$ we have $v_0(t) = C = u_0(1-t)$. Let us verify identity (7.4), say, for $2^{-k-1} \leqslant t < 2^{-k}$:

$$v_{n+1}(t) = av_n(2^{k+1}t - 1) = au_n(2 - 2^{k+1}t) = u_{n+1}(1-t).$$

By definition (7.2)–(7.3) relation (7.5) implies (7.6). Now using (7.6) and (7.1) we calculate averages over the interval $[0, \frac12]$:

$$
\begin{aligned}
\langle u_{n+1}\rangle_{[0,\frac12]} &= 2\int_0^{\frac12} u_{n+1}(t)dt = 2^{-N}\frac{m}{a} + (1-2^{-N})\frac{C}{a}\\
&= \frac{C}{a} - 2^{-N}\frac{C-m}{a}\\
&= \frac{C}{a} - \frac{C}{C-m}\cdot\frac{(a-1)^2}{a^2+1}\cdot\frac{C-m}{a}\\
&= \frac{2C}{a^2+1},\\
\langle v_{n+1}\rangle_{[0,\frac12]} &= 2\int_0^{\frac12} v_{n+1}(t)dt = 2^{-N}am + (1-2^{-N})aC\\
&= aC - 2^{-N}a(C-m)\\
&= aC - \frac{C}{C-m}\cdot\frac{(a-1)^2}{a^2+1}a(C-m)\\
&= \frac{2a^2C}{a^2+1}.
\end{aligned}
$$

Together with (7.4) this proves equalities (7.7).

Adding equalities (7.7) we get (7.5) for the next value of index n. Since (7.5) is clearly true for $n = 0$, this completes the proof of the Lemma by induction. □

Now we calculate the sum from the definition of the Bellman function

$$b_n(N, m, C) = \frac14 \sum_{I\in\mathcal{D}} |I|\,|\langle u_n\rangle_{I^+} - \langle u_n\rangle_{I^-}|\,|\langle v_n\rangle_{I^+} - \langle v_n\rangle_{I^-}|.$$

Lemma 6.

$$b_n(N, m, C) = (C-m)\left(3C - m - (C-m)2^{-N+1}\right)\left(1-(1-2^{-N})^n\right).$$

Proof. We shall use the induction in n. For $n = 0$ we have $u_0(t) = v_0(t) = C$ and b_0=0. To calculate b_{n+1} (assuming that formula for b_n is true) we break the set of all dyadic intervals $\mathcal{D}$ in three groups:

$\mathcal{D}_1$ are all dyadic intervals inside of the interval $[2^{-N-1}, 1-2^{-N-1}]$,

$\mathcal{D}_2 = \{[0, 2^{-k}] : 1 \leqslant k \leqslant N\} \cup \{[1-2^{-k}, 1] : 1 \leqslant k \leqslant N\}$,

$\mathcal{D}_3$ are all other dyadic intervals.

To calculate the sum over $\mathcal{D}_1$ we use the inductive definition of u_{n+1} and v_{n+1} together with the induction hypothesis

$$\begin{aligned}
&\frac{1}{4}\sum_{I\in\mathcal{D}_1}|I|\,|\langle u_{n+1}\rangle_{I+}-\langle u_{n+1}\rangle_{I-}|\,|\langle v_{n+1}\rangle_{I+}-\langle v_{n+1}\rangle_{I-}|\\
&\qquad=(1-2^{-N})b_n\\
&\qquad=(C-m)\left(3C-m-(C-m)2^{-N+1}\right)\left(1-2^{-N}-(1-2^{-N})^{n+1}\right).
\end{aligned}$$

By Lemma 5 and relations (7.2)–(7.3) we have

$$\langle u_{n+1}\rangle_{[2^{-k-1},2^{-k}]}=\frac{C}{a},\qquad \langle v_{n+1}\rangle_{[2^{-k-1},2^{-k}]}=aC,\qquad 1\leqslant k\leqslant N,$$

and

$$\langle u_{n+1}\rangle_{[0,2^{-N-1}]}=\frac{m}{a},\qquad \langle v_{n+1}\rangle_{[0,2^{-N-1}]}=am,$$

whence

$$\begin{aligned}
\langle u_{n+1}\rangle_{[0,2^{-k}]}&=\frac{2^{-N-1}\frac{m}{a}+[2^{-k}-2^{-N-1}]\frac{C}{a}}{2^{-k}}\\
&=\frac{C}{a}-2^{k-N-1}\frac{C-m}{a},\\
\langle v_{n+1}\rangle_{[0,2^{-k}]}&=\frac{2^{-N-1}am+[2^{-k}-2^{-N-1}]aC}{2^{-k}}\\
&=aC-2^{k-N-1}a(C-m).
\end{aligned}\tag{7.8}$$

Due to (7.4) the intervals $[0,2^{-k}]$ and $[1-2^{-k},1]$ give equal terms, therefore

$$\begin{aligned}
&\frac{1}{4}\sum_{I\in\mathcal{D}_2}|I|\,|\langle u_{n+1}\rangle_{I+}-\langle u_{n+1}\rangle_{I-}|\,|\langle v_{n+1}\rangle_{I+}-\langle v_{n+1}\rangle_{I-}|\\
&=\frac{1}{2}\sum_{k=1}^{N}2^{-k}|\langle u_{n+1}\rangle_{[0,2^{-k-1}]}-\langle u_{n+1}\rangle_{[2^{-k-1},2^{-k}]}|\,|\langle v_{n+1}\rangle_{[0,2^{-k-1}]}-\langle v_{n+1}\rangle_{[2^{-k-1},2^{-k}]}|\\
&=\frac{1}{2}\sum_{k=1}^{N}2^{-k}\cdot 2^{k-N}\frac{C-m}{a}\cdot 2^{k-N}a(C-m)\\
&=(C-m)^2 2^{-2N-1}\sum_{k=1}^{N}2^{k}=(C-m)^2 2^{-N}(1-2^{-N}).
\end{aligned}$$

To prepare further calculations let us note that (7.1) implies

$$\frac{(a-1)^2}{a^2+1}=\frac{C-m}{C}2^{-N},\qquad \frac{2a}{a^2+1}=1-\frac{C-m}{C}2^{-N},$$

and therefore,

$$\begin{aligned}
\left(\frac{a^2-1}{a^2+1}\right)^2&=1-\left(\frac{2a}{a^2+1}\right)^2=1-\left(1-\frac{C-m}{C}2^{-N}\right)^2\\
&=\left(2-\frac{C-m}{C}2^{-N}\right)\frac{C-m}{C}2^{-N}.
\end{aligned}\tag{7.9}$$

Finally, we note that in $\mathcal{D}_3$ the only interval $I=[0,1]$ gives a nonzero term, because for all other $I\in\mathcal{D}_3$ we have $I\subset[0,2^{-N-1}]\cup[1-2^{-N-1},1]$, where the functions u_n, v_n are constant. So, using (7.7) and (7.9) we get

$$\begin{aligned}
&\frac{1}{4}\sum_{I\in\mathcal{D}_3}|I|\left|\langle u_{n+1}\rangle_{I+}-\langle u_{n+1}\rangle_{I-}\right|\,\left|\langle v_{n+1}\rangle_{I+}-\langle v_{n+1}\rangle_{I-}\right|\\
&=\frac{1}{4}\left|\langle u_{n+1}\rangle_{[\frac12,1]}-\langle u_{n+1}\rangle_{[0,\frac12]}\right|\,\left|\langle v_{n+1}\rangle_{[\frac12,1]}-\langle v_{n+1}\rangle_{[0,\frac12]}\right|\\
&=C^2\left(\frac{a^2-1}{a^2+1}\right)^2=(C-m)2^{-N}\left(2C-2^{-N}(C-m)\right).
\end{aligned}$$

Gathering all three sums we get

$$\begin{aligned}
b_{n+1}&=(C-m)\left(3C-m-(C-m)2^{-N+1}\right)\left(1-2^{-N}-(1-2^{-N})^{n+1}\right)\\
&\qquad+(C-m)^2 2^{-N}(1-2^{-N})+(C-m)2^{-N}\left(2C-2^{-N}(C-m)\right)\\
&=(C-m)\left(3C-m-(C-m)2^{-N+1}\right)\left(1-(1-2^{-N})^{n+1}\right)\\
&\qquad-(C-m)2^{-N}\Big(3C-m-(C-m)2^{-N+1}\\
&\qquad\qquad-(C-m)(1-2^{-N})-2C+2^{-N}(C-m)\Big)\\
&=(C-m)\left(3C-m-(C-m)2^{-N+1}\right)\left(1-(1-2^{-N})^{n+1}\right).
\end{aligned}$$

□

The following Lemma together with Lemma 3 complete the proof the Theorem for the Bellman function $\mathbf{B}_\mathcal{D}$.

Lemma 7.

$$\mathbf{B}_\mathcal{D}(x;m,M)\geqslant\mathcal{B}(x;m,M).$$

Proof. Fix a point x on the upper boundary of $\Omega_{m,M}$, i.e., $x_1x_2=M^2$, and using the constructed families $u_n(t;N,m,C)$, $v_n(t;N,m,C)$ consider the following functions

$$u(t)=\frac{x_1}{M}u_n(t;N,m,M)\qquad\text{and}\qquad v(t)=\frac{x_2}{M}v_n(t;N,m,M).$$

By Lemma 5 we have $\langle u\rangle_{[0,1]}=x_1$, $\langle v\rangle_{[0,1]}=x_2$. To use these functions as test functions for an estimate $\mathbf{B}_\mathcal{D}(x)$ from below we only need to check that u, v satisfy condition (2.1) for all dyadic intervals I. Since $u(t)v(t)=u_n(t;N,m,M)v_n(t;N,m,M)\geqslant m^2$ (in fact the latter product has only two values: m^2 and M^2), we have

$$\langle u\rangle_I\langle v\rangle_I\geqslant\langle\sqrt{uv}\rangle_I^2\geqslant m^2\qquad\forall I\subset[0,1].$$

To check the upper estimate we fix n and consider the same three types of intervals $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ introduced in the proof of the preceding lemma. For $I\in\mathcal{D}_1$ the question is reduced to the same question for smaller n and therefore the estimate is true by induction, because $\langle u_0\rangle_I\langle v_0\rangle_I=M^2$. For $I\in\mathcal{D}_2$ we use the calculated averages (7.8)

$$\begin{aligned}
\langle u\rangle_I\langle v\rangle_I&=\Big(\frac{M}{a}-2^{k-N-1}\frac{M-m}{a}\Big)\Big(aM-2^{k-N-1}a(M-m)\Big)\\
&\leqslant\frac{M}{a}\cdot aM=M^2.
\end{aligned}$$

For $I=J=[0,1]$ we have $\langle u\rangle_I\langle v\rangle_I=x_1x_2=M^2$, and on all other intervals $I\in\mathcal{D}_3$ the functions u, v are constants and $uv=m^2<M^2$.

The value of the sum in the definition of the Bellman function corresponding to the pair u, v is calculated in Lemma 6, hence for all values of n and N we have

$$\begin{aligned}\mathbf{B}(x;,m,M) &\geqslant b_n(N,m,M)\\ &= (M-m)\left(3M-m-(M-m)2^{-N+1}\right)\left(1-(1-2^{-N})^n\right).\end{aligned}$$

Passing first to the limit as $n\to\infty$ and then as $N\to\infty$ we get

$$\mathbf{B}(x;m,M)\geqslant (M-m)(3M-m)=\mathcal{B}(x;m,M),\qquad \text{if}\quad x_1x_2=M^2\,.$$

So, we get an extremal sequence of pairs u, v for an arbitrary point x on the upper boundary of the domain Ω. It remains to refer to Remark 10, where it was explained how to get an extremal pair for a point inside Ω if such a pair is known for the points of the upper boundary. $\square$

The main difficulty in obtaining a similar assertion for the non-dyadic Bellman function consists in verifying the upper estimate (2.1) for the pair of test function, because we need to check this estimate for *all* subintervals of $[0,1]$. For this more difficult estimate we use the following lemma.

Lemma 8. *If $a<2$, then*

$$\langle u_n\rangle_{[0,t]}\leqslant\frac{a^2+1}{2a}C,\qquad \langle v_n\rangle_{[0,t]}\leqslant\frac{aC}{2-a}; \tag{7.10}$$

$$\langle u_n\rangle_{[t,1]}\leqslant\frac{aC}{2-a},\qquad \langle v_n\rangle_{[t,1]}\leqslant\frac{a^2+1}{2a}C. \tag{7.11}$$

Proof. Since $v_n(t)=u_n(1-t)$, it is sufficient to check only one pair of estimates, for example, (7.10). It will be done by induction. For $n=0$ the estimates are true because $u_0(t)=v_0(t)=C$ and $a>1$. Assume that (7.10) is true for some n and check it for the next value of index using relations (7.2)–(7.3).

For $t\in[0,2^{-N-1}]$ we have

$$\begin{aligned}\langle u_{n+1}\rangle_{[0,t]} &= \frac{m}{a}<\frac{a^2+1}{2a}m<\frac{a^2+1}{2a}C,\\ \langle v_{n+1}\rangle_{[0,t]} &= am<\frac{am}{2-a}<\frac{aC}{2-a}\,.\end{aligned}$$

If $t\in[2^{-k-1},2^{-k}]$, $1\leqslant k\leqslant N$, then

$$\begin{aligned}\langle u_{n+1}\rangle_{[0,t]} &= \frac{1}{t}\Big[2^{-N-1}\frac{m}{a}+(2^{-k-1}-2^{-N-1})\frac{C}{a}+\int_{2^{-k-1}}^{t}u_{n+1}(s)ds\Big]\\ &\leqslant\frac{1}{t}\Big[2^{-k-1}\frac{C}{a}+\frac{1}{a}\int_{2^{-k-1}}^{t}u_n(2^{k+1}s-1)ds\Big]\\ &\leqslant\frac{2^{-k-1}}{t}\Big[\frac{C}{a}+\frac{1}{a}\int_0^{2^{k+1}t-1}u_n(s)ds\Big]\\ &\leqslant\frac{2^{-k-1}}{t}\Big[\frac{C}{a}+(2^{k+1}t-1)\frac{a^2+1}{2a^2}C\Big]\\ &\leqslant\frac{1}{2}\Big[\frac{C}{a}+\frac{a^2+1}{2a^2}C\Big]\leqslant\frac{a^2+1}{2a}C\,;\end{aligned}$$

$$
\begin{aligned}
\langle v_{n+1}\rangle_{[0,t]} &= \frac{1}{t}\Big[2^{-N-1}am + (2^{-k-1}-2^{-N-1})aC + \int_{2^{-k-1}}^{t} v_{n+1}(s)ds\Big] \\
&\leqslant \frac{1}{t}\Big[2^{-k-1}aC + a\int_{2^{-k-1}}^{t} v_n(2^{k+1}s-1)ds\Big] \\
&\leqslant \frac{2^{-k-1}}{t}\Big[aC + a\int_{0}^{2^{k+1}t-1} v_n(s)ds\Big] \\
&\leqslant \frac{2^{-k-1}}{t}\Big[aC + (2^{k+1}t-1)\frac{a^2}{2-a}C\Big] \\
&\leqslant \frac{1}{2}\Big[aC + \frac{a^2}{2-a}C\Big] = \frac{aC}{2-a}\,.
\end{aligned}
$$

If $t\in[1-2^{-k},1-2^{-k-1}]$, $1\leqslant k\leqslant N$, then

$$
\begin{aligned}
\langle u_{n+1}\rangle_{[0,t]} &= \frac{1}{t}\Big[2^{-N-1}\frac{m}{a} + (\frac{1}{2}-2^{-N-1})\frac{C}{a} \\
&\qquad + (\frac{1}{2}-2^{-k})aC + \int_{1-2^{-k}}^{t} u_{n+1}(s)ds\Big] \\
&\leqslant \frac{1}{t}\Big[\frac{C}{2a} + (\frac{1}{2}-2^{-k})aC + (t-1+2^{-k})\frac{a^2+1}{2}C\Big] \\
&\leqslant \frac{1}{1-2^{-k-1}}\Big[\frac{C}{2a} + (\frac{1}{2}-2^{-k})aC + 2^{-k-1}\frac{a^2+1}{2}C\Big] \leqslant \frac{a^2+1}{2a}C\,; \\
\langle v_{n+1}\rangle_{[0,t]} &= \frac{1}{t}\Big[2^{-N-1}am + (\frac{1}{2}-2^{-N-1})aC \\
&\qquad + (\frac{1}{2}-2^{-k})\frac{C}{a} + \int_{1-2^{-k}}^{t} v_{n+1}(s)ds\Big] \\
&\leqslant \frac{1}{t}\Big[\frac{1}{2}aC + (\frac{1}{2}-2^{-k})\frac{C}{a} + (t-1+2^{-k})\frac{C}{2-a}\Big] \\
&\leqslant \frac{1}{1-2^{-k-1}}\Big[\frac{1}{2}aC + (\frac{1}{2}-2^{-k})\frac{C}{a} + 2^{-k-1}\frac{C}{2-a}\Big] \leqslant \frac{aC}{2-a}\,.
\end{aligned}
$$

And, finally, if $t\in[1-2^{-N-1},1]$, where $u_{n+1}(t)=am$ and $v_{n+1}(t)=\frac{m}{a}$ are constant functions, then

$$
\begin{aligned}
C = \langle u_{n+1}\rangle_{[0,1]} &= t\langle u_{n+1}\rangle_{[0,t]} + (1-t)\,\mathrm{const}, \\
C = \langle v_{n+1}\rangle_{[0,1]} &= t\langle v_{n+1}\rangle_{[0,t]} + (1-t)\,\mathrm{const},
\end{aligned}
$$

i.e.,

$$
\langle u_{n+1}\rangle_{[0,t]} = \frac{C-(1-t)\,\mathrm{const}}{t} \qquad \text{and} \qquad \langle v_{n+1}\rangle_{[0,t]} = \frac{C-(1-t)\,\mathrm{const}}{t}
$$

are monotone functions that attained their maximal values at an end of the interval, where the required estimates are true. □

Lemma 9. *If $a<\sqrt{2}$, then*

$$
\langle u_n\rangle_I\langle v_n\rangle_I \leqslant a^6C^2 \qquad \forall I\subset[0,1].
$$

Proof. To make calculation simpler we replace estimates (7.10)–(7.11) by the following

$$
\begin{aligned}
\langle u_n\rangle_{[0,t]} &\leqslant a^3C, \qquad \langle u_n\rangle_{[t,1]} \leqslant a^3C, \\
\langle v_n\rangle_{[0,t]} &\leqslant a^3C, \qquad \langle v_n\rangle_{[t,1]} \leqslant a^3C.
\end{aligned} \tag{7.12}
$$

The inequality

$$
\frac{a^2+1}{2a} < a^3
$$

is true for all a, but the second one

$$\frac{a}{2-a} < a^3$$

is true for a sufficiently close to 1, more precisely, if $a^2 < a+1$. Indeed, multiplying this inequality by $a-1$ we get

$$\begin{gathered} a^3 - a^2 < a^2 - 1 \\ 1 < a^2(2-a) \\ \frac{a}{2-a} < a^3. \end{gathered}$$

Now we start proving the assertion of the Lemma by induction. For $n = 0$ it is evident:

$$\langle u_0 \rangle_I \langle v_0 \rangle_I = C^2 \leqslant a^6 C^2.$$

Let it is true for some n and we shall prove for the next value of index. We introduce the following notation: $I_k^- = [2^{-k-1}, 2^{-k}]$, $I_k^+ = [1-2^{-k}, 1-2^{-k-1}]$, for $1 \leqslant k \leqslant N$, and $I_{N+1}^- = [0, 2^{-N-1}]$, $I_{N+1}^+ = [1-2^{-N-1}, 1]$. First of all we note that if $I \subset I_k^\pm$, then the desired inequality is true by the induction hypothesis for $k \leqslant N$ and is evident for $k = N+1$, where our functions are constants am and $\frac{m}{a}$.

So, we need to prove our estimate only for such I that has not empty intersection with more than one interval $I_k^\pm$. By formulas (7.6) the averages $\langle u_{n+1} \rangle_{I_k^\pm}$ and $\langle v_{n+1} \rangle_{I_k^\pm}$ are either aC or $\frac{C}{a}$. By estimate (7.12) the averages over the end parts of I that are not complete intervals $I_k^\pm$ do not exceed $a^3 C$, hence for the whole interval I we have $\langle u_{n+1} \rangle_I \leqslant a^3 C$, $\langle v_{n+1} \rangle_I \leqslant a^3 C$, and the lemma is proved. □

Lemma 10.

$$\mathbf{B}(x; m, M) \geqslant \mathcal{B}(x; m, M)$$

for all $x \in \Omega_{m,M}$ such that $x_1 x_2 < M^2$.

Proof. Let choose an arbitrary C, $\sqrt{x_1 x_2} < C < M$, and take N big enough to have a sufficiently close to 1, so that $a^3 C \leqslant M$. Then we define two test functions u and v on the interval $J = [-\rho, 1]$, where

$$\rho = \frac{C - \sqrt{x_1 x_2}}{\sqrt{x_1 x_2} - m},$$

as follows

$$u(t) = \begin{cases} \sqrt{\frac{x_1}{x_2}}\, m & t < 0, \\ \sqrt{\frac{x_1}{x_2}}\, u_n(t) & t \geqslant 0, \end{cases} \quad \text{and} \quad v(t) = \begin{cases} \sqrt{\frac{x_2}{x_1}}\, m & t < 0, \\ \sqrt{\frac{x_2}{x_1}}\, v_n(t) & t \geqslant 0. \end{cases}$$

By direct calculation we check the averages of u and v:

$$\langle u \rangle_J = \frac{\rho}{\rho+1} \cdot \sqrt{\frac{x_1}{x_2}}\, m + \frac{1}{\rho+1} \cdot \sqrt{\frac{x_1}{x_2}}\, C = x_1$$

and in a similar way $\langle v \rangle_J = x_2$. In all our examples the lower estimate $\langle u \rangle_I \langle v \rangle_I \geqslant m^2$ is trivial, because always we have $u(t)v(t) \geqslant m^2$. Lemma 9 supplies us with the upper estimate $\langle u \rangle_I \langle v \rangle_I \leqslant M^2$, i.e., this pair satisfies condition (2.1).

Now we chose the following partition of the interval J: $J^- = [-\rho, 0]$, $J^+ = [0, 1]$, and then dyadic partition for $J^\pm$, i.e,

$$\alpha^- = \frac{\rho}{1+\rho} = \frac{C - \sqrt{x_1 x_2}}{C - m}, \qquad \alpha^+ = \frac{1}{1+\rho} = \frac{\sqrt{x_1 x_2} - m}{C - m},$$

in the first generation and $\alpha^- = \alpha^+ = \frac{1}{2}$ in all other generations. Calculating the sum from the definition of $\mathbf{B}$ we take into account that the sum over the interval $[0,1]$ was calculated in Lemma 6:

$$\begin{aligned}
\mathbf{B}(x;m,M) \geqslant& \frac{1}{|J|}\sum_{I\in\mathcal{I}} |(u,h_I)|\,|(v,h_I)| \\
=&\frac{\rho}{1+\rho}\cdot\frac{1}{1+\rho}|\langle u\rangle_{[-\rho,0]}-\langle u\rangle_{[0,1]}|\,|\langle v\rangle_{[-\rho,0]}-\langle v\rangle_{[0,1]}| + \frac{1}{1+\rho}b_n(N,m,C) \\
=&\frac{C-\sqrt{x_1x_2}}{C-m}\cdot\frac{\sqrt{x_1x_2}-m}{C-m}(C-m)^2 + \frac{\sqrt{x_1x_2}-m}{C-m}b_n(N,C,m) \\
=&(C-\sqrt{x_1x_2})(\sqrt{x_1x_2}-m) \\
&\qquad + (\sqrt{x_1x_2}-m)\left(3C-m-(C-m)2^{-N+1}\right)\left(1-(1-2^{-N})^n\right)\,.
\end{aligned}$$

Passing to the limit as $n\to\infty$ and then as $N\to\infty$ we get

$$\begin{aligned}
\mathbf{B}(x;m,M) &\geqslant (C-\sqrt{x_1x_2})(\sqrt{x_1x_2}-m) + (\sqrt{x_1x_2}-m)(3C-m) \\
&= (\sqrt{x_1x_2}-m)(4C-\sqrt{x_1x_2}-m) = \mathcal{B}(x;m,C)\,.
\end{aligned}$$

Since C can be chosen arbitrarily close to M we can pass to the limit $C\to M$ and obtain

$$\mathbf{B}(x;m,M) \geqslant \mathcal{B}(x;m,M)\,.$$

$\square$

The proved lemma completes the proof of Theorem 3 for all interior points of the domain Ω.

8. Concluding remarks

First of all we discuss a bit the obstacle for a similar example of an extremal sequence for the boundary points x with $x_1x_2 = M^2$.

Recall that essentially our inductive procedure of constructing an extremal sequence consists in the following: every interval, where the functions on the preceding step were constant $u_n = x_1$ and $v_n = x_2$, we split in four pieces and replace the pair u_n, v_n by a pair of step functions u_{n+1} with four steps $\{\frac{m}{a}\sqrt{\frac{x_1}{x_2}}, \frac{1}{a}x_1, ax_1, ma\sqrt{\frac{x_1}{x_2}}\}$ and v_{n+1} with four corresponding steps $\{ma\sqrt{\frac{x_2}{x_1}}, ax_2, \frac{1}{a}x_2, \frac{m}{a}\sqrt{\frac{x_2}{x_1}}\}$. However, this procedure never gives a pair satisfying (2.1) for all I, because the product of averages over the interval I being the union of two central parts of four parts of our partition is always greater than M^2. It seems that we can improve situation if we put the points from the lower boundary in the middle of the splitting interval, i.e., take the steps for u_{n+1} in the following order: $\{\frac{1}{a}x_1, \frac{m}{a}\sqrt{\frac{x_1}{x_2}}, ma\sqrt{\frac{x_1}{x_2}}, ax_1\}$, and for v_{n+1} respectively $\{ax_2, ma\sqrt{\frac{x_2}{x_1}}, \frac{m}{a}\sqrt{\frac{x_2}{x_1}}, \frac{1}{a}x_2\}$. Nevertheless, such a change helps not too much. First few steps you get a sequence satisfying condition (2.1), but not up to infinity.

Note that if we have a couple of step functions u and v on J being equal to constants $u^\pm$ and $v^\pm$ on $J^\pm$, then the point $\{\langle u\rangle_I, \langle v\rangle_I\}$ runs over the whole interval with endpoints $(u^\pm, v^\pm)$. Therefore, if condition (2.1) is fulfilled for all interval $I\subset J$ then the points corresponding to neighbor intervals cannot be arbitrary far one form another. But if we fix any interval, corresponding to a point on the lower boundary, and try to check how its neighbor will change in our iterative process, then we see that it is further and further from the starting point. This means that for a fixed N (or a fixed a) we cannot make arbitrary many steps, i.e., we cannot make n arbitrary large to kill the term $(1-2^{-N})^n$ in the expression for b_n. Moreover, a careful calculation shows that if for a given N we take the maximal possible $n = n(N)$ such that the condition (2.1) is still fulfilled then the expression $(1-2^{-N})^{n(N)}$ does not tend to zero, when N tends to infinity.

By this reason we will be not too surprised if it occurs that in the continuous case the Bellman function $\mathbf{B}$ is not continuous on the upper boundary and is smaller than $3M^2 - 4Mm + m^2$.

The second remark we would like to make is related to the proof of the main inequality (4.3) in Lemma 4. We proved it using the specific expression for the candidate in the Bellman function, but methodologically it would be interesting to deduce this global inequality from its infinitesimal form (4.4). It could be done by a direct integration only for the triples x^-, x, x^+ such that the whole straight line segment with the endpoints $x^\pm$ is in Ω.

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