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DIRICHLET TO NEUMANN OPERATOR ON DIFFERENTIAL FORMS

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ABSTRACT

We define the Dirichlet to Neumann operator on exterior differential forms for a compact Riemannian manifold with boundary and derive an explicit formula expressing Betti numbers of the manifold through the DN operator. We express also the Hilbert transform through the DN map. The Hilbert transform connects boundary traces of conjugate co-closed forms.

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1 Introduction

Throughout the paper, M is a connected compact oriented Riemannian manifold of dimension n with nonempty boundary ∂M. Let $\Delta : C^\infty(M) \to C^\infty(M)$ be the Laplace–Beltrami operator. The classical Dirichlet to Neumann (DN) map $\Lambda_{cl} : C^\infty(\partial M) \to C^\infty(\partial M)$ is defined by $\Lambda_{cl}\varphi = \partial\omega/\partial\nu$, where ω is the solution to the Dirichlet problem

$$\begin{cases} \Delta\omega = 0, \\ \omega|_{\partial M} = \varphi \end{cases} \tag{1.1}$$

and ν is the unit outer normal to the boundary. We use the term "classical DN map" and notation Λ_{cl} in order to make a distinction between this operator and the generalization Λ defined below.

In the scope of inverse problems of reconstructing a manifold from boundary measurements, the following question is of great theoretical and applied interest: to what extent are the topology and geometry of M determined by the DN map? It is proved in the two-dimensional case that M is determined by Λ_{cl} up to a conformal equivalence [1, 4]. There is the conjecture that Λ_{cl} determines M up to an isometry in the case of $n \ge 3$. The latter is proved for real analytic manifolds [5]. In the general case, it is proved that the boundary C^∞-jet of the metric is determined by Λ_{cl} for $n \ge 3$ [6].

In [1], an explicit formula is obtained which expresses the Euler characteristic of M through Λ_{cl} in the case of a two-dimensional M with a connected boundary. The Euler characteristic completely determines the topology of M in the latter case. In the three-dimensional case, the vector DN map $\overrightarrow{\Lambda} : C^\infty(T(\partial M)) \to C^\infty(T(\partial M))$ is defined on the space of vector fields in [2], and some formulas are obtained which express the Betti numbers $\beta_1(M)$ and $\beta_2(M)$ in terms of Λ_{cl} and $\overrightarrow{\Lambda}$.

Here we present a multidimensional generalization of the latter results. We define a DN map on the space of differential forms of all degrees and express Betti numbers in terms of the map. As well as in the case of $n = 2, 3$; the background of our formula is the Friedrichs decomposition of the space of harmonic fields. We also consider the Hilbert transform on differential forms and express it in terms of the DN map.

2 Preliminaries

Here, following [7] and mostly adhering to notations of the book, we recall some known facts on differential forms.

Let $\Omega^k(M)$ be the space of smooth real exterior differential forms of degree k and $\Omega(M) = \oplus_{k=0}^n \Omega^k(M)$ the graded algebra of all forms. We use the following standard operators on $\Omega(M)$: the differential d, codifferential δ, Laplace operator $\Delta = d\delta + \delta d$, and Hodge star $\star$. Recall the relations

$$\star\star = (-1)^{k(n-k)}, \quad \star\delta = (-1)^k d\star, \quad \star d = (-1)^{k+1}\delta\star \quad \text{on} \quad \Omega^k(M).$$

The L^2-product on $\Omega(M)$ is defined by $(\alpha, \beta) = \int_M \alpha \wedge \star\beta$ under the agreement that $\int_M \varphi = 0$ for $\varphi \in \Omega^k(M)$ with $k < n$. Recall Green's formula

$$(d\alpha, \beta) - (\alpha, \delta\beta) = \int_{\partial M} j^*(\alpha \wedge \star\beta),$$

where $j : \partial M \to M$ is the embedding.

Elements of the space

$$\mathcal{H}^k(M) = \{\lambda \in \Omega^k(M) \mid d\lambda = 0,\ \ \delta\lambda = 0\}$$

are named harmonic fields. Recall the L^2-orthogonal Hodge–Morrey decomposition

$$\Omega^k(M) = \mathcal{E}^k(M) \oplus \mathcal{C}^k(M) \oplus \mathcal{H}^k(M).$$

Here

$$\mathcal{E}^k(M) = \{d\alpha \mid \alpha \in \Omega^{k-1}(M),\ j^*\alpha = 0\}$$

is the space of exact forms and

$$\mathcal{C}^k(M) = \{\delta\alpha \mid \alpha \in \Omega^{k+1}(M),\ j^*(\star\alpha) = 0\}$$

is the space of co-exact forms.

There are two finite dimensional subspaces distinguished in $\mathcal{H}^k(M)$

$$\mathcal{H}^k_D(M) = \{\lambda \in \mathcal{H}^k(M) \mid j^*\lambda = 0\} \quad \text{and} \quad \mathcal{H}^k_N(M) = \{\lambda \in \mathcal{H}^k(M) \mid j^*(\star\lambda) = 0\}$$

whose elements are named Dirichlet and Neumann harmonic fields respectively. Dimensions of these spaces are expressed by

$$\dim \mathcal{H}^k_N(M) = \dim \mathcal{H}^{n-k}_D(M) = \beta_k(M),$$

where $\beta_k(M)$ is the kth Betti number. There are two L^2-orthogonal Friedrichs decompositions

$$\mathcal{H}^k(M) = \mathcal{H}^k_D(M) \oplus \mathcal{H}^k_{co}(M), \quad \mathcal{H}^k(M) = \mathcal{H}^k_N(M) \oplus \mathcal{H}^k_{ex}(M).$$

Here

$$\mathcal{H}^k_{ex}(M) = \{\lambda \in \mathcal{H}^k(M) \mid \lambda = d\alpha\} \quad \text{and} \quad \mathcal{H}^k_{co}(M) = \{\lambda \in \mathcal{H}^k(M) \mid \lambda = \delta\alpha\}$$

are the spaces of exact harmonic, and co-exact harmonic fields.

The operator $\star$ maps the space $\mathcal{H}^k_D(M)$ isomorphically onto $\mathcal{H}^{n-k}_N(M)$. Introduce the trace spaces

$$j^*\mathcal{H}^k(M) = \{j^*\lambda \mid \lambda \in \mathcal{H}^k(M)\}, \quad j^*\mathcal{H}^k_N(M) = \{j^*\lambda_N \mid \lambda_N \in \mathcal{H}^k_N(M)\}.$$

A Neumann harmonic field λ_N is uniquely determined by its trace $j^*\lambda_N$. Therefore the dimension of the space $j^*\mathcal{H}^k_N(M)$ is equal to $\beta_k(M)$. Let us prove the equality

$$j^*\mathcal{H}^k(M) = \mathcal{E}^k(\partial M) + j^*\mathcal{H}^k_N(M). \tag{2.1}$$

Indeed, any harmonic field $\lambda \in \mathcal{H}^k(M)$ can be represented in the form

$$\lambda = d\eta + \lambda_N, \quad \lambda_N \in \mathcal{H}^k_N(M)$$

by the second Fridrichs decomposition. This implies

$$j^*\lambda = d(j^*\eta) + j^*\lambda_N.$$

Conversely, for $\varphi \in \Omega^{k-1}(\partial M)$ and $\lambda_D \in \mathcal{H}_D^{k+1}(M)$,

$$\int_{\partial M} d\varphi \wedge j^*(\star\lambda_D) = 0$$

since exact forms constitute the ideal in the algebra of closed forms. Thus, the form $\alpha = d\varphi$ satisfies

$$d\alpha = 0 \quad \text{and} \quad \int_{\partial M} \alpha \wedge j^*(\star\lambda_D) = 0 \quad \forall\, \lambda_D \in \mathcal{H}_D^{k+1}(M). \tag{2.2}$$

By Theorem 3.2.5 of [7], (2.2) is the necessary and sufficient condition for the existence of such $\lambda \in \mathcal{H}^k(M)$ that $\alpha = j^*\lambda$.

By the first of Friedrichs decompositions, a harmonic field $\lambda \in \mathcal{H}^k(M)$ can be represented as

$$\lambda = \delta\alpha + \lambda_D, \quad \lambda_D \in \mathcal{H}_D^k(M). \tag{2.3}$$

We will need the following remark: in the representation, the form α can be chosen such that $d\alpha = 0$ and $\Delta\alpha = 0$. Indeed, first consider representation (2.3) with some α and decompose α by Hodge–Morrey

$$\alpha = d\beta + \delta\gamma + \lambda', \quad d\lambda' = \delta\lambda' = 0.$$

This implies $\delta\alpha = \delta d\beta$. Therefore, in (2.3), α can be replaced with the form $\tilde{\alpha} = d\beta$ which satisfies $d\tilde{\alpha} = 0$. Next, if the form α in (2.3) satisfies $d\alpha = 0$, then it satisfies also the equation $\Delta\alpha = 0$ as is seen by applying the operator d to equation (2.3). A similar remark is valid on the second Friedrichs decomposition.

3 DN operator

For any $0 \le k \le n-1$, the DN operator

$$\Lambda : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M) \tag{3.1}$$

is defined as follows. Given $\varphi \in \Omega^k(\partial M)$, the boundary value problem

$$\begin{cases} \Delta\omega = 0, \\ j^*\omega = \varphi, \quad j^*(\delta\omega) = 0 \end{cases} \tag{3.2}$$

is solvable, see Lemma 3.4.7 of [7]. The solution $\omega \in \Omega^k(M)$ is unique up to an arbitrary Dirichlet harmonic field $\lambda_D \in \mathcal{H}_D^k(M)$. Therefore the form

$$\Lambda\varphi = j^*(\star d\omega) = (-1)^{k+1} j^*(\delta \star \omega) \tag{3.3}$$

is independent of the choice of the solution ω.

In the scalar case of $k = 0$, our definition is equivalent to the classical one. Indeed, in this case the boundary value problem (3.2) coincides with (1.1) and definition (3.3) gives

$$\Lambda\varphi = \frac{\partial\omega}{\partial\nu}\mu_\partial \qquad (\varphi \in \Omega^0(\partial M)),$$

where $\mu_\partial \in \Omega^{n-1}(\partial M)$ is the boundary volume form. Thus, in the case of $k = 0$, our operator Λ differs from the classical operator Λ_{cl} by the presence of the factor μ_∂. However, some authors prefer to consider the form-valued operator $\Lambda : \Omega^0(\partial M) \to \Omega^{n-1}(\partial M)$, see, for example, [8].

The boundary value problem (3.2) can be written in a slightly different form as the following statement shows.

Lemma 3.1. *Given $\varphi \in \Omega^k(\partial M)$, let $\omega \in \Omega^k(M)$ be a solution to the boundary value problem (3.2). Then $d\omega \in \mathcal{H}^{k+1}(M)$ and $\delta\omega = 0$. In particular, (3.2) is equivalent to the boundary value problem*

$$\begin{cases} \Delta\omega = 0, \ \delta\omega = 0, \\ j^*\omega = \varphi. \end{cases} \tag{3.4}$$

Proof. Let $\lambda = d\omega \in \Omega^{k+1}(M)$. We state that λ is a harmonic field. Indeed, $d\lambda = dd\omega = 0$. Since d and Δ commute,

$$\Delta\lambda = \Delta d\omega = d\Delta\omega = 0.$$

The boundary conditions

$$j^*\delta\lambda = j^*\delta d\omega = -j^*d\delta\omega = -d(j^*\delta\omega) = 0$$

and

$$j^*(\star d\lambda) = j^*(\star dd\omega) = 0$$

are satisfied. Thus, λ solves the boundary value problem

$$\Delta\lambda = 0, \quad j^*(\star d\lambda) = 0, \quad j^*\delta\lambda = 0.$$

This implies, with the help of Proposition 3.4.5 (iv) of [7], that λ is a harmonic field.

Let now $\varepsilon = \delta\omega$. Then ε is a harmonic field. Indeed, $\delta\varepsilon = \delta\delta\omega = 0$ and $d\varepsilon = d\delta\omega = -\delta d\omega = -\delta\lambda = 0$. Since $j^*\varepsilon = 0$ by the second of the boundary conditions (3.2), ε is a Dirichlet harmonic field. We have thus proved that $\varepsilon = \delta\omega$ is a co-exact harmonic field and it is a Dirichlet harmonic field. This implies, by the first Fridrichs decomposition, that $\varepsilon = 0$. The lemma is proved.

The operator Λ is nonnegative in the following sense: the integral

$$\int_{\partial M} \varphi \wedge \Lambda\varphi$$

is nonnegative for any $\varphi \in \Omega(\partial M)$. This follows from the next more precise statement. Given two forms $\varphi, \psi \in \Omega(\partial M)$, let ω and ε be the corresponding solutions to the boundary value problem (3.2), i.e.,

$$\begin{cases} \Delta\omega = 0, \\ j^*\omega = \varphi, \ j^*(\delta\omega) = 0, \end{cases} \qquad \begin{cases} \Delta\varepsilon = 0, \\ j^*\varepsilon = \psi, \ j^*(\delta\varepsilon) = 0. \end{cases} \tag{3.5}$$

Then

$$\int_{\partial M} \varphi \wedge \Lambda\psi = \int_{\partial M} \psi \wedge \Lambda\varphi = \mathcal{D}(\omega, \varepsilon) := (d\omega, d\varepsilon) + (\delta\omega, \delta\varepsilon). \tag{3.6}$$

Indeed, by Green's formula

$$(d\omega, d\varepsilon) = (\omega, \delta d\varepsilon) + \int_{\partial M} (j^*\omega) \wedge (j^* \star d\varepsilon),$$

$$(\delta\omega, \delta\varepsilon) = (\omega, d\delta\varepsilon) - \int_{\partial M} (j^*\delta\varepsilon) \wedge (j^* \star \omega).$$

Taking the sum of the latter two equations and using (3.5), we obtain (3.6).

To find the dual operator Λ^*, we write the first of equalities (3.6) in the form

$$(\varphi, \star_\partial \Lambda\psi) = (\psi, \star_\partial \Lambda\varphi), \tag{3.7}$$

where $\star_\partial : \Omega(\partial M) \to \Omega(\partial M)$ is the Hodge star on ∂M. Setting $\psi = \star_\partial \psi'$ in (3.7), we obtain

$$(\varphi, \star_\partial \Lambda \star_\partial \psi') = (\star_\partial \psi', \star_\partial \Lambda\varphi) = (\psi', \Lambda\varphi).$$

The last equality holds because $\star_\partial$ is the L^2-isometry of $\Omega(\partial M)$. We have thus obtained

$$\Lambda^* = \star_\partial \Lambda \star_\partial . \tag{3.8}$$

The kernel and range of the operator Λ are described by the following

Lemma 3.2. *The kernel of Λ coincides with the range of Λ. A form $\varphi \in \Omega^k(\partial M)$ belongs to* $\operatorname{Ker} \Lambda = \operatorname{Ran} \Lambda$ *if and only if it is a trace of a harmonic field, i.e.,*

$$\varphi = j^*\lambda \quad \textit{for} \quad \lambda \in \mathcal{H}^k(M).$$

Proof. If $\varphi \in \operatorname{Ker} \Lambda$ and ω is a solution to the boundary value problem (3.2), then $\varphi = j^*\omega$ and

$$\mathcal{D}(\omega, \omega) = \int_{\partial M} \varphi \wedge \Lambda\varphi = 0$$

by (3.6). This means that $\omega \in \mathcal{H}^k(M)$. Conversely, if $\varphi = j^*\omega$ for $\omega \in \mathcal{H}^k(M)$, then ω is a solution to the boundary value problem (3.2) and $\Lambda\varphi = j^*(\star d\omega) = 0$.

A form $\psi \in \Omega^k(\partial M)$ belongs to $\operatorname{Ran} \Lambda$ if and only if the boundary value problem

$$\begin{cases} \Delta\omega = 0, \\ j^*\delta\omega = 0, \quad j^*(\delta \star \omega) = \psi \end{cases}$$

has a solution. By Lemma 3.4.9 of [7], the necessary and sufficient condition for solvability of the latter problem is

$$\int_{\partial M} \psi \wedge j^*\lambda = 0 \quad \forall \lambda \in \mathcal{H}^{n-k-1}(M).$$

With the help of (2.1), we see that the latter condition is equivalent to the following one:

$$d\psi = 0 \quad \text{and} \quad \int_{\partial M} \psi \wedge j^*\lambda_N = 0 \quad \forall \lambda_N \in \mathcal{H}_N^{n-k-1}(M). \tag{3.9}$$

On the other hand, (3.9) is the necessary and sufficient condition for the existence of such $\lambda \in \mathcal{H}^k(M)$ that $j^*\lambda = \psi$ by Theorem 3.2.5 of [7].

Corollary 3.3. *The operator* Λ *possesses the following properties:*

$$\Lambda d = 0, \quad d\Lambda = 0, \quad \Lambda^2 = 0. \tag{3.10}$$

Proof. The first of equalities (3.10) means that any exact form is the trace of a harmonic field. The latter is true by (2.1). The second of equalities (3.10) is equivalent to the obvious fact: the trace of a harmonic field is a closed form. The last of equalities (3.10) follows from the relation $\operatorname{Ker}\Lambda = \operatorname{Ran}\Lambda$.

Corollary 3.4. *The operator* $d\Lambda^{-1}d$ *is well defined, i.e., the equation* $\Lambda\varphi = \psi$ *has a solution* φ *for any exact form* ψ, *and* $d\varphi$ *is uniquely determined by* ψ.

Proof. An exact form ψ is the trace of a harmonic field by (2.1). Therefore $\psi \in \operatorname{Ran}\Lambda$. If $\Lambda\varphi_1 = \Lambda\varphi_2$, then the form $\varphi_1 - \varphi_2$ is closed since it is the trace of a harmonic field. Therefore $d\varphi_1 = d\varphi_2$.

Remark 1. There is some freedom in the definition of the DN operator. One can define the DN map as

$$\tilde{\Lambda} = (-1)^{kn} \star_\partial \Lambda : \Omega^k(\partial M) \to \Omega^k(\partial M).$$

This operator preserves the degree of a form. Moreover, it is a nonnegative self-dual operator, i.e., $(\tilde{\Lambda}\varphi, \varphi) \geq 0$ and $\tilde{\Lambda}^* = \tilde{\Lambda}$ as is seen from (3.6) and (3.8). Thus, the operator $\tilde{\Lambda}$ has more conventional properties than Λ. Nevertheless, we have chosen Λ in our definition of the DN map since we share the opinion by J. Sylvester [8]: the DN operator should transform a k-form to an $(n-k-1)$-form.

Remark 2. Quite different definition of the DN map is chosen in [3]. By this definition, the DN operator maps a form $\varphi \in \Omega^k(\partial M)$ to $\partial\omega/\partial\nu$, where ω is the solution to the boundary value problem (1.1). The main result of [3] is that the full symbol of the latter DN map determines the boundary C^∞-jet of the metric.

4 Betti numbers

If we know the kernel of Λ, we can write down some low bounds for Betti numbers as is seen from the following

Theorem 4.1. *Let* Λ_k *denote the restriction of the operator* Λ *to* $\Omega^k(\partial M)$. *The kernel* $\operatorname{Ker}\Lambda_k$ *contains the space* $\mathcal{E}^k(\partial M)$ *of exact forms and*

$$\dim\left[\operatorname{Ker}\Lambda_k / \mathcal{E}^k(\partial M)\right] \leq \min\{\beta_k(M), \beta_k(\partial M)\}.$$

Proof. Consider the Hodge decomposition for ∂M

$$\Omega^k(\partial M) = \mathcal{C}^k(\partial M) \oplus \mathcal{E}^k(\partial M) \oplus \mathcal{H}^k(\partial M).$$

The space of closed forms coincides with the sum of two last summands of the decomposition.

The kernel $\operatorname{Ker}\Lambda_k$ consists of closed forms by Lemma 3.2 and contains all exact forms by (3.10), i.e.,

$$\mathcal{E}^k(\partial M) \subset \operatorname{Ker}\Lambda_k \subset \mathcal{E}^k(\partial M) \oplus \mathcal{H}^k(\partial M).$$

This implies

$$\dim\left[\operatorname{Ker}\Lambda_k/\mathcal{E}^k(\partial M)\right] \le \dim\mathcal{H}^k(\partial M) = \beta_k(\partial M).$$

By Lemma 3.2 and (2.1),

$$\operatorname{Ker}\Lambda_k = \mathcal{E}^k(\partial M) + j^*\mathcal{H}^k_N(M).$$

Therefore

$$\dim\left[\operatorname{Ker}\Lambda_k/\mathcal{E}^k(\partial M)\right] \le \dim\mathcal{H}^k_N(M) = \beta_k(M).$$

The theorem is proved.

The main result of the article is the following

Theorem 4.2. *For any $0 \le k \le n-1$, the operator*

$$\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d : \Omega^k(\partial M) \to \Omega^{n-k-1}(\partial M)$$

has the finite dimensional range whose dimension is equal to $\beta_{n-k-1}(M)$.

Proof. We will prove that

$$\left(\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d\right)\Omega^k(\partial M) = j^*\mathcal{H}^{n-k-1}_N(M). \tag{4.1}$$

Given $\varphi \in \Omega^k(\partial M)$, let $\omega \in \Omega^k(M)$ be a solution to the boundary value problem (3.2). By Lemma 3.1, $d\omega \in \mathcal{H}^{k+1}(M)$. We apply the first Friedrichs decomposition to $d\omega$

$$d\omega = \delta\alpha + \lambda_D, \quad \text{where} \quad \lambda_D \in \mathcal{H}^{k+1}_D(M). \tag{4.2}$$

As is mentioned at the end of Sec. 2, the form $\alpha \in \Omega^{k+2}(M)$ can be chosen such that

$$d\alpha = 0, \quad \Delta\alpha = 0. \tag{4.3}$$

We set $\beta = \star\alpha \in \Omega^{n-k-2}(M)$. (4.3) implies

$$\delta\beta = 0, \quad \Delta\beta = 0. \tag{4.4}$$

Substituting the value $\alpha = (-1)^{k(n-k)} \star \beta$ into (4.2), we have

$$d\omega = (-1)^{k(n-k)}\delta \star \beta + \lambda_D. \tag{4.5}$$

Apply the operator j^* to equation (4.5)

$$j^*(d\omega) = (-1)^{k(n-k)} j^*(\delta \star \beta). \tag{4.6}$$

Using the relations

$$j^*(d\omega) = d(j^*\omega) = d\varphi$$

and

$$\delta \star \beta = (-1)^{n-k-1} \star d\beta,$$

we rewrite (4.6) in the form

$$d\varphi = (-1)^{kn+n+1} j^*(\star d\beta). \tag{4.7}$$

Formulas (4.4) and (4.7) mean that

$$d\varphi = (-1)^{kn+n+1} \Lambda j^*\beta. \tag{4.8}$$

Next, we apply the operator $\star$ to equation (4.5)

$$\star d\omega = (-1)^{k(n-k)} \star \delta \star \beta + \star\lambda_D$$

and take the restriction to the boundary

$$j^*(\star d\omega) = (-1)^{k(n-k)} j^*(\star\delta \star \beta) + j^*(\star\lambda_D). \tag{4.9}$$

The left-hand side of this formula is equal to $\Lambda\varphi$. Using the relation

$$\star\delta \star \beta = (-1)^{kn} d\beta,$$

we transform the first term on the right-hand side of (4.9) as follows:

$$j^*(\star\delta \star \beta) = (-1)^{kn} j^* d\beta = (-1)^{kn} d(j^*\beta).$$

Thus, (4.9) is equivalent to the equation

$$\Lambda\varphi = (-1)^k d(j^*\beta) + j^*(\star\lambda_D). \tag{4.10}$$

The form $j^*\beta$ can be eliminated from the system of equations (4.8) and (4.10). Indeed, (4.8) implies with the help of Corollary 3.3

$$d(j^*\beta) = (-1)^{kn+n+1}(d\Lambda^{-1}d)\varphi.$$

Inserting this expression into (4.10), we obtain

$$\left(\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d\right)\varphi = j^*(\star\lambda_D).$$

We have thus proved that the left-hand side of (4.1) is a subset of the right-hand side.

To prove the converse inclusion, we first recall that

$$\mathcal{H}^k_D(M) \cap \mathcal{H}^k_N(M) = 0.$$

Together with Friedrichs decompositions, this implies that

$$\mathcal{H}^k(M) = \mathcal{H}^k_{ex}(M) + \mathcal{H}^k_{co}(M),$$

i.e., a harmonic field can be represented as a sum of exact and co-exact harmonic fields.

Given $\lambda_N \in \mathcal{H}_N^{n-k-1}(M)$, the representation

$$\lambda_N = d\alpha + \delta\beta \tag{4.11}$$

exists by the remark of the previous paragraph. The forms α and β can be chosen such that

$$\delta\alpha = 0, \quad \Delta\alpha = 0 \tag{4.12}$$

and

$$d\beta = 0, \quad \Delta\beta = 0. \tag{4.13}$$

The latter statement is proved by the same argument as one used at the end of Sec. 2.

We set

$$\omega = (-1)^{kn+1} \star \beta, \quad \varepsilon = (-1)^{kn+k+n}\alpha.$$

Relations (4.12)–(4.13) imply

$$\delta\omega = 0, \quad \Delta\omega = 0, \tag{4.14}$$

$$\delta\varepsilon = 0, \quad \Delta\varepsilon = 0, \tag{4.15}$$

and equation (4.11) is rewritten in the form

$$\lambda_N = \star d\omega + (-1)^{kn+k+n} d\varepsilon. \tag{4.16}$$

Applying the operator $\star$ to the latter equation, we obtain

$$\star\lambda_N = (-1)^{kn+k+n}(\star d\varepsilon - d\omega). \tag{4.17}$$

We define forms $\varphi, \psi \in \Omega(\partial M)$ by

$$\varphi = j^*\omega, \quad \psi = j^*\varepsilon. \tag{4.18}$$

Restricting equation (4.16) to the boundary, we obtain

$$j^*\lambda_N = j^*(\star d\omega) + (-1)^{kn+k+n} d(j^*\varepsilon). \tag{4.19}$$

Equations (4.14) and the first of equalities (4.18) mean that $j^*(\star d\omega) = \Lambda\varphi$. Therefore (4.19) can be rewritten as

$$j^*\lambda_N = \Lambda\varphi + (-1)^{kn+k+n} d(j^*\varepsilon). \tag{4.20}$$

On the other hand, restricting equation (4.17) to the boundary, we obtain

$$j^*(\star d\varepsilon) = d(j^*\omega). \tag{4.21}$$

Equations (4.15) and the second of equalities (4.18) mean that $j^*(\star d\varepsilon) = \Lambda\psi$. Therefore (4.21) can be rewritten as

$$\Lambda\psi = d\varphi. \tag{4.22}$$

Finally, we eliminate the form ψ from the system of equations (4.20) and (4.22) with the help of Corollary 3.3 and obtain

$$j^*\lambda_N = \left(\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d\right)\varphi.$$

This completes the proof of the theorem.

5 Hilbert transform

One of equivalent definitions of the classical Hilbert transform H on the unit circle $S = \{e^{i\theta}\}$ is as follows. Let $f = \varepsilon + i\omega$ be a holomorphic function in the disc $\{re^{i\theta} \mid 0 \le r \le 1\}$ so that ω and ε are conjugate by Cauchy–Riemann: $d\omega = \star d\varepsilon$. If $\varphi = \omega|_S$ and $\psi = \varepsilon|_S$ are the boundary traces, then $H\frac{d\varphi}{d\theta} = \frac{d\psi}{d\theta}$.

Returning to the general case, we define the Hilbert transform as follows:

$$H = d\Lambda^{-1} : \mathcal{E}^k(\partial M) \to \mathcal{E}^{n-k}(\partial M).$$

This is a well defined operator by Corollary 3.4.

Let $\omega \in \Omega^k(M)$ and $\varepsilon \in \Omega^{n-k-2}(M)$ $(0 \le k \le n-2)$ be two co-closed forms,

$$\delta\omega = 0, \quad \delta\varepsilon = 0.$$

The form ε is named the conjugate form of ω if

$$d\omega = \star d\varepsilon. \tag{5.1}$$

This implies immediately that

$$\Delta\omega = 0, \quad \Delta\varepsilon = 0,$$

and $(-1)^{kn+k+n+1}\omega$ is the conjugate form of ε.

Not any ω satisfying $\Delta\omega = 0$ and $\delta\omega = 0$ has a conjugate form. The remarkable fact is that the existence of the conjugate form can be checked in terms of the trace $\varphi = j^*\omega$ and of the operator Λ.

Theorem 5.1. *A form $\omega \in \Omega^k(M)$ satisfying $\Delta\omega = 0$ and $\delta\omega = 0$ has a conjugate form if and only if the trace $\varphi = j^*\omega$ satisfies*

$$\left(\Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d\right)\varphi = 0. \tag{5.2}$$

In this case, if ε is the conjugate form of ω and $\psi = j^*\varepsilon$, then

$$H d\varphi = d\psi. \tag{5.3}$$

Proof. Necessity. Let a co-closed form $\omega \in \Omega^k(M)$ has a conjugate co-closed form $\varepsilon \in \Omega^{n-k-2}(M)$. Set $\varphi = j^*\omega$ and $\psi = j^*\varepsilon$. The forms ω and ε solve boundary value problems (3.5). Therefore

$$\Lambda\varphi = j^*(\star d\omega), \quad \Lambda\psi = j^*(\star d\varepsilon). \tag{5.4}$$

The second of equalities (5.4) and (5.1) imply

$$\Lambda\psi = j^* d\omega = d(j^*\omega) = d\varphi.$$

Applying the operator $\star$ to (5.1), we get

$$\star d\omega = (-1)^{kn+k+n+1} d\varepsilon.$$

Together with the latter relation, the first of equalities (5.4) gives

$$\Lambda\varphi = (-1)^{kn+k+n+1} j^* d\varepsilon = (-1)^{kn+k+n+1} d(j^*\varepsilon) = (-1)^{kn+k+n+1} d\psi.$$

We have thus proved that

$$\begin{cases} \Lambda\varphi = (-1)^{kn+k+n+1} d\psi, \\ \Lambda\psi = d\varphi. \end{cases} \tag{5.5}$$

Eliminating ψ from the latter system, we obtain (5.2). The second of equations (5.5) is equivalent to (5.3).

Sufficiency. Let a form $\varphi \in \Omega^k(\partial M)$ satisfy (5.2) and ω be a solution to the boundary value problem (3.2). Applying the operator $H = d\Lambda^{-1}$ to equation (5.2), we obtain

$$\left(I + (-1)^{kn+k+n} H^2\right) d\varphi = 0, \tag{5.6}$$

where I is the identity operator.

By Corollary 3.4, the equation

$$\Lambda\psi = d\varphi \tag{5.7}$$

is solvable. Fix a solution ψ to the equation and consider the boundary value problem

$$\begin{cases} d\varepsilon = \chi := (-1)^{kn+k+n+1} \star d\omega, \quad \delta\varepsilon = 0, \\ j^*\varepsilon = \psi. \end{cases} \tag{5.8}$$

By Theorem 3.2.5 of [7], the necessary and sufficient conditions for solvability of the problem are

$$d\chi = 0, \quad j^*\chi = d\psi,$$

and

$$(\chi, \lambda_D) = \int_{\partial M} \psi \wedge j^*(\star\lambda_D) \quad \forall\, \lambda_D \in \mathcal{H}_D^{n-k-1}(M). \tag{5.9}$$

The first condition is satisfied since

$$d\chi = \pm d \star d\omega = \pm \star \delta d\omega = \pm \star \Delta\omega = 0.$$

The second condition holds since, by (5.2),

$$j^*\chi = (-1)^{kn+k+n+1} j^*(\star d\omega) = (-1)^{kn+k+n+1} \Lambda\varphi = d\Lambda^{-1} d\varphi = d\psi.$$

It remains to check (5.9).

The left-hand side of (5.9) is equal to zero for any Dirichlet harmonic field λ_D. Indeed, substituting the value of χ from (5.8), we can write

$$(\chi, \lambda_D) = \pm(\star d\omega, \lambda_D) = \pm(d\omega, \star\lambda_D).$$

The right-hand side of the latter formula is zero by the second Fridrichs decomposition since $d\omega$ is an exact harmonic field and $\star\lambda_D$ is a Neumann harmonic field. Condition (5.9) is thus reduced to the following one:

$$\int_{\partial M} \psi \wedge j^*\lambda_N = 0 \quad \forall\, \lambda_N \in \mathcal{H}_N^{k+1}(M). \tag{5.10}$$

By Theorem 4.2, $j^*\mathcal{H}_N^{k+1}(M)$ coincides with the range of the operator

$$G := \Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d : \Omega^{n-k-2}(\partial M) \to \Omega^{k+1}(\partial M). \tag{5.11}$$

Therefore condition (5.10) can be rewritten as follows:

$$(\psi, \star_\partial G\eta) = \pm \int_{\partial M} \psi \wedge G\eta = 0 \quad \forall\, \eta \in \Omega^{n-k-2}(\partial M).$$

In other words, ψ must belong to the kernel of the operator $(\star_\partial G)^*$,

$$(\star_\partial G)^* \psi = 0. \tag{5.12}$$

One easily obtains from (3.8)

$$G^* = \star_\partial G \star_\partial$$

and

$$(\star_\partial G)^* = \pm \star_\partial G.$$

Therefore (5.12) is equivalent to the equation

$$G\psi = 0. \tag{5.13}$$

Finally, substituting the values $\psi = \Lambda^{-1} d\varphi$ and $G = \Lambda + (-1)^{kn+k+n} d\Lambda^{-1} d$ into (5.13), we see that the latter equation is equivalent to (5.6).

We have thus proved the solvability of the boundary value problem (5.8). The solution ε to the problem is the conjugate form of ω. The theorem is proved.

Corollary 5.2. *For $1 \le k \le n-1$, introduce the space*

$$\mathcal{B}^k(\partial M) = \{d\varphi \mid \varphi \in \Omega^{k-1}(\partial M),\ G\varphi = 0\} \subset \mathcal{E}^k(\partial M),$$

where the operator $G : \Omega(\partial M) \to \Omega(\partial M)$ is defined by (5.11). The Hilbert transform H maps $\mathcal{B}^k(\partial M)$ isomorphically onto $\mathcal{B}^{n-k}(\partial M)$. If

$$\tilde{H}_k : \mathcal{B}^k(\partial M) \to \mathcal{B}^{n-k}(\partial M) \tag{5.14}$$

is the restriction of H to $\mathcal{B}^k(\partial M)$, then $\tilde{H}_k^{-1} = (-1)^{kn+k} \tilde{H}_{n-k}$.

Proof. For a form $\varphi \in \Omega^{k-1}(\partial M)$ satisfying $G\varphi = 0$, let $\omega \in \Omega^{k-1}(M)$ be a solution to the boundary value problem (3.2). Then ω has a conjugate form $\varepsilon \in \Omega^{n-k-1}(M)$ by Theorem 5.1. The trace $\psi = j^*\varepsilon$ satisfies (5.3). The form ε has the conjugate $(-1)^{kn+k}\omega$, and $G\psi = 0$ by Theorem 5.1. Now, (5.3) shoes that $Hd\varphi = d\psi \in \mathcal{B}^{n-k}(\partial M)$. We have thus proved that H maps $\mathcal{B}^k(\partial M)$ to $\mathcal{B}^{n-k}(\partial M)$ and operator (5.14) is well defined. By (5.6), $\tilde{H}_{n-k}\tilde{H}_k = (-1)^{kn+k} I$. Therefore $\tilde{H}_k$ is an isomorphism.

Corollary 5.3. *Let $1 \le k \le n-1$. If $\beta_k(M) = \beta_{n-k}(M) = 0$, then the Hilbert transform H maps $\mathcal{E}^k(\partial M)$ isomorphically onto $\mathcal{E}^{n-k}(\partial M)$. If*

$$H_k : \mathcal{E}^k(\partial M) \to \mathcal{E}^{n-k}(\partial M)$$

is the restriction of H to $\mathcal{E}^k(\partial M)$, then $H_k^{-1} = (-1)^{kn+k} H_{n-k}$.

Proof. If $\beta_k(M) = \beta_{n-k}(M) = 0$, then the operator G vanishes on $\Omega^{k-1}(\partial M)$ and on $\Omega^{n-k-1}(\partial M)$ by Theorem 4.2. Therefore $\mathcal{B}^k(\partial M) = \mathcal{E}^k(\partial M)$ and $\mathcal{B}^{n-k}(\partial M) = \mathcal{E}^{n-k}(\partial M)$. It remains to apply Corollary 5.2.

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