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# ASYMPTOTIC BOUNDS FOR THE SPECTRAL BANDS OF PERIODIC SCHRÖDINGER OPERATORS

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ABSTRACT

We obtain sharp upper and lower bounds for the overlap multiplicity of spectral bands of the multidimensional Schrödinger operator with a rational lattice of periods. These bounds are based on recent results on the lattice counting.

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## 1. Main result

**1.1. Introduction.** The spectral analysis of the Schrödinger operator

$$H = -\Delta + V$$

with a real-valued periodic potential relies on the standard Floquet decomposition, see e.g. [7] or [8]. Let $\Gamma \subset \mathbb{R}^d$, $d \geq 2$ be the period lattice of the potential $V$, that is $V(\mathbf{x} + \boldsymbol{\gamma}) = V(\mathbf{x})$, for all $\boldsymbol{\gamma} \in \Gamma$. By $\mathcal{O}$ we denote the standard fundamental domain of $\Gamma$, that is the parallelepiped spanned by arbitrarily chosen $d$ basis vectors of $\Gamma$. The volume of the domain $\mathcal{O}$ is called the determinant of the lattices and is denoted by $\mathrm{d}(\Gamma)$. The notations $\Gamma^\dagger$ and $\mathcal{O}^\dagger$ are reserved for the dual lattice and its fundamental domain, respectively. If necessary, the dependence on the lattice is reflected in the notation: $\mathcal{O}_\Gamma$, $\mathcal{O}^\dagger_\Gamma$.

Introduce in $\mathfrak{H} = \mathrm{L}^2(\mathcal{O})$ the operators

$$H_0(\mathbf{k}) = (-i\nabla + \mathbf{k})^2, \quad H(\mathbf{k}) = H_0(\mathbf{k}) + V$$

depending on the *quasi-momentum* $\mathbf{k} \in \mathcal{O}^\dagger$, with the natural domain $\mathrm{H}^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}^d/\Gamma$. For each $H(\mathbf{k})$ the spectrum consists of discrete eigenvalues $\lambda_j(\mathbf{k}) = \lambda_j(H(\mathbf{k}))$, $j = 1, 2, \dots$, that we arrange in ascending order counting multiplicity. It is clear that $\lambda_j(\ \cdot\ )$ are continuous functions of $\mathbf{k}$. In general, for any bounded self-adjoint continuous operator-function $B(\mathbf{k})$, the spectrum of $\hat{H}(\mathbf{k}) = H_0(\mathbf{k}) + B(\mathbf{k})$ is discrete, and the eigenvalues $\lambda_j(\hat{H}(\mathbf{k}))$ are continuous functions of $\mathbf{k} \in \mathcal{O}^\dagger$. Introduce the counting functions

$$\begin{aligned} n(\lambda; \hat{H}(\mathbf{k})) &= n_\Gamma\big(\lambda; \hat{H}(\mathbf{k})\big) = \sharp\{j : \lambda_j(\hat{H}(\mathbf{k})) < \lambda\}, \\ N(\lambda; \hat{H}(\mathbf{k})) &= N_\Gamma\big(\lambda; \hat{H}(\mathbf{k})\big) = \sharp\{j : \lambda_j(\hat{H}(\mathbf{k})) \leq \lambda\}, \quad \mathbf{k} \in \mathcal{O}^\dagger. \end{aligned}$$

It is clear that

$$N(\lambda; \hat{H}(\mathbf{k})) = \lim_{\epsilon \downarrow 0} n(\lambda + \epsilon; \hat{H}(\mathbf{k})). \tag{1.1}$$

When $B(\mathbf{k}) = V$, we normally use the notation $n(\lambda; \mathbf{k})$, $N(\lambda; \mathbf{k})$. If $V = 0$ we write $n_0$, $N_0$ instead of $n$, $N$. The images

$$\ell_j = \bigcup_{\mathbf{k} \in \overline{\mathcal{O}^\dagger}} \lambda_j(\mathbf{k}),$$

of the functions $\lambda_j$ are called *spectral bands.* The spectrum of the initial operator $H$ is the union of the bands:

$$\sigma(H) = \cup_j \ell_j.$$

In order to characterise the overlap of different bands, we introduce the *overlap multiplicity*:

$$\mathfrak{m}(\lambda) = \mathfrak{m}_\Gamma(\lambda) = \sharp\{j \in \mathbb{N} : \lambda \in \ell_j\}, \quad \lambda \in \mathbb{R},$$

which shows to how many bands a given value $\lambda$ belongs. The function $\mathfrak{m}(\lambda)$ can be found from the counting functions $n$, $N$. Let $\mathfrak{n}(\lambda; \mathbf{k})$ be either of the two functions $N(\lambda; \mathbf{k})$ or $n(\lambda; \mathbf{k})$. Denote

$$\mathfrak{n}^+(\lambda) = \max_{\mathbf{k}} \mathfrak{n}(\lambda; \mathbf{k}), \ \ \mathfrak{n}^-(\lambda) = \min_{\mathbf{k}} \mathfrak{n}(\lambda; \mathbf{k}). \tag{1.2}$$

Then one easily proves (see [8, 9]) the following relation

$$\mathfrak{m}(\lambda) = N^{+}(\lambda) - n^{-}(\lambda). \tag{1.3}$$

Note that a $\Gamma$-periodic potential is also $\Lambda$-periodic for any sublattice $\Lambda \subset \Gamma$. Let us find out how the multiplicities $\mathfrak{m}_\Gamma(\lambda)$ and $\mathfrak{m}_\Lambda(\lambda)$ are related. A simple calculation shows that

$$\mathfrak{n}_\Lambda(\lambda;\mathbf{k}) = \sum_{\boldsymbol{\nu}\in\Lambda^\dagger/\Gamma^\dagger} \mathfrak{n}_\Gamma(\lambda;\boldsymbol{\nu}+\mathbf{k}), \mathbf{k}\in\mathcal{O}_\Lambda^\dagger.$$

This implies that

$$\mathfrak{n}_\Lambda^{+}(\lambda) \le \frac{\mathrm{d}(\Lambda)}{\mathrm{d}(\Gamma)}\mathfrak{n}_\Gamma^{+}(\lambda), \quad \mathfrak{n}_\Lambda^{-}(\lambda) \ge \frac{\mathrm{d}(\Lambda)}{\mathrm{d}(\Gamma)}\mathfrak{n}_\Gamma^{-}(\lambda),$$

and hence by (1.3)

$$\mathfrak{m}_\Lambda(\lambda) \le \frac{\mathrm{d}(\Lambda)}{\mathrm{d}(\Gamma)}\mathfrak{m}_\Gamma(\lambda). \tag{1.4}$$

Our study of $\mathfrak{m}(\lambda)$ relies on the connection with the counting function for the lattice $\Gamma^\dagger$.

**1.2. The overlap multiplicity and lattice points counting.** For any lattice $\mathrm{M} \subset \mathbb{R}^d$, $d \ge 2$ we denote by $\mathcal{N}[\mathcal{C}] = \mathcal{N}_{\mathrm{M}}[\mathcal{C}]$ the number of lattice points which belong to a bounded set $\mathcal{C} \subset \mathbb{R}^d$, that is

$$\mathcal{N}[\mathcal{C}] = \#\{\boldsymbol{\gamma}\in\mathrm{M} : \boldsymbol{\gamma}\in\mathcal{C}\}.$$

We are interested in counting lattice points inside the "shifted" sets $\mathcal{C}$, i.e., inside

$$\mathcal{C}(\mathbf{k}) = \{\boldsymbol{\xi}\in\mathbb{R}^d : \boldsymbol{\xi}+\mathbf{k}\in\mathcal{C}\},\ \ \mathbf{k}\in\mathbb{R}^d. \tag{1.5}$$

More precisely, it is important for us to have information on the counting function $\mathcal{N}[\,\cdot\,]$ for open balls

$$B(r;\mathbf{k}) = \{\boldsymbol{\xi}\in\mathbb{R}^d : |\boldsymbol{\xi}+\mathbf{k}| < r\},\ \ r > 0.$$

The functions $\mathcal{N}[\,\cdot\,]$ and $n_0(\lambda;\mathbf{k})$ are related in the following way:

$$n_0(\rho^2;\mathbf{k}) = \mathcal{N}_{\Gamma^\dagger}[B(\rho;\mathbf{k})], \rho \ge 0. \tag{1.6}$$

Define for all $r > 0$

$$\mathcal{N}_{\mathrm{M}}^{+}(r) = \max_{\mathbf{k}} \mathcal{N}_{\mathrm{M}}[B(r;\mathbf{k})], \qquad \mathcal{N}_{\mathrm{M}}^{-}(r) = \min_{\mathbf{k}} \mathcal{N}_{\mathrm{M}}[B(r;\mathbf{k})], \tag{1.7}$$

and for any $\lambda > 0$, and $\delta : |\delta| < \lambda$

$$V(\lambda,\delta) = V_{\mathrm{M}}(\lambda,\delta) = \mathcal{N}_{\mathrm{M}}^{+}(\sqrt{\lambda-\delta}) - \mathcal{N}_{\mathrm{M}}^{-}(\sqrt{\lambda+\delta}). \tag{1.8}$$

We call this quantity the *$\delta$-variation of the lattice counting function at the point $\lambda$*. Our aim is to show that the multiplicity $\mathfrak{m}(\lambda)$ can be estimated from both sides by the $\delta$-variation. For the sake of convenience from now on we assume that the average value of the potential is zero, that is

$$\int_{\mathcal{O}} V(\mathbf{x})dx = 0. \tag{1.9}$$

For the general $V$ the results can be recovered in the obvious way.

**Theorem 1.1.** *Let $d \geq 2$ and let $V$ be a continuous real-valued $\Gamma$-periodic function satisfying* (1.9). *Then for any $\delta > 0$*

$$V_{\Gamma^\dagger}(\lambda, \delta) - O(\lambda^{\varkappa_d}) \leq \mathfrak{m}(\lambda) \leq V_{\Gamma^\dagger}(\lambda; -\delta) + O(\lambda^{\varkappa_d}), \tag{1.10}$$

*for large $\lambda$, where*

$$\varkappa_l = \frac{l-3}{2} + \frac{1}{l}, \ l = 1, 2, \dots. \tag{1.11}$$

The proof of this theorem is postponed until the next section. Let us list the known properties of the function $\mathcal{N}[B(\rho; \mathbf{k})]$. We start with a classical bound proved by E. Landau (see [4]) for arbitrary lattice $\mathrm{M} \subset \mathbb{R}^d$, $d \geq 1$:

$$\mathcal{N}_{\mathrm{M}}[B(\rho; \mathbf{k})] = \frac{\mathrm{w}_d}{\mathrm{d}(\mathrm{M})}\rho^d + O(\rho^{2\varkappa_d + 1}), \tag{1.12}$$

uniformly in $\mathbf{k} \in \mathbb{R}^d$. Here where $\mathrm{w}_d$ is the volume of a unit ball in $\mathbb{R}^d$, and $\varkappa_d$ is defined in (1.11). Today more precise results are known (see [2] and references therein). For instance, if $d \geq 5$, then

$$\mathcal{N}_{\mathrm{M}}[B(\rho; \mathbf{k})] = \frac{\mathrm{w}_d}{\mathrm{d}(\mathrm{M})}\rho^d + O(\rho^{d-2}), \tag{1.13}$$

uniformly in $\mathbf{k}$. If one wants to obtain more precise estimates for the remainder in (1.12), one has to differentiate between *rational* and *irrational* lattices.

**Definition 1.2.** *A lattice $\Gamma \subset \mathbb{R}^d$ is said to be rational if for any two vectors $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2 \in \Gamma$ the inner product satisfies the relation*

$$\langle \boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2 \rangle = \beta_\Gamma r_{12}, \tag{1.14}$$

*where $\beta_\Gamma \neq 0$ is a real-valued constant independent of $\boldsymbol{\gamma}_1$, $\boldsymbol{\gamma}_2$, and $r_{12} = r_{21}$ is an integer number. Otherwise the lattice is called irrational.*

It is clear that in order to check the rationality of the lattice it suffices to verify (1.14) only for the basis vectors of $\Gamma$. Note also that for any rational (resp., irrational) lattice $\Gamma$ the dual lattice $\Gamma^\dagger$ is also rational (resp., irrational).

It was shown in [2] that the remainder in (1.13) can be replaced by $o(\rho^{d-2})$ if and only if the lattice M is irrational. For any $\delta > 0$ it follows at once from (1.8) and (1.13) that

$$V_{\mathrm{M}}(\lambda, -\delta) \leq C(\delta)\lambda^{\frac{d-2}{2}}, \quad d \geq 5, \tag{1.15}$$

and

$$\lim_{\delta \to 0} \limsup_{\lambda \to \infty} \lambda^{-\frac{d-2}{2}} V_M(\lambda, -\delta) = 0, \quad \text{if } \ \mathrm{M} \ \text{ is irrational and } \ d \geq 5, \tag{1.16}$$

for large $\lambda$. For rational lattices the corresponding lower bound was proved in [10]:

**Proposition 1.3** [10]**.** *Let M be a rational lattice and let $d \geq 4$. Then there are three positive constants $\delta_0 = \delta_0(\mathrm{M})$, $\lambda_0 = \lambda_0(\mathrm{M})$ and $c_{\mathrm{M}}$, such that for all $\delta \in [0, \delta_0]$ and all $\lambda \geq \lambda_0$, we have*

$$V_{\mathrm{M}}(\lambda, \delta) \geq \begin{cases} c_{\mathrm{M}}\lambda^{\frac{d-2}{2}}, & d \geq 5, \\ c_{\mathrm{M}}(\log\log\lambda)^{-1}\lambda, & d = 4, \\ c_{\mathrm{M}}\lambda, & d = 4, \quad \mathrm{M} = \mathbb{Z}^4. \end{cases} \tag{1.17}$$

The above properties of rational and irrational lattices in combination with Theorem 1.1 lead to the following Theorem which constitutes the main result of the paper.

**Theorem 1.4.** *Let $d \geq 4$ and the lattice $\Gamma$ be rational. Let $V$ satisfy the conditions of Theorem 1.1. Then there exist positive constants $\lambda_0 = \lambda_0(\Gamma, V)$ and $c_\Gamma$, $C_\Gamma$ such that for all $\lambda \geq \lambda_0$*

$$c_\Gamma \lambda^{\frac{d-2}{2}} \leq \mathfrak{m}_\Gamma(\lambda) \leq C_\Gamma \lambda^{\frac{d-2}{2}}, \tag{1.18}$$

*if $d \geq 5$ and*

$$\mathfrak{m}_\Gamma(\lambda) \geq c_\Gamma (\log\log\lambda)^{-1}\lambda, \tag{1.19}$$

*if $d = 4$. If $d = 4$ and $\Gamma$ contains a cubic sublattice, then*

$$\mathfrak{m}_\Gamma(\lambda) \geq c_\Gamma \lambda. \tag{1.20}$$

*The constants $C_\Gamma, c_\Gamma$ depend on the lattice $\Gamma$ and do not depend on $V$.*

Let us state a corresponding result for irrational lattices:

**Theorem 1.5.** *Let $\Gamma \subset \mathbb{R}^d$, $d \geq 5$ be an irrational lattice and let $V$ satisfy the conditions of Theorem 1.1. Then*

$$\mathfrak{m}_\Gamma(\lambda) = o(\lambda^{\frac{d-2}{2}}),\ \lambda \to \infty. \tag{1.21}$$

*Proof of Theorem 1.4.* Let $\delta \in (0, \delta_0]$ be a number such that the estimates (1.17) hold. Then the lower bounds in (1.18), (1.19) follow from Theorem 1.1, since $\varkappa_d < (d-2)/2$. The upper bound in (1.18) follows from Theorem 1.1 and (1.15).

If $d = 4$ and $\Gamma$ contains a cubic sublattice $\Lambda$, we consider $V$ as being $\Lambda$-periodic. By a straightforward scaling transformation $\Lambda$ is reduced to $\mathbb{Z}^4$, and hence the bound (1.17) for $\mathbb{Z}^4$, Theorem 1.1, and (1.4) yield (1.20). $\square$

*Proof of Theorem 1.5.* The argument is much more elementary than in the previous proof. Since $\varkappa_d < (d-2)/2$, according to Theorem 1.1, for any $\delta > 0$ one has

$$\limsup_{\lambda\to\infty} \lambda^{-\frac{d-2}{2}} \mathfrak{m}_\Gamma(\lambda) = \limsup_{\lambda\to\infty} \lambda^{-\frac{d-2}{2}} V_{\Gamma^\dagger}(\lambda, -\delta).$$

As $\delta > 0$ is arbitrary, the bound (1.20) immediately follows from (1.16). $\square$

Comparing (1.21) with (1.18) we see that the overlap multiplicity is sensitive (at least for $d \geq 5$) to the arithmetic properties of the underlying lattice: the lower bound in (1.18) is impossible if the lattice is irrational.

It is instructive to rephrase Theorem 1.4 in the following way:

**Corollary 1.6.** *Let $d \geq 4$ and the conditions of Theorem 1.4 be fulfilled. Then*

$$\lim_{\lambda\to\infty} \frac{\ln \mathfrak{m}(\lambda)}{\ln\lambda} = \frac{d-2}{2},\ d \geq 5, \tag{1.22}$$

$$\liminf_{\lambda\to\infty} \frac{\ln \mathfrak{m}(\lambda)}{\ln\lambda} \geq 1,\ d = 4. \tag{1.23}$$

The estimates (1.22) and (1.23) follow from (1.18) and (1.19), (1.20), respectively. Certainly, rewriting (1.18)–(1.20) using the *ln*-function, leads to a loss of information. On the other hand, the formulae (1.22), (1.23) have a more invariant form than (1.18)–(1.20), as they do not depend on the choice of the rational lattice $\Gamma$.

We emphasise again that the upper bounds in Theorem 1.4 follow from the known asymptotic estimate (1.13), but the lower bounds require the new Proposition 1.3. Lower bounds of the form (1.18), (1.19) were initially proved in [8], and the asymptotics (1.22) was stated as a conjecture (see (0.12), Introduction to [8]). Although the methods of [8] are also based on the lattice point counting, the introduction of the $\delta$-variation in [10] has substantially simplified the proofs of the lower bounds by separating the number-theoretic part of the argument from the spectral one.

Theorem 1.4 implies in particular that the number of gaps in the spectrum of $H$ is finite. This fact, known as the Bethe-Sommerfeld Conjecture, was initially proved for rational lattices and $d \geq 4$ in paper [8], and for arbitrary lattices and $d = 3$ in [9]. For further discussion and references we refer to [8, 9, 5, 6].

## 2. Reduction to a lattice points problem

**2.1. An intermediate estimate.** In this section, we reduce the study of the spectrum counting function $n(\lambda; \mathbf{k})$ to the lattice counting function $\mathcal{N}_{\Gamma^\dagger}(r; \mathbf{k})$.

**Theorem 2.1.** *Let $d \geq 2$ and suppose that $V$ satisfies the conditions of Theorem 1.1. Then for any $\delta > 0$ there are two constants $\lambda_0 = \lambda_0(\delta, V) > 0$ and $C = C(\delta)$ such that for all $\lambda \geq \lambda_0$ the distribution function $n(\lambda; \mathbf{k})$ satisfies the estimate*

$$\mathcal{N}_{\Gamma^\dagger}\big(B(\sqrt{\lambda - \delta}; \mathbf{k})\big) - C\lambda^{\varkappa} \leq n(\lambda; \mathbf{k}) \leq \mathcal{N}_{\Gamma^\dagger}\big(B(\sqrt{\lambda + \delta}; \mathbf{k})\big) + C\lambda^{\varkappa}, \tag{2.1}$$

*where $\varkappa = \varkappa_d$ is defined in* (1.11).

Before proving Theorem 2.1 we show how to derive Theorem 1.1 from it:

*Proof of Theorem 1.1.* Let us use Theorem 2.1 for some fixed $\delta > 0$ and $\delta' < \delta$. By definitions (1.2) and (1.7), we have

$$\mathcal{N}^{-}_{\Gamma^\dagger}(\sqrt{\lambda - \delta}) - C\lambda^{\varkappa} \leq n^{-}(\lambda) \leq \mathcal{N}^{-}_{\Gamma^\dagger}(\sqrt{\lambda + \delta}) + C\lambda^{\varkappa},$$

and in view of (1.1)

$$\mathcal{N}^{+}_{\Gamma^\dagger}(\sqrt{\lambda - \delta}) - C\lambda^{\varkappa} \leq N^{+}(\lambda) \leq \mathcal{N}^{+}_{\Gamma^\dagger}(\sqrt{\lambda + \epsilon + \delta'}) + C\lambda^{\varkappa},$$

for any $\epsilon > 0$. Take $\epsilon = \delta - \delta'$. Now (1.10) follows from (1.3) and (1.9). $\square$

Note in advance that in the proof of Theorem we need only one piece of information for the lattice counting function: the estimate (1.12).

Notice also the following useful observation. If $V(\mathbf{x})$ and and $\tilde{V}(\mathbf{x})$ are two continuous $\Gamma$-periodic potentials and

$$\delta' = \sup_{\mathbf{x}} |V(\mathbf{x}) - \tilde{V}(\mathbf{x})|,$$

then an elementary perturbation argument leads to the estimate

$$N(\lambda - \delta'; H(\mathbf{k})) \leq N(\lambda; \tilde{H}(\mathbf{k})) \leq N(\lambda + \delta'; H(\mathbf{k})).$$

Consequently, if the estimate (2.1) holds with a parameter $\delta = \delta_0 > 0$ for some given potential $V$, then it will also hold for any other potential $\tilde{V}$ with $\delta = \delta_0 + \delta'$. This implies that it suffices to prove Theorem 2.1 for all real trigonometric polynomials with the zero mean value, that is for potentials of the form

$$V(\mathbf{x}) = \sum_{\boldsymbol{\theta}\in\Theta} V_{\boldsymbol{\theta}}(\mathbf{x}),\ \ V_{\boldsymbol{\theta}}(\mathbf{x}) = v_{\boldsymbol{\theta}} e^{i\langle \mathbf{x}, \boldsymbol{\theta}\rangle}, \tag{2.2}$$

where $v_{\boldsymbol{\theta}} = \overline{v_{-\boldsymbol{\theta}}}$ are the Fourier coefficients of the function $V$ and $\Theta \subset \Gamma^{\dagger}$, $\Theta \not\ni \{\mathbf{0}\}$ is a finite subset of the dual lattice.

**2.2. Auxiliary geometrical construction.** Let $\mathrm{M} \subset \mathbb{R}^d$ be an arbitrary lattice, and let $\boldsymbol{\theta} \in \mathrm{M}$, $\boldsymbol{\theta} \neq \mathbf{0}$, be a fixed vector. Introduce the following orthogonal coordinates in $\mathbb{R}^d$:

$$\begin{cases} \boldsymbol{\xi} = (z, \mathbf{y}), & z \in \mathbb{R},\ \ \mathbf{y} \in \mathbb{R}^{d-1},\ \ \langle \mathbf{y}, \boldsymbol{\theta}\rangle = 0 : \\ \boldsymbol{\xi} = z\mathbf{n}(\boldsymbol{\theta}) + \mathbf{y}, & \mathbf{n}(\boldsymbol{\theta}) = \boldsymbol{\theta}|\boldsymbol{\theta}|^{-1}. \end{cases} \tag{2.3}$$

Let us fix two positive numbers $A$, $B$ satisfying the conditions

$$A > |\boldsymbol{\theta}|^2,\ \ 4|\boldsymbol{\theta}|^{-2}A^2 + A < B. \tag{2.4}$$

Introduce the spherical shell

$$S(\lambda) = \{\boldsymbol{\xi} \in \mathbb{R}^d : \lambda - A < |\boldsymbol{\xi}|^2 < \lambda + A\}, \tag{2.5}$$

and split it in two disjoint components:

$$S(\lambda) = \Omega_{\boldsymbol{\theta}}(\lambda) \cup \Lambda_{\boldsymbol{\theta}}(\lambda); \tag{2.6}$$

$$\Omega_{\boldsymbol{\theta}}(\lambda) = \{\boldsymbol{\xi} \in S(\lambda) : |\mathbf{y}|^2 > \lambda - B\}, \tag{2.7}$$

$$\Lambda_{\boldsymbol{\theta}}(\lambda) = \{\boldsymbol{\xi} \in S(\lambda) : |\mathbf{y}|^2 \le \lambda - B\}. \tag{2.8}$$

In all the following lemmas we assume that $\lambda \ge \lambda_0$ with a sufficiently large number $\lambda_0 = \lambda_0(A, B) > 0$. First we estimate the number of lattice points in the set $\Omega_{\boldsymbol{\theta}}(\lambda; \mathbf{k})$ defined as in (1.5):

**Lemma 2.2.** *For any lattice $\mathrm{M} \subset \mathbb{R}^d$, $d \ge 2$ and the numbers $A$, $B$ satisfying* (2.4), *the estimate*

$$\max_{\mathbf{k}\in\mathbb{R}^d} \mathcal{N}_{\mathrm{M}}[\Omega_{\boldsymbol{\theta}}(\lambda; \mathbf{k})] \le C\lambda^{\varkappa_d},$$

*holds with a constant $C = C(\boldsymbol{\theta}, A, B)$ independent of $\lambda$.*

*Proof.* Instead of working with $\Omega_{\boldsymbol{\theta}}$ we estimate the number of lattice points in the set $W_{\boldsymbol{\theta}}(\lambda; \mathbf{k})$ with

$$W_{\boldsymbol{\theta}}(\lambda) = \{\boldsymbol{\xi} \in \mathbb{R}^d : |z|^2 \le A + B,\ \lambda - B < |\mathbf{y}|^2 < \lambda + B\}, \tag{2.9}$$

which contains $\Omega_{\boldsymbol{\theta}}(\lambda; \mathbf{k})$ since $B > A$. Let $\Pi = \Pi_{\boldsymbol{\theta}}$ be the hyper-plane in $\mathbb{R}^d$ defined by the condition $z = 0$, and let $P$ be the orthogonal projection in $\mathbb{R}^d$ onto $\Pi$. By definition of $W_{\boldsymbol{\theta}}$ we have

$$PW_{\boldsymbol{\theta}}(\lambda; \mathbf{k}) = \{\mathbf{y} \in \Pi : \lambda - B < |\mathbf{y} + P\mathbf{k}|^2 < \lambda + B\}.$$

Obviously, the projection $\Lambda = P\mathrm{M}$ is a $d-1$-dimensional lattice in $\Pi$, and the points of $\mathrm{M}$ lie on the straight lines

$$\ell_{\boldsymbol{\eta}} = \{\boldsymbol{\xi} \in \mathbb{R}^d : \boldsymbol{\xi} = \boldsymbol{\eta} + t\boldsymbol{\theta}, t \in \mathbb{R}\}, \quad \boldsymbol{\eta} \in \Lambda.$$

If the intersection $\ell_{\boldsymbol{\eta}}(\lambda, \boldsymbol{\theta}, \mathbf{k}) = \ell_{\boldsymbol{\eta}} \cap W_{\boldsymbol{\theta}}(\lambda; \mathbf{k})$ is not empty, then by definition (2.9), this intersection is a straight segment of length $2\sqrt{A+B}$. Consequently, the number of lattice points $\boldsymbol{\gamma} \in \mathrm{M}$ on $\ell_{\boldsymbol{\eta}}(\lambda, \boldsymbol{\theta}, \mathbf{k})$ does not exceed $O(\sqrt{A+B})$. It remains to show that

$$\sup_{\mathbf{k}\in\mathbb{R}^d} \mathcal{N}_\Lambda[PW_{\boldsymbol{\theta}}(\lambda; P\mathbf{k})] \le C\lambda^{\varkappa_d}. \tag{2.10}$$

It follows from the formula (1.12) applied to the $d-1$-dimensional lattice $\Lambda$ that

$$\begin{aligned}\mathcal{N}_\Lambda[PW_{\boldsymbol{\theta}}(\lambda; P\mathbf{k})] &\le \mathcal{N}_\Lambda\big[B\big(\sqrt{\lambda+B}; P\mathbf{k}\big)\big] - \mathcal{N}_\Lambda\big[B\big(\sqrt{\lambda-B}; P\mathbf{k}\big)\big] \\ &= O(B\lambda^{\frac{d-3}{2}}) + O(\lambda^{\varkappa_d}) = O(\lambda^{\varkappa_d}),\end{aligned}$$

uniformly in $\mathbf{k} \in \mathbb{R}^d$. The proof of (2.10) and that of the Lemma is completed. □

**Lemma 2.3.** *Let $\boldsymbol{\theta} \in \mathrm{M} \subset \mathbb{R}^d$, $|\boldsymbol{\theta}| > 0$, be a fixed vector and let $A$, $B$ satisfy* (2.4). *If $\boldsymbol{\xi} \in \Lambda_{\boldsymbol{\theta}}(\lambda)$, then $\boldsymbol{\xi} \pm \boldsymbol{\theta} \notin S(\lambda)$.*

*Proof.* Using the coordinates (2.3) associated with the vector $\boldsymbol{\theta}$, for $\boldsymbol{\xi} = (z, \mathbf{y})$ we get $\boldsymbol{\xi} + \boldsymbol{\theta} = (z + |\boldsymbol{\theta}|, \mathbf{y})$. Since $\boldsymbol{\xi} \in \Lambda_{\boldsymbol{\theta}}(\lambda)$, we have by definition (2.8) that

$$z^2 = |\boldsymbol{\xi}|^2 - |\mathbf{y}|^2 \ge \lambda - A - \lambda + B = B - A.$$

On the other hand,

$$|\boldsymbol{\xi} + \boldsymbol{\theta}|^2 = |z + |\boldsymbol{\theta}||^2 + |\mathbf{y}|^2 = z^2 + |\mathbf{y}|^2 + 2z|\boldsymbol{\theta}| + |\boldsymbol{\theta}|^2,$$

and hence

$$\begin{aligned}\big||\boldsymbol{\xi} + \boldsymbol{\theta}|^2 - \lambda\big| &\ge 2|z||\boldsymbol{\theta}| - \big||\boldsymbol{\xi}|^2 - \lambda\big| - |\boldsymbol{\theta}|^2 \ge 2\sqrt{B-A}|\boldsymbol{\theta}| - A - |\boldsymbol{\theta}|^2 \\ &\ge 2|\boldsymbol{\theta}|\sqrt{B-A} - 4A + 2A \ge 2A.\end{aligned}$$

Here we have used (2.4). By definition (2.5) this implies that $\boldsymbol{\xi} + \boldsymbol{\theta} \notin S(\lambda)$, as required.

For the vector $\boldsymbol{\xi} - \boldsymbol{\theta}$ the proof is the same with obvious modifications. □

## 3. Proof of Theorem 2.1

Denote by $\mathcal{P}(\mathbf{k}; \mathcal{C})$ the projection in $\mathfrak{H} = \mathrm{L}^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}^d/\Gamma$, on the subspace spanned by the orthonormal family of exponentials

$$\mathbf{e}_{\gamma}(\mathbf{x}) = \frac{1}{\sqrt{\mathrm{d}(\Gamma)}} e^{i\langle\gamma, \mathbf{x}\rangle}, \ \gamma \in \Gamma^\dagger \bigcap \mathcal{C}(\mathbf{k}). \tag{3.1}$$

Let $\mathfrak{H}(\mathbf{k}; \mathcal{C}) = \mathcal{P}(\mathbf{k}; \mathcal{C})\mathfrak{H}$. Suppose that the bounded self-adjoint perturbation $B(\mathbf{k})$ is such that the subspace $\mathfrak{H}(\mathbf{k}; \mathcal{C})$ is an invariant subspace for the operator $\hat{H}(\mathbf{k}) = H_0(\mathbf{k}) + B(\mathbf{k})$. It is natural to define the "partial" counting functions $n(\lambda; \hat{H}(\mathbf{k}), \mathcal{C})$ for the $\hat{H}(\mathbf{k})$ restricted to the subspace $\mathfrak{H}(\mathbf{k}; \mathcal{C})$.

Suppose as in the previous section that $V$ is a trigonometric polynomial (2.2). We begin the proof by reducing the operator $H(\mathbf{k}) = H_0(\mathbf{k}) + V$ to the operator $H_1(\mathbf{k}) = H_0(\mathbf{k}) + \mathcal{P}(\mathbf{k}; S(\lambda))V\mathcal{P}(\mathbf{k}; S(\lambda))$.

**Lemma 3.1.** *Let $V$ be as specified above, and let the set $S(\lambda)$ be as defined in (2.6). Then for any $\delta > 0$ there exist numbers $A_0 = A_0(\delta, V)$ and $\lambda_0 = \lambda_0(\delta)$ such that for all $A \geq A_0$ and $\lambda \geq \lambda_0$ one has*

$$n\big(\lambda - \delta; H_1(\mathbf{k})\big) \leq n(\lambda; H(\mathbf{k})) \leq n\big(\lambda + \delta; H_1(\mathbf{k})\big).$$

*Proof.* Denote for brevity

$$S = S(\lambda), \quad K = \mathbb{R}^d \setminus S, \quad \mathcal{P} = \mathcal{P}(\mathbf{k}; S), \mathcal{Q} = \mathbf{I} - \mathcal{P},$$
$$H = H(\mathbf{k}), \quad H_1 = H_1(\mathbf{k}), \quad H_0 = H_0(\mathbf{k}).$$

Write $H$ in the form

$$H = H_0 + \mathcal{P}V\mathcal{P} + \mathcal{Q}V\mathcal{Q} + \mathcal{Q}V\mathcal{P} + \mathcal{P}V\mathcal{Q}.$$

Estimate the last two terms in the r.h.s. for a function $u \in \mathfrak{H}$:

$$|\big((\mathcal{Q}V\mathcal{P} + \mathcal{P}V\mathcal{Q})u, u\big)| \leq 2\|V\|\|\mathcal{Q}u\| \; \|\mathcal{P}u\| \leq \delta\|\mathcal{P}u\|^2 + \delta^{-1}\|V\|^2\|\mathcal{Q}u\|^2.$$

Consequently,

$$H \leq \mathcal{P}(H_1 + \delta)\mathcal{P} + \mathcal{Q}(H_0 + \|V\| + \delta^{-1}\|V\|^2)\mathcal{Q},$$
$$H \geq \mathcal{P}(H_1 - \delta)\mathcal{P} + \mathcal{Q}(H_0 - \|V\| - \delta^{-1}\|V\|^2)\mathcal{Q}.$$

Denote

$$H^{\pm} = \mathcal{Q}(H_0 \pm \|V\| \pm \delta^{-1}\|V\|^2)\mathcal{Q},$$

so that

$$\begin{cases} n(\lambda; H) \leq n(\lambda; H_1 - \delta, S) + n(\lambda; H^-, K), \\ n(\lambda; H) \geq n(\lambda; H_1 + \delta, S) + n(\lambda; H^+, K). \end{cases} \tag{3.2}$$

Obviously,

$$n(\lambda; H^{\pm}, K) = n(\lambda \mp \|V\| \mp \delta^{-1}\|V\|^2; \mathcal{Q}H_0\mathcal{Q}, K). \tag{3.3}$$

By (2.6), (1.6), and definition of $\mathcal{P}$, the spectrum of the operator $\mathcal{Q}H_0\mathcal{Q}$ on the subspace $\mathfrak{H}(\mathbf{k}; K)$ lies outside the interval $(\lambda - A, \lambda + A)$. Consequently, by an elementary perturbation-theoretic argument, for a sufficiently large $A_0 = A_0(\delta, V)$ the r.h.s. of (3.3) coincides with

$$n(\lambda; \mathcal{Q}H_0\mathcal{Q}, K) = n(\lambda \pm \delta; H_0, K),$$

for all $A \geq A_0$. Therefore (3.2) can be rewritten as follows:

$$n(\lambda; H) \leq n(\lambda + \delta; H_1, S) + n(\lambda + \delta; H_0, K) = n(\lambda + \delta; H_1),$$
$$n(\lambda; H) \geq n(\lambda - \delta; H_1, S) + n(\lambda - \delta; H_0, K) = n(\lambda - \delta; H_1),$$

as required. $\square$

Denote

$$\gamma_+ = \max_{\boldsymbol{\theta} \in \Theta} |\boldsymbol{\theta}|, \quad \gamma_- = \min_{\boldsymbol{\theta} \in \Gamma^\dagger} |\boldsymbol{\theta}|. \tag{3.4}$$

Assume that the positive constants $A$, $B$ satisfy

$$A > \gamma_+^2, \quad 4\gamma_-^{-2}A^2 + A < B, \tag{3.5}$$

so that (2.4) is satisfied for all $\boldsymbol{\theta} \in \Theta$ simultaneously.

The final step of the reduction is made in the following Lemma.

**Lemma 3.2.** *Let $V$ be as in Lemma 3.1 Let $A, B$ be positive constants satisfying (3.5). For all sufficiently large $\lambda$ one has*

$$H_1(\mathbf{k}) = H_0(\mathbf{k}) + \sum_{\boldsymbol\theta\in\Theta} \mathcal{P}\big(\mathbf{k}; \Omega_{\boldsymbol\theta}(\lambda)\big) V_{\boldsymbol\theta} \mathcal{P}\big(\mathbf{k}; \Omega_{\boldsymbol\theta}(\lambda)\big). \tag{3.6}$$

*Proof.* Due to relation (2.2)

$$H_1(\mathbf{k}) = H_0(\mathbf{k}) + \sum_{\boldsymbol\theta\in\Theta} \mathcal{P}(\mathbf{k}; S(\lambda)) V_{\boldsymbol\theta} \mathcal{P}(\mathbf{k}; S(\lambda)),$$

and hence, by definitions (2.7), (2.8) it suffices to check that

$$\mathcal{P}(\mathbf{k}; S(\lambda)) V_{\boldsymbol\theta} \mathcal{P}(\mathbf{k}; \Lambda_{\boldsymbol\theta}(\lambda)) = \mathcal{P}(\mathbf{k}; \Lambda_{\boldsymbol\theta}(\lambda)) V_{\boldsymbol\theta} \mathcal{P}(\mathbf{k}; S(\lambda)) = 0,$$

which is the same as checking the relation

$$\mathcal{P}(\mathbf{k}; S(\lambda)) V_{\pm\boldsymbol\theta} \mathcal{P}(\mathbf{k}; \Lambda_{\boldsymbol\theta}(\lambda)) = 0, \quad \theta \in \Theta.$$

We need to prove only that

$$V_{\pm\boldsymbol\theta} \mathbf{e}_{\boldsymbol\gamma} \notin \mathfrak{H}(\mathbf{k}; S(\lambda)), \tag{3.7}$$

if $\boldsymbol\gamma + \mathbf{k} \in S(\lambda)$, see (3.1) for definition of $\mathbf{e}_{\boldsymbol\gamma}$. By definition of $V_{\boldsymbol\theta}$, see (2.2),

$$V_{\pm\boldsymbol\theta} \mathbf{e}_{\boldsymbol\gamma} = v_{\pm\boldsymbol\theta} \mathbf{e}_{\boldsymbol\gamma\pm\boldsymbol\theta}.$$

In view of (3.5) the parameters $A$, $B$ satisfy (2.4) for all $\boldsymbol\theta \in \Theta$. Consequently, by Lemma 2.3 the vectors $\boldsymbol\gamma + \mathbf{k} \pm \boldsymbol\theta$ do not belong to $S(\lambda)$, that is $\mathbf{e}_{\boldsymbol\gamma\pm\boldsymbol\theta} \notin S(\lambda; \mathbf{k})$ and hence (3.7) is satisfied. $\square$

*Proof of Theorem 2.1.* Recall that it suffices to prove the Theorem for a trigonometric polynomial of the form (2.2). Let $\delta > 0$ be fixed, and let $A_0 = A_0(\delta)$ and $\lambda_0(\delta)$ be as found in Lemma 3.1. Let $A \ge A_0$, $B > 0$ be arbitrary numbers such that (3.5) is satisfied. Since the set $\Theta$ is finite, by Lemma 3.2, the operator $H_1(\mathbf{k})$ is a finite-dimensional perturbation of $H_0(\mathbf{k})$, and in view of Lemma 2.2 one has

$$\dim\big(H_1(\mathbf{k}) - H_0(\mathbf{k})\big) \le \sum_{\boldsymbol\theta\in\Theta} \dim \mathcal{P}(\mathbf{k}; \Omega_{\boldsymbol\theta}(\lambda)) \le C\lambda^{\varkappa_d}, \quad C = C(\delta).$$

Therefore, by Lemma 3.1, we can write the estimate

$$n(\lambda - \delta; H_0(\mathbf{k})) - C\lambda^{\varkappa_d} \le n(\lambda; H(\mathbf{k})) \le n(\lambda + \delta; H_0(\mathbf{k})) + C\lambda^{\varkappa_d},$$

which follows by an elementary perturbation theory. Now (2.1) follows by (1.6). $\square$

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