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Navier-Stokes equations: almost $L_{3,\infty}$-case

G. Seregin

St. Petersburg Department
of Steklov Institute of Mathematics,
27, Fontanka, St. Petersburg 191023, Russia

Abstract A sufficient condition of regularity for solutions to the Navier-Stokes equations is proved. It generalizes the so-called $L_{3,\infty}$-case.

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1 Introduction

The paper is motivated by the following result in [9]. Let us consider two functions v and p defined in $Q_T = \Omega\times]0,T[$, where $\Omega \subset \mathbb{R}^3$ and T is a positive parameter. Assume that they meet three conditions:

$$v \in L_{2,\infty}(Q_T) \cap W^{1,0}_2(Q_T), \quad p \in L_{\frac{3}{2}}(Q_T); \tag{1.1}$$

v and p satisfy the Navier-Stokes equations

$$\left.\begin{array}{c} \partial_t v + v\cdot\nabla\, v - \Delta\, v = -\nabla\, p \\ \operatorname{div} v = 0 \end{array}\right\} \qquad \text{in } Q_T \tag{1.2}$$

in the sense of distributions;
v and p satisfy the local energy inequality

$$\int\limits_{\Omega} \varphi(x,t)|v(x,t)|^2\,dx + 2\int\limits_{\Omega\times]0,t[} \varphi|\nabla v|^2\,dxdt'$$
$$\leq \int\limits_{\Omega\times]0,t[} (|v|^2(\Delta\varphi + \partial_t\varphi) + v\cdot\nabla\varphi(|v|^2+2p))\,dxdt' \tag{1.3}$$

for a.a. $t\in]0,T[$ and for all nonnegative smooth functions, vanishing in the neighborhood of the parabolic boundary $\partial' Q_T$ of the space-time cylinder Q_T.

A pair v and p, having properties (1.1)-(1.3), is also called a suitable weak solution to the Navier-Stokes equations in Q_T. Such kind of solutions was defined and treated by Scheffer [7], [8], Caffarelli-Kohn-Nirenberg [1], and others (see, for instances, [5], [4], [10], and [11]). Our version of the definition of suitable weak solutions is due to F.-H. Lin [5]. It seems to be more convenient to study.

We also say that the space-time point $z_0 = (x_0,t_0)$ with $x_0 \in \Omega$ and $0 < t_0 \leq T$ is a regular point of v (in the Caffarelli-Kohn-Nirenberg sense) if there exists a positive number $r < \min\{\operatorname{dist}\{x_0,\Omega\}, \sqrt{t_0}\}$ such that $v \in L_\infty(Q(z_0,r))$, where $Q(z_0,r) = B(x_0,r)\times]t_0-r^2,t_0[$ and $B(x_0,r)$ is the three-dimensional ball of radius r with the center at the point x_0. The point z_0 is called singular if it is not regular.

We may give different definitions of regular points. For example, we can replace the space $L_\infty(Q(z_0,r))$ with $C(\overline{Q}(z_0,r))$ (or even with $C^\alpha(\overline{Q}(z_0,r))$

for some positive α). However, according to the local regularity theory for the Stokes equations, they turned out to be equivalent.

We may ask the following question: how many singular points are, say, at $t = T$? In [9], it was shown that there exists a positive universal constant ε such that

$$N_T \leq \varepsilon \limsup_{t \uparrow T} \frac{1}{T-t} \int_t^T \int_\Omega |v(x,s)|^3 \, dxds. \tag{1.4}$$

Here, N_T is the number of singular points at $t = T$. Obviously, if the right hand side of (1.4) is finite, then N_T is finite. For example, this can happen if

$$v \in L_{3,\infty}(Q_T). \tag{1.5}$$

The similar result was established earlier by Neustupa [6]. Later, in [12], [2], it was proved that (1.5) implies regularity of v in Q_T and thus $N_T = 0$. Now, it is interesting to figure out what happens if

$$M_T \equiv \limsup_{t \uparrow T} \frac{1}{T-t} \int_t^T \int_\Omega |v(x,s)|^3 \, dxds < +\infty. \tag{1.6}$$

Clearly, (1.6) is less restrictive than (1.5). Nevertheless, the main result of the paper says that the answer is the same.

Theorem 1.1 *Assume that v and p satisfy conditions (1.1)-(1.3). Let, in addition,*

$$m_T = \liminf_{t \uparrow T} \frac{1}{T-t} \int_t^T \int_\Omega |v(x,s)|^3 \, dxds < +\infty. \tag{1.7}$$

Then, points $z = (x,T)$, where $x \in \Omega$, are regular.

To demonstrate how Theorem 1.1 can be used, let us consider the following initial-boundary value problem for the Navier-Stokes equations

$$\left.\begin{array}{c} \partial_t v + v \cdot \nabla v - \Delta v = -\nabla p \\ \operatorname{div} v = 0 \end{array}\right\} \quad \text{in } Q_\infty, \tag{1.8}$$

where $Q_\infty = \Omega \times]0, +\infty[$, Ω is a bounded domain in $\mathbb{R}^3$,

$$v(x,t) = 0, \qquad x \in \partial\Omega, \quad 0 \leq t < \infty, \tag{1.9}$$

$$v(x,0)=a(x), \qquad x\in\Omega. \tag{1.10}$$

It is assumed that

$$a\in C_0^\infty(\Omega), \qquad \operatorname{div} a=0. \tag{1.11}$$

It is well known that problem (1.8)-(1.10) has at least one weak solution which is called the weak Leray-Hopf solution (see monograph [3] for details). One of the challenging problem in mathematical hydrodynamics is to show that the above solution is unique. In turn, among various approaches to this problem, the idea to prove smoothness of the weak Leray-Hopf solutions is quite popular. The following theorem might be regarded as a small step in that direction.

Theorem 1.2 *Let us denote by T the first moment of time when singular points appear. Then*

$$\lim_{t\uparrow T}\frac{1}{T-t}\int\limits_t^T\int\limits_\Omega |v|^3dz=+\infty.$$

PROOF Theorem 1.2 is a direct consequence of Theorem 1.1.

2 Proof of Theorem 1.1

By the natural scaling for the Navier-Stokes equations, it is sufficient to replace the cylinder Q_T with the cylinder $Q=B\times]-1,0[$ and to prove that $z=0$ is regular point. Here, $B(r)=B(0,r)$ and $B=B(1)$. Now, condition (1.7) may be taken in the form

$$M\equiv\liminf_{t\uparrow 0}\ -\frac{2}{t}\int\limits_t^0\int\limits_B |v(x,t)|^3\,dxds<+\infty. \tag{2.1}$$

Next, by the known multiplicative inequality

$$v\in L_{\frac{10}{3}}(Q), \tag{2.2}$$

$$|v|\,|\nabla v|\in L_{\frac98,\frac32}(Q), \tag{2.3}$$

and since

$$p\in L_{\frac32}(Q), \tag{2.4}$$

one can apply the local regularity theory for the Stokes system and conclude that

$$|\partial_t v|+|\nabla^2 v|+|\nabla p|\in L_{\frac{9}{8},\frac{3}{2}}(Q(5/6)). \tag{2.5}$$

Here, $Q(r)=Q(0,r)$ and $Q=Q(1)$. In particular, (2.5) allows us to fix a representative of the function $t\mapsto v(\cdot,t)$ in such a way that

$$\textit{the function } t\mapsto \int\limits_{B(5/6)} v(x,t)\cdot w(x)\,dx \textit{ is continuous on } [-(5/6)^2,0]$$

$$\textit{for any } w\in L_2(B(5/6))$$

and thus, for each $t\in[-(5/6)^2,0]$,

$$\|v(\cdot,t)\|_{L_2(B(5/6))}<+\infty. \tag{2.6}$$

Now, our aim is to show that

$$v(\cdot,0)\in L_3(B(5/6)). \tag{2.7}$$

To this end, we note that, by (2.1), there exists a sequence $t_k\in]-1,0[$ such that $t_k\uparrow 0$ as $k\to\infty$ and, for $k=1,2,...$,

$$-\frac{1}{t_k}\int\limits_{t_k}^{0}\int\limits_{B}|v(x,t)|^3dxdt\leq M. \tag{2.8}$$

Then, we introduce the additional notation

$$g(t)=\|v(\cdot,t)\|_{L_3(B(5/6))},$$

$$E_k=\{t_k<t<0\ \|\ g(t)>10M\},\qquad E'_k=]t_k,0[\setminus E_k,$$

where $k=1,2,....$ By (2.8), we find

$$-\frac{1}{t_k}|E_k|10M\leq -\frac{1}{t_k}\int\limits_{t_k}^{0}g(t)\,dt\leq M.$$

So,

$$|E_k|\leq\frac{|t_k|}{10}$$

and thus

$$|E'_k| = |t_k| - |E_k| \geq \frac{9}{10}|t_k| > 0.$$

Therefore, for each $k = 1, 2, ...$, there exists $s_k \in]t_k, 0[$ such that

$$s_k \uparrow 0 \ as \ k \to \infty \qquad and \qquad g(s_k) \leq 10M. \tag{2.9}$$

On the other hand, according partial regularity theory of the Navier-Stokes equations

$$v(x, s_k) \to v(x, 0), \qquad \forall x \in B \setminus \Sigma,$$

and the 1D Hausdorff measure of Σ is zero. By Fatou's lemma and by (2.9), we have

$$\int\limits_{B(5/6)} |v(x,0)|^3 dx \leq 10M.$$

So, (2.7) is proved.

Now, we split the pressure into two parts:

$$p = p^1 + p^2, \tag{2.10}$$

where p^1 is determined as a unique solution of the following problem

$$\int\limits_B p^1(x,t)\Delta\varphi(x)\, dx = -\int\limits_B v(x,t) \otimes v(x,t) : \nabla^2\varphi(x)\, dx.$$

Here, φ is an arbitrary test function from the space $\{\varphi \in W^2_3(B) \parallel \varphi = 0 \ on \ \partial B\}$. It is well known that p^1 satisfies the estimate

$$\int\limits_B |p^1(x,t)|^{\frac{3}{2}}\, dx \leq c \int\limits_B |v(x,t)|^3\, dx \tag{2.11}$$

and thus

$$-\frac{1}{t_k}\int\limits_{t_k}^0 \int\limits_B |p^1(x,s)|^{\frac{3}{2}}\, dxds \leq cM, \qquad k = 1, 2, ... \tag{2.12}$$

The second component of the pressure is a harmonic function and, therefore, satisfies the estimates:

$$\sup_{x \in B(2/3)} |p^2(x,t)|^{\frac{3}{2}} \leq c \int\limits_B |p^2(x,t)|^{\frac{3}{2}}\, dx$$

$$\leq c\Big(\int\limits_B |p(x,t)|^{\frac32}dx+\int\limits_B |p^1(x,t)|^{\frac32}dx\Big) \tag{2.13}$$

$$\leq c\Big(\int\limits_B |p(x,t)|^{\frac32}dx+\int\limits_B |v(x,t)|^{3}dx\Big).$$

Now, we extend functions v, p^1, and p^2 by zero to the whole space $\mathbb{R}^3\times\mathbb{R}^1$.

Assume that the statement of the theorem is false. Then, we know that

$$\frac{1}{R^2}\int\limits_{Q(R)}\Big(|v|^3+|p|^{\frac32}\Big)\,dz\geq\varepsilon>0,\qquad \forall R\in]0,1] \tag{2.14}$$

for some positive universal constant ε. Fixing $T<-1000$, we may blow up our solution at zero with the help of the following scaling:

$$u^k(y,s)=R_kv(R_ky,R_k^2s),\qquad (y,s)\in\mathbb{R}^3\times\mathbb{R}^1,$$

$$q^{1k}(y,s)=R_k^2p^1(R_ky,R_k^2s),\qquad q^{2k}(y,s)=R_k^2p^2(R_ky,R_k^2s),$$

where $R_k=\sqrt{t_k/T}\to 0$ as $k\to\infty$. We have (remember $t_k=TR_k^2$)

$$-\frac1T\int\limits_T^0\int\limits_{\mathbb{R}^3}|u^k(y,s)|^3dyds=-\frac1T\int\limits_T^0\int\limits_{B(1/R_k)}|u^k(y,s)|^3dyds$$

$$=-\frac{1}{TR_k^2}\int\limits_{TR_k^2}^0\int\limits_B|v(y,s)|^3dyds\leq M. \tag{2.15}$$

q^{1k} can be treated in the similar way:

$$-\frac1T\int\limits_T^0\int\limits_{\mathbb{R}^3}|q^{1k}(y,s)|^{\frac32}dyds=-\frac1T\int\limits_T^0\int\limits_{B(1/R_k)}|q^{1k}(y,s)|^{\frac32}dyds$$

$$=-\frac{1}{TR_k^2}\int\limits_{TR_k^2}^0\int\limits_B|p^1(y,s)|^{\frac32}dyds\leq cM. \tag{2.16}$$

As to q^{2k}, we take into account (2.13) and argue as follows:

$$-\frac{1}{T}\int\limits_{T}^{0}\int\limits_{B(a)}|q^{2k}(y,s)|^{\frac{3}{2}}dyds=-\frac{1}{TR_k^2}\int\limits_{TR_k^2}^{0}\int\limits_{B(aR_k)}|p^2(y,s)|^{\frac{3}{2}}dyds$$

$$\leq -c\frac{(aR_k)^3}{TR_k^2}\int\limits_{TR_k^2}^{0}\int\limits_{B}\Big(|p(x,t)|^{\frac{3}{2}}+|v(x,t)|^3\Big)dxdt$$

$$\leq -c\frac{a^3}{T}R_k\int\limits_{Q}\Big(|p|^{\frac{3}{2}}+|v|^3\Big)dz\to 0 \tag{2.17}$$

as $k\to+\infty$ if $aR_k<2/3$.

Selecting subsequences (still denoted in the same way), we have

$$\begin{aligned} u^k &\rightharpoonup u \qquad in \quad L_3(B(a)\times]T,0[),\\ q^{1k} &\rightharpoonup q \qquad in \quad L_{\frac{3}{2}}(B(a)\times]T,0[),\\ q^{2k} &\to 0 \qquad in \quad L_{\frac{3}{2}}(B(a)\times]T,0[) \end{aligned} \tag{2.18}$$

for any $a>0$. Moreover,

$$|u|^2+|q|\in L_{\frac{3}{2}}(\mathbb{R}^3\times]T,0[). \tag{2.19}$$

It remains to show that the pair u and q is a suitable weak solution to the Navier-Stokes equations on sets of the form $B(a)\times]T,0[$. To this end, we first observe that

$$\|u^k\|_{L_{2,\infty}(B(a)\times]T,0[)}+\|\nabla u^k\|_{L_2(B(a)\times]T,0[)}\leq C(a,T,M)<+\infty \tag{2.20}$$

and, therefore, by known multiplicative inequalities,

$$\||u^k|\,|\nabla u^k|\|_{L_{\frac{9}{8},\frac{3}{2}}(B(a)\times]T,0[)}+\|u^k\|_{L_{\frac{10}{3}}(B(a)\times]T,0[)}\leq C(a,T,M). \tag{2.21}$$

Next, the linear theory says that

$$\||\partial_t u^k|+|\nabla^2 u^k|\|_{L_{\frac{9}{8},\frac{3}{2}}(B(a)\times]T,0[)}\leq C(a,T,M). \tag{2.22}$$

Estimates (2.20)-(2.22), together with known compactness arguments, imply

$$u^k \to u \qquad in \quad L_3(B(a)\times]T,0[,$$

$$u^k \to u \qquad in \quad C([T,0];L_{\frac{9}{8}}(B(a))) \tag{2.23}$$

for any $a>0$.

Now, we can pass to the limit in the Navier-Stokes equations and in the local energy inequality for u^k and q^k on sets of the form $B(a)\times]T,0[$ and conclude that limit functions u and q generate a suitable weak solution to the Navier-Stokes equations on those sets. The function u has the properties

$$u\in L_{2,\infty}(B(a)\times]T,0[)\cap W^{1,0}_2(B(a)\times]T,0[)\cap L_{\frac{10}{3}}(B(a)\times]T,0[),$$

$$|u|\,|\nabla u|+|\partial_t u|+|\nabla^2 u|\in L_{\frac{9}{8},\frac{3}{2}}(B(a)\times]T,0[), \tag{2.24}$$

$$u\in C([T,0];L_{\frac{9}{8}}(B(a))).$$

Let us show that our blow-up solution is not trivial. By scaling and by (2.14),

$$\frac{1}{(aR_k)^2}\int\limits_{Q(aR_k)}\Big(|v|^3+|p|^{\frac{3}{2}}\Big)dz=\frac{1}{a^2}\int\limits_{Q(a)}\Big(|u^k|^3+|q^k|^{\frac{3}{2}}\Big)dz\geq\varepsilon>0 \tag{2.25}$$

for all $a\in]0,1]$ and for all $k=1,2,....$ It follows from (2.23) that

$$\int\limits_{Q(a)}|u^k|^3dz\to\int\limits_{Q(a)}|u|^3dz. \tag{2.26}$$

Going back to the definition of p^1, we find after change of variables:

$$\int\limits_{B(1/R_k)}q^{1k}(y,s)\Delta\psi(y)dy=-\int\limits_{B(1/R_k)}u^k(y,s)\otimes u^k(y,s):\nabla^2\psi(y)dy$$

for any test function $\psi\in W^2_3(B(1/R_k))$ with $\psi=0$ on $\partial B(1/R_k)$ and for any $s\in[-1/R_k^2,0]$. Next, q^{1k} can be split in the following way:

$$q^{1k}=r^{1k}+r^{2k} \qquad in \quad Q(2), \tag{2.27}$$

where

$$\int\limits_{B(2)} r^{1k}(y,s)\Delta\psi(y)dy = -\int\limits_{B(2)} u^k(y,s)\otimes u^k(y,s) : \nabla^2\psi(y)dy$$

for any test function $\psi \in W^2_3(B(2))$ with $\psi = 0$ on $\partial B(2)$ and for any $s \in [-2^2, 0]$. For r^{1k}, we have the estimate

$$\int\limits_{Q(2)} |r^{1k}|^{\frac{3}{2}}dz \leq c\int\limits_{Q(2)} |u^k|^3 dz \to c\int\limits_{Q(2)} |u|^3 dz. \tag{2.28}$$

From (2.27), it follows that

$$\Delta r^{2k}(\cdot, s) = 0 \qquad in \quad B(2)$$

for any $s \in [-2^2, 0]$. By properties of harmonic functions,

$$\sup_{y\in B(1)} |r^{2k}(y,s)|^{\frac{3}{2}} \leq c\int\limits_{B(2)} |r^{2k}(y,s)|^{\frac{3}{2}}dy, \qquad s \in [-2^2, 0].$$

So, letting $e = (y, s)$,

$$\int\limits_{-2^2}^{0} \|r^{2k}(\cdot, s)\|^{\frac{3}{2}}_{L_\infty(B(1))}ds \leq c\int\limits_{Q(2)} |r^{2k}|^{\frac{3}{2}}de \leq$$

$$\leq c\int\limits_{Q(2)} |q^{1k}|^{\frac{3}{2}}de + c\int\limits_{Q(2)} |r^{1k}|^{\frac{3}{2}}de$$

$$\leq c\int\limits_{Q(2)} |q^{1k}|^{\frac{3}{2}}de + c\int\limits_{Q(2)} |u^k|^3 de \leq C(M,T). \tag{2.29}$$

Then, we have (see (2.25) and (2.26))

$$0 < \varepsilon = \limsup_{k\to\infty} \frac{1}{a^2}\int\limits_{Q(a)} \Big(|u^k|^3 + |q^k|^{\frac{3}{2}}\Big)de$$

$$\leq \frac{1}{a^2}\int\limits_{Q(a)} |u|^3 de + c\limsup_{k\to\infty} \frac{1}{a^2}\Big(\int\limits_{Q(a)} |q^{1k}|^{\frac{3}{2}}de + \int\limits_{Q(a)} |q^{2k}|^{\frac{3}{2}}de\Big).$$

By the last relation in (2.18), we have

$$0 < \varepsilon \leq \frac{1}{a^2} \int\limits_{Q(a)} |u|^3 de + c \limsup_{k\to\infty} \frac{1}{a^2} \int\limits_{Q(a)} |q^{1k}|^{\frac{3}{2}} de.$$

Taking into account (2.27)–(2.29), we find

$$0 < \varepsilon \leq \frac{1}{a^2} \int\limits_{Q(a)} |u|^3 de + c \limsup_{k\to\infty} \frac{1}{a^2} \int\limits_{Q(a)} |r^{1k}|^{\frac{3}{2}} de$$

$$+ c \limsup_{k\to\infty} \frac{1}{a^2} \int\limits_{Q(a)} |r^{2k}|^{\frac{3}{2}} de$$

$$\leq \frac{1}{a^2} \int\limits_{Q(a)} |u|^3 de + \frac{c}{a^2} \int\limits_{Q(2)} |u|^3 de$$

$$+ \frac{2}{a^2} |B(a)| \int\limits_{-a^2}^{0} \sup_{y\in B(1)} |r^{2k}(y,s)|^{\frac{3}{2}} ds$$

$$\leq \frac{c}{a^2} \int\limits_{Q(2)} |u|^3 de + C(M,T)a.$$

Choose a sufficiently small so that

$$C(M,T)a \leq \frac{1}{2}\varepsilon.$$

Then

$$0 < \varepsilon a^2 \leq c \int\limits_{Q(2)} |u|^3 de. \tag{2.30}$$

So, u is not trivial.

Now, we are in a position to show that

$$u(\cdot, 0) = 0 \qquad in \quad \mathbb{R}^3. \tag{2.31}$$

To this end, we proceed as follows. For any $a > 0$, we have

$$\frac{1}{a^2} \int\limits_{B(a)} |u(x,0)|\, dx \leq \frac{1}{a^2} \int\limits_{B(a)} |u^k(x,0) - u(x,0)|\, dx$$

$$+\frac{1}{a^2}\int\limits_{B(a)}|u^k(x,0)|\,dx=\beta_k+\frac{1}{a^2}\int\limits_{B(a)}|u^k(x,0)|\,dx$$

$$=\beta_k+\frac{1}{(aR_k)^2}\int\limits_{B(aR_k)}|v(x,0)|\,dx$$

$$\leq\beta_k+c\Big(\int\limits_{B(aR_k)}|v(x,0)|^3\,dx\Big)^{\frac{1}{3}}.$$

By (2.7) and by (2.23), the right hand side of the latter inequality tends to zero as $k\to+\infty$. So, (2.31) is proved.

Other important step in our scheme is to show that, given $T\in]-\infty,0[$, there exists $R(T)\in]0,+\infty[$ such that functions u and q are smooth in spatial variables in $(\mathbb{R}^3\setminus B(R(T)))\times[T/2,0]$. To this end, it is sufficient to prove that

$$\int\limits_{Q(z_0,1)}(|u|^3+|q|^{\frac{3}{2}})dz\to 0,\qquad z_0=(x_0,t_0),\tag{2.32}$$

as $|x_0|\to+\infty$ and $T/2\leq t_0\leq 0$. In fact, (2.32) follows from (2.19) (see similar arguments in [2]). Then, according to Lemma 2.2 in [2], we can state that

$$|u|+|\nabla u|\leq C<+\infty\qquad in\quad(\mathbb{R}^3\setminus B(R(T)))\times[T/2,0].\tag{2.33}$$

As it was shown in [2], (2.31) and (2.33) implies that

$$\omega(x,t)=0,\qquad x\in\mathbb{R}^3,\quad T/2\leq t\leq 0,$$

where $\omega=\nabla\wedge u$ is the vorticity. So, $u(\cdot,t)$ is a harmonic function in $\mathbb{R}^3$ for all $T/2\leq t\leq 0$. On the other hand, it follows from (2.19) that, for a. a. $T/2\leq t\leq 0$, $u(\cdot,t)$ is in $L_3(\mathbb{R}^3)$ and, therefore, $u(\cdot,t)=0$ for the same t. But this is in a contradictions with (2.30). Theorem 1.1 is proved.

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