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ON BOUNDARY CONTROLLABILITY OF DYNAMICAL SYSTEM GOUVERNED BY THE WAVE EQUATION ON A CLASS OF GRAPHS (TREES)

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ABSTRACT

The boundary control problem for the wave equation on a planar graph consisting of strings of variable densities with Dirichlet control at the boundary vertices is considered. The exact controllability in L_2-classes of controls and states is established in the case that the grapf is a tree; a sharp estimate of the time of controllability is given.

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0 Introduction

0.1 About the paper.

Mechanical object inspiring the problem is a planar graph consisting of a finite set of strings of variable densities. Graph can be displaced in direction normal with respect to the plane; displacement of its boundary vertices (the Diriclet boundary control) generates a wave propagating in the graph. We consider the question: Managing boundary control, can one create the waves of prescribed shape? In the case that the graph is a tree (doesn't contain cycles), the answer turns out to be positive: we find a sharp estimate of the time such that the "control $\mapsto$ wave" map becomes surjective (from L_2-space of controls onto L_2-space on the graph) and describe a constructive procedure which inverts this map.

0.2 Statement and main result.

Let $\Omega \subset \mathbf{R}^2$ be a finite connected planar graph with edges $\{e\} = E$ (intervals of straight lines), vertices $\{v_1, \dots, v_m\} = V$ and boundary vertices $\{\gamma_1, \dots, \gamma_n\} = \Gamma \subset V$. The graph is equipped with density ρ which is a function on Ω strictly positive and C^2-smooth on the edges. The value

$$d(\Omega) \colon = \sup_{a,b \in \Omega} \inf_{\pi[a,b]} \int_{\pi[a,b]} \sqrt{\rho}\, |dx|$$

($|dx|$ is the length element on Ω induced by the $\mathbf{R}^2$-metric; inf is taken over the paths $\pi[a,b] \subset \Omega$ connecting a and b) is the optical diameter of the graph.

To the graph one associates the dynamical system

$$\begin{aligned}
&\rho\, u_{tt} - u_{ee} = 0 \qquad in\ \{\Omega \setminus V\} \times (0,T);\\
&\sum_{e \sim v} u_e\,|_{x=v} = 0 \qquad for\ all\ \ v \in V \setminus \Gamma;\\
&u\,|_{t=0} = u_t\,|_{t=0} = 0 \qquad in\ \Omega;\\
&u = f \qquad on\ \Gamma \times [0,T],
\end{aligned}$$

where $(\,\cdot\,)_e$ is differentiation along the edge; the sum is taken over the edges incident to vertex v, $(\cdot)_e\,|_v$ is the derivative along e in direction from v taken at v; $f = f(\gamma, t)$ is the boundary control; $u = u^f(x,t)$ is the solution.

The following is the main result of the paper.

Theorem 1. *If Ω is a tree, for any $T \geq \frac{d(\Omega)}{2}$ and $y \in L_{2,\rho}(\Omega)$ (with measure $\rho|dx|$) there exists a control $f \in L_2(\Gamma \times [0,T])$ providing the equality*

$$u^f(\cdot, T) = y\,.$$

In other words, the dynamical system associated to a tree and gouverned by the wave equation is exactly controllable at times $\geq \frac{d(\Omega)}{2}$. Our proof is constructive: we propose a procedure determining f from y.

0.3 Remarks.

Our statement of the boundary control problem is motivated by inverse problems on graphs (see [2, 3r]); it differs from the traditional one which is to control the complete dynamical state $\{u^f(\cdot,T), u^f_t(\cdot,T)\}$ (see [1, 6, 7]).

In the case that *all* the vertices are available for control, exact controllability of the tree was established in [1] by means of a spectral (frequency-domain) technique. Perhaps, this approach may be extended on the case of boundary control. Nevertheless, we use another (time-domain) approach going back to A. S. Blagovestchenskii (see e.g. [4]): as we guess, it is more natural and, in contrast to the spectral technique, gives a sharp estimate of the time of controllability.

In order to restrict the volume of the paper, we omit some of technical details (mainly in Secs. 2.2–2.4). In particular, we don't prove results placed in Propositions: they are either known or their proofs should present no problems.

1 The graph

1.1 Terminology.

An *edge* is a finite interval of a straight line on the plane $\mathbf{R}^2$:

$$e = \left\{ \mathrm{col}\,\{x^1, x^2\} \;\middle|\; x^i = a^i + b^i s,\;\; a^i, b^i = \mathrm{const},\;\; s \in (0,1) \right\};$$

a *vertex* is a point $v \in \mathbf{R}^2$. A *graph* $\Omega \subset \mathbf{R}^2$ is the disjoint sum $\Omega = E \cup V$ of a finite set of edges $E = \{e\}$ and a finite set of vertices $V = \{v\}$ such that Ω is a closed connected set on the plane.

We say that $v \in V$ and $e \in E$ are *incident* (and write $e \sim v$, $v \sim e$) if $v \in \overline{e}$ (the closure in $\mathbf{R}^2$) and denote $r(v) := \sharp\{e \in E \,|\, e \sim v\}$. Just for simplicity we consider only the graphs satisfying $r(\,\cdot\,) \neq 2$.

The set $\Gamma := \{v \in V \,|\, r(v) = 1\}$ is the *boundary* of Ω; the set $\mathrm{int}\,\Omega := \Omega \setminus \Gamma$ is *interior* points of the graph.

A *path* $\pi[a,b] \subset \Omega$ connecting points $a, b \in \Omega$ is the image of the segment [0,1] through a homeomorphism $\varphi : [0,1] \mapsto \Omega$ such that $\varphi(0) = a$, $\varphi(1) = b$; the notations $\pi\,[a,b)$, $\pi\,(a,b]$, $\pi\,(a,b)$ are also of the clear meaning.

Graph Ω is a *tree* if any $a, b \in \Omega$, $a \neq b$ are connected by the unique path (so, there are no cycles in Ω). In this paper we deal mainly with trees.

A tree may be semiordered in a standard way. Namely, let $\xi \in \Omega$ be fixed and declared to be the *root*; for $a, b \in \Omega$ we write $a \preceq b$ ($a \prec b$) if $a \in \pi\,[\xi, b]$ ($a \in \pi\,[\xi, b)$). Of course, this semiorder is determined by the choice of ξ.

1.2 Optical metric.

A function ρ defined on $\Omega \setminus V$ is said to be a *density* if

(i) ρ is strictly positive: $\rho\,(\,\cdot\,) \geq \rho_0 > 0$;

(ii) at each edge $e \in E$, the function $\rho|_e$ may be extended on $\overline{e}$ up to a function of the class $C^2(\overline{e})$.

Thus, the boundary values $\rho(\gamma)$, $\gamma \in \Gamma$ are well-defined whereas at interior vertices $v \in V \setminus \Gamma$ the density may be considered as an $r(v)$-valued function.

Let $|dx|^2$ be the Euclidean length element induced on Ω by the $\mathbf{R}^2$-metric. The *optical metric* τ on the graph is determined by the element

$$d\tau^2 := \rho(x)|dx|^2\,, \qquad x \in \Omega \setminus V\,;$$

by the properties of ρ this metric is equivalent to the Euclidean one. If Ω is a tree one easily has

$$\tau(a,b)=\begin{cases}\int\limits_{\pi[a,b]}\sqrt{\rho}\,|dx|, & a\neq b;\\ 0, & a=b.\end{cases}$$

We denote

$$B_r[a]:=\{x\in\Omega\,\Big|\,\tau(a,x)<r\};$$

the value

$$d(\Omega):=\sup_{a,b\,\in\,\Omega}\tau(a,b);$$

is called by the (optical) diameter of the graph; the value

$$T_*:=\frac{d(\Omega)}{2}$$

is its radius. The point $x_*\in\Omega$ satisfying $\tau(x_*,\Gamma)=T_*$ is called by the (optical) *center* of the graph.

Proposition 1. *The center of a tree is unique.*

1.3 Spaces and classes on graph.

Introduce the (real) Hilbert space $\mathcal{H}=L_{2,\rho}(\Omega)$,

$$(y,w)_{\mathcal{H}}:=\int\limits_{\Omega}yw\,\rho\,|dx|=\sum_{e\in E}\int\limits_{e}yw\,\rho\,|dx|.$$

We assigne a function y on Ω to the class $\mathcal{H}^1_0$ if

(i) $y\in C(\Omega)$;

(ii) $y\,|_{\overline{e}}$ belongs to the Sobolev class $H^1(\overline{e})$ on every edge e;

(iii) $y=0$ on Γ,

and denote

$$(y,w)_{\mathcal{H}^1_0}:=\int\limits_{\Omega}\nabla y\cdot\nabla w\,|dx|=\sum_{e\in E}\int\limits_{e}\frac{dy}{de}\frac{dw}{de}\,|dx|,$$

where ∇ is the Euclidean gradient on Ω, $\frac{d}{de}$ is the differentiation with respect to Euclidean length in any (but one and the same for y and w) direction along e.

We also introduce the tripple (the rigged Hilbert space)

$$\mathcal{H}^1_0\ \subset\ \mathcal{H}\ \subset\mathcal{H}^{-1}$$

where $\mathcal{H}^{-1}$ is dual to $\mathcal{H}^1_0$ with respect to the product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$. As is customary, we denote by $(h,y)_{\mathcal{H}}$ the value of the functional $h\in\mathcal{H}^{-1}$ on the element $y\in\mathcal{H}^1_0$. The well known fact is that the Dirac measures defined by $(\delta_\xi,y)_{\mathcal{H}}=y(\xi)$ belong to $\mathcal{H}^{-1}$.

Let us say that function $y\in\mathcal{H}$ belongs to the class $\mathcal{K}$ if

(i) $y\in C(\Omega)$;

(ii) $y\,|_{\overline{e}}\in C^2(\overline{e})$ on every $e\in E$;

(iii) for each interior vertex $v \in V \setminus \Gamma$ the equality (the Kirchhoff's law)

$$\sum_{e \sim v} \frac{dy}{de}(v) = 0 \tag{1}$$

holds where

$$\frac{dy}{de}(v) := \lim_{x \in e,\, x \to v} \frac{y(x) - y(v)}{|x - v|}.$$

At last, let us agree to say that function y is *piecewise smooth* if there exists a finite set of points $A_y := \{a_1, \ldots, a_N\} \supset V$ such that for any open interval $\omega \subset \Omega \setminus A$ the reduction $y\,|_\omega$ may be extended on $\overline{\omega}$ up to the function of the class $C^2(\overline{\omega})$.

2 Waves on the graph

2.1 Dynamical system.

Speaking about dynamical system associated to the graph, we mean the initial boundary value problem of the form

$$\rho\, u_{tt} - u_{ee} = 0 \qquad in\ \{\Omega \setminus V\} \times (0, T); \tag{2.1}$$

$$u(\,\cdot\,, t) \in \mathcal{K}, \qquad for\ \ all\ \ t \in\ [0, T]; \tag{2.2}$$

$$u\,|_{t=0} = u_t\,|_{t=0} = 0 \quad in\ \Omega; \tag{2.3}$$

$$u = f \quad on\ \Gamma \times [0, T], \tag{2.4}$$

with the (Dirichlet) boundary control $f = f(\gamma, t)$; the solution $u = u^f(x, t)$ describes a wave initiated at Γ and propagating in Ω. Recall the well-known fact: by virtue of the conditions on the density ρ, if f is C^2-smooth with respect to t and vanishes near $t = 0$ the problem has the unique classical solution u^f.

The space of controls $\mathcal{F}^T := L_2\,(\Gamma \times [0, T])$ with the inner product

$$(f, g)_{\mathcal{F}^T} = \sum_{\gamma \in \Gamma} \int_0^T f(\gamma, t)\, g(\gamma, t)\, dt$$

is called by the *outer space* of the system (2.1)–(2.4).

The space $\mathcal{H}$ is the *inner space*; the waves (states) $u^f(\,\cdot\,, t)$ are time-dependent elements of $\mathcal{H}$.

The "input $\to$ state" correspondence is realized by the *control operator* $W^T : \mathcal{F}^T \mapsto \mathcal{H}$,

$$W^T f := u^f(\cdot, T)$$

defined on controls producing the classical solutions to (2.1)–(2.4).

2.2 Propagation of singularities.

All the results of Secs. 2.2 and 2.3 are valid for any graphs (not only for trees).

Fundamental solution. Fix $\gamma \in \Gamma$ and denote

$$\delta_\gamma(\gamma') := \begin{cases} 0, & \gamma' \neq \gamma; \\ 1, & \gamma' = \gamma. \end{cases}$$

Take $\chi \in C^2[0,T]$ such that $\chi \geq 0$, $\operatorname{supp}\chi \subset (0,T)$, and $\int\limits_0^T \chi(t)dt = 1$. Choose a small $\varepsilon > 0$ and denote

$$\chi_\varepsilon(t) := \begin{cases} \frac{1}{\varepsilon}\chi(\frac{t}{\varepsilon}), & 0 \leq t \leq \varepsilon T; \\ 0, & \varepsilon T < t < T; \end{cases}$$

let $u^{\delta_\gamma \chi_\varepsilon}$ be the classical solution to (2.1)–(2.4) corresponding to the control $f(\gamma', t) = \delta_\gamma(\gamma')\chi_\varepsilon(t)$. As may be shown, tending $\varepsilon \to 0$ (so that χ_ε tends to the Dirac δ-function) the limit passage gives a function

$$u^{\delta_\gamma \delta} := \lim_{\varepsilon \to 0} u^{\delta_\gamma \chi_\varepsilon} \in C\left([0,T];\ \mathcal{H}^{-1}\right)$$

which is the *fundamental solution* to the problem (2.1)–(2.4). This solution describes the wave initiated by the instanteneous source supported at γ. In the rest of Sec. 2.2 we consider the properties of $u^{\delta_\gamma \delta}$ in more detail.

First edge. Recall that $\tau(a,b)$ is the optical distance between $a, b \in \Omega$. Let e be the edge incident to γ, and $v \in V$ the second vertex incident to e. For times $0 < t \leq \tau(\gamma, v)$ the well-known representation (time-domain WKB expansion)

$$u^{\delta_\gamma \delta}(x,t) = \left[\frac{\rho(x)}{\rho(\gamma)}\right]^{-\frac{1}{4}} \delta\Big(t - \tau(x,\gamma)\Big) + u_\gamma(x,t)\,, \ \ x \in \Omega \tag{2.5}$$

holds, where u_γ is a bounded function C^2-smooth on $[\operatorname{supp} u_\gamma] \setminus \{(\gamma, 0)\}$ and satisfying $\operatorname{supp} u_\gamma(\,\cdot\,,t) \subseteq \overline{B_t[\gamma]} \subseteq \overline{e}$ (see e.g. [4]). The first term in (2.5) is interpreted as the leading singularity which moves from γ towards v along e with variable speed $\rho^{-1/2}$: see the point l on Fig. 1a.

Passing through the first vertex. At the moment $t = \tau(\gamma, v)$ the singularity reaches v and, then, passes through v. A simple analysis using (2.5) and the Kirchhoff law (1.1) gives the following representation for times $\tau(\gamma, v) < t \leq \tau(\gamma, v) + \varepsilon$ (with small $\varepsilon > 0$):

$$u^{\delta_\gamma \delta}(x,t) = \alpha(x)\,\delta\Big(t - \tau(\gamma, v) - \tau(v, x)\Big) + u_\gamma(x,t)\,, \ \ x \in \Omega \tag{2.6}$$

with a bounded u_γ satisfying $\operatorname{supp} u_\gamma(\,\cdot\,,t) \subseteq \overline{B_t[\gamma]}$ and piecewise smooth on $\operatorname{supp} u_\gamma(\,\cdot\,,t)$. The *amplitude* α is a piecewise smooth function strictly positive on another edges $e' \sim$

v , $e' \neq e$ incident to v.

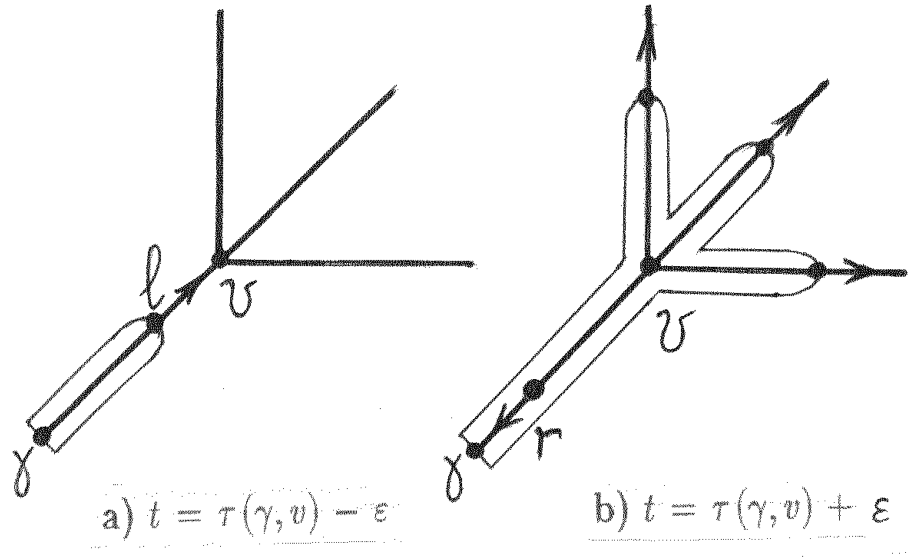

FIG. 1. PASSING THROUGH v

Note in addition that $\alpha\,|_e$ can be negative. From the physical point of view, this means that the reflected part of the singularity moving back to γ (point r on Fig. 1b) can have the phase opposite to one of the incident singularity (point l on Fig. 1a; in both cases the ball $B_t[\gamma]$ is contoured).

First reflection from the boundary. Let $\gamma' \in \Gamma$ be a boundary vertex nearest to v:

$$\tau(\gamma', v) = \min_{\gamma'' \in \Gamma} \tau(\gamma'', v)$$

(may be $\gamma = \gamma'$). As $t \to \tau(\gamma, v) + \tau(v, \gamma') - 0$, the singularity passed through v (and, perhaps, through another vertices or reflected from v back to γ) approaches to γ' (see the point l on Fig. 2a). Then the singularity is reflected by γ' (the point r on Fig. 2b). A simple analysis using the condition

$$u^{\delta_\gamma \delta}(\,\cdot\,, t)\,|_\Gamma\; = 0\,, \quad t > 0$$

leads to the representation

$$u^{\delta_\gamma \delta}(x, t) = \alpha(x)\,\delta\Big(t - \tau(\gamma, v) - \tau(v, \gamma') - \tau(\gamma', x)\Big) + u_\gamma(x, t) \tag{2.7}$$

which is valid for times $\tau(\gamma, v) + \tau(v, \gamma') < t \leq \tau(\gamma, v) + \tau(v, \gamma') + \varepsilon$ at points x close to γ', with a bounded piecewise smooth u_γ. If $\gamma' \neq \gamma$, the amplitude α is negative which means that reflection from the boundary inverts the phase of the incident singularity.

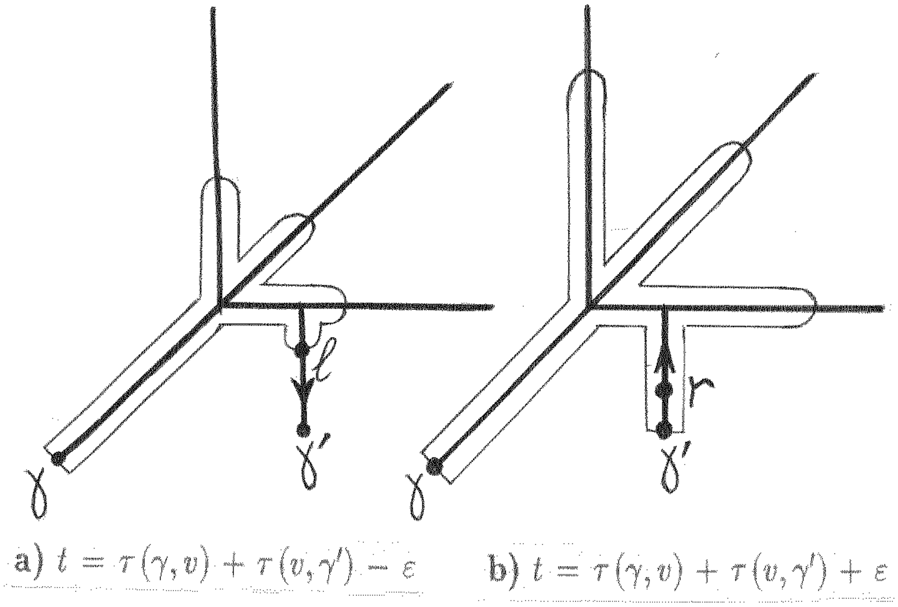

FIG. 2. REFLECTING FROM $\gamma' \neq \gamma$

Arbitrary times. Representations (2.5)–(2.7) are sifficient for further analysis of propagation of singularities in Ω. Omitting technical details, let us summarize the results.

In the space-time domain $\Omega \times \overline{\mathbf{R}}_+$, for a fixed (x_0, t_0) let us define the *characteristic cone*

$$\mathcal{C}_{(x_0,t_0)} := \Big\{(x,t)\,|\,t - t_0 = \tau(x, x_0)\Big\};$$

for a subset $A \subset \Omega \times \overline{\mathbf{R}}_+$ we denote

$$\mathcal{C}_A := \bigcup_{(x,t)\,\in\, A} \mathcal{C}_{(x,t)} \quad .$$

Introduce also the *characteristic set* $\mathcal{T}_{(x_0,t_0)}$ which is defined by the following recurrent procedure:

Step 0: put

$$\mathcal{T}^{[0]}_{(x_0,t_0)} := \mathcal{C}_{(x_0,t_0)}$$

and

$$\mathcal{V}^{[0]}_{(x_0,t_0)} := \{\,(x,t) \in \mathcal{T}^{[0]}_{(x_0,t_0)}\,|\, x \in V\};$$

....

Step j = 1,2,... : put

$$\mathcal{T}^{[j]}_{(x_0,t_0)} := \mathcal{T}^{[j-1]}_{(x_0,t_0)} \bigcup \mathcal{C}_{\mathcal{V}^{[j-i]}_{(x_0,t_0)}}$$

and

$$\mathcal{V}^{[j]}_{(x_0,t_0)} := \{ (x,t) \in \mathcal{T}^{[j]}_{(x_0,t_0)} \,|\, x \in V \} ;$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ;$$

define

$$\mathcal{T}_{(x_0,t_0)} := \bigcup_{j=0}^{\infty} \mathcal{T}^{[j]}_{(x_0,t_0)} \quad .$$

Note that $\mathcal{T}_{(x_0,t_0)}$ can be also characterized as the minimal subset in $\Omega \times \overline{\mathbf{R}}_+$ sutisfying the conditions:

(i) $\mathcal{C}_{(x_0,t_0)} \subset \mathcal{T}_{(x_0,t_0)}$;

(ii) if $v \in V$ and $t_v \in \mathbf{R}_+$ are such that $(v,t_v) \in \mathcal{T}_{(x_0,t_0)}$ then $\mathcal{C}_{(v,t_v)} \subset \mathcal{T}_{(x_0,t_0)}$.

This set is of the form of a space-time graph; Fig.3 illustrates the case of $x_0 = \gamma$, $t_0 = 0$.

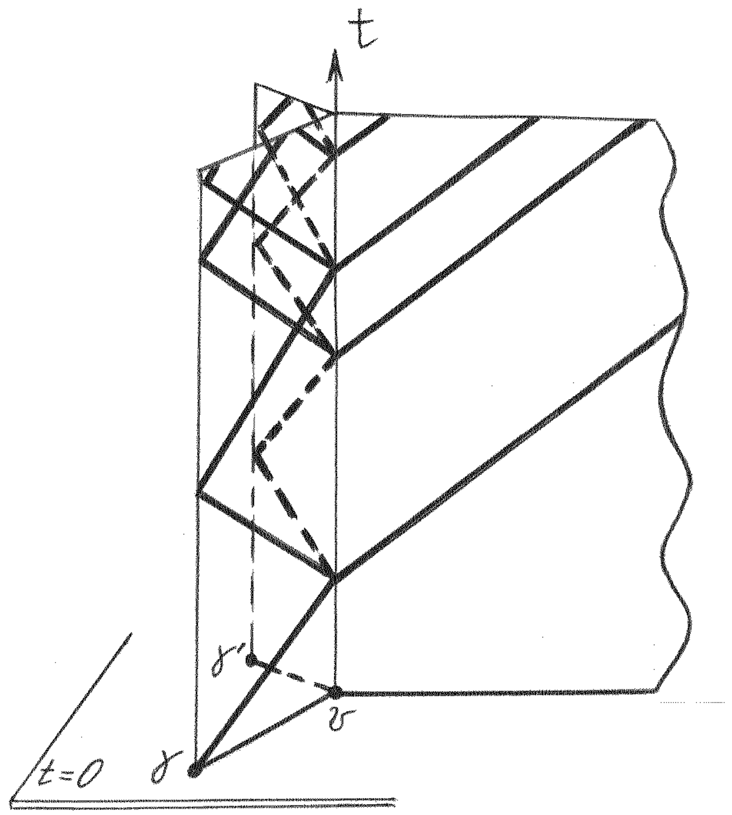

FIG. 3. THE SET $C_{(\gamma,0)}$

We also denote by

$$\mathcal{C}^s_{(x_0,t_0)} := \{ x \in \Omega \,|\, (x,s) \in \mathcal{C}_{(x_0,t_0)} \} ; \quad \mathcal{T}^s_{(x_0,t_0)} := \{ x \in \Omega \,|\, (x,s) \in \mathcal{T}_{(x_0,t_0)} \}$$

the crossections by the plane $t = s$; notice the simple equality

$$\mathcal{C}^s_{(x_0,t_0)} = \{ x \in \Omega \,|\, \tau(x,x_0) = s - t_0 \} .$$

Proposition 2. *The leading singularities of* $u^{\delta_\gamma \delta}$ *(as a space-time distribution) are located on the characteristic set* $\mathcal{T}_{(\gamma,0)}$. *For any* $t > 0$ *the representation*

$$u^{\delta_\gamma \delta}(\,\cdot\,,t) = \sum_{\xi \in \mathcal{C}^t_{(\gamma,0)}} \alpha_\gamma(\xi)\,\delta_\xi(\,\cdot\,) + \sum_{\eta \in \mathcal{T}^t_{(\gamma,0)} \setminus \mathcal{C}^t_{(\gamma,0)}} \beta_\gamma(\eta,t)\,\delta_\eta(\,\cdot\,) + u_\gamma(\,\cdot\,,t) \tag{2.8}$$

holds in Ω *where:*

$\delta_\xi \in \mathcal{H}^{-1}$ *are the Dirac measures;*

the amplitude function α_γ *is strictly positive on* Ω *piecewise smooth and* C^2*-smooth out of the interior vertices; the value* $\alpha_\gamma(\xi)$ *is determined by the density* $\rho\,|_{\pi[\gamma,\xi]}$ *;*

β_γ *is a piecewise smooth function defined on* $\mathcal{T}_{(\gamma,0)} \setminus \mathcal{C}_{(\gamma,0)}$ *;*

$u_\gamma(\,\cdot\,,t)$ *is a bounded piecewise smooth function (with possible discontinuities on* $\mathcal{T}^t_{(\gamma,0)}$*) satisfying* $\operatorname{supp} u_\gamma(\,\cdot\,,t) \subset \overline{B_t[\gamma]}$.

The first sum in (2.8) selects the so-called *forward* singularities which lie on the forward front of the wave $u^{\delta_\gamma \delta}(\,\cdot\,,t)$ moving from γ. These singularities are present only if $t < \max_{x\in\Omega} \tau(x,\gamma)$ whereas $\alpha_\gamma(\xi)$ is the amplitude of the first (in time) singularity arriving at ξ from γ at the moment $t = \tau(\xi,\gamma)$.

The second sum contains the *secondary* singularities appearing in result of reflections from vertices and supported into $B_t[\gamma]$.

A typical picture is shown on Fig. 4 where the points $\xi_1, \dots, \xi_5$ support the forward singularities whereas $\xi_6, \dots, \xi_9$ support the secondary ones (the ball $B_t[\gamma]$ is contoured).

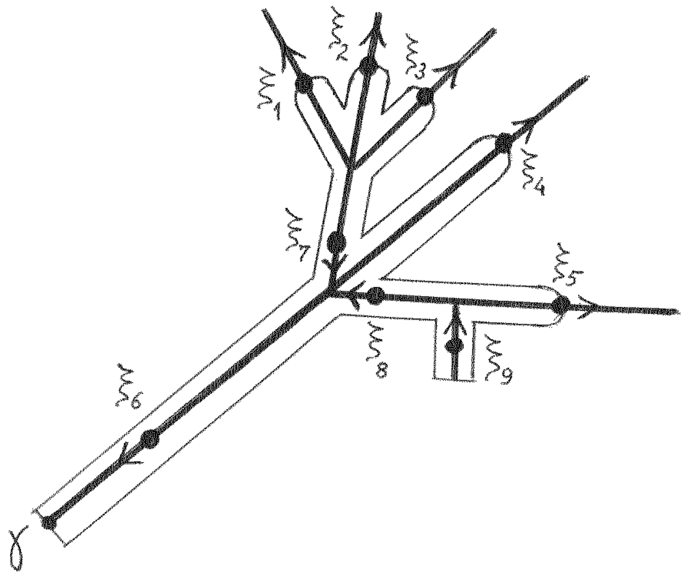

Fig. 4. Singularities of $u^{\delta_\gamma \delta}(\cdot,t)$

2.3 Generalized solutions.

Here we describe some of the properties of the solutions to the problem (2.1)–(2.4) which may be easily derived from the representation (2.8).

Let

$$\mathcal{F}^T_\gamma := \left\{ f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \gamma \times [0,T] \right\}$$

be the subspace of controls acting from $\gamma \in \Gamma$; introduce the classes

$$(\mathcal{F}^T)^1_0 := \left\{ f \in \mathcal{F}^T \mid f(\,\cdot\,,t) = \sum_{\gamma\,\prime \in \Gamma} \delta_{\gamma\,\prime}(\,\cdot\,) g_{\gamma\,\prime}(t)\,, \quad g_{\gamma\,\prime} \in H^1_0\,[0,T] \right\}$$

(so that controls belonging to this class satisfy $f(\,\cdot\,,0) = f(\,\cdot\,,T) = 0$) and

$$(\mathcal{F}^T_\gamma)^1_0 := (\mathcal{F}^T)^1_0 \bigcap \mathcal{F}^T_\gamma\,.$$

Let $\mathcal{H}^1 := H^1(\Omega) \supset \mathcal{H}^1_0$ be the Sobolev class of functions on Ω satisfying (i), (ii), Sec. 1.4.

(i) (*The Duhamel formula*). If f is such that u^f is classical, the representation

$$u^f(x,t) \;=\; \sum_{\gamma \in \Gamma} \left[u^{\delta_\gamma \delta}(x,\,\cdot\,) * f(\gamma,\,\cdot\,) \right] (\,t\,) \tag{2.9}$$

(the convolution with respect to time) holds.

(ii) (*Generalized solutions*). For any $f \in \mathcal{F}^T$ the right hand side of (2.9) is well-defined and belongs to the class $C\,([0,T]\,;\,\mathcal{H})$; it defines the (generalized) solution u^f to (2.1)–(2.4) of this class. As result, the control operator $W^T \,:\, \mathcal{F}^T \mapsto \mathcal{H}$ is continuous that is the well-known fact (see e.g. [1]).

(iii) (*Localization of solutions*). For controls $f \in \mathcal{F}^T_\gamma$ the inclusion

$$\operatorname{supp} u^f(\,\cdot\,,t) \subset \overline{B_t[\gamma]}\,, \quad t \ge 0$$

holds.

(iv) (*Delayed solutions*). For $f \in \mathcal{F}^T$ denote

$$f_{t_0}(\,\cdot\,,t) := \begin{cases} 0, & 0 \le t < t_0\,; \\ f(\,\cdot\,,t - t_0), & t_0 \le t \le T\,. \end{cases}$$

Since the graph and its density do not depend on time, one has

$$u^{f_{t_0}}(\,\cdot\,,t) = u^f(\,\cdot\,,t - t_0)\,. \tag{2.10}$$

As result, if $f(\,\cdot\,,t) = 0$ for $0 \le t \le t_0$ then one has

$$\operatorname{supp} u^f(\,\cdot\,,t) \subset \overline{B_{t-t_0}[\gamma]}\,, \quad t \ge 0\,. \tag{2.11}$$

(v) (*Smoothness*). In the case of $f \in (\mathcal{F}^T)^1_0$ one has $u^f(\,\cdot\,,t) \in \mathcal{H}^1$ for all $t \in [0,T]$ and $u^f(\,\cdot\,,T) \in \mathcal{H}^1_0$.

3.2 Boundary control problem.

Here we consider the question: Managing the boundary control, can one create the waves of prescribed shape? In rigorous formulation, this is a variant of the boundary control problem for the system (2.1)–(2.4) (see e.g. [1]): given function $y \in \mathcal{H}$ to find a control $f \in \mathcal{F}^T$ providing

$$u^f(\,\cdot\,, T) \,=\, y\,. \tag{3.1}$$

In the case that Ω is a tree, Theorem 1 gives a positive answer: for times $T \geq T_*$ the equation (3.1) turns out to be solvable with respect to f. For the reader's convenience, the proof is devided on a few parts presented in sec. 3.2–3.6 Recall that radius T_* and center x_* of the tree were introduced in Sec. 1.2.

3.2 Control f_ξ.

In what follows we assume Ω semiordered with respect to the root x_* and say that b lies back of a if $b \succ a$. To each $a \in \operatorname{int}\Omega,\ a \neq x_*$ we associate the subtree

$$\Omega_a := \{x \in \Omega \,|\, x \succeq a\} \,=\, \{x \in \Omega \,|\, \tau(x, x_*) \geq \tau(a, x_*)\}$$

and denote $\Gamma_a := \Omega_a \bigcap \Gamma$.

Fix a point $\xi \in \Omega \backslash V,\ \xi \neq x_*$; let Ω_ξ be the corresponding subtree, and the vertex $\gamma_\xi \in \Gamma_\xi$ such that

$$\tau(\xi, \gamma_\xi) = \max_{\gamma \in \Gamma_\xi} \tau(\xi, \gamma)\,;$$

we call γ_ξ by the *controlling* vertex. Note the simple relations $\tau(\xi, \gamma_\xi) < T_*$ and $\overline{B_{\tau(\xi,\gamma_\xi)}[\gamma_\xi]} \subseteq \Omega_\xi$.

Return to the system (2.1)–(2.4) and take $T = T_*$. In this part of the proof we will establish that there exists a (singular) control f_ξ such that the relation

$$u^{f_\xi}(\,\cdot\,, T_*) \,=\, \alpha(\xi)\, \delta_\xi(\,\cdot\,) + u_\xi(\,\cdot\,) \quad in\ \ \Omega \tag{3.2}$$

holds with a piecewise smooth strictly positive α and a piecewise smooth u_ξ satisfying $\operatorname{supp} u_\xi \subset \Omega_\xi$. The control f_ξ will be found in the form

$$f_\xi(\gamma, t) = \delta_{\gamma_\xi}(\gamma)\, \delta\left(t - [T_* - \tau(\xi, \gamma_\xi)]\right) +$$

$$+ \sum_{\gamma' \in \Gamma} \delta_{\gamma'}(\gamma) \sum_{j=1}^{N(\gamma')} \kappa^j_{\gamma'}\, \delta\left(t - [T_* - t^j_{\gamma'}]\right) \tag{3.3}$$

with coefficients κ^j_γ and times t^j_γ satisfying $t^j_\gamma < \tau(\xi, \gamma_\xi)$, i.e. in the form of a combination of delayed pointwise instanteneous sources. Correspondingly, u^{f_ξ} will be a linear combination of the delayed fundamental solutions.

Step 1 (creating the singularity at ξ). Take

$$f^{(0)}_\xi(\gamma, t) \,:=\, \delta_{\gamma_\xi}(\gamma)\, \delta\Big(t - [T_* - \tau(\xi, \gamma_\xi)]\Big)\,.$$

In accordance with the delay relation (2.10) one has

$$u^{f_\xi^{(0)}}(\,\cdot\,,T_*) = u^{\delta_{\gamma_\xi}\delta}(\,\cdot\,,\tau(\xi,\gamma_\xi))$$

whereas (2.8) gives the representation

$$u^{f_\xi^{(0)}}(\,\cdot\,,T_*) \;=\; \alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,) + u_\xi^{(0)}(\,\cdot\,) \tag{3.4}$$

where the term $u_\xi^{(0)} \in \mathcal{H}^{-1}$ satisfies $\operatorname{supp} u_\xi^{(0)} \subset \overline{B_{\tau(\xi,\gamma_\xi)}[\gamma_\xi]} \subset \overline{\Omega_\xi}$ and is regular (doesn't contain Dirac measures) in a neighbourhood of ξ. The control $f_\xi^{(0)}$ is considered as a zero iteration of the required f_ξ: it creates the wanted singularity at the point ξ.

However, in general case, the term $\alpha_{\gamma_\xi}\delta_\xi$ in (3.4) is not the unique singularity of $u^{f_\xi^{(0)}}(\cdot,T_*)$: control $f_\xi^{(0)}$ can produce also another singularities in Ω_ξ (see (2.8)). Setting the goal to provide (3.2), these additional singularities are undesirable and must be removed.

Step 2 (removing the nearest singularity). Detalizing $u_\xi^{(0)}$ in (3.4), one can write

$$u^{f_\xi^{(0)}}(\,\cdot\,,T_*) \;=\; \alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,) + \beta\,\delta_\eta(\,\cdot\,) + u_\xi^{(00)}(\,\cdot\,) \tag{3.5}$$

where $\eta \in \Omega_\xi$ is a point which supports the singularity of $u^{f_\xi^{(0)}}(\,\cdot\,,T_*)$ nearest (in optical metric) to ξ and lying back of ξ, whereas $u_\xi^{(00)}$ is regular on $\pi\,[\xi,\eta]$.

Consider the subtree $\Omega_\eta \subset \Omega_\xi$, choose its controlling vertex γ_η, and define

$$g_\eta^{(0)}(\gamma,t) := -\frac{\beta}{\alpha_{\gamma_\eta}(\eta)}\,\delta_{\gamma_\eta}(\gamma)\,\delta\Big(t-[T_*-\tau(\eta,\gamma_\eta)]\Big)\,.$$

By virtue of the evident inequality $\tau(\eta,\gamma_\eta) < \tau(\xi,\gamma_\xi)$ the control $g_\eta^{(0)}$ is delayed with respect to $f_\xi^{(0)}$. In accordance with (2.8), (2.10), (2.11), such the control creates the wave

$$u^{g_\eta^{(0)}}(\,\cdot\,,T_*) \;=\; -\beta\,\delta_\eta(\,\cdot\,) + u_\eta^{(0)}(\,\cdot\,) \tag{3.6}$$

with $u_\eta^{(0)}$ supported in $\overline{\Omega_\eta}$ and regular near η.

Denoting

$$f_\xi^{(1)} := f_\xi^{(0)} + g_\eta^{(0)}$$

and comparing (3.5) with (3.6) one easily obtains

$$u^{f_\xi^{(1)}}(\,\cdot\,,T_*) \;=\; \alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,) + u_\xi^{(1)}(\,\cdot\,) \tag{3.7}$$

where $u_\xi^1 := u_\xi^{(00)} + u_\eta^{(0)}$ is supported in $\overline{\Omega_\xi}$ and is regular on $\pi\,[\xi,\eta]$. Thus, the first iteration $f_\xi^{(1)}$ preserves the singularity wanted at ξ and removes the unnecessary one at η. In other words, we succeded to clean of singularities a part of $\Omega_\xi\backslash\{\xi\}$ containing the segment $\pi\,(\xi,\eta]$.

Step 3 (extending the part cleaned). Detalizing (3.7), one has

$$u^{f_\xi^{(1)}}(\,\cdot\,, T_*) \;=\; \alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,) + \beta'\,\delta_{\eta'}(\,\cdot\,) + u_\xi^{(11)}(\,\cdot\,)$$

where $\eta' \in \Omega_\xi$ is a point which supports a singularity of $u^{f_\xi^{(1)}}(\,\cdot\,, T_*)$ nearest to ξ and, obviously, satisfies $\tau(\eta', \xi) \geq \tau(\eta, \xi)$ whereas $u_\xi^{(11)}$ is regular on $\pi\,[\xi, \eta']$. Applying the trick of Step 2 to the subtree $\Omega_{\eta'}$, one can remove $\beta'\,\delta_{\eta'}$ and clean $\pi\,(\xi, \eta']$ of singularities.

Continuing this procedure, on each step we preserve $\alpha_{\gamma_\xi}\,\delta_\xi$ and extend the part of $\Omega_\xi \setminus \{\xi\}$ free of singularities. At the same time, removing a next singularity, we create the new ones lying back of the removed. Therefore, one needs to justify that $\Omega_\xi \setminus \{\xi\}$ will be exhausted in a whole after a finite number of elementary operations described in *Steps 1–3.*

3.3 Justification.

(i) The nearest singularity at η mentioned in Step 2 can not lie too close to ξ; namely, the estimate

$$\tau(\xi, \eta) \;\geq\; 2\min\left\{\tau(\xi, v_\xi)\,,\; \min_{e\in E} d(e)\right\} \tag{3.8}$$

holds where $v_\xi \in \Omega_\xi \cap V$ is the vertex nearest to ξ (back of ξ). Let us prove this inequality.

Begin with the case that η is a secondary singularity of the wave $u^{f_\xi^{(0)}}(\,\cdot\,, T_*)$ lying at the same edge as ξ (see Fig. 5a). The unique possible mechanism giving rise such the secondary singularity is the reflection from endpoins of the edges of Ω_ξ which either abut on or are incorporated into the path $\pi[\gamma_\xi, \xi]$. The forward singularity initiated at γ_ξ moves along this path and, when passing through vertices $v_1, v_2, \dots, v_\xi \in \pi[\gamma_\xi, \xi]$, produces singularities moving along the edges mentioned above. The second endpoints of the edges reflect these singularities; they return back to $v_1, v_2, \dots, v_\xi$, pass through and, eventually, turn out to be injected into $\pi[\gamma_\xi, \xi]$. Then such the singularities move towards ξ in the wake of the forward one (see Fig. 5a: the solid line marks the trajectory of the forward singularity; the dotted line is the trajectory giving rise the secondary singularity at η). Taking into account the travel time along the edges and back, we have

$$\tau(\xi, \eta) \;\geq\; 2\min\{d(e)\,|\,\overline{e} \cap \pi[\gamma_\xi, \xi] \;\neq\; \{\emptyset\}\,\} \;\geq\; \lambda \tag{3.9}$$

where

$$\lambda \;:=\; 2\,\min_{e\in E} d(e)\,.$$

At the same time, if ξ is close enough to v_ξ, the nearest singularities (forward and secondary) at $\eta, \eta', \dots$ are separated from ξ with the distance

$$\tau(\xi, \eta) \;=\; 2\,\tau(\xi, v_\xi) \tag{3.10}$$

(see Fig. 5b). Comparing (3.9) with (3.10) we get (3.8).

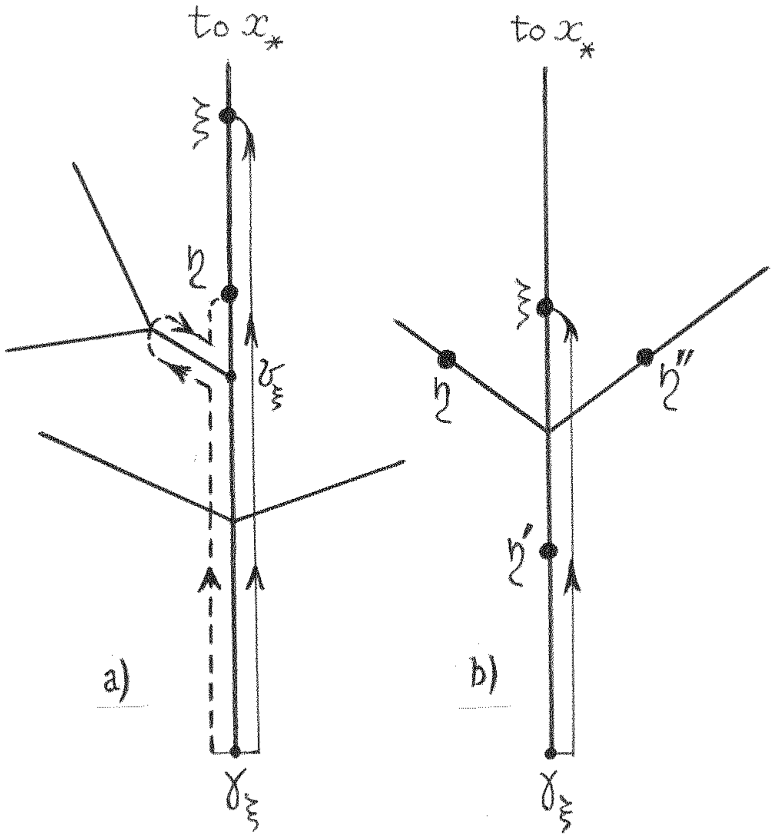

FIG. 5. SEPARATION OF SINGULARITIES

(ii) Assume that the edge $e_\xi \ni \xi$ contains secondary singularities of $u^{f_\xi^{(0)}}(\,\cdot\,, T_*)$ (see Fig. 5a); let us estimate the number of elementary operations required to remove them (to clean e_ξ).

Recall that all the singularities of $u^{f_\xi^{(0)}}(\,\cdot\,, T_*)$ can be located only at the points of the crossection of the characteristic set $\mathcal{T}^{T_*}_{(\gamma_\xi, T_*-\tau(\gamma_\xi,\xi))}$. Introduce the "density of singularities"

$$p := \max_{\substack{(\gamma,t)\in\Gamma\times[0,T_*)\\ a,b\,\in\Omega:\ \tau(a,b)=\lambda}} \sharp\left\{\mathcal{T}^{T_*}_{(\gamma,t)} \bigcap \pi\,[\,a\,,\,b\,]\right\},$$

and represent

$$\pi\,[\,\xi\,,\,v_\xi\,)\;=\;\bigcup_{j=1}^{n_\xi+1}\pi\,[\,\xi_{j-1}\,,\,\xi_j\,)\,,$$

where

$$\xi =: \xi_0 \prec \xi_1 \prec \cdots \prec \xi_{n_\xi+1} := v_\xi\,; \quad \tau(\xi_{j-1},\xi_j)\;=\;\lambda\,,\;\; j=1,\ldots,$$

$$n_\xi\;:=\;\mathrm{Ent}\,\frac{\tau(\xi, v_\xi)}{\lambda}$$

(Ent is the integer part), so that $\tau(\xi, v_\xi) = n_\xi\,\lambda + \tau(\xi_{n_\xi}, \xi_{n_\xi+1})$ (Fig. 6 corresponds to $n_\xi = 3$).

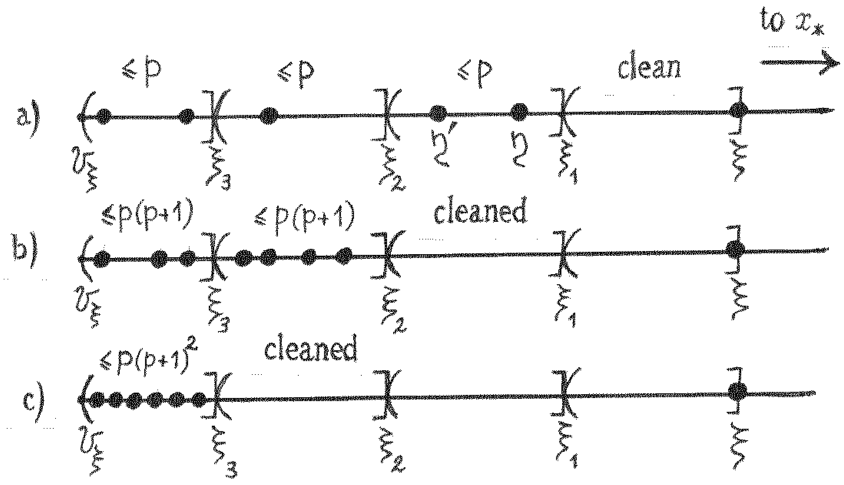

FIG. 6. CLEANING PROCEDURE

Returnt to the solution $u^{f_\xi^{(0)}}(\,\cdot\,, T_*)$; at the beginning of the cleaning procedure the situation is the following. By virtue of (3.8), the part $\pi\,[\,\xi_0\,,\,\xi_1\,)$ contains the unique (forward) singularity at $\xi_0 = \xi$ whereas each of the parts $\pi\,[\,\xi_j\,,\,\xi_{j+1}\,)$, $j = 1, 2, \dots, n_\xi$ can contain at most p secondary singularities (see η, η' on Fig.6a). Therefore, to remove the secondary singularities from $\pi\,[\,\xi_1\,,\,\xi_2\,)$ one needs no more than p elementary operations.

Every single operation can induce additional singularities back of the one being removed but, by virtue of (3.8), only into the parts $\pi\,[\,\xi_2\,,\,\xi_3\,)$, $\pi\,[\,\xi_3\,,\,\xi_4\,)$ ... and not more than p singularities per each of these parts. Therefore, once $\pi\,[\,\xi_1\,,\,\xi_2\,)$ has been cleaned, the picture turns out to be the following (see Fig. 6b). Each of the parts $\pi\,[\,\xi_2\,,\,\xi_3\,)$, $\pi\,[\,\xi_3\,,\,\xi_4\,), \dots$ contains at most $p + p\cdot p = p(p+1)$ singularities, and to clean $\pi\,[\,\xi_2\,,\,\xi_3\,)$ one needs at most $p(p+1)$ operations. Realizing them, we add some a number of singularities on the parts $\pi\,[\,\xi_3\,,\,\xi_4\,), \dots$; in result, on each of $\pi\,[\,\xi_j\,,\,\xi_{j+1}\,)$, $j > 3$ it turns out to be no more than $p(p+1) + p\cdot p(p+1) = p(p+1)^2$ singularities (see Fig. 6c).

Going on the procedure, we can easily count that to create $\alpha\delta_\xi$ and to clean $\pi\,(\,\xi\,,\,v_\xi\,)$ one needs at most

$$1 + p + p(p+1) + p(p+1)^2 + \cdots + p(p+1)^{n_\xi - 1} =$$
$$= 1 + p\,\frac{(p+1)^{n_\xi} - 1}{(p+1) - 1} = (p+1)^{n_\xi}$$

elementary operations.

(iii) After e_ξ one needs to clean another edges incident to v_ξ which already contain singularities appeared when e_ξ was being cleaned, and so on, cleaning the rest of Ω_ξ. A simple count of elementary operations gives the following (rather rough) estimate: total number of them doesn't exceed

$$(\,p + 1\,)^A, \quad A := \mathrm{Ent}\left\{\frac{\sum_{e\in E} d(e)}{\lambda}\right\} + 1\,.$$

So, our procedure cleans $\Omega_\xi \setminus \{\xi\}$ in a finite number of operations. The cases $\xi = x_*$ and/or $\xi \in V$ can be considered quite analogously.

This completes the construction of the control f_ξ providing the relation (3.2). Let us give an illustration for a simple graph.

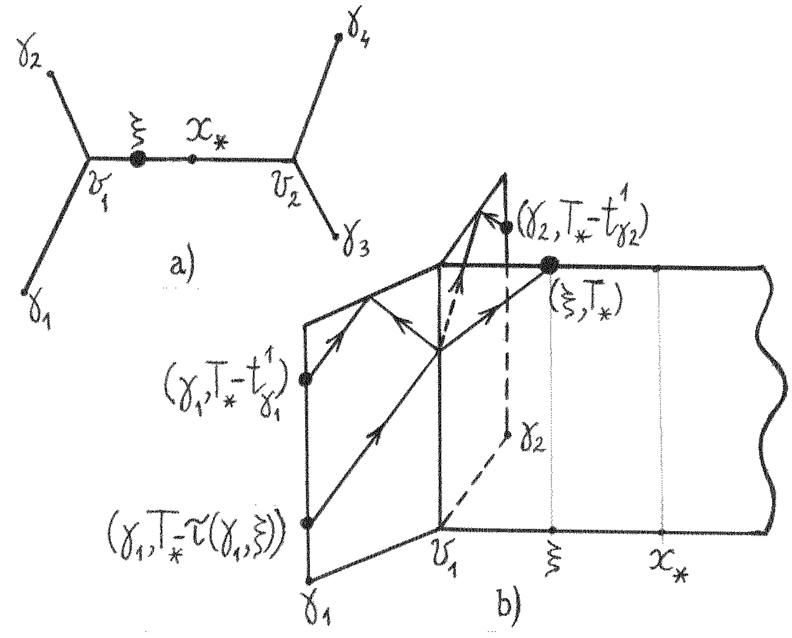

Fig. 7. Control f_ξ

For ξ shown at Fig.7a, the controlling vertex is γ_1 and the control (3.3) takes the form:

$$f_\xi(\gamma, t) = \delta_{\gamma_1}(\gamma)\,\delta\Big(t - [T_* - \tau(\gamma_1, \xi)]\Big) + \kappa^1_{\gamma_1}\delta_{\gamma_1}(\gamma)\,\delta\Big(t - [T_* - t^1_{\gamma_1}]\Big) +$$

$$+ \kappa^1_{\gamma_2}\delta_{\gamma_2}(\gamma)\,\delta\Big(t - [T_* - t^1_{\gamma_2}]\Big)\,.$$

The required singularity at (ξ, T_*) is created by the forward singularity starting from $(\gamma_1, T_* - \tau(\gamma_1, \xi))$. Passing through v_1, it produces two additional singularities moving from v_1 to γ_1 and to γ_2. These (unnecessary) singularities must be removed; at the moment $t = T_*$ they are compensated by the singularities arriving from $(\gamma_1, T_* - t^1_{\gamma_1})$ and $(\gamma_2, T_* - t^1_{\gamma_2})$.

In conclusion, let us notice a pecularity of the procedure described above. Once an edge has been cleaned, beginning to clean a next one, we need to choose a next controlling vertex which can be not unique. Up to this choice, the procedure is well-determined and gives a certain f_ξ. Nevertheless, it is not difficult to see that, in the case of arbitrary tree, another versions of the procedure solving the same problem can exist. In other words, in general case, the control f_ξ which produces the wave supported in $\overline{\Omega_\xi}$ and possessing the required singularity at ξ is not unique.

Note in addition that a more delicate analysis shows that, adding regular terms to the right-hand side of (3.3), one can construct f_ξ such that (3.2) holds with $u_\xi \in C^k(\Omega)$ with any prescribed $k = 0, 1, \ldots$.

3.4 Dependence on ξ.

Return again to the illustration on Fig. 7. Playing the role of parameters of f_ξ, the values $\tau(\gamma_1,\xi)\,,\kappa^1_{\gamma_1}\,,\kappa^1_{\gamma_2}\,,t^1_{\gamma_1}\,,t^1_{\gamma_2}\,,$ depend on ξ in themselves. As is easy to see, they are piecewise smooth functions of $\xi\in\Omega$ smooth out of the vertices v_1, v_2, and the relations

$$-\,\frac{d\,\tau(\gamma_1,\xi)}{d\,\xi}\;=\;\left|\frac{d\,t^1_{\gamma_i}}{d\,\xi}\right|\;=\;1$$

hold where $\frac{d}{d\,\xi}\;:=\;\frac{d}{d\,\tau(x_*,\xi)}$. These relations show that the maps $\xi\mapsto\tau(\gamma_1,\xi)\,,\;\xi\mapsto t^1_{\gamma_i}(\xi)$ are local diffeorphisms from Ω into $\mathbf{R}$.

A simple analysis shows that such the dependence on ξ is a fact of general character.

Proposition 3. *Out of a finite set of points $\Xi=\{\xi_1,\dots,\xi_q\}\subset\Omega$ including the interior vertices of the tree, the parameters γ_ξ, $\tau(\gamma_\xi,\xi)$, $N_\gamma=N_\gamma(\xi)$, $\kappa^j_{\gamma_i}=\kappa^j_{\gamma_i}(\xi)$, $t^j_{\gamma_i}=t^j_{\gamma_i}(\xi)$ entering into representation (3.3) depend on ξ smoothly. The maps $\xi\mapsto\gamma_\xi$ from Ω into Γ and $\xi\mapsto N_\gamma(\xi)$ from Ω into $\mathbf{R}$ are piecewise constant. The functions $\tau(\gamma_\xi,\xi)$, $\kappa^j_\gamma(\xi)$, $t^j_\gamma(\xi)$ of $\xi\in\Omega$ are piecewise smooth on their natural domains of definitions whereas the relations*

$$-\,\frac{d\,\tau(\gamma_\xi,\xi)}{d\,\xi}\;=\;\left|\frac{d\,t^j_\gamma}{d\,\xi}\right|\;=\;1$$

are valid for $\xi\notin\Xi$, so that $\xi\mapsto\tau(\gamma_\xi,\xi)$ and $\xi\mapsto t^j_\gamma(\xi)$ are the local diffeomorphisms.

Let $\omega=\pi(\xi',\xi'')\subset\Omega$ be a small enough open interval which doesn't contain exeptional points $\xi_1,\dots,\xi_q$ and let $\gamma_\xi\equiv\tilde\gamma$, $N_\gamma(\xi)\equiv N_\gamma$ for $\xi\in\omega$ whereas $t^j_\gamma(\,\cdot\,):\omega\mapsto\mathbf{R}$ are diffeomorphisms. Taking $\psi\in\mathcal{H}$ such that $\operatorname{supp}\psi\subset\omega$, one can define a superposition $\int\limits_\omega f_\xi\,\psi(\xi)\,|d\xi|$ of functionals (distributions) f_ξ by setting

$$\Bigl(\int\limits_\omega f_\xi\,\psi(\xi)\,|d\xi|\,,\,g\Bigr)_{\mathcal{F}^{T_*}}\;:=\;\int\limits_\omega (f_\xi,g)_{\mathcal{F}^{T_*}}\,\psi(\xi)\,|d\xi|$$

for test functions $g\in(\mathcal{F}^{T_*})^1_0$. Taking into account Proposition 3, representation (3.3) easily gives:

$$\int\limits_\omega f_\xi(\gamma,t)\,\psi(\xi)\,|d\xi|\;=\;\delta_{\tilde\gamma}(\gamma)\,\psi\Bigl(\tau^{-1}(T_*-t,\tilde\gamma)\Bigr)+$$

$$+\sum_{\gamma'\in\Gamma}\delta_{\gamma'}(\gamma)\sum_{j=1}^{N(\gamma')}s^j_{\gamma'}\,\kappa^j_{\gamma'}\Bigl((t^j_{\gamma'})^{-1}(T_*-t)\Bigr)\,\psi\Bigl((t^j_{\gamma'})^{-1}(T_*-t)\Bigr)\,,\qquad(\gamma,t)\in\Gamma\times[0,T]\,,$$

where $\tau^{-1}(\,\cdot\,,\tilde\gamma)$, $(t^j_\gamma)^{-1}$ are the functions inverse to $\tau(\,\cdot\,,\tilde\gamma)$, t^j_γ; $s^j_\gamma=\pm1$. Using this explicit expression, it is not difficult to show that the map $\psi\mapsto\int\limits_\omega f_\xi\,\psi(\xi)\,|d\xi|$ is continuous from $L_2(\omega)$ into $\mathcal{F}^{T_*}$.

Covering $\Omega\setminus\Xi$ by intervals ω, we easily arrive to the following result. For any $\psi\in\mathcal{H}$ the superposition $\int\limits_\Omega f_\xi\,\psi(\xi)\,|d\xi|$ is well-posed as an element of $\mathcal{F}^{T_*}$ whereas the map

$$Q^{T_*}\;:\;\psi\mapsto\int\limits_\Omega f_\xi\,\psi(\xi)\,|d\xi|$$

is a linear continuous operator from $\mathcal{H}$ into $\mathcal{F}^{T_*}$.

Using this fact, it is not a problem to justify the following calculation:

$$W^{T_*}Q^{T_*}\psi \;=\; W^{T_*}\int\limits_{\Omega} f_\xi\,\psi(\xi)\,|d\xi| \;=\; \int\limits_{\Omega}\Big[W^{T_*}f_\xi\Big]\psi(\xi)\,|d\xi| \;=$$

$$= \int\limits_{\Omega} u^{f_\xi}(\,\cdot\,,T_*)\,\psi(\xi)\,|d\xi| \;=\; \langle\,\text{see}\,(3.2)\,\rangle \;=\; \alpha\,\psi \;+\; \int\limits_{\Omega} u_\xi(\,\cdot\,)\,\psi(\xi)\,|d\xi|\,. \tag{3.11}$$

3.5 Triangularity.

In the inner space $\mathcal{H}$ consider the increasing family of subspaces (the *nest*)

$$\mathcal{H}^\mu := \Big\{\,y\in\mathcal{H}\;\Big|\;\operatorname{supp} y\subset\Omega\setminus B_{T_*-\mu}[x_*]\Big\}\,,\quad 0\le\mu\le T_*\,,$$

so that $\mathcal{H}^0=\{0\}\,,\ \mathcal{H}^{T_*}=\mathcal{H}$. As is easy to check, this nest is continuons:

$$P^{\,\mu} \;=\; s-\lim_{\nu\to\mu}\,P^{\,\nu}\,,$$

where $P^{\,\mu}$ is the (orthogonal) projection in $\mathcal{H}$ onto $\mathcal{H}^\mu$.

Introduce the integral operator $V:\mathcal{H}\mapsto\mathcal{H}$,

$$(Vy)(x) \;:=\; \int\limits_{\Omega} u_\xi(x)\,y(\xi)\,|d\xi|\,,\qquad x\in\Omega$$

where the kernel is the regular term $u_\xi(x)$ of representation (3.2). This kernel is a piecewise smooth function on $\Omega\times\Omega$; its principle property which, basically, motivates the construction of control f_ξ is the following. Since $\operatorname{supp} u_\xi\subset\Omega_\xi$, one has $u_\xi(x)\equiv 0$ for $x\prec\xi$. Therefore, the integral defining V in fact is taken only over the path connecting x with the center of the tree:

$$(Vy)(x) = \int\limits_{\pi\,[x,x_*]} u_\xi(x)\,y(\xi)\,|d\xi|$$

which implies the *triangularity* of V with respect to the family $\{\mathcal{H}^\mu\}$:

$$V\,\mathcal{H}^\mu \;\subset\; \mathcal{H}^\mu\,,\qquad 0\le\mu\le T_*\,.$$

Due to the piecewise smoothness of its kernel, V is a compact operator in $\mathcal{H}$. Triangularity with respect to continuous nest plus compactness provide V to be a *Volterra* operator : its spectrum consists of the unique point $\lambda=0$ (see [5]).

3.6 Completing the proof of Theorem 1.

Return again to (3.2) and introduce the operator $C:\mathcal{H}\mapsto\mathcal{H}\,$,

$$(Cy)(x) := \int\limits_{\Omega} u^{f_\xi}\,(x,T_*)\,y(\xi)\,|d\xi| =$$

$$= \alpha(x)\, y(x) + \int\limits_{\Omega} u_\xi(x)\, y(\xi)\, |d\xi| = ([A+V]\, y)(x)\,, \quad x \in \Omega\,,$$

where the operator A multiplies functions by the amplitude α separated from zero. Since A is boundedly invertible, whereas V is a Volterra operator, the operator C turns out to be boundedly invertible.

Comparing the definition of C with (3.11), one has the operator equality

$$W^{T_*} Q^{T_*} \,=\, A \,+\, V$$

leading to the relation

$$W^{T_*} Q^{T_*} [\,A \,+\, V\,]^{-1} \,=\, \mathbb{I}\,.$$

The relation implies that for a given $y \in \mathcal{H}$ the control $f = Q^{T_*} [\,A \,+\, V\,]^{-1} y$ satisfies the equality $W^{T_*} f = y$, i.e. solves the boundary control problem (3.1) at the time $T = T_*$.

Since the reachable set $\{u^f(\,\cdot\,, T) \mid f \in \mathcal{F}^T\,\}$ doesn't decrease as T grows, the problem is solvable for any $T > T_*$. Theorem 1 is proved.

3.7 Comments.

- In terms of Control Theory, Theorem 1 establishes the *exact boundary controllability* of the system (2.1)–(2.3) and gives an estimate of the time of controllability $T_{\rm contr} \le \frac{d\,(\Omega)}{2}$. The example of a tree consisting of unique edge controlled from both of its endpoints (with obvious $T_{\rm contr} = \frac{d\,(\Omega)}{2}$) shows that this estimate is sharp. At the same time, it is not difficult to give examples realizing $T_{\rm contr} < \frac{d\,(\Omega)}{2}$.
- Our scheme may be easily adapted to the case of more general wave equation $\rho\, u_{tt} - (\mu u_e)_e \,+\, qu \,=\, 0$ with variable ρ, μ, q. The only modification required is to define the optical metric as $d\tau^2 = \frac{\rho}{\mu}|dx|^2$.
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