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DEFORMATION OF ORTHOSYMPLECTIC LIE SUPERALGEBRA $osp(1|4)$

E. Celeghini and P. P. Kulish

Dipartimento di Fisica, Universita di Firenze
and INFN Sezione di Firenze
Via G. Sansone 1, Sesto, 50019, Firenze, Italy

St. Petersburg Department of Steklov Mathematical Institute,
Fontanka 27, St. Petersburg 191023, Russia

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ABSTRACT

Triangular deformation of the orthosymplectic Lie superalgebra $osp(1|4)$ is defined by chains of twists. Corresponding classical r-matrix is obtained by a contraction procedure from the trigonometric r-matrix. The carrier space of the constant r-matrix is the Borel subalgebra.

1. Introduction

Orthosymplectic Lie superalgebras $osp(m|2n)$ have variety of applications in gauged supergravity, supersymmetric quantum mechanics, integrable $\mathbb{Z}_2$-graded spin chains etc. Similar applications with corresponding changes can have quantum deformations of these superalgebras. In this paper a triangular deformation of the basic Lie superalgebra $g = osp(1|4)$ is given by explicite construction of a universal twist element $\mathcal{F} \in \mathcal{U}(g) \otimes \mathcal{U}(g)$ [1]. It is a natural extension of known triangular deformation of rank 1 Lie superalgebra $osp(1|2)$ [2, 3] along the lines presented in [4] about twists with deformed carrier subspaces. The constructed twist includes generators of the Borel subalgebra $B_+ \subset osp(1|4)$. Deformed quantum algebra can be used to define a noncommutative version of the super anti-de Sitter space (s-AdS) generatizing [5–8], deformation of superconformal quantum mechanics [9] etc.

2. Classical r-matrices

The direction of twist deformation is given by a classical triangular r-matrix $r \in g \wedge g$, being a solution of classical (super) Yang–Baxter equation on $g \otimes g \otimes g$ [10]

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = 0, \tag{1}$$

where in the super-case the commutators and the tensor products are $\mathbb{Z}_2$-graded:

$$[a, b] = ab - (-1)^{p(a)p(b)} ba,$$
$$(a \otimes b)(c \otimes d) = (-1)^{p(b)p(c)}(ac \otimes bd).$$

The parity of a homogeneous element b of a $\mathbb{Z}_2$-graded space $V = V_0 \oplus V_1$ is denoted by $p(b)$. The classical r-matrix we are looking for can be obtained by a contraction procedure from the trigonometric r-matrix of basic classical Lie superalgebras

$$r(\lambda - \mu) = (r_0 e^{\lambda-\mu} + r_0^{21})/(e^{\lambda-\mu} - 1),$$

where r_0 is the super-analog of the Drinfeld–Jimbo constant r-matrix, solution to the cYBE,

$$r_0 = \frac{1}{2}\sum k_i \otimes k_i + \sum_{\alpha\in\Delta_+} e_\alpha \otimes e_{-\alpha},$$

where $\{k_i\}$ is an orthonormal basis of Cartan subalgebra and Δ_+ is the set of positive (even and odd) roots. The sum

$$c_2^{\otimes} := r_0 + r_0^{21} = \sum k_i \otimes k_i + \sum_{\alpha\in\Delta_+} (e_\alpha \otimes e_{-\alpha} + (-1)^{p(\alpha)} e_{-\alpha} \otimes e_\alpha)$$

is an invariant element of $U(g) \otimes U(g)$ with respect to the adjoint action:

$$[x \otimes 1 + 1 \otimes x, c_2^{\otimes}] = 0,$$

$x \in g$ (a tensor Casimir element). Take the long root generator e_θ and consider adjoint transformation by the group element $\exp(te_\theta)$:

$$Ad(\exp(te_\theta))^{\otimes 2} r(\lambda - \mu) = \frac{1}{2}\Bigg(coth\frac{\lambda-\mu}{2} \cdot c_2^{\otimes} +$$
$$+ \sum_{\alpha\in\Delta_+} (e_\alpha \otimes e_{-\alpha} - (-1)^{p(\alpha)} e_{-\alpha} \otimes e_\alpha) + t \sum_{\alpha\in\Delta_+} e_\alpha \wedge [e_\theta, e_{-\alpha}]\Bigg).$$

Scaling the spectral parameter $\lambda \to \varepsilon\lambda$, $\mu \to \varepsilon\mu$ and $t \to t/\varepsilon$, one gets after contraction

$$\begin{aligned} &\lim_{\varepsilon\to 0} \varepsilon Ad(\exp(te_\theta/\varepsilon))^{\otimes 2} r(\varepsilon(\lambda-\mu)) = \\ &= (\lambda-\mu)^{-1} c_2^{\otimes} + t \sum_{\alpha\in\Delta_+} e_\alpha \wedge [e_\theta, e_{-\alpha}]. \end{aligned} \tag{2}$$

Due to the Lie algebra nature of the cYBE (1), the resulting expression satisfies this equation as well and both terms satisfy it separatly. The second term will be used to deform the universal enveloping algebra $\mathcal{U}(osp(1|4))$. The Cartan–Weyl basis of the orthosymplectic Lie superalgebra $osp(1|4)$ has the following generators: H, J (a Cartan subalgebra), $v_\pm$, $w_\pm$ (odd root generators), $X_\pm = \pm(v_\pm)^2$, $Y_\pm = \pm(w_\pm)^2$, $U_\pm$, $Z_\pm$. There are two $osp(1|2)$ subalgebras $\{H, v_\pm, X_\pm\}$ and $\{J, w_\pm, Y_\pm\}$, and the even root generators define Lie subalgebra $sp(4) \simeq so(5)$. Constructing a universal twist element $\mathcal{F}$ we will use only commutators of the Borel subalgebra generators with subscripts $+$. Some of these commutation relations are

$$\begin{gathered} [H, X_+] = 2X_+, \quad [H, v_+] = v_+, \quad [H, U_+] = U_+, \\ [H, Z_+] = Z_+, \quad [Z_+, U_+] = 2X_+, \quad [Z_+, Y_+] = U_+, \\ [Z_+, w_+] = v_+, \quad [v_+, w_+] = U_+. \end{gathered}$$

The generators Z_+, w_+ correspond to the simple roots. X_+ corresponds to the long root and it commutes with all positive root generators and with the generator J of the Cartan subalgebra. The root θ in (2) is the long one, hence all generators in the constant r-matrix (2) correspond to positive roots and one generator belongs to the Cartan subalgebra $[X_\theta, X_{-\theta}] = H \in g$. In the case of $osp(1|4)$ this constant r-matrix is ($X_\theta = X_+$)

$$r = H \wedge X_+ - v_+ \otimes v_+ + Z_+ \wedge U_+. \tag{3}$$

Solution of the cYBE defines a cobracket δ on the Lie (super) algebra: a map $\delta : g \to g \wedge g$

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], \quad x \in g.$$

It is important to point out that the dimension of the kernel of δ is equal to the number of independent primitive elements in the quantum $\mathcal{U}_\xi(g)$ which is a quantization of the bialgebra $\mathcal{U}(g)$. Proof can be given by transition to the dual Poisson–Lie group using quantum duality principle [11, 12].

The kernel of cobrachet defined by the r-matrix (3) is generated by elements X_+, J, Y_+, w_+. It is easy to see that the kernel of the cobrachet δ is a Lie subalgebra $\mathrm{Ker}\,\delta \subset g$. If there is a triangular r-matrix with its carrier in $\mathrm{Ker}\,\delta$, then the sum of this additional r-matrix and the one defined δ, satisfies the cYBE as well. In the case we are interested in there is a super-jordanian r-matrix [3]

$$r^{(sj)} = J \wedge Y_+ - w_+ \wedge w_+. \tag{4}$$

Finally, we will construct a universal twist corresponding to the r-matrix:

$$r = H \wedge X_+ - v_+ \otimes v_+ + Z_+ \wedge U_+ + J \wedge Y_+ - w_+ \otimes w_+. \tag{5}$$

(One can add also an abelian term $X_+ \wedge Y_+$, and introduce few independent parameters in front of different r-matrices.) This form of the classical r-matrix can be generalized to the case of the universal enveloping algebra of the orthosymplectic Lie superalgebra $osp(1|2n)$. It has (positive) simple root even generators Z_k, $k = 1, 2, \ldots, n-1$, $\alpha_k = \varepsilon_k - \varepsilon_{k+1}$, and one odd simple root generator v_n, $\alpha_n = \varepsilon_n$. Other positive root generators are $X_j = v_j^2$, $j = 1, 2, \ldots, n$, Z_{kj} and U_{kj} related to positive roots $\varepsilon_k - \varepsilon_j$ and $\varepsilon_k + \varepsilon_j$, $1 \leq k < j \leq n$ [13]. Commutation relations

$$[H_k, X_k] = 2X_k, \quad [H_k, v_k] = v_k,$$
$$[Z_{kj}, U_{kj}] = 2X_k, \quad [v_k, v_j] = U_{kj},$$
$$[Z_{kj}, v_j] = v_k, \quad [H_k, Z_{kj}] = Z_{kj}, \quad [H_k, U_{kj}] = U_{kj}$$

are used to prove that

$$r = \sum_{k=1}^{n} a_k (H_k \wedge X_k + \sum_{j>k}^{n} Z_{kj} \wedge U_{kj} - v_k \otimes v_k) \tag{6}$$

satisfies the cYBE. Similar to the symplectic algebra $sp(2n)$ [18] the sum of r-matrices (6) corresponds to the sequence of injections

$$osp(1|2) \subset osp(1|4) \subset \cdots \subset osp(1|2n).$$

3. Universal twist element of $B_+ \subset osp(1|4)$

Construction of explicite form of the twist corresponding to the triangular r-matrix (5) will be realized by a recipe known as chains of twists [14, 15]. The classical r-matrix is decomposed into few terms which generate factors of the universal twist. This decomposition consists of the jordanian r-matrix

$$r^{(j)} = H \otimes X_+ - X_+ \otimes H := H \wedge X_+, \tag{7}$$

extended jordanian $r^{(ej)}$ and super-jordanian r-matrices

$$r^{(ej)} = r^{(j)} + Z_+ \wedge U_+, \quad r^{(sj)} = r^{(j)} - v_+ \otimes v_+. \tag{8}$$

Twist elements corresponding to $r^{(ej)}$ and $r^{(sj)}$ are known [14, 3], and they are represented in a factorized form

$$\mathcal{F}^{(ej)} = \mathcal{F}^{(e)} \mathcal{F}^{(j)}, \quad \mathcal{F}^{(sj)} = \mathcal{F}^{(s)} \mathcal{F}^{(j)}, \tag{9}$$

where

$$\mathcal{F}^{(j)} = \exp(H \otimes \sigma), \quad \mathcal{F}^{(e)} = \exp(\frac{1}{2} Z_+ \otimes U_+ e^{-\sigma}),$$
$$\mathcal{F}^{(s)} = \exp(-v_+ \otimes v_+ f(\sigma \otimes 1, 1 \otimes \sigma)), \quad \sigma = \frac{1}{2} \ln(1 + X_+).$$

The function f is symmetric. It can be represented in series expension $\sum f_n$ [3], or through factorization of $\mathcal{F}^{(s)}$ as

$$\mathcal{F}^{(s)} = (1 - v_+ \otimes v_+ f_1(\sigma) \otimes f_1(\sigma)) \mathcal{F}^{(c)}, \quad f_1(\sigma) = (e^{\sigma} + 1)^{-1}$$

with an appropriate coboundary twist [9]

$$\mathcal{F}^{(c)} = u \otimes u\Delta(u^{-1}), \quad u = (\frac{1}{2}(e^{\sigma}+1))^{1/2}.$$

Due to commutativity of X_+ or σ and v_+ with Z_+, U_+, the twishing by $\mathcal{F}^{(s)}$ after $\mathcal{F}^{(j)}$ does not change the coproducts of Z_+ and U_+, and vise versa. Hence, one can combine these twists into an extended superjordanian twist

$$\mathcal{F}^{(esj)} = \mathcal{F}^{(s)}\mathcal{F}^{(e)}\mathcal{F}^{(j)} = \mathcal{F}^{(e)}\mathcal{F}^{(s)}\mathcal{F}^{(j)}, \tag{10}$$

corresponding to the r-matrix (3).

To take into account further extension of the r-matrix (5) by a super-jordanian term of the second $osp(1|2)$ subalgebra, we have to use a deformed carrier space transforming the generators Y_+ and w_+ according to [4, 14].

The generators v_+, U_+ and $\sigma = \frac{1}{2}\ln(1+X_+)$ are entering into this transformation. After the twist (10) their coproducts are

$$\begin{gathered}
\Delta^{(esj)}(\sigma) = \mathcal{F}^{(esj)}\Delta(\sigma)(\mathcal{F}^{(esj)})^{-1} = \Delta_0(\sigma) = \sigma\otimes 1 + 1\otimes\sigma, \\
\Delta^{(esj)}(v_+) = \Delta^{(sj)}(v_+) = v_+\otimes 1 + e^{\sigma}\otimes v_+, \\
\Delta^{(esj)}(U_+) = \Delta^{(e_j)}(U_+) = U_+\otimes e^{\sigma} + e^{2\sigma}\otimes U_+.
\end{gathered}$$

The twisted coproducts of the Borel subalgebra generators J, Y_+, w_+ of the second $osp(1|2)$ are

$$\begin{gathered}
\Delta^{(esj)}(J) = \Delta_0(J) = J\otimes 1 + 1\otimes J, \\
\Delta^{(esj)}(Y_+) = \Delta_0(Y_+) + \frac{1}{2}U_+\otimes U_+e^{-\sigma} + \frac{1}{4}X_+\otimes U_+^2 e^{-2\sigma},
\end{gathered}$$

and a rather lengthy expession for $\Delta^{(esj)}(w_+)$.

The deformed generator $\widetilde{Y}_+$ with the primitive coproduct is given by an adjoint transformation [18]

$$\widetilde{Y}_+ = Ad\exp(-Z_+U_+\sigma/2X_+)Y_+ = Y_+ - \frac{1}{4}U_+^2 e^{-2\sigma}. \tag{11}$$

Hence, the second jordanian factor is similar to the $sp(4)\simeq so(5)$ case [4] $\exp(J\otimes\sigma(\widetilde{Y}_+))$. Although the deformed coproduct of the odd generator w_+ is rather cumbersome after the transformation (11) the coproduct of $\tilde{w}_+$

$$\tilde{w}_+ = Ad\exp(-Z_+U_+\sigma/2X_+)w_+ = w_+ - \frac{1}{2}v_+U_+e^{-\sigma}/(e^{\sigma}+1)$$

is primitive as well.

A unversal twist with carrier space including all generators of the Borel subalgebra $B_+\subset osp(1|4)$, is given by the product

$$\mathcal{F} = \mathcal{F}^{(sj2)}\mathcal{F}^{(esj)}.$$

It is natural to conjecture that a similar chain of twists can be constructed for $\mathcal{U}(osp(1|2n))$ as it was in the case of $\mathcal{U}(sp(2n))$ and $\mathcal{U}(so(2n+1))$ [18, 19]. This would extend in a sense an analogy between algebras $so(2n+1)$ and $osp(1|2n)$ [16].

One can introduce more convenient set of generators of the twisted Borel subalgebra $B \subset osp(1|4)$ using the FRT-formalism [20], and the upper triangular form of the generators of B in the defining 5×5 irreducible representation ρ. The new generators are the entries of the universal L-operator: $L = (\rho \otimes id)\mathcal{R}$. Using a symmetric grading $(0,0,1,0,0)$ of the rows and columns one gets

$$L = \begin{pmatrix} T^{-1} & u & V & z & h \\ 0 & K^{-1} & W & j & \tilde{z} \\ 0 & 0 & 1 & \widetilde{W} & \widetilde{V} \\ 0 & 0 & 0 & K & \tilde{u} \\ 0 & 0 & 0 & 0 & T \end{pmatrix}.$$

The coproduct of these generators is given by the matrix product of the L-operators. The commutation relations follow from the RTT-relation: $RL_1L_2 = L_2L_1R$ [20], taking into account extra signs in the Z_2-graded tensor product [10]. The R-matrix can be obtained using a super-version of the $r^3 = 0$ theorem: $R = \exp(\eta r_\rho)$, where $r_\rho = (\rho \otimes \rho)r$. Similar to the twisting of $osp(1|2)$ [3] one can express new generators in terms of the initial ones.

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