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Euler structure and Gysin homomorphism in oriented homology theories

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Abstract

It is proved the homological Self-intersection formula, Grothendieck type formula and Excess-formula for oriented homology theory.

1 Introduction

Many results concerning relations between algebraic topology and algebraic geometry were published at last time. Several results about bijections between Chern structures, Thom structures and push-forwards in cohomology were proved in [Pa1]. Several interesting cohomological formulas, well-known in topology, were transferred in the context of algebraic geometry. For example, Grothendieck type formula and excess formula were announced in [PS] and proved in [Sol].

The natural continue of this research is to get the same results for homology theories. Unfortunately, we have no well-defined product in homology, and we have to consider homomorphisms instead of elements. The analogue results about bijections were proved in [Pi].

According to the formulas in [Sol], the aim of this article is to find the homological analogue for the Grothendieck type formula and for the excess formula. The relations between [Pi] and the present paper the same that the relations between [Pa1] and [Sol]. We recall here the main theorems from [Sol] above the same homologycal theorems.

The present article consists of three parts. In the first part we recall the main definitions from [Pi] and prove useful theorems for future. The Splitting principle, Self-intersection formula and Projection type formula are proved in the first part. It is convenient to use Euler (not Chern) structure in this article, and we define Euler structure and prove that the natural relation between Euler and Chern structures is bijection.

The proof of Grothendieck type formula takes all the second part. First of all, we prove the theorem for the "convenient" universal bundle case. Then we consider the case of line bundle over affine variety. Use Jounolou trick (see [J]) we can transfer the formula to the line bundle over arbitrary variety. And we reduce the common case to the line bundle case in the last subsection.

The aim of the third part is to formulate and prove Excess formula. We consider the special case and then reduce our formula to this special case.

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Let us recall the main notation using below.

Let k be a field. Everywhere below the term "variety" means a quasi-projective variety over k; Sm denotes the category of smooth varieties; $SmOp$ denotes the category of pairs (X, U) with smooth X and open U in X.

Since $\emptyset$ is open, we may identify the category of smooth varieties with a full subcategory $SmOp$ assigning the pair $(X, \emptyset)$ to a variety X.

Everywhere $\mathbb{P}^n$ denotes the projective space of the dimension n. Let V be a vector space, then $\mathbb{P}(V)$ is the space of lines in V.

Let E be a vector bundle over X, then $\mathbb{P}(E)$ denotes the space of lines in a vector bundle E. $\mathcal{O}_E(-1)$ denotes the tautological vector bundle on $\mathbb{P}(E)$, $\mathcal{O}_E(1)$ is the dual bundle to the $\mathcal{O}_E(-1)$.

1.1 Homology theories

Definition 1. *A homology theory is a covariant functor $\mathcal{A} : SmOp \to Ab$ together with a functor morphism $\partial : \mathcal{A}(X, U) \to \mathcal{A}(U)$ satisfying the following properties:*

1. *Localization: the sequence*
   $\mathcal{A}(U) \xrightarrow{j_\mathcal{A}} \mathcal{A}(X) \xrightarrow{i_\mathcal{A}} \mathcal{A}(X, U) \xrightarrow{\partial} \mathcal{A}(U) \xrightarrow{j_\mathcal{A}} \mathcal{A}(X)$
   is exact for each pair (X, U) where $j : U \hookrightarrow X$ and $i : (X, \emptyset) \hookrightarrow (X, U)$ are the natural inclusions;

2. *Homotopy invariance: the operator $\mathcal{A}(X \times \mathbb{A}^1) \to \mathcal{A}(X)$ induced by the projection $X \times \mathbb{A}^1 \to X$ is an isomorphism;*

3. *Excision: the operator $\mathcal{A}(X', U') \to \mathcal{A}(X, U)$, induced by a morphism $e : (X', U') \to (X, U)$, is an isomorphism, if the morphism e is etale and for $Z = X \setminus U, Z' = X' \setminus U'$ one has $e^{-1}(Z) = Z'$ and $e : Z' \to Z$ is an isomorphism;*

We denote $\mathcal{A}(X,U) = \mathcal{A}^{X\setminus U}(X)$ and call such cohomology as cohomology with support $X \setminus U$. The operator

$$\mathcal{A}(X) \xrightarrow{i_*} \mathcal{A}^Z(X)$$

is called the support restriction operator for the pair $(X, X\setminus Z)$.

Now we list some important properties of homology theories.

1. $\mathcal{A}(X,X) = \mathcal{A}^{\emptyset}(X) = 0, \mathcal{A}(\emptyset) = 0$;
2. $\mathcal{A}(X \sqcup Y) = \mathcal{A}(X) \oplus \mathcal{A}(Y)$;
3. Localization sequence for a triple. Let $V \subset U \subset X$ be open subsets. Then there is an exact sequence
$$\mathcal{A}(X,V) \longrightarrow \mathcal{A}(X,U) \xrightarrow{\partial} \mathcal{A}(U,V) \longrightarrow \mathcal{A}(X,V) \longrightarrow \mathcal{A}(X,U)$$
4. Mayer-Vietoris sequence. Let $X = U_1 \cup U_2, U_{12} = U_1 \cap U_2$. Then there exists an exact sequence
$$\mathcal{A}(U_{12}) \longrightarrow \mathcal{A}(U_1) \oplus \mathcal{A}(U_2) \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{A}(U_{12}) \longrightarrow \mathcal{A}(U_1) \oplus \mathcal{A}(U_2).$$
5. Strong homotopy invariance. Let $p : T \to X$ be an affine bundle and let $V \subset X$ be a closed subset. Then $p_{\mathcal{A}} : \mathcal{A}^V(T) \to \mathcal{A}^V(X)$ is an isomorphism.
6. Deformation to the normal cone. Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N. Let X'_t be the blow-up of $X \times \mathbb{A}^1$ with the center $Y \times 0$. Let $\tilde{X}$ be the proper preimage of $X \times 0$ under the blow-up map, and let $X_t = X'_t \setminus \tilde{X}$.

   Then there exists a morphism $p_t : X_t \to \mathbb{A}^1$ and a closed imbedding $i_t : Y \times \mathbb{A}^1 \hookrightarrow X_t$ such that the map $p_t \circ i_t$ coincides with the projection $Y \times \mathbb{A}^1 \to \mathbb{A}^1$ and:

   1) the fiber of p_t over $1 \in \mathbb{A}^1$ is canonically isomorphic to X and the base change of i_t by means of the imbedding $1 \hookrightarrow \mathbb{A}^1$ coincides with the imbedding $i : Y \hookrightarrow X$;

   2) the fiber of p_t over $0 \in \mathbb{A}^1$ is canonically isomorphic to N and the base change of i_t by means of the imbedding $1 \hookrightarrow \mathbb{A}^1$ coincides with the zero section $Y \hookrightarrow N$.

   Thus we have a diagram
   $$(N, N\setminus Y) \xrightarrow{i_0} (X_t, X_t \setminus Y \times \mathbb{A}^1) \xleftarrow{i_1} (X, X\setminus Y)$$
   The following theorem can be found in [Pi] (Theorem 2.1.8).

   Theorem 1. *The following diagram consists of isomorphisms*
   $$\mathcal{A}^Y(N) \xrightarrow{i_{0\,\mathcal{A}}} \mathcal{A}^{Y\times\mathbb{A}^1}(X_t) \xleftarrow{i_{1\,\mathcal{A}}} \mathcal{A}^Y(X).$$

1.2 Orientation and Gysin homomorphism

In this section we recall some definitions and constructions from [Pi].

Definition 2. *An orientation on the theory $\mathcal{A}$ is the rule assigning to each smooth variety X,to each closed subset $Z \subset X$ and to each vector bundle E over X an isomorphism*

$$th^Z(E) : \mathcal{A}^Z(E) \to \mathcal{A}^Z(X)$$

which satisfies the following properties:

1. *Invariance: for each vector bundle isomorphism $\varphi : E \to F$ the diagram commutes*

$$\begin{array}{ccc} \mathcal{A}^Z(E) & \xrightarrow{th^Z(E)} & \mathcal{A}^Z(X) \\ {\scriptstyle\varphi_{\mathcal{A}}}\downarrow & & \| \\ \mathcal{A}^Z(F) & \xrightarrow{th^Z(F)} & \mathcal{A}^Z(X); \end{array}$$

2. *Base change: for each morphism $f : (X', X' \setminus Z') \to (X, X \setminus Z)$ with closed subsets Z and Z' and for each vector bundle E over X and for its pull-back E' over X' and for the projection $g : E' = E \times_X X' \to E$ the diagram commutes*

$$\begin{array}{ccc} \mathcal{A}^{Z'}(E') & \xrightarrow{th^{Z'}(E')} & \mathcal{A}^{Z'}(X') \\ {\scriptstyle g_{\mathcal{A}}}\big\downarrow & & {\scriptstyle f_{\mathcal{A}}}\big\downarrow \\ \mathcal{A}^{Z}(E) & \xrightarrow{th^{Z}(E)} & \mathcal{A}^{Z}(X); \end{array}$$

3. *For each vector bundles $p : E \to X$ and $q : F \to X$ the following diagram commutes*

$$\begin{array}{ccc} \mathcal{A}^{Z}(E \oplus F) & \xrightarrow{th^{Z}(p^*F)} & \mathcal{A}^{Z}(E) \\ {\scriptstyle th^{Z}(q^*E)}\big\downarrow & & {\scriptstyle th^{Z}(E)}\big\downarrow \\ \mathcal{A}^{Z}(F) & \xrightarrow{th^{Z}(F)} & \mathcal{A}^{Z}(X) \end{array}$$

   and both compositions coincide with the operator $th^Z(E \oplus F)$.

The operators $th^Z(E)$ are called Thom isomorphisms. The theory $\mathcal{A}$ is called orientable if there exists an orientation of $\mathcal{A}$. The theory $\mathcal{A}$ is called oriented if an orientation is chosen and fixed.

The next definition will be useful below.

Definition 3. *Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties. Consider a Cartesian square (in the category of schemes)*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} \\ {\scriptstyle \tilde{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{i} & X \end{array}$$

consisting of smooth varieties. This square is called transversal if the canonical morphism $\tilde{\varphi}^*(N) \to \tilde{N}$ is an isomorphism (here N and $\tilde{N}$ are the normal bundle to Y in X and to $\tilde{Y}$ in $\tilde{X}$).

Definition 4. *Let $\mathcal{A}$ be an oriented homology theory. Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with normal bundle N. We can define Gysin homomorphism $i^{gys} : \mathcal{A}(X) \to \mathcal{A}(Y)$ as the following composition*

$$\mathcal{A}(X) \longrightarrow \mathcal{A}^Y(X) \xrightarrow{\sim} \mathcal{A}^{Y \times \mathbb{A}^1}(X_t) \xleftarrow{\sim} \mathcal{A}^Y(N) \xrightarrow{th^Y(N)} \mathcal{A}^Y(Y) = \mathcal{A}(Y)$$

when the second and the third arrows are the deformation to the normal cone.

It is proved in [Pi] that Gysin homomorphism satisfies the following properties:

1. $(f \circ g)^{gys} = g^{gys} \circ f^{gys}$ for any projective morphisms $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ of smooth varieties;
2. for the transversal square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} \\ {\scriptstyle \tilde{\varphi}}\big\downarrow & & \big\downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{i} & X \end{array}$$

the following diagram commutes

$$\begin{array}{ccc} \mathcal{A}(\tilde{Y}) & \xleftarrow{i^{\tilde{gys}}} & \mathcal{A}(\tilde{X}) \\ {\scriptstyle \tilde{\varphi}_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \varphi_{\mathcal{A}}} \\ \mathcal{A}(Y) & \xleftarrow{i^{gys}} & \mathcal{A}(X) \end{array}$$

3. normalization: for any smooth variety X one has $(id_X)^{gys} = id_{\mathcal{A}(X)}$.
4. localization: for any closed imbedding of smooth varieties $i : Y \hookrightarrow X$ and the inclusion $j : X - Y \hookrightarrow X$ the sequence
$$\mathcal{A}(X \setminus Y) \overset{j_{\mathcal{A}}}{\to} \mathcal{A}(X) \overset{i^{gys}}{\to} \mathcal{A}(X)$$
is exact. This sequence is often called Gysin sequence below in the text.

1.3 Chern and Euler structures

The aim of this subsection is to observe the main properties of Euler and Chern homomorphism.

Definition 5. *For each smooth variety X, closed subset $Z \subset X$ and each vector bundle E over X we define Euler homomorphism*
$$e^Z(E) : \mathcal{A}^Z(X) \to \mathcal{A}^Z(X)$$
as the following composition:
$$\mathcal{A}^Z(X) \xrightarrow{z_{\mathcal{A}}} \mathcal{A}^Z(E) \xrightarrow{th^Z(E)} \mathcal{A}^Z(X)$$
where $z : X \to E$ be the zero section.

It follows from [Pi] and from appendix in the end of this subsection that Euler homomorphism is the top Chern homomorphism and hence it satisfies the following properties:

1. Functoriality: for any morphism $f : X' \to X$ and vector bundle E/X the following diagram is commutative
$$\begin{array}{ccc} \mathcal{A}^{Z'}(X') & \xrightarrow{e^{Z'}(f^*E)} & \mathcal{A}^{Z'}(X') \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ \mathcal{A}^Z(X) & \xrightarrow{e^Z(E)} & \mathcal{A}^Z(X) \end{array}$$
when $Z \subset X$ is a closed subset and $Z' = f^{-1}(Z)$,
$e^Z(E_1) = e^Z(E_2)$ for isomorphic E_1 and E_2,
e commutes with ∂;
2. Nondegeneracy: the operator
$$(p_{\mathcal{A}}, p_{\mathcal{A}} \circ e) : \mathcal{A}(X \times \mathbb{P}^1) \to \mathcal{A}(X) \oplus \mathcal{A}(X)$$
is an isomorphism where $e = e(q^*\mathcal{O}_1(-1))$, $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ and $p : X \times \mathbb{P}^1 \to X$ are projections;
3. Additivity: for each short exact sequence of vector bundles
$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$
one has
$$e^Z(E) = e^Z(E_1)e^Z(E_2);$$

4. Vanishing: $e^Z(\mathbf{1}_X) = 0$.

Now we can formulate Projective Bundle Theorem (see [Pi], Theorem 2.3.1).

Theorem 2. ***PBT***
Let $\mathcal{A}$ be an oriented homology theory. Let X be a smooth variety and let E be a vector bundle over X with $rk(E) = n+1$. For the endomorphism $c = e(\mathcal{O}_E(-1)) : \mathcal{A}(\mathbb{P}(E)) \to \mathcal{A}(\mathbb{P}(E))$ we have an isomorphism

$$(p_{\mathcal{A}}, p_{\mathcal{A}} \circ c, \dots, p_{\mathcal{A}} \circ c^n) : \mathcal{A}(\mathbb{P}(E)) \to \mathcal{A}(X) \oplus \mathcal{A}(X) \oplus \dots \oplus \mathcal{A}(X)$$

where $p : \mathbb{P}(E) \to X$ is a projection.

The Splitting principle will be used often below. The same cohomological result can be found in [Pa1] (Lemma 3.5.1). To prove it we need a lemma.

Lemma 1. *Let S be a smooth variety and let F_1, F_2 be two vector bundles over S and let $\alpha : F_1 \to F_2$ be a vector bundle epimorphism and let $K = ker(\alpha)$. Then there is an affine bundle $g : T \to S$ such that the epimorphism $g^*(\alpha)$ splits.*

This lemma is proved in [Pa1]. We repeat proof of it.

Proof. Let $Hom(F_2, F_1)$ be the scheme representing the sheaf $\mathcal{H}om(F_2, F_1)$ and let $\phi : Hom(F_2, F_1) \to Hom(F_2, F_2)$ be the morphism corresponding to the morphism $\mathcal{H}om(F_2, F_1) \to \mathcal{H}om(F_2, F_2)$ induced by the vector bundle map $\alpha : F_1 \to F_2$. Let $id : S \to Hom(F_2, F_2)$ be the section of the projection $Hom(F_2, F_2) \to S$ corresponding to the identity map $F_2 \to F_2$. Let $Sect(\alpha) = \phi^{-1}(id(S))$ be a closed subscheme of the scheme $Hom(F_2, F_1)$ and let $g : T = Sect(\alpha) \to S$ be the projection. The scheme T represents the sheaf of sections of the sheaf epimorphism α. Thus there exists a canonical section $s : g^*(F_2) \to g^*(F_1)$ of the epimorphism $g^*(\alpha) : g^*(F_1) \to g^*(F_2)$. This section gives rise by a standard way to a vector bundle isomorphism $g^*(F_1) = g^*(F_2) \oplus g^*(K)$.

To prove lemma it remains that the variety $T = Sect(\alpha)$ is a torsor under the vector bundle $Hom(F_2, K)$. The lemma is proved. □

Theorem 3. ***Splitting principle for homology.***
Let E be a rank n vector bundle over a smooth variety X. Then there exists a smooth variety T and a morphism $r : T \to X$ such that the vector bundle $r^*(E)$ is a direct sum of line bundles and for each closed subset Z of X and for $S = r^{-1}(Z)$ the operator $r_{\mathcal{A}} : \mathcal{A}^S(T) \to \mathcal{A}^Z(X)$ is a split surjection.

Proof. Let E be a rank n vector bundle over a smooth variety X and let $p : \mathbb{P}(E) \to X$ be the associated projective bundle over X. Then there is the canonical short exact sequence of vector bundles on $\mathbb{P}(E)$ with the rank $n-1$ vector bundle E'

$$0 \longrightarrow \mathcal{O}_E(-1) \longrightarrow p^*(E) \longrightarrow E' \longrightarrow 0$$

and the operator $p_{\mathcal{A}} : \mathcal{A}^{p^{-1}(Z)}(\mathbb{P}(E)) \to \mathcal{A}^Z(X)$ is a split surjection by the Projective Bundle theorem. Repeating this construction several times we get a smooth variety Y, a morphism $q : Y \to X$ and a filtration $0 \subset E_1 \subset E_2 \subset \dots \subset E_n = q^*(E)$ of the vector bundle $q^*(E)$ such that all the quotients E_i/E_{i-1} are line bundles. Moreover the operator $q_{\mathcal{A}} : \mathcal{A}^{q^{-1}(Z)}(Y) \to \mathcal{A}^Z(X)$ is a split surjection.

Now we use the previous lemma. Take $S = Y$ and consider the filtration $0 \subset E_1 \subset E_2 \subset \dots \subset E_n = q^*(E)$ of the vector bundle $q^*(E)$. Applying several times lemma we get an affine bundle $g : T \to S$ such that we have the direct sum decomposition $g^*(q^*(E)) = \oplus_{i=1}^n (g^*(E_i/E_{i-1}))$ of the vector bundle $(q \circ g)^*(E)$. The proof is complete. □

1.3.1 The correspondence between Chern and Euler structures

The aim of this appendix is to define one-to-one correspondence between Chern and Euler structures.

First of all, we recall the definition of Chern structure from [Pi].

Definition 6. *A Chern structure on $\mathcal{A}$ is an assignment which associate to each smooth X, closed subset $Z \subset X$ and each line bundle L on X a homomorphism $c^Z(L) : \mathcal{A}^Z(X) \to \mathcal{A}^Z(X)$ satisfying the following properties:*

1. *Functoriality: let $\varphi : (X', U') \to (X, U)$ be a morphism of varieties, $Z = X \setminus U$, $Z' = X' \setminus U'$ and L be a linear bundle on X. Then the following diagram commutes*

$$\begin{array}{ccc} \mathcal{A}^{Z'}(X') & \xrightarrow{c^{Z'}(\varphi^* L)} & \mathcal{A}^{Z'}(X') \\ {\scriptstyle \varphi_{\mathcal{A}}}\downarrow & & {\scriptstyle \varphi_{\mathcal{A}}}\downarrow \\ \mathcal{A}^Z(X) & \xrightarrow{c^Z(L)} & \mathcal{A}^Z(X), \end{array}$$

   $c^Z(L_1) = c^Z(L_2)$ for isomorphic L_1 and L_2;

   Chern homomorphisms commutes with ∂ from a long localization sequence;

2. *Nondegeneracy: the operator*

$$(p_{\mathcal{A}}, p_{\mathcal{A}} \circ c) : \mathcal{A}(X \times \mathbb{P}^1) \to \mathcal{A}(X) \oplus \mathcal{A}(X)$$

   is an isomorphism where $c = c(q^*\mathcal{O}_1(-1))$, $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ and $p : X \times \mathbb{P}^1 \to X$ are projections;

3. *Vanishing: $c^Z(\mathbf{1}_X) = 0$.*

The Chern structure on a homology theory $\mathcal{A}$ is said to be commutative if for any smooth variety X, closet subset $Z \subset X$ and line bundles L_1 and L_2 over X the Chern homomorphisms $c^Z(L_1)$ and $c^Z(L_2)$ commute with each other.

Definition 7. *One says that $\mathcal{A}$ is endowed with the Euler structure if for each smooth variety X, closed subset $Z \subset X$ and each vector bundle E over X there is given a homomorphism*

$$e^Z(E) : \mathcal{A}(X) \to \mathcal{A}(X)$$

satisfying the following properties:

1. *Functoriality: for any morphism $f : X' \to X$ and vector bundle E/X the following diagram is commutative*

$$\begin{array}{ccc} \mathcal{A}^{Z'}(X') & \xrightarrow{e^{Z'}(f^* E)} & \mathcal{A}^{Z'}(X') \\ {\scriptstyle f_*}\downarrow & & {\scriptstyle f_*}\downarrow \\ \mathcal{A}^Z(X) & \xrightarrow{e^Z(E)} & \mathcal{A}^Z(X) \end{array}$$

   when $Z \subset X$ is a closed subset and $Z' = f^{-1}(Z)$,

   $e^Z(E_1) = e^Z(E_2)$ for isomorphic E_1 and E_2,

   e commutes with ∂;

2. *Nondegeneracy: the operator*

$$(p_{\mathcal{A}}, p_{\mathcal{A}} \circ e) : \mathcal{A}(X \times \mathbb{P}^1) \to \mathcal{A}(X) \oplus \mathcal{A}(X)$$

   is an isomorphism where $e = e(q^*\mathcal{O}_1(-1))$, $q : X \times \mathbb{P}^1 \to \mathbb{P}^1$ and $p : X \times \mathbb{P}^1 \to X$ are projections;

3. *Additivity: for each short exact sequence of vector bundles*

$$0 \longrightarrow E_1 \longrightarrow E \longrightarrow E_2 \longrightarrow 0$$

one has

$$e^Z(E) = e^Z(E_1)e^Z(E_2);$$

4. *Vanishing:* $e^Z(\mathbf{1}_X) = 0$.

It is obvious that the restriction of Euler structure on line bundles gives us commutative Chern structure.

Let us recall the inverse construction from [Pi]. Let c be a commutative Chern structure. Let X be a smooth variety, $Z \subset X$ be a closed subset and E/X be a vector bundle with $rkE = n$. Let $Z' = p^{-1}(Z)$. Set $\xi = c^{Z'}(\mathcal{O}_E(-1)) \in End(\mathcal{A}^{Z'}(\mathbb{P}(E))$. By PBT the homomorphism ξ can be uniquely represented by $n \times n$ matrix $(\xi_{ij})_{i,j=0}^{n-1}$ with entries in $End(\mathcal{A}^Z(X))$. More detail, any element $a \in \mathcal{A}^{Z'}(\mathbb{P}(E))$ can be written by PBT as a column $(a_0, \ldots, a_{n-1})^T$ where $a_k = p_{\mathcal{A}}(\xi^k(a)) \in \mathcal{A}^Z(X)$. Then by definition of ξ_{ij} we have formula (the usually matrix multiplication) $(\xi(a))_k = \Sigma_{j=0}^{n-1} \xi_{kj}(a_j)$. Taking $k = n-1$ we write

$$p_{\mathcal{A}}(\xi^n(a)) = \xi_{n,n-1}(p_{\mathcal{A}}(a)) + \xi_{n,n-2}(p_{\mathcal{A}}(\xi(a))) + \cdots + \xi_{n,0}(p_{\mathcal{A}}(\xi(a))).$$

Set $c_0^Z(E) = id$, $c_k^Z(E) = (-1)^{k+1}\xi_{n,k-1}$. These elements are called Chern homomorphisms of vector bundle E. We set $e^Z(E) = c_n^Z(E)$. It is proved in [Pi] that the top Chern homomorphism satisfies all axiom of Euler structure.

Now we can prove the main theorem of this subsection.

Theorem 4. *The above constructions are inverse to each other.*

Proof.

It is obvious that the composition $c^Z(L) \to c_n^Z(E) \to c^Z(L)$ is identity. Now we have to prove that the composition $e^Z(E) \to c^Z(L) \to c_n^Z(E)$ is identity.

The line bundle case is obvious because the top Chern homomorphism for a line bundle coincides with Chern structure.

The case $E = \oplus L_i$ is obvious by the property of Euler homomorphism $e(L_1)e(L_2) = e(L_1 \oplus L_2)$.

Let E/X be a vector bundle. By Splitting principle there exists a smooth variety T and a morphism $r : T \to X$ such that the vector bundle $r^*(E)$ is a direct sum of line bundles and for each closed subset Z of X and for $S = r^{-1}(Z)$ the operator $r_{\mathcal{A}} : \mathcal{A}^S(T) \to \mathcal{A}^Z(X)$ is a split surjection. For bundle $r^*(E)$ over T the constructed Euler homomorphism coincides with the original. As $r_{\mathcal{A}}$ is a split surjection then there is unique homomorphism $i : \mathcal{A}(X) \to \mathcal{A}(T)$ such that $r_{\mathcal{A}} \circ i = id_{\mathcal{A}(X)}$. Hence, for a homomorphism $c_n^{Z'}(r^*(E)) = e^{Z'}(r^*(E)) : \mathcal{A}^{Z'}(T) \to \mathcal{A}^{Z'}(T)$ there exists the unique homomorphism $f : \mathcal{A}^Z(X) \to \mathcal{A}^Z(X)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{A}^{Z'}(T) & \xrightarrow{e^{Z'}(r^*(E))} & \mathcal{A}^{Z'}(T) \\ {\scriptstyle r_{\mathcal{A}}}\downarrow & & {\scriptstyle r_{\mathcal{A}}}\downarrow \\ \mathcal{A}^Z(X) & \xrightarrow{f} & \mathcal{A}^Z(X). \end{array}$$

This homomorphism is unique but $e^Z(E)$ and $c_n^Z(E)$ both satisfies the above property. Then they coincide. $\square$

1.4 Self-intersection formula

First of all we discuss cohomological analogue of this formula. The following theorem has been formulated in [PS] and proved in [Sol].

Theorem 5. ***Cohomological self-intersection formula.***
Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with normal bundle N of rank d. Let $(\mathcal{A}^*, \cup, c)$ be a ring cohomology theory endowed with a Chern structure. Then

$$(i^{\mathcal{A}} \circ i_{gys})(x) = x \cup c_d(N).$$

Another words, the following diagram commutes

$$\begin{tikzcd} \mathcal{A}^*(Y) \arrow[r, "i_{gys}"] \arrow[d, equal] & \mathcal{A}^*(X) \arrow[d, "i^{\mathcal{A}}"] \\ \mathcal{A}^*(Y) \arrow[r, "\cup c_d(N)"] & \mathcal{A}^*(Y) \end{tikzcd}$$

All our homological formulas will be formulated in diagram language.

Theorem 6. ***Self-intersection for Homology.***
Let $\mathcal{A}$ be an oriented homology theory. Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with normal bundle N. Then the following diagram commutes

$$\begin{tikzcd} \mathcal{A}(Y) \arrow[r, "i_{\mathcal{A}}"] \arrow[d, equal] & \mathcal{A}(X) \arrow[d, "i^{gys}"] \\ \mathcal{A}(Y) \arrow[r, "e(N)"] & \mathcal{A}(Y). \end{tikzcd}$$

Proof. We have to prove that the following diagram commutes

$$\begin{tikzcd} \mathcal{A}(Y) \arrow[r] \arrow[d, equal] & \mathcal{A}(X) \arrow[d] \\ \mathcal{A}(Y) \arrow[d, equal] & \mathcal{A}^Y(X) \arrow[d, "\simeq"] \\ \mathcal{A}(Y) \arrow[d] & \mathcal{A}^{Y\times\mathbb{A}^1}(X_t) \\ \mathcal{A}(N) \arrow[r] \arrow[d, shift right, "p_{\mathcal{A}}"'] & \mathcal{A}^Y(N) \arrow[u, "\simeq"'] \arrow[d, "th(N)"] \\ \mathcal{A}(Y) \arrow[r, "e(N)"] \arrow[u, shift right, "z_{\mathcal{A}}"'] & \mathcal{A}(Y). \end{tikzcd}$$

The bottom square is commutative because $p_{\mathcal{A}} : \mathcal{A}(N) \to \mathcal{A}(Y)$ and $z_{\mathcal{A}} : \mathcal{A}(Y) \to \mathcal{A}(N)$ are inverse to each other.

To prove that the top diagram is commutative is to prove that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{A}^Y(Y) \arrow[r] \arrow[d] & \mathcal{A}^Y(X) \arrow[d] \\ \mathcal{A}^Y(N) \arrow[r] & \mathcal{A}^{Y\times\mathbb{A}^1}(X_t). \end{tikzcd}$$

The last diagram is induced by commutative diagram

$$\begin{tikzcd}
N \arrow[r] & X_t & X \arrow[l] \\
Y \arrow[u] \arrow[r, "i_0"] & Y \times \mathbb{A}^1 \arrow[u] & Y \arrow[u] \arrow[l, "i_1"']
\end{tikzcd}$$

and commutative up to homotopy diagram

$$\begin{tikzcd}
Y \arrow[r, "i_0"] \arrow[d, equal] & Y \times \mathbb{A}^1 \arrow[d, "p"] & Y \arrow[l, "i_1"'] \arrow[d, equal] \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y.
\end{tikzcd}$$

The theorem is proved. □

1.5 Projection type formula

The next theorem will be useful in future.

Theorem 7. *Let $\mathcal{A}$ be an oriented homology theory. Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties and let $E \to X$ be a vector bundle. Then*

$$i^{gys} e(E) = e(i^*(E)) i^{gys}$$

Proof. Let N be the normal bundle over Y in X. Consider the diagram

$$\begin{tikzcd}
\mathcal{A}(X) \arrow[r] \arrow[d, "e(E)"'] & \mathcal{A}^Y(X) \arrow[r, "\sim", "i_1"'] \arrow[d, "e^Y(E)"'] & \mathcal{A}^{Y\times\mathbb{A}^1}(X_t) \arrow[d, "e^{Y\times\mathbb{A}^1}(E')"'] & \mathcal{A}^Y(N) \arrow[l, "\sim"'] \arrow[r, "th^Y(N)"] \arrow[d, "e^Y(p^*i^*(E))"'] & \mathcal{A}(Y) \arrow[d, "e(i^*(E))"'] \\
\mathcal{A}(X) \arrow[r] & \mathcal{A}^Y(X) \arrow[r, "\sim", "i_1"'] & \mathcal{A}^{Y\times\mathbb{A}^1}(X_t) & \mathcal{A}^Y(N) \arrow[l, "\sim"'] \arrow[r, "th^Y(N)"] & \mathcal{A}(Y)
\end{tikzcd}$$

where E' is the pull-back of E under the composition $X_t \hookrightarrow X'_t \xrightarrow{\sigma} X$.

To prove the commutativity of this diagram we need to check that the right hand square commutes (the other squares commutes because Euler class is functorial). Consider the right hand square.

$$\begin{tikzcd}
\mathcal{A}^Y(N) \arrow[r, "th(N)"] \arrow[d] & \mathcal{A}(Y) \arrow[d] \\
\mathcal{A}^Y(p^*i^*(E)) \arrow[r, "th^Y(p^*i^*(E))"] \arrow[d, "th^Y(E)"'] & \mathcal{A}^Y(N) \arrow[d, "th(i^*(E))"'] \\
\mathcal{A}^Y(N) \arrow[r, "th(N)"] & \mathcal{A}(Y)
\end{tikzcd}$$

The top square commutes by functoriality and the bottom square commutes by the base change of Thom homomorphism. The theorem is proved. □

2 Grothendieck type formula

In this section we formulate and prove Grothendieck type formula. The cohomological analogue of it was announced in [PS] and proved in [Sol].

Theorem 8. ***Cohomological Grothendieck type formula.***

Let $\mathcal{A}^*$ be an oriented cohomology theory. Let X be a smooth variety and let E/X be a vector bundle and let $s : X \to E$ be a section of E such that the scheme theoretic vanishing locus Y of the section s is a smooth subvariety in X of pure codimension d. Then

$$i_{gys}(1_Y) = e(E)$$

where $i : Y \to X$ is the natural imbedding.

In the diagram language the same result looks as the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{A}^*(X) \arrow[r, "i^{\mathcal{A}}"] \arrow[d, "\cup e(E)"'] & \mathcal{A}^*(Y) \arrow[d, equal] \\
\mathcal{A}^*(X) & \mathcal{A}^*(Y). \arrow[l, "i_{gys}"']
\end{tikzcd}$$

Theorem 9. ***Homological Grothendieck type formula.***

Let $\mathcal{A}$ be an oriented homology theory. Let X be a smooth variety and let E/X be a vector bundle and let $s : X \to E$ be a section of E such that the scheme theoretic vanishing locus Y of the section s is a smooth subvariety in X of pure codimension d. Then the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}(X) & \mathcal{A}(Y) \arrow[l, "i_{\mathcal{A}}"'] \arrow[d, equal] \\
\mathcal{A}(X) \arrow[u, "e(E)"] \arrow[r, "i^{gys}"] & \mathcal{A}(Y).
\end{tikzcd}$$

The proof of this theorem consists of four parts.

2.1 The Universal bundle case

In this subsection we prove the Grothendieck type formula for the case

$$X = \mathbb{P}^n = \{[x_0 : x_1 : \cdots : x_n] | \exists i : x_i \neq 0\};$$

$$\mathcal{O}_n(1) = \mathbb{P}^{n+1} \setminus \{[1 : 0 : \cdots : 0]\};$$

$$p_n : \mathcal{O}_n(1) \to \mathbb{P}^n, \quad p_n([x_0 : x_1 : \cdots : x_{n+1}]) = [x_1 : \cdots : x_{n+1}]$$

is the line bundle;

$$z_n : \mathbb{P}^n \to \mathcal{O}_n(1), \quad z_n([x_0 : x_1 : \cdots : x_n]) = [0 : x_0 : x_1 : \cdots : x_n]$$

is zero section;

$$s_n : \mathbb{P}^n \to \mathcal{O}_n(1), \quad s_n([x_0 : x_1 : \cdots : x_n]) = [x_0 : x_1 : \cdots : x_n : 0]$$

is the section transversal to z. Then $Y = \mathbb{P}^{n-1}$.

We need to prove that $s_{n,\ \mathcal{A}} \circ s_n^{gys} = e(\mathcal{O}_n(1))$. The Universal bundle case is good because $N_{\mathbb{P}^{n+1}/\mathbb{P}^n} = \mathcal{O}_n(1)$ and we can use Self-intersection formula.

Consider the transversal square

$$\begin{tikzcd}
\mathbb{P}^n \arrow[r, "z_n"] & \mathbb{P}^{n+1} \\
\mathbb{P}^{n-1} \arrow[u, "s_{n-1}"] \arrow[r, "z_{n-1}"] & \mathbb{P}^n \arrow[u, "s_n"']
\end{tikzcd}$$

and write the Self-intersection formula

$$s_n^{gys} \circ z_{n\ \mathcal{A}} = e(\mathcal{O}_n(1)).$$

As z_n and s_n are homotopic then $z_{n\ \mathcal{A}} = s_{n\ \mathcal{A}}$ and $z_n^{gys} = s_n^{gys}$. The Universal bundle case is proved.

2.2 The case of the line bundle over affine variety

Let V be vector space of the dimension n, V^* be the dual space, $\mathbb{P}(V)$ be the space of lines in V, it is the same that the space of hyperplanes in V^*. Let W be the hyperplane in V, then there exists the natural inclusion

$$i_W : \mathbb{P}(W) \hookrightarrow \mathbb{P}(V).$$

Consider the bundle

$$p : \mathbb{P}(\mathbf{1} \oplus V) \setminus \mathbb{P}(\mathbf{1}) \to \mathbb{P}(V) :$$

the line which is not equal to $\mathbf{1}$ maps to the projection on V. The similar bundle is the bundle $N_{\mathbb{P}(V)/\mathbb{P}(W)}$ — this is the space $\mathbb{P}(V)\setminus\mathbb{P}(L)$ with the natural projection on $\mathbb{P}(W)$ where L is such line that $W \oplus L = V$. Consider the section

$$s_V^{\mathbb{P}}(V) \to \mathbb{P}(L \oplus V) \setminus \mathbb{P}(L), \qquad s(l) = [p_L(l) : l],$$

when p_L is the projection on L. In this case $\mathbb{P}(W) = \{s_V = 0\}$. It is proved in the previous subsection that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}(\mathbb{P}(V)) & \mathcal{A}(\mathbb{P}(W)) \arrow[l, "i_{W,\mathcal{A}}"'] \\
\mathcal{A}(\mathbb{P}(V)) \arrow[u, "e(\mathbb{P}(L\oplus V)\setminus\mathbb{P}(L))"] \arrow[r, "i_W^{gys}"] & \mathcal{A}(\mathbb{P}(W)). \arrow[u, equal]
\end{tikzcd}$$

Now we prove the Grothendieck type formula for the line bundle over affine variety X.

Let $p : E \to X$ be a line bundle. As X is affine that there is an epimorphism of vector bundles

$$\tilde{\varphi} : X \times W \to E$$

where $X \times W$ is a trivial vector bundle. Let L be a one-dimensional vector space, let $V = W \oplus L$ and let $\varphi : X \times V \to E$ be the following map:

$$\varphi(x, w, \alpha a) = \tilde{\varphi}(x, w) + \alpha s(x),$$

where a is the basis vector of L.

Consider the map

$$f_E : X \to \mathbb{P}(V^*), \quad f_E(x) = Ker\varphi(x).$$

We recall that $Ker\varphi(x)$ is a hyperplane in V, i.e. a line in V^*. Define

$$E^\vee = \mathbb{P}(L \oplus V^*) \setminus \mathbb{P}(L)$$

and

$$\tilde{E} = f_E^*(E^\vee) = \{(x, P) \in X \times \mathbb{P}(V) | Ker\varphi(x) = p_{\mathbb{P}(V^*)}(P)\},$$

where $p_{\mathbb{P}(V^*)}$ is a projection on $\mathbb{P}(V^*)$.

Now we define the following map $h^E \to \tilde{E}$. Let $e \in E$, then there exists $(x, v) \in X \times V$ such that $\varphi(x, v) = e$. Consider the map

$$g : \mathbb{P}(V^*) \times V \longrightarrow E^\vee, \ \ g(f, v) = [f(v)a : f]$$

(recall that a is the basis of L). Define $h(e) = (x, g(f_E(x), v))$.

Let us verify that h is an isomorphism of vector bundles E and $\tilde{E}$.

First of all we prove that $(x, g(f_E(x), v)) \in \tilde{E}$, i.e. $Ker\varphi(x) = p_{\mathbb{P}(V^*)}(g(f_E(x), v))$. It is true because $Ker\varphi(x) = f_E(x)$ and

$$p_{\mathbb{P}(V^*)}(g(f_E(x), v)) = p_{\mathbb{P}(V^*)}([f_E(x)(v)a : f_E(x)]) = p_{\mathbb{P}(V^*)}([Ker\varphi(x)(v)a : Ker\varphi(x)] = Ker\varphi(x).$$

$$g(f_E(x),v)=0 \Longleftrightarrow f_E(x)(v)=0 \Longleftrightarrow v\in Ker\varphi(x)=f_E(x).$$

We check that $Ker\varphi(x)=Kerg(f_E(x))$ and so we verify that h is a well-defined isomorphism.

Note that $s_V\circ f_E=\tilde{f}_E\circ h\circ s$ where $\tilde{f}_E:\tilde{E}\to E^\vee$ is induced by f_E. To check the last equality we consider the vector $(x,0,a)\in X\times W\oplus L$, then $\varphi(x,0,a)=s(x)$ and

$$h(s(x))=(x,g(f_E(x),0,a))=(x,[f_E(x)(0,a)a:f_E(x)]).$$

Hence

$$\tilde{f_E}(x,[f_E(x)(0,a)a:f_E(x)])=[f_E(x)(0,a)a:f_E(x)]=s_V(f_E(x)).$$

It is obvious that $f_E^*(N_{\mathbb{P}(V^*)/\mathbb{P}(W^*)})=N_{X/Y}$. Then

$$f_E|_{Y\,\mathcal{A}}\circ i^{gys}=i_W^{gys}\circ f_{E,\,\mathcal{A}}$$

and the following diagram commutes

$$\begin{array}{ccc}
\mathcal{A}(X) & \xrightarrow{f_{E\,\mathcal{A}}} & \mathcal{A}(\mathbb{P}(V^*)) \\
{\scriptstyle i^{gys}}\downarrow & & {\scriptstyle i_W^{gys}}\downarrow \\
\mathcal{A}(Y) & \xrightarrow{f_E|_{Y\,\mathcal{A}}} & \mathcal{A}(\mathbb{P}(W^*)) \\
{\scriptstyle i_{\mathcal{A}}}\downarrow & & {\scriptstyle i_{\mathcal{A}}}\downarrow \\
\mathcal{A}(X) & \xrightarrow{f_{E\,\mathcal{A}}} & \mathcal{A}(\mathbb{P}(V^*)).
\end{array}$$

Right hand composition coincides with $e(N_{\mathbb{P}(V^*)/\mathbb{P}(W^*)})$. As Euler class is functorial then the commutativity of the desired diagram is checked.

2.3 The case of the line bundle over quasi-projective variety

Let $L\to X$ be a line bundle over variety X, $s:X\longrightarrow L$ be a section transversal to the zero section.

Now we use result from [J]: there exists an affine variety X' and affine bundle $p:X'\to X$.

Consider the induced bundle $L'=p^*(L)=\{(x,m)\in X'\times L|pr(m)=p(x)\}$ and the section $s':X'\to L'$, $s'(x)=(s(p(x)),x)$. Recall that Y is the set of zeros of s (see theorem 8). As p is an affine bundle, $Y'=p^{-1}(Y)$ has pure codimension 1 in X. It is obvious that s' is transversal to zero section.

Let us prove that the following square

$$\begin{array}{ccc}
X' & \xrightarrow{p} & X \\
\uparrow & & \uparrow \\
Y' & \xrightarrow{p} & Y
\end{array}$$

is transversal. We know that the squares

$$\begin{array}{ccccc}
s'(X') & \longrightarrow & L' & \longrightarrow & L \\
\uparrow & & \uparrow & & \uparrow \\
Y' & \longrightarrow & X & \xrightarrow{p} & X
\end{array}$$

are transversal (the second square is transversal because $L'=p^*(L)$), and hence their composition is transversal. This composition coincides with the another composition:

$$\begin{array}{ccccc}
s'(X') & \xrightarrow{p'} & s(X) & \xrightarrow{j} & L \\
\uparrow & & \uparrow & & \uparrow \\
Y' & \longrightarrow & Y & \longrightarrow & X.
\end{array}$$

If the composition of two squares is transversal and one of these is transversal then another square is transversal too, because

$$N_{s'(X')/Y'} = (jp')^* N_{L/z(X)} = p^* j^* N_{L/z(X)} = p^* N_{s(X)/Y}.$$

Hence the following square

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ {\scriptstyle i'}\uparrow & & {\scriptstyle i}\uparrow \\ Y' & \xrightarrow{p} & Y \end{array}$$

is transversal.

In the previous subsection we have proved that the diagram

$$\begin{array}{ccc} \mathcal{A}(X') & \xleftarrow{i'_{\mathcal{A}}} & \mathcal{A}(Y') \\ {\scriptstyle e(L')}\uparrow & & \| \\ \mathcal{A}(X') & \xrightarrow{i'^{gys}} & \mathcal{A}(Y') \end{array}$$

commutes. As p is an affine bundle then $p_{\mathcal{A}}$ is an isomorphism and $p|_{Y\,\mathcal{A}}$ is isomorphism. Then the previous diagram is isomorphic to the next:

$$\begin{array}{ccc} \mathcal{A}(X) & \xleftarrow{i_{\mathcal{A}}} & \mathcal{A}(Y) \\ {\scriptstyle e(L)}\uparrow & & \| \\ \mathcal{A}(X) & \xrightarrow{i^{gys}} & \mathcal{A}(Y). \end{array}$$

2.4 Reduction to the line bundle

Use theorem 3 (Splitting Principle) it is enough to consider the case $E = L_1 \oplus \cdots \oplus L_n$ where L_i are line bundles.

Let E_1 and E_2 be bundles with the sections s_1 and s_2 which both are transversal to zero section and which are transversal to each other. Let Y_1 and Y_2 be zero sets of s_1 and s_2 respectively. And let (E_1, s_1, Y_1) and (E_2, s_2, Y_2) both satisfies the Grothendieck Type Formula. Let us prove that $(E_1 \oplus E_2, s_1 \oplus s_2, Y_1 \oplus Y_2)$ satisfies the Grothendieck Type Formula too.

Let $Y = Y_1 \cap Y_2$. Consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{j_1} & Y_1 \\ {\scriptstyle j_2}\downarrow & & {\scriptstyle i_1}\downarrow \\ Y_2 & \xrightarrow{i_2} & X \end{array}$$

where j_1 and j_2 are the natural inclusions. We have the commutative diagram

$$\begin{array}{ccc} \mathcal{A}(Y_1) & \xleftarrow{j_{1\,\mathcal{A}}} & \mathcal{A}(Y) \\ {\scriptstyle e(L_2|_Y)}\uparrow & & \| \\ \mathcal{A}(Y_1) & \xrightarrow{i_1^{gys}} & \mathcal{A}(Y) \end{array}$$

and analogous diagram for Y_2. Collecting this diagram and the diagram for (X, s_1, Y_1) we get the desired diagram.

The Grothendieck Type Formula is proved.

3 Excess formula for homology theories

In this section we prove the excess formula.

Let $\sigma : \widehat{X} \to X$ be the blowing-up with the center Y, and let $\mathcal{A}$ be an oriented homology theory. Consider the diagram

$$\begin{array}{ccc} \mathbb{P}(N) & \xrightarrow{j} & \widehat{X} \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \sigma} \\ Y & \xrightarrow{i} & X \end{array}$$

and consider the associated diagram

$$\begin{array}{ccc} \mathcal{A}(\mathbb{P}(N)) & \xleftarrow{j^{gys}} & \mathcal{A}(\widehat{X}) \\ \downarrow{\scriptstyle p_{\mathcal{A}}} & & \downarrow{\scriptstyle \sigma^{\mathcal{A}}} \\ \mathcal{A}(Y) & \xleftarrow{i^{gys}} & \mathcal{A}(X). \end{array}$$

The last diagram is not commute; our excess formula shows it.

As L_N is subbundle in $p^*(N)$, then there exists E over $\mathbb{P}(N)$ and the short exact sequence

$$0 \to L_N \to p^*(N) \to E \to 0.$$

The bundle E is called excess bundle.

Theorem 10. *(Excess formula in cohomology.) Let $\mathcal{A}$ be an oriented cohomology theory. Then (in the previous context)*

$$\sigma^{\mathcal{A}} \circ i_{gys}(x) = j_{gys}(p^{\mathcal{A}}(x) \cup c_{n-1}(E)).$$

Theorem 11. *(Excess formula in homology.) Let $\mathcal{A}$ be an oriented homology theory. Then (in the previous context)*

$$i^{gys} \circ \sigma = p_{\mathcal{A}} \circ e(E) \circ j^{gys}.$$

In other words, the next diagram commutes

$$\begin{array}{ccccc} \mathcal{A}(\mathbb{P}(N)) & \xleftarrow{e(E)} & \mathcal{A}(\mathbb{P}(N)) & \xleftarrow{j^{gys}} & \mathcal{A}(\widehat{X}) \\ {\scriptstyle p_{\mathcal{A}}}\downarrow & & & & {\scriptstyle \sigma_{\mathcal{A}}}\downarrow \\ \mathcal{A}(Y) & & \xleftarrow{i^{gys}} & & \mathcal{A}(X) \end{array}$$

3.1 The retraction case

Lemma 2. *Let Y is a subvariety in X, such that there exists a retraction $r : X \to Y$. Let $\mathcal{N} = r^*(N)$ and let X' be a variety, such that $\mathbb{P}(N)$ is a subvariety in X', and X' is subvariety in $\mathbb{P}(\mathcal{N})$ of pure codimension d. Consider the commutative diagram of varieties*

$$\begin{array}{ccccc} \mathbb{P}(N) & \xrightarrow{j} & X' & \xrightarrow{k} & \mathbb{P}(\mathcal{N}) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle P} \\ Y & \xrightarrow{i} & X & = & X, \end{array}$$

where j and k are the natural inclusions. Then there exists a vector bundle E of rank d over $\mathbb{P}(N)$ such that

$$i^{gys} \circ \sigma = p_{\mathcal{A}} \circ e(E) \circ j^{gys}.$$

Proof. Consider the transversal square

$$\begin{tikzcd}
\mathbb{P}(\mathcal{N}) \arrow[r, "P"] & X \\
\mathbb{P}(N) \arrow[u, "k\circ j"] \arrow[r, "p"] & Y. \arrow[u, "i"']
\end{tikzcd}$$

Then

$$p_{\mathcal{A}} \circ (k \circ j)^{gys} = i^{gys} \circ P_{\mathcal{A}}.$$

As Gysin homomorphism is functorial we have

$$p_{\mathcal{A}} \circ j^{gys} \circ k^{gys} = i^{gys} \circ P_{\mathcal{A}}.$$

Add $k_{\mathcal{A}}$:

$$p_{\mathcal{A}} \circ j^{gys} \circ k^{gys} \circ k_{\mathcal{A}} = i^{gys} \circ P_{\mathcal{A}} \circ k_{\mathcal{A}}.$$

Use self-intersection formula:

$$p_{\mathcal{A}} \circ j^{gys} \circ e(N_{\mathbb{P}(\mathcal{N})/X'}) = i^{gys} \circ P_{\mathcal{A}} \circ k_{\mathcal{A}}.$$

Use projection type formula:

$$p_{\mathcal{A}} \circ e(E) \circ j^{gys} = i^{gys} \circ P_{\mathcal{A}} \circ k_{\mathcal{A}}$$

where $E = j^*(N_{\mathbb{P}(\mathcal{N})/X'})$. As $P_{\mathcal{A}} \circ k_{\mathcal{A}} = \sigma_{\mathcal{A}}$ we get the desired equation. □

3.2 Reduction to the normal bundle

We need to prove the commutativity of the diagram:

$$\begin{tikzcd}
\mathcal{A}(Y) & \mathcal{A}(\mathbb{P}(N)) \arrow[l, "p_{\mathcal{A}}"'] & \mathcal{A}(\mathbb{P}(N)) \arrow[l, "e(E)"'] \\
\mathcal{A}(X) \arrow[u, "i^{gys}"'] \arrow[r, equal] & \mathcal{A}(X) & \mathcal{A}(\widehat{X}) \arrow[l, "\sigma_{\mathcal{A}}"'] \arrow[u, "j^{gys}"']
\end{tikzcd}$$

We would like to use the previous lemma but we need the retraction $r : X \to Y$. We can get N instead of X (with the natural projection). Let us prove this reduction.

Consider the desired diagram:

$$\begin{tikzcd}
\mathcal{A}(Y) & \mathcal{A}(\mathbb{P}(N)) \arrow[l, "p_{\mathcal{A}}"'] & \mathcal{A}(\mathbb{P}(N)) \arrow[l, "e(E)"'] \\
\mathcal{A}(N) \arrow[u, "p_{\mathcal{A}}"'] & & \mathcal{A}(L_N) \arrow[u, "p_{\mathcal{A}}"'] \\
\mathcal{A}^Y(N) \arrow[u, "th(N)"'] \arrow[r, equal] \arrow[d, "i_{\mathcal{A}\,0}", "\simeq"'] & \mathcal{A}^Y(N) & \mathcal{A}^{\mathbb{P}(N)}(L_N) \arrow[l, "\widetilde{\sigma}_{\mathcal{A}}"'] \arrow[u, "th(L_N)"'] \arrow[d, "I_{\mathcal{A}\,0}", "\simeq"'] \\
\mathcal{A}^{\mathbb{A}^1\times Y}(X_t) \arrow[r, equal] & \mathcal{A}^{\mathbb{A}^1\times Y}(X_t) & \mathcal{A}^{\mathbb{A}^1\times \mathbb{P}(N)}(\widehat{X}_t) \arrow[l, "\sigma_{\mathcal{A}\,t}"'] \\
\mathcal{A}^Y(X) \arrow[u, "i_{\mathcal{A}\,1}", "\simeq"'] \arrow[r, equal] & \mathcal{A}^Y(X) & \mathcal{A}^{\mathbb{P}(N)}(\widehat{X}) \arrow[l, "\sigma_{\mathcal{A}}"'] \arrow[u, "I_{\mathcal{A}\,1}", "\simeq"'] \\
\mathcal{A}(X) \arrow[u, "i_{\mathcal{A}}"'] \arrow[r, equal] & \mathcal{A}(X) & \mathcal{A}(\widehat{X}) \arrow[l, "\sigma_{\mathcal{A}}"'] \arrow[u, "i_{\mathcal{A}}"']
\end{tikzcd}$$

The three bottom squares commutes as induced by the next commutative diagram:

$$\begin{array}{ccc}
(N, N\setminus Y) & \xleftarrow{\ \widetilde{\sigma}\ } & (L_N, L_N \setminus \mathbb{P}(N)) \\
\Big\downarrow{\scriptstyle i_0} & & \Big\downarrow{\scriptstyle I_0} \\
(X_t, X_t \setminus \mathbb{A}^1\times Y) & \xleftarrow{\ \sigma_t\ } & (\widehat{X}_t, \widehat{X}_t \setminus \mathbb{A}^1 \times \mathbb{P}(N)) \\
\Big\uparrow{\scriptstyle i_1} & & \Big\uparrow{\scriptstyle I_1} \\
(X, X\setminus Y) & \xleftarrow{\ \sigma\ } & (\widehat{X}, \widehat{X}\setminus \mathbb{P}(N)) \\
\Big\uparrow{\scriptstyle i} & & \Big\uparrow{\scriptstyle i} \\
X & \xleftarrow{\ \sigma\ } & \widehat{X}
\end{array}$$

In the top square we deal with normal bundles only.

Let us describe the map σ_t. Consider the map $\sigma\times id : \widehat{X}\times\mathbb{A}^1 \to X\times\mathbb{A}^1$. Then (see [Ha] Ch.2, prop. 7.15) there exists the map $\widehat{X\times\mathbb{A}^1}_{\mathbb{P}(N)\times 0} \to \widehat{X\times\mathbb{A}^1}_{Y\times 0}$ and we get the desired map after excision.

Note that if $i : Y \to N$ is zero section that i^{gys} coincides with the composition

$$\mathcal{A}(Y) \overset{th(N)}{\longleftarrow} \mathcal{A}^Y(N) \overset{i^{\mathcal{A}}}{\longleftarrow} \mathcal{A}(N).$$

3.3 The end of the proof

Now we need to check the conditions of our lemma and to prove that the bundle E from lemma is coincides with the excess bundle.

Let $r : N \to Y$ be the projection. Consider $\mathcal{N} = r^*(N)$. Then the desired diagram is the next:

$$\begin{array}{ccccc}
\mathbb{P}(N) & \xrightarrow{\ j\ } & L_N & \xrightarrow{\ k\ } & \mathbb{P}(\mathcal{N}) \\
\Big\downarrow{\scriptstyle p} & & \Big\downarrow{\scriptstyle \widetilde{\sigma}} & & \Big\downarrow{\scriptstyle P} \\
Y & \xrightarrow{\ i\ } & N & = & N
\end{array}$$

To use lemma we need to define k. Note that

$$\mathcal{N} = \{(x,y) \in N\times N | p(x) = p(y)\} = \mathbb{P}(N)\times_Y N.$$

Consider the projection $\mathbb{P}(\mathcal{N})$ on $\mathbb{P}(N)$ we can identify $\mathbb{P}(\mathcal{N})$ with $p^*(N)$. Then we get

$$\mathbb{P}(\mathcal{N}) = p^*(N) \supset L_N.$$

Now it is enough to prove that $E = j^*(N_{\mathbb{P}(\mathcal{N})/L_N})$. Consider the diagram which rows are short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N_{L_N/\mathbb{P}(N)} & \longrightarrow & N_{\mathbb{P}(\mathcal{N})/\mathbb{P}(N)} & \longrightarrow & j^*N_{\mathbb{P}(\mathcal{N})/L_N} & \longrightarrow & 0 \\
 & & \Big\| & & \Big\| & & & & \\
0 & \longrightarrow & L_N & \longrightarrow & p^*(N) & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

Then $E = j^*(N_{\mathbb{P}(\mathcal{N})/L_N})$. Excess formula is proved.

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