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Chern and Thom elements in the representable cohomology theories

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Abstract

It is proved that to define Chern (or Thom) structure on a representable cohomology theory it is sufficient to define Chern (or Thom) element with the certain properties.

1 Introduction

Recent publications show an increasing interest in studying relations between algebraic topology and algebraic geometry. Several results concerning bijections between Chern structures, Thom structures, and push-forwards were announced in [PS] and proved in [Pa1].

According to a well known Universal Bundle Theorem in topology, to define a Chern structure on line bundles one can define it on the universal bundle. There are no similar results in algebraic geometry. Therefore different methods are required to define a Chern and Thom structures in the context of algebraic geometry.

Now we discuss our main result. Let $\mathcal{A}$ be a representable ring cohomology theory (for example, algebraic cobordism). We will show that to define a Chern structure, or Thom structure it is sufficient to define the corresponding element on the universal bundle. Actually all we need for our construction is to have an element $c \in \mathcal{A}(\mathbb{P}^\infty)$ with certain natural properties. As easy consequences of our construction we get the following uniqueness result: If two structures (Chern or Thom) coincide on the universal bundle then they coincides tautologically.

The paper consists of seven sections. In Sections 2 and 3, we recall some definitions and collect necessary results from [PS] and [Pa1]. In Section 4 we construct a homotopy category. Note that our definition of quasi-homotopy, which is convenient for the purposes of this paper, slightly differs from the definition used in [PS]. In section 5 we construct the isomorphism between the Picard group of X and the group of homotopical classes $[X, \mathbb{P}^\infty]$. This isomorphism allows us to transfer a Chern element from the universal bundle to an arbitrary line bundle. In Section 7 we deal with Thom elements and Thom structures. Our construction of the Thom structure is not direct. First, using a Thom element we find a Chern element, which as we have shown before, defines a Chern structure. The latter also defines a Thom structure since there exists a one-to-one correspondence between Chern structures and Thom structures.

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Let us recall the main notation using below.

Let k be a field. Everywhere below the term "variety" means a quasi-projective variety over k; Sm denotes the category of smooth varieties; $SmOp$ denotes the category of pairs (X, U) with smooth X and open U in X.

Since $\emptyset$ is open, we may identify the category of smooth varieties with a full subcategory $SmOp$ assigning the pair $(X, \emptyset)$ to a variety X.

Everywhere $\mathbb{P}^n$ denotes the projective space of the dimension n. Let V be a vector space, then $\mathbb{P}(V)$ is the space of lines in V.

Let E be a vector bundle over X, then $\mathbb{P}(E)$ denotes the space of lines in a vector bundle E. $\mathcal{O}_E(-1)$ denotes the tautological vector bundle on $\mathbb{P}(E)$, $\mathcal{O}_E(1)$ is the dual bundle to the $\mathcal{O}_E(-1)$.

Everywhere $\mathcal{O}_n(-1)$ denotes the tautological vector bundle on $\mathbb{P}^n$.

2 Ring cohomology theories

In Sections 2 and 3 we recall some definitions and results from [PS] and [Pa1].

In this paper we deal with ring cohomology theories only, so everywhere below *a cohomology theory* means *a ring cohomology theory.*

Definition 1. *A ring cohomology theory is a contravariant functor $\mathcal{A} : SmOp \to Ab$ together with a functor morphism $\partial : \mathcal{A}(U) \to \mathcal{A}(X,U)$ and bilinear $\cup : \mathcal{A}(X,U_1) \times \mathcal{A}(X,U_2) \to \mathcal{A}(X,U_1 \cap U_2)$, which is functorial in both variables, satisfying the following properties:*

1. *Localization: the sequence*
   $\mathcal{A}(X) \xrightarrow{j^{\mathcal{A}}} \mathcal{A}(U) \xrightarrow{\partial} \mathcal{A}(X,U) \xrightarrow{i^{\mathcal{A}}} \mathcal{A}(X) \xrightarrow{j^{\mathcal{A}}} \mathcal{A}(U)$
   is exact for each pair (X,U) where $j : U \hookrightarrow X$ and $i : (X,\emptyset) \hookrightarrow (X,U)$ are the natural inclusions;

2. *Homotopy invariance: the operator $\mathcal{A}(X) \to \mathcal{A}(X \times \mathbb{A}^1)$ induced by the projection $X \times \mathbb{A}^1 \to X$ is an isomorphism;*

3. *Excision: the operator $\mathcal{A}(X,U) \to \mathcal{A}(X',U')$, induced by a morphism $e : (X',U') \to (X,U)$, is an isomorphism, if the morphism e is etale and for $Z = X \setminus U, Z' = X' \setminus U'$ one has $e^{-1}(Z) = Z'$ and $e : Z' \to Z$ is an isomorphism;*

4. *Associativity: $a \cup (b \cup c) = (a \cup b) \cup c$;*

5. *Unit element: there is an element $1 \in \mathcal{A}(X)$ such that $1 \cup x = x$ and 1 is functorial;*

6. *Partial Leibnitz rule: $\partial(a \cup b) = \partial a \cup b$.*

Note that cohomology theories considered in this paper are not necessarily graded and the boundary operator is not necessarily of degree $+1$.

We denote $\mathcal{A}(X,U) = \mathcal{A}_{X\setminus U}(X)$ and call such cohomology as cohomology with support $X \setminus U$.

Now we list some important properties of ring cohomology theories.

1. $\mathcal{A}(X,X) = \mathcal{A}_{\emptyset}(X) = 0, \mathcal{A}(\emptyset) = 0$;

2. $\mathcal{A}(X \sqcup Y) = \mathcal{A}(X) \oplus \mathcal{A}(Y)$;

3. Localization sequence for a triple. Let $V \subset U \subset X$ be open subsets. Then there is an exact sequence
   $\mathcal{A}(X,U) \longrightarrow \mathcal{A}(X,V) \longrightarrow \mathcal{A}(U,V) \longrightarrow \mathcal{A}(X,U) \longrightarrow \mathcal{A}(X,V)$

4. Mayer-Vietoris sequence. Let $X = U_1 \cup U_2, U_{12} = U_1 \cap U_2$. Then there exists an exact sequence
   $\mathcal{A}(X) \longrightarrow \mathcal{A}(U_1) \oplus \mathcal{A}(U_2) \longrightarrow \mathcal{A}(U_{12}) \longrightarrow \mathcal{A}(X) \longrightarrow \mathcal{A}(U_1) \oplus \mathcal{A}(U_2).$

5. Strong homotopy invariance. Let $p : T \to X$ be an affine bundle and let $V \subset X$ be a closed subset. Then $p^{\mathcal{A}} : \mathcal{A}_V(X) \to \mathcal{A}_V(T)$ is an isomorphism.

6. Let there is a diagram of inclusions

$$\begin{array}{ccc} (Y,C) & \xrightarrow{i_1} & (X,A) \\ \downarrow & & \downarrow \\ (Y,C\cup D) & \xrightarrow{i} & (X,A\cup B) \\ \uparrow & & \uparrow \\ (Y,D) & \xrightarrow{i_2} & (X,B) \end{array}$$

then $i^{\mathcal{A}}(x \cup y) = i_1^{\mathcal{A}} x \cup i_2^{\mathcal{A}} y$.

3 Chern and Thom structures

Definition 2. *Let $\mathcal{A}$ be a cohomology theory. A Chern structure on $\mathcal{A}$ is an assignment $L \mapsto c(L)$ which assosiate to each smooth X and each line bundle L/X a universally central element $c(L) \in \mathcal{A}(X)$ satisfying the following properties:*

1. *Functoriality: $f^{\mathcal{A}}(c(L)) = c(f^*(L))$ for each morphism $f : Y \to X$; $c(L_1) = c(L_2)$ for isomorphic L_1 and L_2;*

2. *Nondegeneracy: the operator $(1, \xi) : \mathcal{A}(X) \oplus \mathcal{A}(X) \to \mathcal{A}(X \times \mathbb{P}^1)$ is an isomorphism where $\xi = c(\mathcal{O}(-1))$ and $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbb{P}^1$;*

3. *Vanishing: $c(\mathbf{1}_X) = 0 \in \mathcal{A}(X)$ for each smooth X.*

The element $c(L) \in \mathcal{A}(X)$ is called (the first) Chern scass of the line bundle L.

Definition 3. *We say that $\mathcal{A}$ is endowed with a Thom structure if for every smooth variety X and every line bundle L/X there is chosen a universally central element $th(L) \in \mathcal{A}(L, L \setminus X)$ satisfying the following properties:*

1. *Functoriality: $\varphi^{\mathcal{A}}(th(L_2)) = th(L_1)$ for each isomorphism $\varphi : L_1 \to L_2$ of line bundles;*

   $f_L^{\mathcal{A}}(th(L)) = th(L_Y)$ for each morphism $f : Y \to X$ and each line bundle L/X, where $L_Y = L \times_X Y$ is the line bundle over Y and $f_L : L_Y \to L$ is the projection to L;

2. *Nondegeneracy: the cup-product $\cup th(1) : \mathcal{A}(X) \to \mathcal{A}(X \times \mathbf{A}^1, X \times \mathbf{A}^1 \setminus X)$ is an isomorphism (here X is identified with $X \times \{0\}$).*

We recall that for every line bundle L/X there are maps $i : L \to (L, L \setminus X)$ and $z : X \to L$. Let $L \mapsto th(L)$ be a Thom structure and let $c(L) = z^{\mathcal{A}} \circ i^{\mathcal{A}}(th(L))$. Then we can define an assignment $L \mapsto c(L)$.

The following proposition (See 2.2.10 from [Pa1]) will be used often below.

Proposition 1. *Let $\mathcal{A}$ be a cohomology theory. There exists a short exact sequence*

$$0 \longrightarrow \mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathbf{1}\oplus L)) \longrightarrow \mathcal{A}(\mathbb{P}(\mathbf{1}\oplus L)) \xrightarrow{i^{\mathcal{A}}} \mathcal{A}(\mathbb{P}(\mathbf{1}\oplus L)\setminus \mathbb{P}(\mathbf{1})) \longrightarrow 0.$$

By excision we have

$$\mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathbf{1}\oplus L)) \simeq \mathcal{A}_X(L),$$

by homotopy invariance we have $\mathcal{A}(\mathbb{P}(\mathbf{1}\oplus L)\setminus \mathbb{P}(\mathbf{1})) = \mathcal{A}(\mathbb{P}(L))$. Thus we can write the following exact sequence

$$0 \longrightarrow \mathcal{A}_X(L) \longrightarrow \mathcal{A}(\mathbb{P}(\mathbf{1}\oplus L)) \xrightarrow{i^{\mathcal{A}}} \mathcal{A}(\mathbb{P}(L)) \longrightarrow 0.$$

In particular, for each variety X there exists a splitting short exact sequence

$$0 \longrightarrow \mathcal{A}_{X\times\{0\}}(X\times\mathbb{A}^1) \longrightarrow \mathcal{A}(X\times\mathbb{P}^1) \longrightarrow \mathcal{A}(X) \longrightarrow 0.$$

If $X = pt$ then there exists an isomorphism $\mathcal{A}(\mathbb{P}^0)\oplus\mathcal{A}_{\{0\}}(\mathbb{A}^1) \xrightarrow{\sim} \mathcal{A}(\mathbb{P}^1)$.

Proposition 1 shows that to define

$$th(L) \in \mathcal{A}_X(L),$$

we may equivalently define

$$\bar{th}(L) \in \mathcal{A}(\mathbb{P}(\mathbf{1}\oplus L))$$

such that

$$i^{\mathcal{A}}(\bar{th}(L)) = 0.$$

To get an element having these properties, we put

$$\bar{th}(L) = c(\mathcal{O}_{\mathbf{1}\oplus L}(1)\otimes p^*(L)),$$

where

$$p : \mathbb{P}(\mathbf{1}\oplus L) \to X$$

is the projection and $\mathcal{O}_{\mathbf{1}\oplus L}(1)$ is the dual bundle to the tautological bundle $\mathcal{O}_{\mathbf{1}\oplus L}(-1)$ over $\mathbb{P}(\mathbf{1}\oplus L)$.

The following theorem can be found in [Pa1] (Theorem 3.2.4).

Theorem 1. *The assignments introduced above are inverse to each other one-to-one correspondences between Chern and Thom structures.*

4 Homotopy category

In this section we recall the definition of the homotopy cathegory from [PS].

Definition 4. *A morphism $\pi : X \to Y$ is called a quasi-affine bundle if it can be represented as a composition of several morphisms*

$$X = X^{(1)} \to X^{(2)} \to \cdots \to X^{(n)} = Y$$

such that each arrow is an affine bundle.

A base change of a quasi-affine bundle is a quasi-affine bundle and a composition of two quasi-affine bundles is a quasi-affine bundle. If $\pi_1 : X_1 \to X$ and $\pi_2 : X_2 \to X$ are quasi-affine bundles then the map $X_1 \times_X X_2 \to X$ is a quasi-affine bundle.

Definition 5. *The diagram*

$$X \xleftarrow{p} X' \xrightarrow{f} Y$$

is called a quasi-morphism if p is a quasi-affine bundle and f is a usual morphism.

We say that quasi-morphisms $X \overset{p}{\leftarrow} X' \overset{f}{\to} Y$ and $X \overset{p_1}{\leftarrow} X'' \overset{f_1}{\to} Y$ are equivalent if there exists an isomorphism $\alpha : X' \to X''$ such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[d, equal] & X' \arrow[l, "p"'] \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[d, equal] \\
X & X'' \arrow[l, "p_1"'] \arrow[r, "f_1"] & Y
\end{tikzcd}$$

Now we construct the cathegory QSm. The objects of QSm are smooth varieties and the morphisms are the equivalent classes of quasi-morphisms. A composition of quasi-morphisms $X \overset{\pi}{\leftarrow} X' \overset{f}{\to} Y$ and $Y \overset{p}{\leftarrow} Y' \overset{g}{\to} Z$ is defined by the following diagram

$$\begin{tikzcd}
X' \times_Y Y' \arrow[r, "P"] \arrow[d, "\Pi"'] & Y' \arrow[r, "g"] \arrow[d, "p"] & Z \\
X' \arrow[r, "f"] \arrow[d, "\pi"'] & Y & \\
X & &
\end{tikzcd}$$

Since $\pi \circ \Pi$ is a quasi-affine bundle, the diagram $X \overset{\pi\circ\Pi}{\leftarrow} X' \times_Y Y' \overset{g\circ P}{\to} Y$ is a quasi-morphism.

Definition 6. *We say that quasi-morphisms f/π and f'/π' are elementary homotopic if there exists a quasi-morphism*

$$H : \mathbb{A}^1 \times X \leftarrow \mathcal{X} \rightarrow Y$$

such that $H \circ i_0 = f/\pi$ and $H \circ i_1 = f'/\pi'$ when $i_t : t \times X \hookrightarrow \mathbb{A}^1 \times X, t = 0, 1$.

Definition 7. *We say that quasi-morphisms f'/π' and f''/π'' are homotopic if there exists a family of quasi-morphisms $f_0/\pi_0, \dots, f_r/\pi_r$ such that $f'/\pi' = f_0/\pi_0, f''/\pi'' = f_r/\pi_r$ and for each index $i = 0, \dots, r-1$ the quasi-morphisms f_i/π_i and f_{i+1}/π_{i+1} are elementary homotopic.*

The set of all homotopy classes of quasi-morphisms X to Y will be denoted by $[X, Y]$. The homotopy class of a quasi-morphism f/π will be denoted by $[f, \pi]$.

It is obvious that a composition of quasi-morphisms induces a composition of their homotopy classes.

5 The isomorphism between homotopy classes and Picard group

Definition 8. *For a given inclusion $V \subset W \subset k^\infty$ of finite-dimensional subspaces, let $i_{V,W} : \mathbb{P}(V) \hookrightarrow \mathbb{P}(W)$ denote the inclusion of the associated projective spaces. The assignment $f/\pi \mapsto (i_{V,W} \circ f)/\pi$ defines a map of sets $[X, \mathbb{P}(V)] \to [X, \mathbb{P}(W)]$. We set $[X, \mathbb{P}^\infty] = \varinjlim [X, \mathbb{P}(V)]$, where the inductive system is taken over all finite-dimensional subspaces V of k^∞.*

We have $Pic(\mathbb{P}^\infty) = \mathbb{Z}$. Let $[\mathcal{O}(-1)]$ denote the unique element of $Pic(\mathbb{P}^\infty)$, which maps to $[\mathcal{O}(-1)]$ under the canonical map $Pic(\mathbb{P}^\infty) \to Pic(\mathbb{P}(V))$.

Now we introduce a group structure on $[X, \mathbb{P}^\infty]$. Let V_1 and V_2 be two finite dimensional vector spaces over the field k and let $V_1 \otimes V_2$ be the tensor product of the spaces. Let

$$s_{V_1,V_2} : \mathbb{P}(V_1) \times \mathbb{P}(V_2) \to \mathbb{P}(V_1 \otimes V_2)$$

be the Segre map, which takes the point $(\{l_1\}, \{l_2\})$ to the point $\{l_1 \otimes l_2\}$. Let $X \overset{\pi'}{\leftarrow} X' \overset{f'}{\rightarrow} \mathbb{P}(V)$ and $X \overset{\pi''}{\leftarrow} X'' \overset{f''}{\rightarrow} \mathbb{P}(W)$ be quasi-morphisms. The sum of (f'/π') and (f''/π'') is defined to be a composition

$$X \xleftarrow{\pi' \times_X \pi''} X' \times_X X'' \xrightarrow{f' \times f''} \mathbb{P}(V) \times \mathbb{P}(W) \xrightarrow{s_{V,W}} \mathbb{P}(V \otimes W).$$

Lemma 1. *$[X, \mathbb{P}^\infty]$ is a well-defined monoid.*

Proof.

1. Correctness. It is enough to prove the following particular case. Let $(f'/\pi') : X \leftarrow X' \rightarrow Y$ and $(f''/\pi'') : X \leftarrow X'' \rightarrow Y$ be two elementary

homotopic quasi-morphisms with homotopy $\mathbb{A}^1 \times X \xleftarrow{H_1} \mathcal{X} \xrightarrow{H_2} \mathbb{P}^\infty$ and let $X \xleftarrow{p} X_1 \xrightarrow{f} \mathbb{P}^\infty$ be a quasi-morphism. Then $(f'/\pi')+(f/p)$ and $(f''/\pi'')+(f/p)$ are elementary homotopies. Such homotopy can be constructed as a sum

$$(H_2/H_1) + (id \times f \circ pr/p),$$

where $pr : \mathbb{A}^1 \times X \to X$ is a natural projection.

2. Associativity is obvious:

$$X_1 \times_X (X_2 \times_X X_3) = (X_1 \times_X X_2) \times_X X_3.$$

3. Zero element. The equality $X \times_X X' = X'$ shows that the quasi-morphism $(const/id)$ is a zero element of our semigroup. □

A quasi-affine bundle $p : X' \to X$ induces the isomorphism

$$p^* : Pic(X') \to Pic(X).$$

Therefore for every quasi-morphism $(f/p) : X \leftarrow X' \to \mathbb{P}^n$ we constructed a line bundle

$$cl_X(f/p) = (p^*)^{-1} f^* \mathcal{O}(-1).$$

This defines a map

$$cl_X : [X, \mathbb{P}^\infty] \to Pic(X).$$

Lemma 2. *cl_X is a surjective homomorphism of monoids.*

Proof.

1. Correctness. We have to prove that if (f'/p') and f''/p'' are elementary homotopic with homotopy

$$(P/H) : \mathbb{A}^1 \times X \leftarrow \mathcal{X} \to \mathbb{P}^n,$$

then

$$(p'^*)^{-1} f'^* \mathcal{O}(-1) \simeq (p''^*)^{-1} f''^* \mathcal{O}(-1).$$

Since pr induces an isomorphism of Picard group it is enough to prove that

$$pr^* (p'^*)^{-1} f'^* \mathcal{O}(-1) \simeq pr^* (p''^*)^{-1} f''^* \mathcal{O}(-1),$$

or equivalently to prove that

$$i_0^* (P^*)^{-1} H^* \mathcal{O}(-1) \simeq i_1^* (P^*)^{-1} H^* \mathcal{O}(-1),$$

where $pr : \mathbb{A}^1 \times X \to X$ is the natural projection and $i_t : X \to \mathbb{A}^1 \times X$, $t = 0, 1$ are the natural inclusions. The latter equality is obvious.

2. Homomorphism. To prove that

$$cl_X((f/\pi)+(g/p)) = cl_X(f/\pi)\otimes cl_X(g/p),$$

we consider a diagram

$$X \xleftarrow{\pi\times p} X'\times_X X'' \xrightarrow{f\times g} \mathbb{P}^n\times\mathbb{P}^m \xrightarrow{s} \mathbb{P}^\infty.$$

Since

$$s^*\mathcal{O}_\infty(-1)=\mathcal{O}_n(-1)\otimes\mathcal{O}_m(-1)$$

and

$$(f\times g)^*(\mathcal{O}_n(-1)\otimes\mathcal{O}_m(-1)) = L_1'\otimes L_2',$$

where $L_1'=(f\times g)^*(\mathcal{O}_n(-1))$ and $L_2'=(f\times g)^*(\mathcal{O}_m(-1))$, we deduce that cl_X is a homomorphism.

3. Surjection. We have to prove that for any line bundle L over X there exists a quasi-morphism $f/p : X\leftarrow X'\rightarrow\mathbb{P}^n$ such that $f^*\mathcal{O}_n(-1)\simeq p^*L$. Using a Jounolou trick ([J])we can find an affine bundle $p: X'\rightarrow X$, where X' is an affine variety. Let $L'=p^*L$. Now we include L' in $\mathbf{1}^{n+1}$ and define $f(x)=L(x)$, where $L(x)$ is the line, which constitute L. □

Lemma 3. $Ker(cl_X)=\{(f/p)\in[X,\mathbb{P}^\infty]|cl_X(f/p)=0\}=0.$

Proof. Let $f/p : X\leftarrow X'\rightarrow\mathbb{P}(V)$ be a quasi-morphism and let

$$(p^*)^{-1}f^*\mathcal{O}(-1)\simeq\mathbf{1}_X.$$

Then $f^*\mathcal{O}(-1)\simeq\mathbf{1}_{X'}$. Therefore it is sufficient to prove that if $f: X\rightarrow\mathbb{P}(V)$ is a morphism such that $f^*\mathcal{O}(-1)\simeq\mathbf{1}_X$, then f is homotopic to a constant map. There exists a section $s: X'\rightarrow V$ of $f^*\mathcal{O}(-1)$ such that $s(x)\neq 0$ for any $x\in X'$. Therefore $f: X'\rightarrow\mathbb{P}(V)$ is a composition of $s: X'\rightarrow V\setminus\{0\}$ and $pr: V\setminus\{0\}\rightarrow\mathbb{P}(V)$. We may assume that f is not surjective (otherwise we consider $V\subset W$ and one can easily see that the map $X\xrightarrow{f}\mathbb{P}(V)\hookrightarrow\mathbb{P}(W)$ is not surjective). Then there exists a line $\{l_0\}\in\mathbb{P}(V)$ such that for every $x\in X$, $s(x)\notin\{l_0\}$. From this we deduce that s is homotopic to a constant map with the homotopy

$$H:\mathbf{1}\times X\rightarrow V\setminus\{0\},\ H(t,x)=tv+(1-t)s(x),$$

where $v\in l_0\setminus\{0\}$. □

Theorem 2. *$[X,\mathbb{P}^\infty]$ is a group and cl_X is the natural group isomorphism.*

Proof. First we construct an inverse element to $f/p\in[X,\mathbb{P}^\infty]$. Let us consider the diagram

$$\mathbb{P}(V)\xleftarrow{p_1}\mathbb{P}(V)\times\mathbb{P}(V^*)\setminus\{(v,\varphi)|\varphi(v)=0\}\xrightarrow{p_2}\mathbb{P}(V^*).$$

It is well-known that p_1 and p_2 are affine bundles and $p_1^*(\mathcal{O}(-1)) \simeq p_2^*(\mathcal{O}(1))$. We claim that the composition of f/p with p_2/p_1 gives the inverse element to f/p, which we denote by $(f/p)^{-1}$. To prove the claim, we consider the equality

$$cl_X((f/p)^{-1}(f/p)) = cl_X(f/p)^{-1} \otimes cl_X(f/p) = cl_X(f/p)^* \otimes cl_X(f/p) = \mathbf{1}_X,$$

where $cl_X(f/p)^*$ is the dual bundle to $cl_X(f/p)$. Using Lemma 4 we get $(f/p)^{-1} + (f/p) = 0$.

Since cl_X is a surjective group homomorphism and $Ker(cl_X) = 0$, cl_X is an isomorphism and $[X, \mathbb{P}^\infty]$ is an abelian group. The proof is complete. □

It is obvious that cl_X is the natural isomorphism, i.e. that for every morphism $g : Y \to X$ the following diagram commutes:

$$\begin{array}{ccc} [Y, \mathbb{P}^\infty] & \xrightarrow{cl_Y} & Pic(Y) \\ \uparrow{\scriptstyle g^*} & & \uparrow{\scriptstyle g^*} \\ [X, \mathbb{P}^\infty] & \xrightarrow{cl_X} & Pic(X) \end{array}$$

6 Chern elements

Let $\mathcal{A}$ be a representable cohomology theory. In this section we prove a one-to-one correspondence between Chern elements and Chern structures of $\mathcal{A}$.

Definition 9. *A universally central element $c \in \mathcal{A}(\mathbb{P}^\infty)$ is called a Chern element if for a flag $L \subset V \overset{i_V}{\hookrightarrow} k^\infty$ consisting of 1- and 2-dimensional vector subspases the element $c_V = i_V^{\mathcal{A}}(c)$ vanishes in $\mathcal{A}(\mathbb{P}(L)) = \mathcal{A}(pt)$ and the operator*

$$(\cup 1, \cup c_V) : \mathcal{A}(pt) \oplus \mathcal{A}(pt) \to \mathcal{A}(\mathbb{P}(V))$$

is an isomorphism.

Let c be a Chern element. It was proved in the previous section that for every line bundle there exists a unique (up to homotopy) quasi-morphism $f/p : X \leftarrow X' \to \mathbb{P}^\infty$ such that $(p^*)^{-1} f^*(\mathcal{O}(-1)) = L$. This induces the following diagram

$$\mathcal{A}(X) \xrightarrow[\sim]{p^{\mathcal{A}}} \mathcal{A}(X') \xleftarrow{f^{\mathcal{A}}} \mathcal{A}(\mathbb{P}^\infty),$$

where $p^{\mathcal{A}}$ is an isomorphism. Let $c(L) = (p^*)^{-1} f^*(c)$.

Theorem 3. *Let $\mathcal{A}$ be a representable ring cohomology theory. For a Chern element c there exists a unique Chern structure $L \mapsto c(L)$ such that the restriction $c(L)$ to $\mathcal{O}(-1)$ is c.*

If two Chern structures $L \mapsto c^{(1)}(L)$ and $L \mapsto c^{(2)}(L)$ coincide on $\mathcal{O}(-1)$, they coincide on every line bundle.

Proof.

1. Correctness. Let L/X be a line bundle. Since there exists a unique (up to homotopy) element f/p such that $cl_X(f/p) = L$ it is enough to prove that elementary homotopies of quasi-morphisms

$$(f'/p') : X \leftarrow X' \rightarrow \mathbb{P}^\infty$$

and

$$(f''/p'') : X \leftarrow X'' \rightarrow \mathbb{P}^\infty$$

with the quasi-homotopy

$$\mathbb{A}^1 \times X \leftarrow \mathcal{X} \rightarrow \mathbb{P}^\infty$$

give the same Chern classes. Since $i_0 : X \rightarrow \mathbb{A}^1 \times X$ and $i_1 : X \rightarrow \mathbb{A}^1 \times X$ both induce the same cohomology map we have

$$(f'/p')^{\mathcal{A}} = (f''/p'')^{\mathcal{A}},$$

which proves correctness.

2. Functoriality is easily deduced from theorem 2.

3. Nondegeneracy. Since $\mathcal{A}$ is representable, there exists an element

$$\sigma \in \mathcal{A}_{\{0\}}(\mathbb{A}^1)$$

such that

$$\cup\sigma : \mathcal{A}(\mathbb{P}^0) \rightarrow \mathcal{A}_{\{0\}}(\mathbb{A}^1)$$

is an isomorphism. Using proposition 1, we deduce that

$$(1, \cup\sigma) : \mathcal{A}(\mathbb{P}^0) \oplus \mathcal{A}(\mathbb{P}^0) \rightarrow \mathcal{A}(\mathbb{P}^0) \oplus \mathcal{A}_{\{0\}}(\mathbb{A}^1) \tilde{\rightarrow} \mathcal{A}(\mathbb{P}^1)$$

is an isomorphism.

Let c be a Chern element. Then there exists an isomorphism

$$(1, \cup c_1) : \mathcal{A}(\mathbb{P}^0) \oplus \mathcal{A}(\mathbb{P}^0) \rightarrow \mathcal{A}(\mathbb{P}^1).$$

Therefore we have two $\mathcal{A}(\mathbb{P}^0)$-module isomorphisms $(1, \cup\sigma)$ and $(1, \cup c_1)$. Let

$$(1, u) = (1, \cup c_1)^{-1}(1, \cup\sigma)(1, 1).$$

Then

$$\sigma = uc_1.$$

It is clear that u is a central unit element.

Let X be a variety. Let us consider the elements

$$\sigma_X = pr^{\mathcal{A}}(\sigma)$$

and

$$c_X = pr'^{\mathcal{A}}(c_1),$$

where $pr : X \times \mathbb{A}^1 \to \mathbb{A}^1$ and $pr' : X \times \mathbb{P}^1 \to \mathbb{P}^1$ are usual projections. Since $\mathcal{A}$ is representable there is an isomorphism

$$(1, \cup\sigma_X) : \mathcal{A}(X) \oplus \mathcal{A}(X) \to \mathcal{A}(X) \oplus \mathcal{A}_{X\times\{0\}}(X \times \mathbb{A}^1) \to \mathcal{A}(X \times \mathbb{P}^1).$$

Since $\sigma_X = uc_X$ and u is unit we have an isomorphism

$$(1, \cup c_X) : \mathcal{A}(X) \oplus \mathcal{A}(X) \to \mathcal{A}(X \times \mathbb{P}^1).$$

4. Vanishing. This property is obvious.

5. Uniqueness follows from the uniqueness property of quasi-morphism $f/p : X \leftarrow X' \to \mathbb{P}^\infty$ and functoriality property of the Chern class. □

7 Thom elements

Definition 10. *A universally central element $th \in \mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}_\infty(-1))$ is called a Thom element if for every 1-dimensional subspace $l \subset k^\infty$ the operator*

$$\cup th_l : \mathcal{A}(\mathbb{P}(l)) \to \mathcal{A}_{\mathbb{P}(l)}(\mathcal{O}_l(-1)) = \mathcal{A}_{\mathbb{P}(l)}(l)$$

is an isomorphism.

The aim of this section is to establish a relation between Thom elements and Thom structures. First, we show that having a Thom element we can easily find a Chern element.

Theorem 4. *Let $z : \mathbb{P}^\infty \to \mathcal{O}_\infty(-1)$ be the zero section and let $i^{\mathcal{A}} : \mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}(-1)) \to \mathcal{A}(\mathcal{O}(-1))$ be the support extension operator. Let $c = (z^{\mathcal{A}} \circ i^{\mathcal{A}})(th)$, where th is a Thom element. Then c is a Chern element.*

Proof. To prove the vanishing property, we have to show that for every line l the restriction of a Thom element onto $\mathcal{A}(\mathbb{P}(l))$ is zero. Let us consider the following diagram of inclusions:

$$\begin{array}{ccc} \{1\} & \longrightarrow & l \\ \downarrow & & \downarrow \\ (l \setminus \{0\}, l \setminus \{0\}) & \longrightarrow & (l, l \setminus \{0\}) \end{array}$$

Note that the composition of induced maps is zero because $\mathcal{A}(l\setminus\{0\}, l\setminus\{0\}) = 0$. This shows that the restriction th onto $\mathbb{P}(l) = \{1\}$ is zero. The vanishing property is proved.

To prove nondegeneracy of c, we consider the isomorphisms

$$\mathcal{A}(\mathbb{P}(l)) \oplus \mathcal{A}(\mathbb{P}(l)) \overset{(1,\cup th)}{\to} \mathcal{A}(\mathbb{P}(l)) \oplus \mathcal{A}_{\{0\}}(l) \overset{\sim}{\to} \mathcal{A}(\mathbb{P}(V)),$$

where l is a line and V is a two-dimensional subspace. Proposition 1 gives us the latest isomorphism. We deduce that there is an isomorphism

$$\mathcal{A}(\mathbb{P}(l)) \oplus \mathcal{A}(\mathbb{P}(l)) \overset{(1,\cup\sigma)}{\to} \mathcal{A}(\mathbb{P}(V))$$

when σ is the image of the Thom element under the following maps

$$\mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}_\infty(-1)) \to \mathcal{A}_{\mathbb{P}^0}(\mathcal{O}_0(-1)) \overset{\sim}{\leftarrow} \mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathcal{O}_0(-1)\oplus\mathbf{1})) \hookrightarrow \mathcal{A}(\mathbb{P}(\mathcal{O}_0(-1)\oplus\mathbf{1})) = \mathcal{A}(\mathbb{P}^1).$$

The end of our proof one can deduce from the next lemma. □

Lemma 4. *In the notation above one has*

$$\sigma = -c.$$

To prove this lemma we recall the proposition 6.5.2 from [PY].

Proposition 2. *Let $\mathcal{A}$ be a cohomology theory. Let $s_i : \mathbb{P}^i \to \mathbb{P}(\mathcal{O}_i(-1) \oplus \mathbf{1})$ be the zero sections, $p_i : \mathbb{P}(\mathcal{O}_i(-1) \oplus \mathbf{1}) \to \mathbb{P}^i$ be the natural projections, $j : \mathbb{P}(\mathcal{O}_0(-1) \oplus \mathbf{1}) \to \mathbb{P}(\mathcal{O}_1(-1) \oplus \mathbf{1})$ is the map inducing by the natural inclusion. Then for every element $a \in \mathcal{A}(\mathbb{P}(\mathcal{O}_1(-1) \oplus \mathbf{1}))$ vanishing after restriction to the subvariety $\mathbb{P}(\mathcal{O}_1(-1)) \subset \mathbb{P}(\mathcal{O}_1(-1) \oplus \mathbf{1})$ one has the relation in $\mathcal{A}(\mathbb{P}^1)$*

$$s_1^{\mathcal{A}}(a) = -j^{\mathcal{A}}(a).$$

Proof of lemma. Let a be an image th under the map

$$\mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}_\infty(-1)) \overset{\sim}{\leftarrow} \mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathcal{O}_\infty(-1)\oplus\mathbf{1})) \to \mathcal{A}(\mathbb{P}(\mathcal{O}_\infty(-1)\oplus\mathbf{1})) \to \mathcal{A}(\mathbb{P}(\mathcal{O}_1(-1)\oplus\mathbf{1})).$$

Use the above proposition we deduce the desired equality. □

Now our aim is to establish one-to-one correspondence between Thom elements and Thom structures.

We have the following diagram

$$\begin{array}{ccc} \text{Thom elements} & \to & \text{Chern elements} \\ \uparrow & & \updownarrow \\ \text{Thom structures} & \leftrightarrow & \text{Chern structures} \end{array}$$

when the vertical up arrows are the natural restriction, the vertical down arrow is defined in the earlier section and the top arrow is described in this section. We know from [Pa1] that the bottom arrow is bijection, a section earlier we prove that the right hand side arrow is bijection. Now we can prove that each round trip coincides with the identity. To prove this we need a lemma.

Lemma 5. *Let $\mathcal{A}$ be an oriented cohomology theory. Then the composition operator*

$$\mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}_\infty(1)) \longrightarrow \mathcal{A}(\mathcal{O}_\infty(1)) \longrightarrow \mathcal{A}(\mathbb{P}^\infty)$$

is a monomorphism, and the composition operator

$$\mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}_\infty(-1)) \longrightarrow \mathcal{A}(\mathcal{O}_\infty(-1)) \longrightarrow \mathcal{A}(\mathbb{P}^\infty)$$

is a monomorphism.

To prove the lemma we use some propositions (Corollary 1.2.5, lemma 1.3.4 and lemma 1.3.5) from [Pa2].

Proposition 3. *1. An $\mathcal{A}(pt)$-algebra homomorphism*

$$\mathcal{A}(pt)[[t]] \to \mathcal{A}(\mathbb{P}^\infty)$$

which takes the variable t to the class $c(\mathcal{O}_\infty(1))$ is an isomorphism.

2. *For a line bundle L over X set $t\bar{h}(L) = c(\mathcal{O}_{\mathbf{1}\oplus L}(1) \otimes p^*(L))$. Then the operator $(\cup t\bar{h}(L)) \circ p^{\mathcal{A}} : \mathcal{A}(X) \to \mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathbf{1} \oplus L))$ is an isomorphism.*

3. *The diagram commutes*

$$\begin{tikzcd}
\mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathbf{1} \oplus L)) \arrow[r, "s^{\mathcal{A}}"] & \mathcal{A}(X) \\
\mathcal{A}(X) \arrow[u, "\cup t\bar{h}(L)"] \arrow[r, equal] & \mathcal{A}(X). \arrow[u, "\cup c(L)"']
\end{tikzcd}$$

Proof of lemma. Take $X = \mathbb{P}^\infty$ and $L = \mathcal{O}_\infty(1)$ and consider the commutative diagram from the last proposition. By the first claim of the proposition 3, the operator $\cup c(L) : \mathcal{A}(X) \to \mathcal{A}(X)$ is injective. The operator $\cup t\bar{h}(L)$ is an isomorphism we deduce that the operator $s^{\mathcal{A}} : \mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathbf{1} \oplus L)) \to \mathcal{A}(X)$ is injective.

Consider the commutative diagram

$$\begin{tikzcd}
\mathcal{A}_{\mathbb{P}(\mathbf{1})}(\mathbb{P}(\mathbf{1} \oplus \mathcal{O}_\infty(1))) \arrow[r, "s^{\mathcal{A}}"] \arrow[d, "\simeq"'] & \mathcal{A}(\mathbb{P}^\infty) \\
\mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}_\infty(1)) \arrow[r] & \mathcal{A}(\mathcal{O}_\infty(1)). \arrow[u, "z^{\mathcal{A}}"]
\end{tikzcd}$$

The left arrow is an isomorphism by excision the upper arrow is injective. Then the operator

$$\mathcal{A}_{\mathbb{P}^\infty}(\mathcal{O}_\infty(1)) \longrightarrow \mathcal{A}(\mathcal{O}_\infty(1)) \longrightarrow \mathcal{A}(\mathbb{P}^\infty)$$

is monomorphism. Similarly the second operator from our lemma is a monomorphism. □

Theorem 5. *Let $\mathcal{A}$ be a representable cohomology theory. Then in the diagram*

$$\begin{array}{ccc}
\textit{Thom elements} & \overset{\alpha}{\to} & \textit{Chern elements} \\
\beta \uparrow & & \gamma \downarrow \uparrow \gamma^{-1} \\
\textit{Thom structures} & \overset{\delta}{\leftarrow} & \textit{Chern structures}
\end{array}$$

each arrow is a bijection and each round trip coincides with the identity.

Proof. The last lemma shows that α is injective. In the previous section we proved that γ is bijective. In [Pa1] it is proved that δ is bijective. Since $\gamma^{-1} \circ \delta^{-1} = \alpha \circ \beta$ we see that β is injective. Using the relation $\alpha \circ \beta = \gamma^{-1} \circ \delta^{-1}$ once again we get a chain of relations

$$\alpha \circ \beta \circ \delta \circ \gamma = \gamma^{-1} \circ \delta^{-1} \circ \delta \circ \gamma = id,$$

$$\gamma \circ \alpha \circ \beta \circ \delta = \gamma \circ \gamma^{-1} \circ \delta^{-1} \circ \delta = id,$$

$$\delta \circ \gamma \circ \alpha \circ \beta = \delta \circ \gamma \circ \gamma^{-1} \circ \delta^{-1} = id.$$

The last one shows that β is surjective (and hence β is bijective). It proves that $\beta \circ \delta \circ \gamma \circ \alpha = id$. □

Thus we prove the main theorem of this section.

Theorem 6. *Let $\mathcal{A}$ be a representable ring cohomology theory and let th be a Thom element.*

1. *There exists a unique Thom structure $L \mapsto th(L)$ such that $th(\mathcal{O}_\infty(-1)) = th$.*

2. *If two Thom structures coincide on $\mathcal{O}_\infty(-1)$, they coincide.*

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