PDMI PREPRINT — 05/2004

SOME REMARKS ON IMPEDANCE TOMOGRAPHY PROBLEM FOR 3d-MANIFOLDS

M. I. Belishev

Russian Academy of Sciences
Steklov Mathematical Institute

191011, St. Petersburg, Fontanka, 27

belishev@pdmi.ras.ru
April 25, 2004

ABSTRACT

We derive the formulas expressing the topological characteristics (Betti numbers) of 3d-manifold with boundary through its Dirichlet-to-Neumann maps associated with scalar and vector harmonic fields.

Supported by the RFBR grants 02–01–00260 and NS–2261.2003.1

0 Introduction

0.1. About the paper. As was shown by Lassas and Uhlmann [3], a smooth two-dimensional compact orientable Riemannian manifold is determined by its Dirichlet-to-Neumann (DN) map up to conformal equivalence. In [1], this result was obtained by another technique (the BC-method); in the same paper a simple formula linking the Euler characteristic of the manifold to its DN map has been derived.

Here the 3d-analogs of this formula are presented; namely, we express the dimensions of the Dirichlet and Neumann subspaces of harmonic vector fields (the Betti numbers) in terms of the scalar and vector DN maps (see (3.11), (3.15)). In contrast to the 2d-case these dimensions do not determine the topology of the manifold but, nevertheless, give a substantive information it on. The background of our 3d-formulas is the Friedrichs decomposition of the space of harmonic vector fields [6].

If a manifold of any dimension ≥ 3 is real analytic it is determined by its scalar DN map up to isometry [4]; so, roughly speaking, this map determines everything including the Betti numbers. However, for finding them by [4] one needs at first to recover the manifold, i.e., to solve the inverse problem whereas our formulas express the Betti numbers through the inverse data directly. One more point is that analyticity is not welcomed in this kind of problems, and the most interesting and challenging question of reconstruction in the general (nonanalytic) case remains open.

1 Vector analysis

1.1. Operations in Ω. In Sec. 1, we recall some of the definitions of vector analysis (see [6, Chapter 3]).

Let Ω be a smooth[1] compact orientable Riemannian 3d-manifold with connected boundary Γ, g the metric tensor, μ the volume 3-form.

For a (vector) field a given in Ω one defines a conjugate 1-form $a_\sharp$ by $a_\sharp(b) = g(a, b)$; for an 1-form ω a conjugate field $\omega^\sharp$ is defined by $g(\omega^\sharp, b) = \omega(b)$.

The scalar product "$\cdot$": $\{\text{fields}\} \times \{\text{fields}\} \to \{\text{functions}\}$ is defined pointwise by $a \cdot b = g(a, b)$. The vector product $\times$: $\{\text{fields}\} \times \{\text{fields}\} \to \{\text{fields}\}$ is defined pointwise by $g(a \times b, c) = \mu\,(a, b, c)$.

The gradient ∇: $\{\text{functions}\} \to \{\text{fields}\}$ and the divergence div: $\{\text{fields}\} \to \{\text{functions}\}$ are defined in a standard way (see, e.g., [6]).

The curl is defined as a map curl: $\{\text{fields}\} \to \{\text{fields}\}$, $\operatorname{curl} a = (\star\, \mathrm{d}\, a_\sharp)^\sharp$, where d is the exterior derivative and $\star$ is the Hodge operator.

Recall the basic identities $\operatorname{div}\operatorname{curl} = 0$ and $\operatorname{curl}\nabla = 0$.

The scalar Laplacian Δ: $\{\text{functions}\} \to \{\text{functions}\}$ is $\Delta := \operatorname{div}\nabla$. The vector Laplacian $\vec{\Delta}$: $\{\text{fields}\} \to \{\text{fields}\}$ is $\vec{\Delta} := \nabla \operatorname{div} - \operatorname{curl}\operatorname{curl}$.

Let ν be the outward unit normal on Γ, μ_Γ the (induced) surface form on Γ; recall the Green formulas

$$\int_\Omega \operatorname{div} a\, u\, \mu = \int_\Gamma a \cdot \nu\, u\, \mu_\Gamma - \int_\Omega a \cdot \nabla u\, \mu \tag{1.1}$$

and

$$\int_\Omega \operatorname{curl} a\, \cdot\, b\, \mu = \int_\Gamma \nu \times a\, \cdot\, b\, \mu_\Gamma + \int_\Omega a \cdot \operatorname{curl} b\, \mu. \tag{1.2}$$

The surface integral in (1.2) may be written in the form $\int_\Gamma \mu\,(\nu, a, b)\, \mu_\Gamma$.

[1] Everywhere in the paper "smooth" means "C^∞-smooth up to the boundary."

1.2. Operations on Γ. Each field a may be represented at the boundary as

$$a \,=\, a_\theta \,+\, a\cdot\nu\,\nu,$$

where $a_\theta := a - a\cdot\nu\,\nu$ is a tangential component, $a_\theta \perp \nu$.

Considering Γ as a Riemannian manifold equipped with the metric $g\,|_\Gamma$ and the volume element μ_Γ by ∇_Γ and $\operatorname{div}_\Gamma$ we shall mean the corresponding gradient and divergence. Note the equality[2]

$$(\nabla\, u)_\theta \,=\, \nabla_\Gamma\,(u\,|_\Gamma) \quad \text{on} \quad \Gamma;$$

recall the well-known relation

$$\operatorname{curl} a\cdot\nu \,=\, -\operatorname{div}_\Gamma \nu\times a \quad \text{on} \quad \Gamma \tag{1.3}$$

and the identity

$$\operatorname{div}_\Gamma \nu\times\nabla_\Gamma \,=\, 0. \tag{1.4}$$

In what follows for a class A of functions on Γ we denote $\dot{A} := \{\, f \,\in\, A \mid \int\limits_\Gamma f\mu_\Gamma \,=\, 0\}$; $H^s(\dots)$ and $\vec{H}^s(\dots)$ are the Sobolev classes of functions and fields. The class of potential fields $\mathcal{P}_\Gamma\colon\, = \{\nabla_\Gamma f \mid f\in H^1(\Gamma)\}$ is considered as a subspace of $\vec{L}_2(\Gamma)$.

Introduce the "integration operators" $J\colon \mathcal{P}_\Gamma \to \dot{L}_2(\Gamma)$ defined by

$$J\,\nabla_\Gamma\, f \,=\, f, \qquad f\in\dot{H}^1(\Gamma) \tag{1.5}$$

and $\vec{J}\colon\, \dot{L}_2(\Gamma) \,\to \mathcal{P}_\Gamma$ defined by

$$\vec{J}\operatorname{div}_\Gamma j \,=\, -\,j, \quad j\in\mathcal{P}_\Gamma\cap\vec{H}^1(\Gamma). \tag{1.6}$$

As is easy to see, both of the operators are injective and compact; the relations

$$\operatorname{Ran} J \,=\, \dot{H}^1(\Gamma); \quad \operatorname{Ran}\vec{J} \,=\, \mathcal{P}_\Gamma\cap\vec{H}^1(\Gamma); \quad J^* \,=\, \vec{J};$$

$$J^{-1} \,=\, \nabla_\Gamma; \quad \vec{J}^{-1} \,=\, -\operatorname{div}_\Gamma$$

hold.

2 Helmholtz type decompositions

2.1. Decomposition of $\vec{L}_2(\Omega)$. Here we also recall the well–known facts (see [6]).

The space of the vector fields $\vec{L}_2(\Omega)$ with the inner product

$$(\,a,\,b\,)_{\vec{L}_2(\Omega)} \,=\, \int\limits_\Omega a\cdot b\;\mu$$

may be represented in the form of an orthogonal sum:

$$\vec{L}_2(\Omega) \,=\, \mathcal{P}^0 \,\oplus\, \mathcal{S} \tag{2.1}$$

[2] Here and below, for a vector field $a(\cdot)$ and a point $\gamma\in\Gamma$ we identify the tangential component $a[(\gamma)]_\theta$ to the corresponding element of $T_\gamma\Gamma$.

of the subspace of potential fields

$$\mathcal{P}^0 := \{\nabla p \,|\, p \in H^1(\Omega),\, p|_\Gamma = 0\}$$

and the subspace of solenoidal fields

$$\mathcal{S} := \{\, s \in \vec{L}_2(\Omega) \,|\, \operatorname{div} s = 0\,\}$$

that is the classical Helmholtz decomposition.

The Hodge–Morrey decomposition detalizes the second summand in (2.1):

$$\mathcal{S} = \mathcal{H} \oplus \mathcal{C}^0$$

where

$$\mathcal{H} := \{\, a \in \vec{L}_2(\Omega) \,|\, \operatorname{div} a = 0, \;\; \operatorname{curl} a = 0\,\}$$

is the subspace of *harmonic fields* whereas

$$\mathcal{C}^0 := \{\, \operatorname{curl} b \,|\, b \in \vec{H}^1(\Omega), \;\; b_\theta = 0\,\}$$

is the subspace of curls. So, (2.1) takes the form

$$\vec{L}_2(\Omega) = \mathcal{P}^0 \oplus \mathcal{H} \oplus \mathcal{C}^0. \tag{2.2}$$

In what follows, for a class of fields $\mathcal{A}$ we denote by $\mathcal{A}^\infty$ the (sub)class of the smooth elements of $\mathcal{A}$. As is well-known, the classes $\mathcal{P}^{0\,\infty}$, $\mathcal{S}^\infty$, $\mathcal{H}^\infty$, $\mathcal{C}^{0\,\infty}$ are dense in the corresponding subspaces.

2.2. Harmonic fields. The second summand in (2.2) is of particular interest and here we list some of the properties of its elements (see, e.g., [6]).

(i) Harmonic fields are smooth into Ω: $\mathcal{H} \subset \vec{C}^\infty_{\rm loc}(\Omega)$; recall that $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

(ii) A harmonic field possesses a trace at the boundary: the map $\operatorname{tr} : a \mapsto a|_\Gamma$ acts continuously from $\mathcal{H}$ to $\vec{H}^{-\frac{1}{2}}(\Gamma)$. By the well-known uniqueness theorem this map is injective, i.e., a harmonic field is determined by its trace.

(iii) The subspace $\mathcal{H}$ may be represented in the form of the summ

$$\mathcal{H} = \mathcal{C} \oplus \mathcal{D} \tag{2.3}$$

of the subspace of curls

$$\mathcal{C} := \{a \in \mathcal{H} \,|\, a = \operatorname{curl} h\}$$

and the subspace of the Dirichlet fields

$$\mathcal{D} := \{d \in \mathcal{H} \,|\, \nu \times d = 0\},$$

or as the summ

$$\mathcal{H} = \mathcal{G} \oplus \mathcal{N} \tag{2.4}$$

of the subspace of gradients

$$\mathcal{G} := \{a \in \mathcal{H} \,|\, a = \nabla u\}$$

and the subspace of the Neumann fields

$$\mathcal{N} := \{n \in \mathcal{H} \,|\, n \cdot \nu = 0\}.$$

Representations (2.3), (2.4) are known as the Friedrichs decompositions. The smooth classes $\mathcal{C}^\infty$ and $\mathcal{G}^\infty$ are dense in $\mathcal{C}$ and $\mathcal{G}$.

The elements of the Dirichlet and Neumann subspaces are smooth; their dimensions (the Betti numbers of the manifold Ω) $\beta_1 = \dim \mathcal{N}$, $\beta_2 = \dim \mathcal{D}$ are finite and determined by topology of Ω (see [6]). Recall that the boundary Γ is assumed connected; in this case the following holds.

Lemma 1. *The inequality* $\dim \mathcal{N} \ge \dim \mathcal{D}$ *is valid.*

Proof. To prove this inequality is to show that $d \in \mathcal{D}$ and $d \perp N$ implies $d = 0$.

Since $d \perp \mathcal{N}$, by virtue of (2.4) one has $d = \nabla u$; hence $\Delta u = \operatorname{div} d = 0$ in Ω. At the same time, $\nu \times \nabla u = \nu \times d = 0$ so that u is a harmonic function inΩ whereas ∇u is parallel to ν on Γ. As Γ is connected the last yields $u|_\Gamma = \text{const}$. Therefore $u = \text{const}$ in Ω and $d = \nabla u = 0$. $\square$

The next lemma specifies a "positional relationship" of the subspaces occuring in the Friedrichs decompositions. We denote by $P_{\mathcal{A}}$ the orthogonal projection in $\vec{L}_2(\Omega)$ on a subspace $\mathcal{A}$.

Lemma 2. *The relations*

$$1)\ \mathcal{D} \cap \mathcal{N} = \{0\}; \qquad 2)\ \operatorname{clos} P_{\mathcal{G}}\mathcal{C} = \mathcal{C}; \qquad 3)\ P_{\mathcal{N}}\mathcal{C} = \mathcal{N};$$
$$4)\ P_{\mathcal{D}}\mathcal{G} = \mathcal{D}; \qquad 5)\ \dim P_{\mathcal{N}}\mathcal{D} = \dim \mathcal{D} \tag{2.5}$$

are valid.

Proof.

1) If $a \in \mathcal{D} \cap \mathcal{N}$, then $a \times \nu = 0$ and $a \cdot \nu = 0$ on Γ, i.e., $\operatorname{tr} a = 0$. By injectivity of tr we get $a = 0$.

2) If $a \in \mathcal{G} \ominus P_{\mathcal{G}}\mathcal{C}$, then $a = \nabla u$ and $\nabla u \perp \mathcal{C}$. By (2.3) the latter leads to $\nabla u \in \mathcal{D}$ yielding $\nu \times \nabla u = 0$ in Ω and, since Γ is connected, $u|_\Gamma = \text{const}$. Hence $u = \text{const}$ and $a = \nabla u = 0$.

3) If $n \in \mathcal{N} \ominus P_{\mathcal{N}}\mathcal{C}$, then $n \perp \mathcal{C}$ and, by virtue of (2.3), one has $n \in \mathcal{D}$. So, $n \in \mathcal{N} \cap \mathcal{D}$ and we get $n = 0$.

4) If $d \in \mathcal{D} \ominus P_{\mathcal{D}}\mathcal{G}$, then $d \perp \mathcal{G}$ and, by virtue of (2.4), one has $d \in \mathcal{N}$. So, $d \in \mathcal{D} \cap \mathcal{N}$ and we get $d = 0$.

5) If $\dim P_{\mathcal{N}}\mathcal{D} < \dim \mathcal{D}$, there exists a nonzero $d \in \mathcal{D}$ orthogonal to $\mathcal{N}$. By (2.4) this orthogonality implies $d \in \mathcal{G}$, i.e., $g = \nabla u$. Therefore $\Delta u = 0$ in Ω and ∇u is parallel to ν on Γ. As Γ is connected this yields $u = \text{const}$ and $d = \nabla u = 0$. Thus, the inequality of the dimensions leads to a contradiction. $\square$

Note in addition that in a possible case of $\dim \mathcal{N} > \dim \mathcal{D}$ one has $\operatorname{clos} P_{\mathcal{C}}\mathcal{G} \ne \mathcal{C}$.

2.3. Decompositions on Γ. The space of vector fields $\vec{L}_2(\Gamma)$ contains the subspace of potential fields

$$\mathcal{P}_\Gamma := \{\nabla_\Gamma f \,|\, f \in H^1(\Gamma)\},$$

the subspace of solenoidal fields

$$\mathcal{S}_\Gamma := \{\sigma \in \vec{L}_2(\Gamma) \,|\, \operatorname{div}_\Gamma \sigma = 0\},$$

the subspace

$$\mathcal{P}_\Gamma^\nu := \{\nu \times \nabla_\Gamma f \,|\, f \in H^1(\Gamma)\},$$

and the harmonic subspace

$$\mathcal{H}_\Gamma := \{\eta \in \vec{L}_2(\Gamma) \,|\, \operatorname{div}_\Gamma \eta = 0,\ \operatorname{div}_\Gamma \nu \times \eta = 0\}.$$

The identity (1.4) obviously implies $\mathcal{P}_\Gamma^\nu \subset \mathcal{S}_\Gamma$. The subspace $\mathcal{H}_\Gamma$ is of finite dimension determined by topology of Γ. The smooth classes $\mathcal{P}_\Gamma^\infty$ and $\mathcal{S}_\Gamma^\infty$ are dense in the corresponding subspaces whereas $\mathcal{H}_\Gamma \subset \vec{C}^\infty(\Gamma)$.

The Helmholtz and Hodge–Morrey decompositions on Γ are of the well-known form

$$\vec{L}_2(\Gamma) = \mathcal{P}_\Gamma \oplus \mathcal{S}_\Gamma = \mathcal{P}_\Gamma \oplus \mathcal{H}_\Gamma \oplus \mathcal{P}_\Gamma^\nu.$$

3. Formulas

3.1. Electric DN map. Let $u = u^f(x)$ be a solution to the problem of electrostatics

$$\Delta u = 0 \qquad in \quad \operatorname{int}\Omega; \tag{3.1}$$

$$u = f \qquad on \quad \Gamma \tag{3.2}$$

with a smooth function f. With this problem one associates the DN map $\Lambda : L_2(\Gamma) \to L_2(\Gamma)$, $\operatorname{Dom}\Lambda = C^\infty(\Gamma)$,

$$\Lambda f := \frac{\partial u^f}{\partial \nu} = \nabla u^f \cdot \nu \qquad on \quad \Gamma;$$

recall some of its properties.

(i) The relations $\operatorname{Ker}\Lambda = \text{const}\ \operatorname{Ran}\Lambda = \dot{C}^\infty(\Gamma)$ hold.

(ii) Integration by parts gives

$$\int_\Omega \nabla u^{f'} \cdot \nabla u^{f''}\ \mu = \langle \text{see } (1.1), (3.1), (3.2) \rangle = \int_\Gamma \Lambda f'\ f''\ \mu_\Gamma$$

and shows that Λ is a nonnegative operator. The well-known fact is that Λ is an elliptic first order pseudodifferential operator.

(iii) Recall that the integrations J and $\vec{J}$ have been defined in Sec. 1.2. Introduce a transform $H : \mathcal{P}_\Gamma \to \dot{L}_2(\Gamma)$, $\operatorname{Dom} H = \mathcal{P}_\Gamma^\infty$,

$$H := \Lambda J.$$

It is not difficult to check that $\operatorname{Ker} H = \{0\}$ and $\operatorname{Ran}\Lambda = \dot{C}^\infty(\Gamma)$. By standard arguments of elliptic theory H turns out to be a bounded and boundedly invertible operator with the inverse $H^{-1} = \langle see\ (1.7)\rangle = \nabla_\Gamma \Lambda^{-1}$.

Given one of the operators Λ or H one can characterize the traces of harmonic gradients as follows. By virtue of the obvious $\mathcal{G}^\infty = \{\nabla u^f \,|\, f \in \dot{C}^\infty(\Gamma)\}$ we have

$$\operatorname{tr}\mathcal{G}^\infty = \{\nabla_\Gamma f + (\Lambda f)\nu \,|\, f \in \dot{C}^\infty(\Gamma)\} = \{\nabla_\Gamma f + (H\nabla_\Gamma f)\nu \,|\, \nabla_\Gamma f \in \mathcal{P}_\Gamma^\infty\}. \tag{3.3}$$

3.2. Magnetic DN map. The problem of magnetostatics is of the form

$$\vec{\Delta} h = 0, \quad \operatorname{div} h = 0 \quad \text{in} \quad \operatorname{int}\Omega; \tag{3.4}$$

$$-\nu \times h = j \qquad on \quad \Gamma \tag{3.5}$$

with $j \in \vec{C}^\infty(\Gamma)$. This problem is solvable but not uniquely: a field h is a solution to (3.4), (3.5) with $j = 0$ iff $h \in \mathcal{D}$ ([6, Lemma 3.5.6]). In what follows we denote by h^j the (unique) solution satisfying $h^j \perp \mathcal{D}$.

With the problem (3.4), (3.5) one associates the DN map $\vec{\Lambda} : \vec{L}_2(\Gamma) \to \vec{L}_2(\Gamma)$, $\operatorname{Dom}\vec{\Lambda} = \vec{C}^\infty(\Gamma)$,

$$\vec{\Lambda} j := (\operatorname{curl} h^j)_\theta \qquad on \quad \Gamma;$$

let us list some of its properties.

(i) The relations

$$\int_\Omega \operatorname{curl} h^{j'} \cdot \operatorname{curl} h^{j''}\ \mu = \langle \text{see } (1.2), (3.4), (3.5) \rangle = \int_\Gamma \vec{\Lambda} j' \cdot j''\ \mu_\Gamma$$

show that the magnetic DN map is a nonnegative operator.

(ii) Recall that the solenoidal subspace $\mathcal{S}_\Gamma$ was introduced in Sec. 2.3.

Lemma 3. *The relations*

$$\operatorname{Ker}\vec{\Lambda} \subset \mathcal{S}_\Gamma, \quad \operatorname{Ran}\vec{\Lambda} \supset \mathcal{P}_\Gamma^\infty \tag{3.6}$$

hold.

Proof. If $j \in \operatorname{Ker}\vec{\Lambda}$, then

$$0 = \int\limits_\Gamma \vec{\Lambda} j \cdot j \;\; \mu_\Gamma = \langle \text{see}\,(i) \rangle = \int\limits_\Omega |\operatorname{curl} h^j|^2 \;\; \mu;$$

hence, $\operatorname{curl} h^j = 0$ in Ω. Therefore, at the boundary one has

$$0 = \nu \cdot \operatorname{curl} h^j = \langle \text{see (1.3), (3.5)} \rangle = \operatorname{div}_\Gamma j,$$

so that $j \in \operatorname{Ker}\vec{\Lambda}$ leads to $j \in \mathcal{S}_\Gamma$.

Take an arbitrary $f \in C^\infty(\Gamma)$ and represent

$$\nabla u^f = \langle \text{see (2.3)} \rangle = \operatorname{curl} h + d$$

with a smooth $h \perp \mathcal{D}$, $\operatorname{div} h = 0$ and $d \in \mathcal{D}$. Passing to the traces on Γ one gets

$$(\nabla u^f)_\theta = \nabla_\Gamma f = (\operatorname{curl} h)_\theta = \vec{\Lambda} j$$

where $j = -\nu \times \operatorname{tr} h$. So, $\operatorname{Ran}\vec{\Lambda}$ covers the class $\mathcal{P}_\Gamma^\infty$ and we arrive at (3.6). $\square$

We omit the proof of the following relation specifying a structure of $\operatorname{Ran}\vec{\Lambda}$:

$$\operatorname{Ran}\vec{\Lambda} = \mathcal{P}_\Gamma^\infty \dot{+} \operatorname{tr}\{\mathcal{N} \ominus P_\mathcal{N}\mathcal{D}\}.$$

By virtue of (2.2), 5) this representation easily leads to an interesting equality

$$\dim \left\{ \operatorname{Ran}\vec{\Lambda} \,/\, \mathcal{P}_\Gamma^\infty \right\} = \dim \mathcal{N} - \dim \mathcal{D}$$

which, nevertheless, is not too rich in content for tomography: as may be shown, the difference $\dim\mathcal{N} - \dim\mathcal{D}$ is determined by topology of Γ.

Note in addition that relations (3.6) provide the composition $\operatorname{div}_\Gamma \vec{\Lambda}^{-1} \nabla_\Gamma$ to be well-defined on $C^\infty(\Gamma)$.

(iii) Introduce a transform $\vec{H} : \dot{L}_2(\Gamma) \to \vec{L}_2(\Gamma), \operatorname{Dom}\vec{H} = \dot{C}_\Gamma^\infty$,

$$\vec{H} := \vec{\Lambda}\vec{J}.$$

As is easy to see, this transform is injective; by arguments of elliptic theory $\vec{H}$ is a bounded and boundedly invertible operator with the inverse $\vec{H}^{-1} = \langle \text{see (1.7)} \rangle = -\operatorname{div}_\Gamma \vec{\Lambda}^{-1}$.

Given one of the operators $\vec{\Lambda}$ or $\vec{H}$ we can characterize the traces of harmonic curls. By virtue of the obvious $\mathcal{C}^\infty = \{\operatorname{curl} h^j \,|\, j \in \mathcal{P}_\Gamma^\infty\}$ and $\dot{C}^\infty = \{\operatorname{div}_\Gamma j \,|\, j \in \mathcal{P}_\Gamma^\infty\}$ one has

$$\operatorname{tr}\mathcal{C}^\infty = \{ \vec{\Lambda} j + (\operatorname{div}_\Gamma j)\,\nu \,|\, j \in \mathcal{P}_\Gamma^\infty \} = \langle \text{see (1.7)} \rangle =$$

$$= \{ -\vec{H} \operatorname{div}_\Gamma j + (\operatorname{div}_\Gamma j)\,\nu \,|\, \operatorname{div}_\Gamma j \in \dot{C}^\infty(\Gamma)\}. \tag{3.7}$$

First formula. Here we express $\beta_1 = \dim \mathcal{N}$ in terms of the DN maps.

Theorem 1. *The representations*

$$\operatorname{tr} \mathcal{N} \;=\; [\,\vec{\Lambda} \,-\, \nabla_\Gamma\, \Lambda^{-1} \operatorname{div}_\Gamma\,]\; \mathcal{P}_\Gamma^\infty \;=\; [\,\vec{H} \,+\, H^{-1}\,]\; \dot{C}^\infty(\Gamma) \tag{3.8}$$

are valid.

Proof. Take $j \in \mathcal{P}_\Gamma^\infty$ and decompose $\operatorname{curl} h^j$ by (2.4):

$$\operatorname{curl} h^j \;=\; \nabla u \,+\, n.$$

As j runs over $\mathcal{P}_\Gamma^\infty$ the left hand side runs over $\mathcal{C}^\infty$ whereas the summands n cover the subspace $\mathcal{N}$ by virtue of (2.5), 3).

Passing to the traces and separating tangential and normal components one has

$$(\operatorname{curl} h^j\,)_\theta \;=\; \vec{\Lambda}\, j \;=\; \nabla_\Gamma\, f \,+\, \operatorname{tr} n \tag{3.9}$$

where $f = u\,|_\Gamma \in \dot{C}^\infty(\Gamma)$ and

$$\nu \,\cdot\, \operatorname{curl} h^j \;=\; \nu \,\cdot\, \nabla\, u,$$

the last being equivalent to $\operatorname{div}_\Gamma j \;=\; \Lambda\, f$ or, the same, to

$$f \;=\; \Lambda^{-1} \operatorname{div}_\Gamma j. \tag{3.10}$$

Substituting (3.10) in (3.9) one obtains

$$\operatorname{tr} n \;=\; \vec{\Lambda}\, j \;-\; \nabla_\Gamma\, \Lambda^{-1} \operatorname{div}_\Gamma j\,, \quad j \in \mathcal{P}_\Gamma^\infty$$

that is equivalent to the first of the relations (3.8).

Returning to (3.9), the term $\vec{\Lambda}\, j$ may be written in the form

$$\vec{\Lambda}\, j \;=\; \langle \text{see (1.6)}\,\rangle \;=\; -\,\vec{\Lambda}\, \vec{J} \operatorname{div}_\Gamma j \;=\; -\,\vec{H} \operatorname{div}_\Gamma j$$

whereas (3.10) yields

$$\nabla_\Gamma\, f \;=\; \nabla_\Gamma\, \Lambda^{-1} \operatorname{div}_\Gamma j \;=\; \langle \text{see (1.5)} \rangle \;=\; H^{-1} \operatorname{div}_\Gamma j.$$

Thereafter we can write (3.9) as

$$\operatorname{tr} n \;=\; -[\,\vec{H} \,+\, H^{-1}\,] \operatorname{div}_\Gamma j\,, \quad j \in \mathcal{P}_\Gamma^\infty$$

that leads to the second equality in (3.8) by virtue of the evident $\{\operatorname{div}_\Gamma j \,|\, j \in \mathcal{P}_\Gamma^\infty\} = \dot{C}^\infty(\Gamma)$. $\square$

By the injectivity of tr one has $\dim \operatorname{tr} \mathcal{N} \;=\; \dim \mathcal{N}$; hence, (3.8) leads to

$$\dim \mathcal{N} \;=\; \dim\,[\,\vec{\Lambda} \,-\, \nabla_\Gamma\, \Lambda^{-1} \operatorname{div}_\Gamma\,]\; \mathcal{P}_\Gamma^\infty \;=\; \dim\,[\,\vec{H} \,+\, H^{-1}\,]\; \dot{C}^\infty(\Gamma) \tag{3.11}$$

that is the formula declared.

Second formula. Here we find $\beta_2 = \dim \mathcal{D}$ from the DN maps. Recall the remark made at the last paragraph of (ii), Sec. 3.2.

Theorem 2. *The representations*

$$\operatorname{tr}\mathcal{D} \,=\, \{\,[\,\Lambda \,-\, \operatorname{div}_\Gamma \vec{\Lambda}^{-1}\,\nabla_\Gamma\,]\ \dot{C}^\infty(\Gamma)\,\}\,\nu \,=\, \{\,[\,H \,+\, \vec{H}^{-1}\,]\ \mathcal{P}^\infty_\Gamma\,\}\,\nu \tag{3.12}$$

are valid.

Proof. Take $f \in \dot{C}^\infty(\Gamma)$ and decompose ∇u^f by (2.3):

$$\nabla u^f \,=\, \operatorname{curl} h \,+\, d.$$

As f runs over $\dot{C}^\infty(\Gamma)$ the left hand side runs over $\mathcal{G}^\infty$ whereas the summands d cover the subspace $\mathcal{D}$ by virtue of (2.5), 4).

Passing to the traces and separating normal and tangential components one has

$$\Lambda\, f \,=\, \operatorname{div}_\Gamma j \,+\, \nu\,\cdot\,d \tag{3.13}$$

where $j = -\nu \times h \in \vec{C}^\infty(\Gamma)$ and

$$\nabla_\Gamma\, f \,=\, \vec{\Lambda}\, j,$$

the last being equivalent to

$$\operatorname{div}_\Gamma j \,=\, \operatorname{div}_\Gamma \vec{\Lambda}^{-1}\,\nabla_\Gamma\, f. \tag{3.14}$$

Substituting (3.14) in (3.13) one obtains

$$\nu\,\cdot\,d = \Lambda\, f \,-\, \operatorname{div}_\Gamma \vec{\Lambda}^{-1}\,\nabla_\Gamma\, f\,, \quad f \in \dot{C}^\infty(\Gamma)$$

that is equivalent to the first of the relations (3.12).

Returning to (3.13), the term $\Lambda\, f$ may be written in the form

$$\Lambda\, f \,=\, \langle\text{see } (1.5)\rangle \,=\, \Lambda\, J\,\nabla_\Gamma\, f \,=\, H\,\nabla_\Gamma\, f$$

whereas (3.14) yields

$$\operatorname{div}_\Gamma j \,=\, \langle\text{see } (1.6)\rangle \,=\, -\vec{H}^{-1}\,\nabla_\Gamma\, f.$$

Thereafter one can write (3.13) as

$$\nu\,\cdot\,d = [\,H \,+\, \vec{H}^{-1}\,]\,\nabla_\Gamma\, f\,, \quad f \in \dot{C}^\infty(\Gamma)$$

that leads to the second equality in (3.12). $\square$

By the injectivity of tr one has $\dim\operatorname{tr}\mathcal{D} \,=\, \dim\mathcal{D}$; hence, (3.12) leads to a formula for the 2nd Betti number:

$$\dim\mathcal{D} \,=\, \dim\,[\,\Lambda \,-\, \operatorname{div}_\Gamma \vec{\Lambda}^{-1}\,\nabla_\Gamma\,]\ \dot{C}^\infty(\Gamma) \,=\, \dim\,[\,H \,+\, \vec{H}^{-1}\,]\ \mathcal{P}^\infty_\Gamma. \tag{3.15}$$

3.3. Harmonic quaternion fields. In conclusion, let us clarify the role of the operators H, $\vec{H}$ as natural 2d-analogs of the classical Hilbert transform on the unique circle.

One of the equivalent ways of introducing $H_{\rm class}$ is the following. Let $w = u + {\rm i}\, u_*$ be a function analytic and smooth in the disc $D := \{ z \in \mathbb{C} \,|\, |z| \leq 1 \}$ (so that u and u_* are conjugated by Cauchy–Riemann: ${\rm d}\, u_* = \star {\rm d}\, u$), $T := \partial D$; γ the polar angle on T, $f := u\,|_T$, $f_* := u_*\,|_T$. The transform $H_{\rm class}$ maps $\frac{{\rm d} f}{{\rm d}\gamma}$ to $\frac{{\rm d} f_*}{{\rm d}\gamma}$ and may be represented as $H_{\rm class} = \Lambda J$ where Λ is the electric DN map of D, J is an integration: $\frac{\rm d}{{\rm d}\gamma} J = {\rm id}$ (see [1]).

Returning to a 3d-manifold Ω we say that a function u and a solenoidal field h are conjugated (and write $h = u_*$, $u = h_*$) if $\nabla u = {\rm curl}\, h$ in Ω or, equivalently, ${\rm d}\, h_* = \star {\rm d}\, h_\sharp$. This definition immediatelly implies $\Delta u = 0$ and ${\rm curl}\,{\rm curl}\, h = 0$, i.e., ∇u, ${\rm curl}\, h \in \mathcal{G} \cap \mathcal{C} \subset \mathcal{H}$.

If at least $\mathcal{N} \neq \{0\}$, so that Ω is of nontrivial topology, not each u satisfying $\Delta u = 0$ as well as not each h satisfying ${\rm curl}\,{\rm curl}\, h = 0$ has a conjugate. However, a remarkable fact is that the existence of the conjugates may be checked in terms of the traces of u and h through the operators H and $\vec{H}$. We omit the proof of the following result which is a very simple consequence of representations (3.3) and (3.7).

Theorem 3.

(i) *For $f \in \dot{C}^\infty(\Gamma)$ the function u^f has a conjugate $h^j = (u^f)_*$ iff*

$$[\,\mathbb{1} + \vec{H} H\,]\, \nabla_\Gamma f = 0;$$

in this case the equality

$${\rm div}_\Gamma\, j = H\, \nabla_\Gamma f$$

holds and determines $j \in \mathcal{P}_\Gamma^\infty$.

(ii) *For $j \in \mathcal{P}_\Gamma^\infty$ the field h^j has a conjugate $\nabla u^f = (h^j)_*$ iff*

$$[\,\mathbb{1} + H \vec{H}\,]\, {\rm div}_\Gamma\, j = 0;$$

in this case the equality

$$\nabla_\Gamma f = -\vec{H}\, {\rm div}_\Gamma\, j$$

holds and determines $f \in \dot{C}^\infty(\Gamma)$.

A pair $q = \{u, h\}$ may be considered as a "quaternion field" in Ω (see, e.g., [2]). We say q to be harmonic and assign it to the class $\mathcal{Q}$ if $h = u_*$. As we expect, it is the class $\mathcal{Q}$ which will play a key role in reconstruction of Ω through DN maps. Theorem 3 shows that the set ${\rm tr}\, \mathcal{Q}$ is determined by the operators Λ, $\vec{\Lambda}$ and may be explicitly characterized in terms of the Hilbert transforms H, $\vec{H}$. Namely, with regard to (3.3), (3.7) one easily has:

$${\rm tr}\, \mathcal{Q}^\infty = \left\{ \{J\nabla_\Gamma f,\ \nabla_\Gamma f + (H\nabla_\Gamma f)\,\nu\} \,|\, \nabla_\Gamma f \in {\rm Ker}\,[\mathbb{1} + \vec{H} H] \right\} =$$

$$= \left\{ \{-J\vec{H}\, {\rm div}_\Gamma\, j, -\vec{H}\, {\rm div}_\Gamma\, j + ({\rm div}_\Gamma\, j)\,\nu\} |\ {\rm div}_\Gamma\, j \in {\rm Ker}\,[\mathbb{1} + H\vec{H}] \right\}.$$

3.4. Comments.

- Recently L. N. Pestov has elaborated a "microlocal" version of $H_{\rm class}$ and applied it for solving the 2d kinematic inverse problem [5]. Perhaps, our H and $\vec{H}$ could be useful for a 3d-generalization (see also [2]).
- If Ω is homeomorphic to a ball in $\mathbf{R}^3$ one has $\dim \mathcal{N} = \dim \mathcal{D} = 0$, so that (3.11) yields $\vec{\Lambda} = \nabla_\Gamma \Lambda^{-1} {\rm div}_\Gamma$ and $\vec{H} = -H^{-1}$. An open question is whether the electric DN map determines the magnetic one (or conversely) in the general case.
- I'm obliged to G. M. Henkin for valuable discussions and consultations. I'd like to thank S. V. Belisheva for assistance in preparing the manuscript.

References

1. M. I. Belishev, "The Calderon problem for two-dimensional manifolds by the BC-method," *SIAM J. Math. Anal.*, **35** (1), 172–182 (2003).
2. J. E. Gilbert and M. A. M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambrige Studies Adv. Math., **26**, Cambridge Univ. Press (1991), 334 p.
3. M. Lassas and G. Uhlmann, "On determining the Riemann manifold from the Dirichlet-to-Neumann map," *Ann. Sci. Ecole Norm. Sup.*, **4**, 34, 771–787 (2001).
4. M. Lassas, M. Taylor, and G. Uhlmann, "The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundart," *Comm. Anal. Geom.*, **11**, No 2, 207–221 (2003).
5. L. N. Pestov, "Questions of well-posedness of the Ray tomography problems" [in Russian], *Sib. Nauchn. Izd.*, Novosibirsk (2003), 110 p.
6. G. Schwarz, "Hodge decomposition – a method for solving boundary value problems," *Lect. Notes Math.*, **1607**, Springer–Verlag, Berlin (1995).