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A PROPERTY OF SOME MEROMORPHIC FUNCTION

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ABSTRACT

We regard functions Φ that are meromorphic in the unit disk $\mathbb{D}$ and that are the ratio of an inner function and a finite Blaschke product, where the degree of an inner function is no less that degree of finite Blaschke product. Let $\Omega_\Phi = \{z \in \mathbb{D} : |\Phi(z)| > 1\}$. Then the function Φ can be extended to a continuous function on $\operatorname{clos}\Omega_\Phi$ and the image under Φ of the intersection of $\partial\Omega_\Phi$ and of the unit circle is of zero Lebesgue measure if and only if the intersection of $\partial\Omega_\Phi$ and of the unit circle is of zero Lebesgue measure. .

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Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the unit disk, m the normalized Lebesgue measure on $\mathbb{T}$. In this paper we shall consider meromorphic functions Φ, that are represented in the form

$$\Phi = \varphi/\psi,$$

where ψ is a finite Blaschke product, $\deg\psi = N < \infty$, and φ is an inner function, $\deg\varphi \geq N$ (a degree of an inner function φ is the number of Blaschke factors, if φ is a finite Blaschke product, and ∞ in any other case). We prove the following theorem.

Theorem 1. *Let ψ be a finite Blaschke product, $\deg\psi = N < \infty$, φ inner, $N \leq \deg\varphi \leq \infty$, and $\Phi = \varphi/\psi$. Then Φ has meromorphic continuation to $\mathbb{D}$ given by the formula $\Phi(z) = \varphi(z)/\psi(z)$, $z \in \mathbb{D}$. We put $\Omega_\Phi = \{z \in \mathbb{D} : |\Phi(z)| > 1\}$. Then Φ can be extended to continuous function on $\operatorname{clos}\Omega_\Phi$ and $m(\Phi(\partial\Omega_\Phi \cap \mathbb{T})) = 0$ if and only if $m(\partial\Omega_\Phi \cap \mathbb{T}) = 0$.*

We need the following lemmas.

Lemma 2. *Let φ be an inner function, ψ a finite Blaschke product, and $\deg\psi = N$. Then for any $\lambda \in \mathbb{C}$, $|\lambda| > 1$, the equation $\varphi - \lambda\psi = 0$ has exactly N zeros in $\mathbb{D}$ (account to multiplicity).*

This lemma is a consequence of the Rouché theorem.

Lemma 3 *Let $\mathcal{C} \subset \mathbb{D}$ be a connected set. Then the set $\{\zeta \in \operatorname{clos}\mathcal{C} \cap \mathbb{T} : \zeta$ is not angular limit of points in $\mathcal{C}\}$ is no more than countable.*

Proof. We use the idea from the proof of Lemma 3.2 of [2]. We denote the set from Lemma 3 by τ. We fixe a number s, $0 < s < \pi/2$, and we denote by $\mathcal{S}(\zeta, s)$ the sector with vertex ζ, of angle $2s$, symmetric with respect to the ray from 0 through ζ. By assumption, there exist two rational numbers $a_\zeta, b_\zeta \in (0,1)$ such that the set $\mathcal{S}(\zeta) = \{t\zeta : a_\zeta < t < 1\} \cup \{z \in \mathbb{D} : |z - a_\zeta\zeta| \leq b_\zeta\}$ is contained in $\mathbb{D}\setminus\mathcal{C}$. We suppose that τ is uncountable. then there exist a uncountable set $\tau_0 \subset \tau$ and two numbers a, b such that $a_\zeta = a$ and $b_\zeta = b$ for all $\zeta \in \tau_0$. Clearly then we can find $\zeta_1, \zeta_2, \zeta_3 \in \tau_0$ such that $\mathcal{S}(\zeta_1) \cap \mathcal{S}(\zeta_2) \cap \mathcal{S}(\zeta_3) \neq \emptyset$. Evidently, $\mathbb{D}\setminus\mathcal{S}(\zeta_1) \cap \mathcal{S}(\zeta_2) \cap \mathcal{S}(\zeta_3) = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$, where $\mathcal{G}_j$ are open sets such that $\mathcal{G}_j \cup \mathcal{G}_\ell = \emptyset$, if $j \neq \ell$, $j, \ell = 1, 2, 3$. Since $\mathcal{C}$ is connected, there exists j, $j = 1$, 2, or 3, such that $\mathcal{C} \subset \mathcal{G}_j$. Thus, $\tau_0 \subset \operatorname{clos}\mathcal{C} \subset \operatorname{clos}\mathcal{G}_j$, hence, $\zeta_1, \zeta_2, \zeta_3 \in \operatorname{clos}\mathcal{G}_j$ for one j only. This is a contradiction. $\square$

We shall use the following notation. Let $\gamma: [0,1) \to \mathbb{C}$ be a continuous mapping. The cluster set for γ is the set $\mathcal{C}(\gamma) = \{w \in \mathbb{C} :$ there exists $t_n \to_n 1$ such that $\gamma(t_n) \to_n w\}$. $\mathcal{C}(\gamma)$ is a closed connected set (see [3, ch.1]).

We put $\mathcal{A} = \{\lambda \in \widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D} : \Phi - \lambda$ has a multiply zero in $\mathbb{D}\}$. It is evident that $\mathcal{A}$ consists of separated points. Futher, if $\Gamma: [0,1] \to (\widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}) \setminus \mathcal{A}$ is a simple curve, then $\Phi^{-1}(\Gamma[0,1]) = \cup_{j=1}^N \gamma_j[0,1]$, where $\gamma_j: [0,1] \to \mathbb{D}$ are simple curves such that $\gamma_i[0,1] \cap \gamma_j[0,1] = \emptyset$, if $i \neq j$, $i, j = 1, \ldots, N$. Therefore, if $\{\lambda_n\}_{n=1}^\infty \subset (\widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}) \setminus \mathcal{A}$, $\lambda_0 \in \mathbb{T}$ and $\lambda_n \to_n \lambda_0$, then there exists a simple curve $\Gamma: [0,1] \to \big((\widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}) \setminus \mathcal{A}\big) \cup \{\lambda_0\}$ such that $\lambda_n = \Gamma(t_n)$, $n = 1, 2, \ldots$, $\lambda_0 = \Gamma(1)$, $t_n \to_n 1$, and $\Phi^{-1}(\Gamma[0,1)) = \cup_{j=1}^N \gamma_j[0,1)$, where $\gamma_j: [0,1) \to \mathbb{D}$ are continuous mappings such that $\gamma_i[0,1) \cap \gamma_j[0,1) = \emptyset$, if $i \neq j$, $i, j = 1, \ldots, N$.

We shall show that $\mathcal{C}(\gamma_j)$ consists of one point. We have $\lim_{t\to 1} \Phi(\gamma_j(t)) = \lambda_0$. Therefore, if $\mathcal{C}(\gamma_j) \cap \mathbb{D} \neq \emptyset$, then $\Phi(\mathcal{C}(\gamma_j) \cap \mathbb{D}) = \{\lambda_0\}$. If we suppose that $\mathcal{C}(\gamma_j) \cap \mathbb{D}$ consists of more than one point, then $\mathcal{C}(\gamma_j)\cap\mathbb{D}$ has a cluster point in $\mathbb{D}$, but Φ is a meromorphic function, therefore Φ must be a constant, this is a contradiction. Thus, if $\mathcal{C}(\gamma_j)\cap\mathbb{D}\neq\emptyset$, then $\mathcal{C}(\gamma_j)\cap\mathbb{D}$ consists of one point. From the connectiness of $\mathcal{C}(\gamma_j)\cap\mathbb{D}$ we conclude that $\mathcal{C}(\gamma_j)\cap\mathbb{D}=\mathcal{C}(\gamma_j)$. Now we regard the case where $\mathcal{C}(\gamma_j)\subset\mathbb{T}$. If we suppose that $\mathcal{C}(\gamma_j)$ consists of more than one point, we must conclude that $\mathcal{C}(\gamma_j)$ is a closed arc, and, consequently, has positive measure. Let us regard the set of points ζ from $\mathcal{C}(\gamma_j)$ such that ζ is an angular limit of points from γ_j and there exists an angular boundary value $\Phi(\zeta)$ at ζ. By Lemma 3 and from the properties of Φ this set has positive measure. By construction $\Phi(\zeta)=\lambda_0$ for ζ from this set, consequently, Φ must be a constant. This is a contradiction.

Thus, $\mathcal{C}(\gamma_j)$ consists of one point. We denote this point by w_j, we put $\gamma_j(1)=w_j$, then $\gamma_j\colon[0,1]\to\operatorname{clos}\mathbb{D}$ is a continuous mapping. By construction, $\lim_{t\to1}\Phi(\gamma_j(t))=\lambda_0$, $j=1,\dots,N$.

Now we shall prove that Φ has angular boundary value at every point of $\partial\Omega_\Phi\cap\mathbb{T}$ and that the function Φ defined on $\partial\Omega_\Phi\cap\mathbb{T}$ by its angular boundary values is continuous on $\operatorname{clos}\Omega_\Phi$.

Lemma 4. *Let $\{z_n\}_{n=1}^\infty\subset\Omega_\Phi$, let $\zeta_0\in\mathbb{T}$, and let $z_n\to_n\zeta_0$. Then Φ has angular boundary value $\Phi(\zeta_0)$ at ζ_0 and $\lim_n\Phi(z_n)=\Phi(\zeta_0)$.*

Proof. We put $\lambda_n=\Phi(z_n)$. First, we regard the case $\{\lambda_n\}_n\subset(\widehat{\mathbb{C}}\setminus\operatorname{clos}\mathbb{D})\setminus\mathcal{A}$. We have $1<|\lambda_n|=|\varphi(z_n)|/|\psi(z_n)|<1/|\psi(z_n)|\to_n 1$, that is, $\lim_n|\lambda_n|=1$. Therefore we can choose a subsequence $\{n_\ell\}_{\ell=1}^\infty$ such that $\lim_\ell\lambda_{n_\ell}$ exists. We denote this limit by λ_0. We have $\{\lambda_{n_\ell}\}_{\ell=1}^\infty\subset(\widehat{\mathbb{C}}\setminus\operatorname{clos}\mathbb{D})\setminus\mathcal{A}$, $\lambda_0\in\mathbb{T}$ and $\lambda_{n_\ell}\to_\ell\lambda_0$. For $\{\lambda_{n_\ell}\}_\ell$ and λ_0 we construct Γ as was desribed above, then we obtain γ_j, $j=1,\dots,N$ as above. We have $\{z_{n_\ell}\}_\ell\subset\cup_{j=1}^N\gamma_j[0,1)$. From the property $\gamma_i[0,1)\cap\gamma_j[0,1)=\emptyset$, if $i\neq j$, we conclude that there exists a number ν, $1\le\nu\le N$, such that the set of ℓ for that $z_{n_\ell}\in\gamma_\nu[0,1)$ is infinite. Hence, $\gamma_\nu(1)=\zeta_0$. Futher, $\lim_{t\to1}\Phi(\gamma_\nu(t))=\lambda_0$, therefore there exists $\lim_{t\to1}\varphi(\gamma_\nu(t))$, consequently ([3, Theorem 2.3]), φ has angular boundary value at ζ_0. Thus, Φ has angular boundary value λ_0 at ζ_0. Now, if we suppose that there exists another subsequence $\{n_s\}_s$ such that $\lim_s\lambda_{n_s}$ exists and is equal to a some point μ_0, we must conclude that the angular boundary value $\Phi(\zeta_0)$ at ζ_0 is equal to μ_0. We obtain that $\lim_n\lambda_n$ exists. The lemma is proved in the case $\lambda_n\notin\mathcal{A}$, $n=1,2,\dots$.

In the arbitrary case, we take a sequence $\{\varepsilon_n\}_n$ such that $\varepsilon>0$ and $\varepsilon\to_n0$, and we choose $\{z_{*n}\}_n\subset\Omega_\Phi$ such that $\Phi(z_{*n})\notin\mathcal{A}$, $|z_n-z_{*n}|<\varepsilon_n$, and $|\lambda_n-\Phi(z_{*n})|<\varepsilon_n$. Then we apply already proved part of the lemma to $\{z_{*n}\}_n$. $\square$

Corollary 5. *We define Φ on $\partial\Omega_\Phi\cap\mathbb{T}$ by its angular boundary values. Then Φ is continuous on* $\operatorname{clos}\Omega_\Phi$.

Corollary 5 is deduced from Lemma 4 by a standard way.

It is well-know (and can be obtained from the properties of curves Γ and γ_j introduced in the beginning of this section) that the quantity of components of the open set Ω_Φ is finite. Moreover, if Ω is a component of Ω_Φ, then there exists a natural number N_Ω such that $\Phi-\lambda$ has exactly N_Ω zeros in Ω (account to multiplicity) for any $\lambda\in\widehat{\mathbb{C}}\setminus\operatorname{clos}\mathbb{D}$. It is well-know that Ω is a finite-connected domain. We need to regard such Ω for which $\partial\Omega\cap\mathbb{T}\neq\emptyset$ only. For such Ω we define a simple-connected domain $\mathcal{O}(\Omega)$ as the union of Ω, of components K of $\partial\Omega$ for which $K\cap\mathbb{T}=\emptyset$, and of components $\mathcal{U}$ of $\mathbb{C}\setminus\Omega$ such that $\partial\mathcal{U}\cap K\neq\emptyset$ for some K described just above. It is evident that $\Omega\subset\mathcal{O}(\Omega)\subset\mathbb{D}$ and $\partial\Omega\cap\mathbb{T}=\partial\mathcal{O}(\Omega)\cap\mathbb{T}$. Let $\omega\colon\mathbb{D}\to\mathcal{O}(\Omega)$ be a comformal mapping. It is easy to see that there exists r, $0 < r < 1$, such that $R = \{z \in \mathbb{C} : \ r < |z| < 1\} \subset \omega^{-1}(\Omega)$. To convenience, we choose r such that the poles of Φ do not contain in R, too.

We put $h = \Phi \circ \omega$, $h: \omega^{-1}(\Omega) \to \widehat{\mathbb{C}} \setminus \operatorname{clos}\mathbb{D}$. Then h is a meromorphic function. We obtain that h can be extended to the function continuous on $\operatorname{clos}\omega^{-1}(\Omega)$ in the way similar to one that was used to obtain continuity of Φ on $\operatorname{clos}\Omega_\Phi$. We have $h(\mathbb{T}) \subset \mathbb{T}$, therefore h can be extended to function analytic in $\{z \in \mathbb{C} : \ r < |z| < 1/r\}$ by the formula $h(z) = 1/\overline{h(1/\bar{z})}$, $1 < |z| < 1/r$. Futher, $h(R) \subset \mathbb{T}$, therefore the argument of $h|_\mathbb{T}$ is a strongly increasing function. Thus, there exist constants c_h, C_h such that $0 < c_h < C_h < \infty$ and $c_h m(\tau) \leq m(h(\tau)) \leq C_h m(\tau)$ for any measurable set $\tau \subset \mathbb{T}$.

We note that ω is continuous on $\operatorname{clos}\mathbb{D}$. Indeed, let $\gamma: [0,1] \to \operatorname{clos}\mathbb{D}$ be a continuous mapping such that $\gamma(0,1) \subset \mathbb{D}$. We have $\Phi(\mathcal{C}(\omega(\gamma))) = h(\gamma(1))$, that is, $\Phi(\mathcal{C}(\omega(\gamma)))$ consists of one point, therefore $\mathcal{C}(\omega(\gamma))$ consists of one point. The remained part of the proof of continuity of ω is standard.

By [6, VIII.30], there exists a constant C_ω such that $m(\omega^{-1}(E)) \leq C_\omega m(E)$ for any measurable subset E of $\mathbb{T}$ such that $E \subset \partial\Omega \cap \mathbb{T}$. Let E be a such set. We put $\tau = \omega^{-1}(E)$. We have $m(\Phi(E)) = m((\Phi \circ \omega)(\omega^{-1}(E)) = m(h(\tau)) \leq C_h m(\tau) \leq C_h C_\omega m(E)$. We obtain that $m(\Phi(E)) \leq C_h C_\omega m(E)$ for any measurable set $E \subset \partial\Omega \cap \mathbb{T}$. Because of the quantity of components Ω of Ω_Φ is finite, there exists a constant $C > 0$ such that

$$m(\Phi(E)) \leq C m(E) \tag{1}$$

for any measurable set $E \subset \partial\Omega_\Phi \cap \mathbb{T}$. In particular, $m(\Phi(\partial\Omega_\Phi \cap \mathbb{T})) \leq C m(\partial\Omega_\Phi \cap \mathbb{T})$.

We remember that $\Phi = \varphi/\psi$, where ψ is a finite Blaschke product and φ is an inner function. The inner function φ can be represented in the form

$$\varphi(z) = \prod_n \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \bar{a}_n z} \exp\Big(-\int_\mathbb{T} \frac{\zeta + z}{\zeta - z} d\mu(\zeta)\Big), \qquad z \in \mathbb{D}, \tag{2}$$

where $\{a_n\}_n \subset \mathbb{D}$, $\sum_n (1 - |a_n|) < \infty$ and μ is a positive Borel singular (with respect to m) measure on $\mathbb{T}$. We shall show that φ has an angular derivative $\varphi'(\xi)$ at every point $\xi \in \partial\Omega_\Phi \cap \mathbb{T}$. By [1, Theorem 2],

$$|\varphi'(\xi)| = \sum_n \frac{1 - |a_n|^2}{|\xi - a_n|^2} + 2\int_\mathbb{T} \frac{d\mu(\zeta)}{|\zeta - \xi|^2}, \qquad \xi \in \mathbb{T}, \tag{3}$$

and $\varphi'(\xi)$ exists if and only if the right part of (3) is finite.

Lemma 6. *In the notation introduced above*

$$|\varphi'(\xi)| \leq |\psi'(\xi)| \text{ for } \xi \in \partial\Omega_\Phi \cap \mathbb{T}.$$

Proof. Let $z \in \Omega_\Phi$. By definition of Ω_Φ we have $|\varphi(z)| > |\psi(z)|$, therefore $\log|\varphi(z)| > \log|\psi(z)|$. Futher,

$$\log|\varphi(z)| = \sum_n \log\left|\frac{a_n - z}{1 - \bar{a}_n z}\right| - \int_\mathbb{T} \frac{1 - |z|^2}{|\zeta - z|^2} d\mu(\zeta) =$$

$$= \frac{1}{2}\sum_n \log\left|\frac{a_n - z}{1-\bar{a}_n z}\right|^2 - (1-|z|^2)\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - z|^2}.$$

From the inequality $1-x < -\log x$ for $0<x<1$ we write

$$\frac{1}{2}\sum_n \left(\left|\frac{a_n - z}{1-\bar{a}_n z}\right|^2 - 1\right) - (1-|z|^2)\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - z|^2} =$$

$$= (|z|^2-1)\left(\frac{1}{2}\sum_n \frac{1-|a_n|^2}{|1-\bar{a}_n z|} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - z|^2}\right) > \log|\varphi(z)| > \log|\psi(z)|,$$

therefore

$$\frac{1}{2}\sum_n \frac{1-|a_n|^2}{|1-\bar{a}_n z|} + \int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta - z|^2} < \frac{-\log|\psi(z)|}{1-|z|^2}, \qquad z\in\Omega_\Phi. \tag{4}$$

Because of ψ is a finite Blaschke product, $\frac{-\log|\psi(z)|}{1-|z|^2} \to_{z\to\xi} \frac{1}{2}|\psi'(\xi)|$, where $\xi\in\mathbb{T}$.

We put $\mu_0 = \frac{1}{2}\sum_n (1-|a_n|^2)\delta_{a_n} + \mu$, where δ_{a_n} is Dirac measure in the point a_n. Then the left part of (4) is equal to

$$\int_{\operatorname{clos}\mathbb{D}} \frac{1}{|1-\bar{\zeta}z|^2} d\mu_0(\zeta).$$

Let $\xi\in\partial\Omega_\Phi\cap\mathbb{T}$, then there exists a sequence $\{z_n\}_n$ such that $z_n\in\Omega_\Phi$ and $z_n\to_n \xi$. By Fatou's lemma,

$$\int_{\operatorname{clos}\mathbb{D}} \frac{1}{|1-\bar{\zeta}\xi|^2} d\mu_0(\zeta) \le \liminf_n \int_{\operatorname{clos}\mathbb{D}} \frac{1}{|1-\bar{\zeta}z_n|^2} d\mu_0(\zeta) \le$$

$$\le \liminf_n \left(\frac{-\log|\psi(z_n)|}{1-|z_n|^2}\right) = \frac{1}{2}|\psi'(\xi)|,$$

that is,

$$\sum_n \frac{1-|a_n|^2}{|\xi - a_n|^2} + 2\int_{\mathbb{T}} \frac{d\mu(\zeta)}{|\zeta-\xi|^2} \le |\psi'(\xi)| \qquad \xi\in\partial\Omega_\Phi\cap\mathbb{T}. \tag{5}$$

Now the lemma follows from (2) and (5). $\square$

Corollary 7. *Let* $\xi\in\partial\Omega_\Phi\cap\mathbb{T}$. *Then* $\prod_n \frac{|a_n|}{a_n}\frac{a_n-\xi}{1-\bar{a}_n\xi}$ *and* $\int_{\mathbb{T}} \frac{\zeta+\xi}{\zeta-\xi} d\mu(\zeta)$ *convergence, and angular boundary values of* φ *on* $\partial\Omega_\Phi\cap\mathbb{T}$ *are given by the formula*

$$\varphi(\xi) = \prod_n \frac{|a_n|}{a_n}\frac{a_n-\xi}{1-\bar{a}_n\xi} \exp\left(-\int_{\mathbb{T}} \frac{\zeta+\xi}{\zeta-\xi} d\mu(\zeta)\right), \qquad \xi\in\partial\Omega_\Phi\cap\mathbb{T}. \tag{6}$$

The proof of Corollary 7 consists of well-known standard computation and is omitted.

Corollary 8. *In the notation introduced above* $\mu(\partial\Omega_\Phi\cap\mathbb{T}) = 0$.

Proof. It is well-know (see [4, II.6.2]) that $\lim_{\varepsilon\to 0}\mu(I(\xi,\varepsilon))/\varepsilon=\infty$ for a.e. $\xi\in\mathbb{T}$ with respect to μ, where $I(\xi,\varepsilon)=\{\zeta\in\mathbb{T}: |\arg\zeta-\arg\xi|<\varepsilon\}$. We have

$$\frac{1}{\varepsilon}\mu(I(\xi,\varepsilon))\le\frac{1}{\varepsilon^2}\mu(I(\xi,\varepsilon))=\int_{I(\xi,\varepsilon)}\frac{d\mu(\zeta)}{\varepsilon^2}\le$$

$$\le\int_{I(\xi,\varepsilon)}\frac{d\mu(\zeta)}{|\zeta-\xi|^2}\le\frac{1}{2}|\psi'(\xi)|$$

for all $\xi\in\partial\Omega_\Phi\cap\mathbb{T}$, therefore $\lim_{\varepsilon\to 0}\mu(I(\xi,\varepsilon))/\varepsilon\le\frac{1}{2}|\psi'(\xi)|<\infty$ for all $\xi\in\partial\Omega_\Phi\cap\mathbb{T}$. Thus, $\mu(\partial\Omega_\Phi\cap\mathbb{T})=0$. $\square$

Lemma 9. *Let $\{\varepsilon_\nu\}_{\nu=1}^\infty$ be a sequence such that $\varepsilon_\nu>0$ and $\varepsilon\to_\nu 0$. We put $I_\nu=\{\zeta\in\mathbb{T}:\ \operatorname{dist}(\zeta,\partial\Omega_\Phi\cap\mathbb{T})<\varepsilon_\nu\}$ and*

$$\varphi_\nu(z)=\prod_{n=1}^{\nu}\frac{|a_n|}{a_n}\frac{a_n-z}{1-\bar a_n z}\exp\left(-\int_{\mathbb{T}\setminus I_\nu}\frac{\zeta+z}{\zeta-z}d\mu(\zeta)\right),\qquad z\in\mathbb{D}\cup I_\nu.$$

Then $\varphi_\nu(\xi)\to_{\nu\to\infty}\varphi(\xi)$, $\xi\in\partial\Omega_\Phi\cap\mathbb{T}$.

This is the consequence of Corollaries 7 and 8.

We need the following two lemmas.

Lemma 10. *Let $K\subset\mathbb{T}$ be a compact set, let functions $g_\nu, g\colon K\to\mathbb{C}$, $\nu=1,2,\dots,$ be continuous, and let $g_\nu\to_\nu g$ on K. Then there exists a sequence $\{K_n\}_{n=1}^\infty$ such that K_n are compact sets, $K_n\subset K_{n+1}\subset K$, $m(K\setminus\cup_{n=1}^\infty K_n)=0$ and $g_\nu\to_\nu g$ uniformly on K_n, $n=1,2,\dots$.*

Proof. We take a sequence $\{\varepsilon_n\}_{n=1}^\infty$ such that $\varepsilon_n>0$ and $\varepsilon\to_n 0$. By Egoroff theorem, there exists a measurable set τ_1 such that $\tau_1\subset K$, $m(\tau_1)<\varepsilon_1$ and $g_\nu\to_\nu g$ uniformly on $K\setminus\tau_1$. We put $K_1=\operatorname{clos}(K\setminus\tau_1)$. From continuity of g_ν and g we have $g_\nu\to_\nu g$ uniformly on K_1. Now we apply Egoroff theorem to the set $K\setminus K_1$ and ε_2. We obtain the measurable set τ_2 such that $\tau_2\subset K\setminus K_1$, $m(\tau_2)<\varepsilon_2$ and $g_\nu\to_\nu g$ uniformly on $(K\setminus K_1)\setminus\tau_2$. We put $K_2=K_1\cup\operatorname{clos}((K\setminus K_1)\setminus\tau_2)$. We have that $g_\nu\to_\nu g$ uniformly on K_2 and $m(K\setminus K_2)<\varepsilon_2$. Continuing in a such way we obtain the sequence $\{K_n\}_{n=1}^\infty$ such that $K_n\subset K_{n+1}\subset K$, $g_\nu\to_\nu g$ uniformly on K_n, and $m(K\setminus K_n)<\varepsilon_n$. Therefore $m(K\setminus\cup_{n=1}^\infty K_n)=\lim_{n\to\infty}m(K\setminus K_n)\le\lim_n\varepsilon_n=0$. $\square$

Lemma 11. *Let $K\subset\mathbb{T}$ be a compact set, and let $g_n,g:K\to\mathbb{T}$, $n=1,2,\dots,$ be continuous functions such that $g_n\to_n g$ uniformly on K. Then $m(g(K))\ge\limsup_n m(g_n(K))$.*

Proof. We have $m(g(K))=\lim_{\varepsilon\to 0}m(G_\varepsilon)$, where $G_\varepsilon=\{\zeta\in\mathbb{T}:\ \operatorname{dist}(\zeta,g(K))<\varepsilon\}$. From uniformly convergence of $\{g_n\}_n$ we have that for any $\varepsilon>0$ there exists a number ν such that $g_n(K)\subset G_\varepsilon$ for all $n>\nu$. Therefore $\limsup_n m(g_n(K))\le m(G_\varepsilon)$ for any $\varepsilon>0$, and we conclude that $\limsup_n m(g_n(K))\le\lim_{\varepsilon\to 0}m(G_\varepsilon)$. $\square$

Now we are ready to prove the following theorem.

Theorem 12. *In the notation introduced above*

$$m(\Phi(\partial\Omega_\Phi\cap\mathbb{T}))\ge\frac{1}{N}\int_{\partial\Omega_\Phi\cap\mathbb{T}}\big(|\psi'|-|\varphi'|\big)dm.$$

Proof. We put $\Phi_\nu = \varphi_\nu/\psi$, $\nu = 1, 2, \dots$, where φ_ν are defined in Lemma 9. We have that Φ_ν are smooth functions on I_ν and $\Phi_\nu(\xi) \to_\nu \Phi(\xi)$ for all $\xi \in \partial\Omega_\Phi \cap \mathbb{T}$. We apply Lemma 10 to the set $\partial\Omega_\Phi \cap \mathbb{T}$ and the functions Φ_ν, $\nu = 1, 2, \dots$, and Φ. We obtain the sequence $\{K_n\}_{n=1}^\infty$ of compact sets such that $K_n \subset K_{n+1} \subset \partial\Omega_\Phi \cap \mathbb{T}$, $\Phi_\nu \to_\nu \Phi$ uniformly on K_n, and $m\big((\partial\Omega_\Phi \cap \mathbb{T}) \setminus \cup_{n=1}^\infty K_n\big) = 0$. From the last equality and (1) we have $m(\Phi(\partial\Omega_\Phi \cap \mathbb{T})) = m(\Phi(\cup_{n=1}^\infty K_n)) = \lim_n m(\Phi(K_n))$. By Lemma 11, $m(\Phi(K_n)) \geq \limsup_\nu m(\Phi_\nu(K_n))$, $n = 1, 2, \dots$.

Futher, $\Phi'_\nu = (\varphi'_\nu\psi - \varphi_\nu\psi')/\psi^2$ on I_ν, and, by Julia-Carathéodory theorem (see [1, 5]), $\varphi'_\nu(\xi) = \bar{\xi}\varphi_\nu(\xi)|\varphi'_\nu(\xi)|$, $\psi'(\xi) = \bar{\xi}\psi(\xi)|\psi'(\xi)|$ for $\xi \in I_\nu$, therefore $|\Phi'_\nu| = |\psi'_\nu| - |\varphi'_\nu|$ on I_ν. Let K be a compact set such that $K \subset \partial\Omega_\Phi \cap \mathbb{T}$. Using usually change of variables and taking into account the fact that every point of $\mathbb{T}$ has no more than N preimages on $\partial\Omega_{\Phi_\nu} \cap \mathbb{T}$ we have that $m(\Phi_\nu(K)) \geq \frac{1}{N}\int_K (|\psi'| - |\varphi'_\nu|)dm$. By Levi's theorem, $\int_K |\varphi'_\nu| dm \to_\nu \int_K |\varphi'| dm$. Thus,

$$\limsup_\nu m(\Phi_\nu(K_n)) \geq \limsup_\nu \frac{1}{N}\int_{K_n} (|\psi'| - |\varphi'_\nu|)dm = \frac{1}{N}\int_{K_n} (|\psi'| - |\varphi'|)dm,$$

therefore

$$m(\Phi(K_n)) \geq \frac{1}{N}\int_{K_n} (|\psi'| - |\varphi'|)dm.$$

Now we tend n to ∞ in the both parts of the last inequality, and the result follows. □

Corollary 13. *If $m(\partial\Omega_\Phi \cap \mathbb{T}) > 0$, then $m(\Phi(\partial\Omega_\Phi \cap \mathbb{T})) > 0$.*

Proof. By Lemma 6, we have that $|\psi'| - |\varphi'| \geq 0$ on $\partial\Omega_\Phi \cap \mathbb{T}$, therefore

$$\int_{\partial\Omega_\Phi \cap \mathbb{T}} \big(|\psi'| - |\varphi'|\big)dm$$

may be zero if and only if $|\psi'| = |\varphi'|$ a.e. on $\partial\Omega_\Phi \cap \mathbb{T}$. Let us regard the function $h = \varphi'\psi - \varphi\psi'$. The function h is analytic on $\mathbb{D}$. By Julia-Carathéodory theorem and Lemma 6, h has angular boundary values $h(\xi) = \bar{\xi}\varphi(\xi)\psi(\xi)(|\varphi'(\xi)| - |\psi'(\xi)|)$ for all $\xi \in \partial\Omega_\Phi \cap \mathbb{T}$. If $m(\partial\Omega_\Phi \cap \mathbb{T}) > 0$ and $|\psi'| = |\varphi'|$ a.e. on $\partial\Omega_\Phi \cap \mathbb{T}$, we must conclude that the set of points $\xi \in \mathbb{T}$ such that $h(\xi) = 0$ has positive measure, therefore h must be equal to zero on $\mathbb{D}$ ([3, Theorem 8.1]). This contradict with the form of h. □

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