PDMI PREPRINT — 23/2004

BOUNDARY BEHAVIOR IN VECTOR-VALUED STAR-INVARIANT SUBSPACES

Vladimir Kapustin and Alexei Poltoratski

Steklov Mathematical Institute
at St. Petersburg
Fontanka 27, St. Petersburg 191011
Russia
Dept. of Mathematics,
Texas A&M University,
College Station, TX 77843, USA

December 26, 2004

ABSTRACT

In [1], D. Clark considered certain families of positive singular measures on the unit circle associated to inner functions θ in the unit disk. These measures were shown to be the spectral measures of unitary rank-one perturbations of the model operator acting on the (scalar) model space K_θ, a subspace of the Hardy space H^2. In particular, it was shown that K_θ can be mapped unitarily onto $L^2(\sigma)$ for any of such measures σ. Later it was proved [2] that this mapping, which could be viewed as a generalization of the classical Fourier transform, actually takes functions from K_θ to their angular boundary values, which exist σ-almost everywhere for any such σ. In this paper we present an analog of the above-mentioned results for vector-valued model spaces.

The first author is supported by RFFI grant 02-01-00264. The second author is supported by NSF grant 0200699.

The spaces K_Θ

Let H be a separable Hilbert space, and suppose that $U : H \to H$ is a singular unitary operator. Consider the spectral resolution of U:

$$H = \int \oplus H(z) d\mu(z),$$

where μ is a non-negative finite singular measure on the unit circle. Elements of H will be viewed as square summable $H(z)$-valued functions. The operator U acts on such functions as the multiplication by z.

Let E be an auxiliary Hilbert space, and let $R : E \to H$ be an operator such that there exist operators $R(z)$ for which

$$(Re)(z) = R(z)e, \quad e \in E; \qquad \int ||R(z)||^2 d\mu(z) < \infty. \tag{1}$$

It is well known that if $R \in \mathfrak{S}_2$ then there exist operators $R(z)$ for which (1) is fulfilled. We also assume that the minimal subspace of H that reduces U and contains the range of R coincides with H.

Define a function Θ in the unit disk, whose values are operators on E, by the formula

$$\int \frac{1 + \bar{z}\lambda}{1 - \bar{z}\lambda} R(z)^* R(z) d\mu(z) = (I - \Theta(\lambda))^{-1}(I + \Theta(\lambda)). \tag{2}$$

Note that for every λ the operator on the left hand side (denote it by A) has nonnegative real part $\frac{A+A^*}{2}$. It is easy to check that then the operator $\Theta(\lambda)$ is well defined on the range of $I + A$, which is a dense subset of E, and $||\Theta(\lambda)|| \leqslant 1$; by continuity $\Theta(\lambda)$ is extended to an operator on E. Since μ is a singular measure, Θ is an inner function, i.e., boundary values of Θ on the unit circle are unitary operators almost everywhere relative to the Lebesgue measure.

The scalar situation [1] is a special case of this construction. Namely, suppose that all spaces $H(z)$ and E are one-dimensional and coincide with $\mathbb{C}$, $R(z)1 = 1$. We then have $H = L^2(\mu)$, $R1 = 1$, and the function θ is detrermined by

$$\int \frac{1 + \bar{z}\lambda}{1 - \bar{z}\lambda} d\mu(z) = \frac{1 + \theta(\lambda)}{1 - \theta(\lambda)}.$$

The family of measures σ_α (where $|\alpha| = 1$) appears as μ in this formula if we consider the function $\bar{\alpha}\theta$ in place of θ.

The space K_Θ is defined as the orthogonal difference

$$K_\Theta = H^2(E) \ominus \Theta H^2(E),$$

where $H^2(E)$ is the standard Hardy space of E-valued analytic functions in the unit disk $\mathbb{D}$. Functions from K_Θ are *pseudocontinuable*, which means that for any $f \in K_\Theta$ there exists a funcion g from the Hardy space in the complement of $\mathbb{D}$ and an inner funciton B such $f = Bg$ almost everywhere on the unit circle $\mathbb{T}$.

Any such space K_θ is spanned by its reproducing kernels, the functions of the form

$$k_{\lambda,e}(z) = \frac{(I - \Theta(z)\Theta(\lambda)^*)e}{1 - \bar{\lambda} z},$$

where λ runs over the unit disk and e runs over E. Indeed, these vectors are projections onto K_Θ of the vectors $\frac{e}{1-\bar{\lambda}z}$, which form a complete family in $H^2(E)$.

To establish the connection between U and the model contraction acting on K_Θ we must realize U as an operator acting on K_Θ. Just like in the scalar case, this can be done via a natural unitary operator (generalized Fourier transform) sending K_Θ to H.

Define an operator $V : K_\Theta \to H$ on the dense set by

$$V k_{\lambda,e} = h_{\lambda,e},$$

where

$$h_{\lambda,e}(z) = R(z)\frac{(I - \Theta(\lambda)^*)e}{1 - \bar{\lambda} z}.$$

Theorem 1. *V is a unitary operator, and the inverse operator acts by the formula*

$$(V^{-1}h)(\alpha) = (I - \Theta(\alpha)) \int \frac{R(z)^* h(z) d\mu(z)}{1 - \bar{z}\alpha} = (I - \Theta(\alpha)) R^* (I - \alpha U^*)^{-1} h. \qquad (3)$$

Proof. The second identity in (3) is trivial; to prove the first, note that the linear span of the functions $h_{\lambda,e}$ is dense in H. Hence, to prove the theorem, it suffices to check that V is an isometry and verify formula (3) for $h = h_{\lambda,e}$. We have

$$\begin{aligned}
&(I - \Theta(\alpha)) \left(\int \frac{R(z)^* R(z) d\mu(z)}{(1 - \bar{\lambda} z)(1 - \alpha\bar{z})} \right) (I - \Theta(\lambda)^*) \\
&= \frac{I - \Theta(\alpha)}{2(1 - \bar{\lambda}\alpha)} \left(\int \left(\frac{1 + z\bar{\lambda}}{1 - z\bar{\lambda}} + \frac{1 + \bar{z}\alpha}{1 - \bar{z}\alpha} \right) R(z)^* R(z) d\mu(z) \right) (I - \Theta(\lambda)^*) \\
&= \frac{I - \Theta(\alpha)}{2(1 - \bar{\lambda}\alpha)} \left((I + \Theta(\lambda)^*)(I - \Theta(\lambda)^*)^{-1} + (I - \Theta(\alpha))^{-1}(I + \Theta(\alpha)) \right) (I - \Theta(\lambda)^*) \qquad (4) \\
&= \frac{1}{2(1 - \bar{\lambda}\alpha)} \left((I - \Theta(\alpha))(I + \Theta(\lambda)^*) + (I + \Theta(\alpha))(I - \Theta(\lambda)^*) \right) \\
&= \frac{I - \Theta(\alpha)\Theta(\lambda)^*}{1 - \bar{\lambda}\alpha}.
\end{aligned}$$

To prove that V is an isometry, we need to establish the identity

$$(k_{\lambda,e}, k_{\alpha,e'})_{K_\theta} = (h_{\lambda,e}, h_{\alpha,e'})_H.$$

The latter follows from the relations

$$(k_{\lambda,e}, k_{\alpha,e'})_{K_\theta} = \left(k_{\lambda,e}, \frac{e'}{1 - \bar{\alpha} z} \right)_{H^2} = \frac{((I - \Theta(\alpha)\Theta(\lambda)^*)e, e')}{1 - \bar{\lambda}\alpha}$$

and

$$\begin{aligned}
(h_{\lambda,e}, h_{\alpha,e'})_H &= \int \left(R(z)\frac{(I-\Theta(\lambda)^*)e}{1-\bar\lambda z}, R(z)\frac{(I-\Theta(\alpha)^*)e'}{1-\bar\alpha z} \right) d\mu(z) \\
&= \left((I-\Theta(\alpha)) \left(\int \frac{R(z)^*R(z)}{(1-\bar\lambda z)(1-\alpha\bar z)} d\mu(z) \right) (I-\Theta(\lambda)^*)e, e' \right) \\
&= \left(\frac{I-\Theta(\alpha)\Theta(\lambda)^*}{1-\bar\lambda\alpha} e, e' \right),
\end{aligned}$$

where we used (4). Now we check formula (3) for $h = h_{\lambda,e}$. Applying (4) again, we obtain

$$\begin{aligned}
&(1-\Theta(\alpha)) \int \frac{R(z)^* h_{\lambda,e}(z) d\mu(z)}{1-\bar z\alpha} \\
&= (1-\Theta(\alpha)) \int \frac{R(z)^*R(z)\frac{(I-\Theta(\lambda)^*)e}{1-\bar\lambda z} d\mu(z)}{1-\bar z\alpha} \\
&= (1-\Theta(\alpha)) \left(\int \frac{R(z)^*R(z)d\mu(z)}{(1-\bar\lambda z)(1-\bar z\alpha)} \right) (I-\Theta(\lambda)^*)e \\
&= \frac{(I-\Theta(\alpha)\Theta(\lambda)^*)e}{1-\bar\lambda\alpha} = k_{\lambda,e}(\alpha),
\end{aligned}$$

as required. □

Convergence

Next we study the convergence of funcitons from K_Θ to their boundary values trying to establish an analogue of a result from [2].

As was shown in [2], in the scalar situation the generalized Fourier transform V defined in the previous section simply sends any function from K_θ into its boundary values, which conveniently exist σ_α-almost everywhere for any $\alpha \in \mathbb{T}$. The vector-valued situation turns out to be more complicated. The following simple example shows that even in the two-dimensional case not every K_θ-function posesses non-tangential boundary values μ-almost everywhere (keeping the notations of the last section).

Example. Let θ_1, θ_2 be two scalar inner functions, $\theta_1(0) = \theta_2(0) = 0$, $\Theta = \theta_1 \oplus \theta_2$, and let probability measures τ_1, τ_2 be determined by the relations

$$\frac{1+\theta_i}{1-\theta_i}(z) = \int_{\mathbb{T}} \frac{1+\bar\xi z}{1-\bar\xi z} d\tau_i(\xi), \qquad i = 1, 2.$$

Define $\mu = \tau_1 + \tau_2$. One can check that then μ and Θ satisfy (2). On the other hand, the existence of angular boundary values for functions from K_θ μ-a.e. is equivalent to the existence of such values for all functions from $K_{\theta_i}, i = 1, 2$. Thus, to construct an example of Θ for which there exists $f \in K_\Theta$ whose boundary values fail to exist μ-almost everywhere, it suffices to take any inner function θ_1 and a measure τ_2 such that functions from K_{θ_1} fail to have boundary values τ_2-almost everywhere. For instance, take the unit point mass at 1 as τ_2 (hence $\theta_2(z) = z$). The condition under which all functions from

K_{θ_1} have angular limit at 1 can be found in [3]; it may be rewritten as $\frac{\alpha-\theta_1(z)}{1-z} \notin L^2$ for any complex α. It remains to take any function θ_1, for which this property is not fulfilled.

Our next theorem shows that, nevertheless, an analogue of the scalar result exists. From here on we assume that for μ-a.e. ξ the operator $R(\xi)^*R(\xi)$ has a closed image. Even though this does not seem to interfere with any forseable applications, it would be interesting to see how much one can aleviate this restriction.

Theorem 2. *Let $f \in K_\Theta$. Then for μ-almost all ξ limits $h(\xi) = \lim R(\xi)f(z)$ exist as $z \underset{\triangleleft}{\longrightarrow} \xi$. Moreover, $\int ||h(\xi)||^2 d\mu(\xi) = ||f||^2_{K_\Theta}$ and $Vf = h$.*

If μ is a complex measure on $\mathbb{T}$ and $f \in L^1(\mu)$, we denote by $Pf\mu$ and $Qf\mu$ the Poisson and the conjugate Poisson integrals correspondingly:

$$Pf\mu(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|\xi - z|^2} f(\xi) d\mu(\xi)$$

and

$$Qf\mu(z) = \int_{\mathbb{T}} \frac{2\operatorname{Im}\bar{\xi}z}{|\xi - z|^2} f(\xi) d\mu(\xi).$$

The notation $Kf\mu$ stands for the Cauchy integral

$$Kf\mu(z) = \frac{1}{2}(1 + Pf\mu(z) + iQf\mu(z)) = \int_{\mathbb{T}} \frac{f(\xi)d\mu(\xi)}{1-\bar{\xi}z}.$$

If f is a (vector-valued) function in $\mathbb{D}$ we denote by Mf its nontangential maximal funciton on $\mathbb{T}$:

$$Mf(\xi) = \sup_{z\in\Gamma_\xi} ||f(z)||.$$

Lemma. *Let μ be a positive (scalar) measure on $\mathbb{T}$ and let L be a Hilbert space.*

1) For any L-valued function $g \in L^2(L,\mu)$ the maximal function $M\frac{Kg\mu}{K\mu}$ is in $L^{2,\infty}(\mu)$.

2) If $g \in L^1(L,\mu)$ then $\frac{Kg\mu}{K\mu}(\lambda)$ tends to $g(z)$ as $\lambda \underset{\triangleleft}{\longrightarrow} z$ in L-norm for μ-almost all z.

Proof. 1) Consider the scalar inner function θ in $\mathbb{D}$ such that $\mu = \sigma_1 \in M_\theta$. Denote $\nu = \sigma_{-1}$. Without loss of generality, μ is a probability measure. Then $\theta(0) = 0$ and ν is a probability measure. Let $g \in L^2(L,\mu)$. Choose an orthonormal basis $e_1, e_2, \ldots$ of L. Then $g(z) = \sum e_n g_n(z)$ for some scalar funcitons $g_n \in L^2(\mu)$, $\sum ||g_n||^2_{L^2(\mu)} = ||g||^2_{L^2(L,\mu)}$.

Now from scalar Clark's theory we have

$$\frac{Kg\mu}{K\mu} = \sum e_n \frac{Kg_n\mu}{K\mu} = \sum e_n \frac{Kf_n\nu}{K\nu}$$

for some scalar funcitons $f_n \in L^2(\nu)$, $||f_n||_{L^2(\nu)} = ||g_n||_{L^2(\mu)}$. Without loss of generality, we can assume that all g_n are real valued and satisfy $\int g_n d\mu = 0$. Then all f_n are pure imaginary and satisfy $\int f_n d\nu = 0$, see [2]. Denote $G_n = \frac{Kg_n\mu}{K\mu} = \frac{Kf_n\nu}{K\nu}$. Then

$$\begin{aligned} Pg_n\mu = \operatorname{Re} Kg_n\mu = \operatorname{Re} G_n/(1-\theta) = \operatorname{Re}\frac{Kf_n\nu}{(1-\theta)K\nu} = \operatorname{Re}\frac{1+\theta}{1-\theta}Kf_n\nu \\ = \operatorname{Re}\frac{1+\theta}{1-\theta}\operatorname{Re} Kf_n\nu - \operatorname{Im}\frac{1+\theta}{1-\theta}\operatorname{Im} Kf_n\nu = iP\mu Qf_n\nu + iQ\mu Pf_n\nu. \quad (5)\end{aligned}$$

Let us denote $f = \sum e_n f_n$. Then $f \in L^2(L, \nu)$, $||f||_{L^2(L,\nu)} = ||g||_{L^2(L,\mu)}$ and $Kf\nu/K\nu = Kg\mu/K\mu$. Thus (5) can be transformed into

$$Pg\mu = iP\mu Qf\nu + iQ\mu Pf\nu. \tag{6}$$

Notice that

$$I(\xi) = M\frac{Pg\mu}{P\mu}(\xi) \leqslant M\frac{P||g(\cdot)||_L\mu}{P\mu}(\xi)$$

and therefore, by the Hardy-Littlewood maximal theorem, I is in $L^2(\mu)$ with $||I|| \leqslant C||g||_{L^2(L,\mu)}$. Dividing (6) by $P\mu$ we get

$$||Qf\nu(z) + \frac{Q\mu Pf\nu}{P\mu}(z)||_L < I(\xi) \tag{7}$$

for any $z \in \Gamma_\xi$.

The integral $Pf\nu$ can be estimated using the Cauchy-Schwartz inequality. Let us denote by F the function $||f(z)||_L \in L^2(\nu)$. Then

$$||Pf\nu||_L \leqslant PF\mu \leqslant (PF^2\nu P\nu)^{1/2}. \tag{8}$$

Denote $J(\xi) = M(PF^2\nu/P(\mu+\nu))$. Since $F^2 \in L^1(\nu)$, by the Hardy-Littlewood theorem $J(\xi) \in L^{1,\infty}(\mu)$. Using (8) we obtain

$$||Pf\nu(z)||_L \leqslant \sqrt{J(\xi)P(\mu+\nu)P\nu} \leqslant C\sqrt{J(\xi)}\frac{1-|\theta|^2}{|1-\theta^2||1+\theta|} \tag{9}$$

for any $z \in \Gamma_\xi$. This allows us to estimate the second summand in (7):

$$\begin{aligned}\left\|\frac{Q\mu Pf\nu}{P\mu}(z)\right\|_L &= \left|\frac{Q\mu}{P\mu}(z)\right| \, ||P\mu(z)||_L \\ &\leqslant C\frac{\operatorname{Im}\frac{1+\theta(z)}{1-\theta(z)}}{\operatorname{Re}\frac{1+\theta(z)}{1-\theta(z)}}\sqrt{J(\xi)}\frac{1-|\theta(z)|^2}{|1-\theta(z)^2||1+\theta(z)|} \leqslant \frac{2C}{|1+\theta(z)|}\sqrt{J(\xi)}\end{aligned}$$

for any $z \in \Gamma_\xi$.

Now from (7) we obtain that

$$||Qf\nu(z)||_L \leqslant I(\xi) + \frac{2C}{|1+\theta(z)|}\sqrt{J(\xi)}$$

where $||I||_{L^2(\mu)} < C||g||_{L^2(L,\mu)}$, for any $z \in \Gamma_\xi$. Together with (9) we get

$$\left\|\frac{Kg\mu}{K\mu}(z)\right\|_L = \left\|\frac{Kf\nu}{K\nu}(z)\right\|_L = \frac{1}{2}||(1+\theta(z))(Pf\nu + Qf\nu)(z)||_L \leqslant 4C\sqrt{J(\xi)} + I(\xi)$$

for any $z \in \Gamma_\xi$.

Since $I \in L^2(\mu)$ and $J \in L^{1,\infty}(\mu)$, this finishes 1).

2) Let first $g \in L^2(L,\mu)$. Since $Kg_n/K\mu$ converges to g_n nontangentially μ-a.e. [2], we have

$$\frac{K(\sum_1^N e_n g_n)\mu}{K\mu}(z) \to \sum_1^N e_n g_n(\xi)$$

as $z \underset{\triangleleft}{\longrightarrow} \xi$ for μ-almost all ξ for any N. For the remainder of the sum,

$$\frac{K(\sum_{N+1}^\infty e_n g_n)\mu}{K\mu}(z),$$

we can use the maximal estimate from the previous part to obtain the statement.

In the general case let us denote $G(\xi) = ||g(\xi)||_L$. Then $G \in L^1(\mu)$ and

$$\frac{Kg\mu}{K\mu} = \frac{K\frac{g}{G}(G\mu)}{KG\mu}\frac{KG\mu}{K\mu}.$$

To the first factor converges by the previous argument because $\frac{g}{G}$ is bounded: one just needs to replace μ there with $G\mu$. The second factor converges by the scalar result from [2]. $\square$

Proof of Theorem 2. Let $f(z) = (I - \Theta(z))KR^*h\mu(z)$. All we need is to prove that the non-tangential limit of $R(\xi)f(z)$ is equal $h(\xi)$ for μ-almost all ξ, and the rest will follow from Theorem 1.

Without loss of generality we can assume that $h(\xi) = 0$ (otherwise subtruct $R(\cdot)R^{-1}h(\xi)$). By the lemma, this (almost always) means that $||KR^*h\mu(z)|| = o(K\mu(z))$.

Let $f(z) = p(z)+q(z)$, where $p(z) \in \operatorname{Ker} R(\xi)^*R(\xi) = \operatorname{Ker} R(\xi)$ and $q(z) \perp \operatorname{Ker} R(\xi)^*R(\xi)$. Then

$$R(\xi)f(z) = R(\xi)(p(z) + q(z)) = R(\xi)q(z)$$

and all that is left is to show that $||q(z)|| \to 0$.

Suppose that $||q(z)|| \not\to 0$. Set

$$P(z) = \begin{cases} p(z) \text{ if } ||f(z)|| < 1 \\ p(z)/||f(z)|| \text{ if } ||f(z)|| > 1 \end{cases}$$

and similarly

$$Q(z) = \begin{cases} q(z) \text{ if } ||f(z)|| < 1 \\ q(z)/||f(z)|| \text{ if } ||f(z)|| > 1. \end{cases}$$

Put $F = P + Q$.

Then $||Q(z)|| \not\to 0$. By the lemma

$$\frac{KR^*R\mu(z)}{K\mu(z)}$$

tends to $(R^*R)(\xi)$ for μ-a.e. ξ in norm (just replace L with the vector space of linear operators on E). Since $R(\xi)^*R(\xi)$ is invertible on $\operatorname{Ker}^\perp(R(\xi)^*R(\xi))$ and $Q \perp \operatorname{Ker} R(\xi)^*R(\xi)$, for all z close enough to ξ,

$$||Q(z)|| < C||[(R^*R)(\xi)]Q(z)|| < 2C\left\|\left[\frac{(KR^*R\mu)(z)}{K\mu(z)}\right]Q(z)\right\|,$$

and so

$$\frac{||(KR^*R\mu)(z)Q(z)||}{K\mu(z)} \not\to 0.$$

By a similar reasoning

$$\frac{||(KR^*R\mu)(z)P(z)||}{K\mu(z)} = ||R(\xi)^*R(\xi)P(z)|| + o(1) = o(1)$$

since $P(z) \in \operatorname{Ker} R(\xi)^*R(\xi)$ and $(KR^*R\mu)(z)/K\mu(z) \to R(\xi)^*R(\xi)$.

At the same time

$$\begin{aligned} ||(KR^*R\mu)(z)(P(z)+Q(z))|| &= ||(KR^*R\mu)(z)F(z)|| \\ &\leqslant ||(KR^*R\mu)(z)f(z)|| = ||KR^*h\mu(z)|| = o(K\mu(z)) \end{aligned}$$

which leads to a contradiction. $\square$

References

1. D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math. **25** (1972), 169–191.
2. A. Poltoratski, *The boundary behavior of pseudocontinuable functions*, St.Petersburg Math. J. **5** (1994), no. 2, 389–406.
3. P. R. Ahern, D. N. Clark, *Radial limits and invariant subspaces*, Amer. J. Math. **92** (1970), no. 2, 332–342.