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Non-Unitary Spin-Charge Separation in One-Dimensional Fermion Gas

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Abstract

In this Letter we report exact results on the infrared asymptotics of one-particle dynamical correlation function of the gas of impenetrable spin 1/2 fermions at infinitesimal temperature. The correlation function shows signs of spin-charge separation with scaling behavior in the charge part and exponential decay as a function of the space coordinate in the spin part. Surprisingly, the anomalous dimensions in the charge part do not correspond to any unitary conformal field theory. We find that the fermion spectral weight has a power law divergency at low energy with the anomalous exponent $-1/2$.

Significant advance of the theory of strongly correlated one dimensional electron systems has been made possible by two important peculiarities of one dimensional physics: universality of low energy properties summarized by the conformal field theory (CFT) [1] and the existence of exactly solvable strongly coupled models like the Hubbard model or the Thirring model [2]. As far as their low energy properties are concerned, most physically important one dimensional electron systems belong to a certain universality class usually referred to as the Luttinger liquid (LL) [3]. By bosonization the LL theory is mapped onto a theory of free massless compactified bosonic fields. The compactification radii play the role of phenomenological parameters defining the effective theory. Classification and phenomenology of such effective theories is often called g-ology [4]. Due to a relatively simple analytic structure of the LL theory, g-ology is a powerful calculational tool relating all dynamical information about a strongly correlated system (such as its correlation functions, spectral weights, transport coefficients, stability with respect to perturbations etc.) to a few phenomenological constants, which can be obtained from e.g. its thermodynamical properties. For exactly solvable models this phenomenological constants are usually found from the analysis of thermodynamics by means of the thermodynamic Bethe ansatz [5].

Despite the success of g-ology, a direct calculation of infrared asymptotics of correlation functions in exactly solvable models has so far been performed only for a few special cases [2, 6–10] and as a general problem still remains a challenge. Such direct calculations are important for two reasons. First, they provide a test bed for the hypotheses underlying the LL theory. Second, they give insight into the situations where the low energy physics is not described by the LL.

An interesting example of a physically relevant model, where exact calculation of dynamical correlation functions is possible is a model of non-relativistic impenetrable spin 1/2 fermions. This model is the infinite U limit of the system of spin 1/2 fermions described by the Hamiltonian

$$H = \int dx \left[- \sum_{\sigma=\uparrow,\downarrow} \psi^\dagger_\sigma(x) \partial_x^2 \psi_\sigma(x) + U n_\downarrow(x) n_\uparrow(x) \right]. \tag{1}$$

The model (1) can be viewed as a continuum limit of the Hubbard model and is the simplest example of interacting fermions in one dimension. This model has a long history. The eigenstates and the spectrum of Hamiltonian (1) for an arbitrary U were found in [11]. Thermodynamics of this model was studied in [12]. Parameters of the low energy LL theory for the Hubbard model were calculated in [13].

The infinite U limit of the model was studied in [6–9, 14–16]. This is an interesting limit, where the ground state of the system becomes infinitely degenerate in spin and the CFT description fails. On the other hand, in this limit a specific factorization of Yang's wave function occurs [7], opening the way to a direct calculation of spectral weights and correlation functions of the model through their Fredholm determinant representation [6, 14]. Due to the above mentioned ground state degeneracy the limit $U \to \infty$ in correlation functions of the model is nontrivial in the sense that it does not commute with the limit of vanishing temperature $T \to 0$. Indeed, taking the limit $T \to 0$ first one obtains results [7, 8, 15] consistent with a naive infinite U limit of the corresponding CFT [2, 16]. In particular, the one-particle momentum distribution function shows scaling behavior near the Fermi momentum k_F with the anomalous exponent 1/8 [7]. On the other hand, the one-particle density matrix (i.e. the equal time one-particle correlation function) decays exponentially as a function of coordinate [6] if one takes the $U \to \infty$ limit first. Due to this exponential decay the momentum distribution function shows no scaling near k_F.

In this Letter we report our results on the asymptotics of the one-particle dynamical correlation function

$$G(x,t) = \left\langle \psi_\uparrow^\dagger(x,t)\psi_\uparrow(0,0) \right\rangle \tag{2}$$

when the limit $U \to \infty$ is taken before the limit $T \to 0$. Due to the rotational invariance of Hamiltonian (1) the correlation function of spin down fermions coincides with (2).

We begin our discussion with first recalling how the infinite U limit in correlation functions is taken at zero temperature and then we discuss the opposite order of limits qualitatively. Then we present our exact result for the asymptotics of the correlation function (2) and discuss its structure and physical meaning.

Assume that in the ground state of the system there is a finite fermion density $n_c = n_\uparrow + n_\downarrow$, controlled by the chemical potential μ. At any finite U the low energy physics of this model can be described by the LL theory. According to this theory the low energy spectrum is described by the effective Hamiltonian

$$H_{\text{eff}} = H_c + H_s \tag{3}$$

where H_c describes charge fluctuations, H_s describes spin fluctuations and $[H_c, H_s] = 0$. Independent dynamics of spin and charge is called spin-charge separation [1]. Hamiltonians

H_c and H_s are, in fact, Tomonaga-Luttinger Hamiltonians corresponding to $c=1$ conformal field theories with global $U(1)$ symmetry. Hamiltonian H_s also possess global $SU(2)$ symmetry inherited from the rotational symmetry of (1). This additional constraint on H_s fixes the structure of its spectrum completely up to a non-universal constant v_s, which is the spin propagation velocity depending on the microscopic parameters. The fermion operator in the LL theory is represented as a sum of anticommuting "left" and "right" fermion fields

$$\psi_\sigma(x) = \psi_{L,\sigma}(x)e^{-ik_Fx} + \psi_{R,\sigma}(x)e^{ik_Fx}, \quad \sigma = \uparrow, \downarrow \tag{4}$$

where k_F is the Fermi momentum. The "left" and "right" fermion fields are products of spin and charge vertex operators

$$\psi_{L,\sigma} = C_L S_{L,\sigma}, \quad \psi_{R,\sigma} = C_R S_{R,\sigma}, \tag{5}$$

where the spin operators satisfy $S_{L(R),\downarrow} = S^\dagger_{L(R),\uparrow}$. Operators S and C commute with each other and satisfy $[S, H_c] = [C, H_s] = 0$. Formulas (3), (4) and (5) only apply to the low energy properties of the system. That is, the energy must be much smaller than a certain cutoff scale Λ, determined by μ and U.

At zero temperature, the correlation function (2) predicted by the Luttinger theory has the form

$$\begin{aligned} G(x,t) = &\mathcal{S}(x - v_s t)\mathcal{C}(x - v_c t, x + v_c t)e^{-ik_Fx} \\ &+\mathcal{S}(x + v_s t)\mathcal{C}(x + v_c t, x - v_c t)e^{ik_Fx} \end{aligned} \tag{6}$$

where

$$\mathcal{S}(x - v_s t) = \left\langle S^\dagger_R(x,t) S_R(0,0) \right\rangle = \frac{\text{const}}{(x - v_s t)^{1/2}} \tag{7}$$

and

$$\begin{aligned} \mathcal{C}(x - v_c t, x + v_c t) &= \left\langle C^\dagger_R(x,t) C_R(0,0) \right\rangle \\ &= \frac{\text{const}}{(x - v_c t)^{2\Delta_c}(x + v_c t)^{2\bar{\Delta}_c}}, \end{aligned} \tag{8}$$

where v_s and v_c are the propagation velocities of spin and charge excitations respectively. Correlation functions (7), (8) do not contain any length parameter. This reflects the scaling invariance of the LL theory. According to (7) the spin operator $S_{R,\sigma}$ is a right mover with fixed anomalous dimension 1/4. This is a consequence of the constraint imposed on the effective Hamiltonian H_s by the global $SU(2)$ symmetry. At the same time, the correlation function of the "right" charge operator C_R splits into a product of left-moving (depending on $x+v_ct$) and right-moving (depending on $x-v_ct$) parts with different positive anomalous dimensions Δ and $\bar{\Delta}$. These dimensions are determined by the Luttinger parameter K, which can be found from the charge compressibility $\kappa = \partial n_c/\partial\mu$ of the system as follows

$$K = \frac{\pi v_c}{2}\kappa. \tag{9}$$

One has [1]

$$\begin{aligned} \Delta_c &= \frac{1}{16}\left(\sqrt{K}+\frac{1}{\sqrt{K}}\right)^2 \quad , \\ \bar{\Delta}_c &= \frac{1}{16}\left(\sqrt{K}-\frac{1}{\sqrt{K}}\right)^2 . \end{aligned} \tag{10}$$

For any finite U parameters v_c and κ can be calculated from the thermodynamic Bethe ansatz. For infinite U fermions become impenetrable, which from the point of view of thermodynamics means that they satisfy the Pauli principle independently of their spin orientation. For such a system parameters v_c and κ are calculated in the same way as for a system of spinless non-interacting fermions [16, 17]. Calculating the right hand side of (9) for spinless fermions one finds

$$K = \frac{1}{2}. \tag{11}$$

For the anomalous dimensions in the charge sector this gives

$$\Delta_c = \frac{9}{32}, \quad \bar{\Delta}_c = \frac{1}{32}. \tag{12}$$

Correlation function for impenetrable fermions in the form (6) with anomalous exponents in the charge sector given by (12) was suggested in [2]. It was also discussed in [15], where the $U\to\infty$ limit was analyzed microscopically.

Next, consider the infinite U limit taken at finite temperature. For large U there happens a separation of energy scales in the problem. This separation is controlled by a small dimensionless parameter k_F/U. While μ remains the characteristic cutoff energy in the charge sector, the characteristic cutoff energy in the spin sector becomes of the order of $\mu k_F/U$. The same happens with the velocities. While the propagation velocity for the charge $v_c \to 2\sqrt{\mu}$ as $U \to \infty$, the velocity of spin excitations vanishes, $v_s \to 0$. In the limit of infinite U the spin degrees of freedom completely loose dynamics and the ground state of the system becomes infinitely degenerate with respect to spin flips. From the point of view of the low energy physics it is interesting to consider a situation where

$$\frac{\mu k_F}{U} \ll T \ll \mu \tag{13}$$

In this situation all spin configurations have equal statistical weight and the dynamics of the system is effectively averaged over spin configurations. At the same time, the charge degrees of freedom are not strongly affected by temperature, because the latter is much lower than the corresponding cutoff energy μ. Thus in the temperature range (13) correlation function (2) should not strongly depend on temperature and its behavior should be in this sense universal. The infinite U limit of the correlation function taken at a fixed small temperature $T \ll \mu$ thus corresponds to taking the infinite U limit first and then taking the limit $T \to 0$. One can try to conjecture the structure of the one-particle correlation function (2) in this situation using formulas (7), (8) and the general results of the CFT. For finite temperatures and for $x > v_s/T$, the spin part of the correlation function predicted by the LL theory is

$$\mathcal{S}_R(x - v_s t) \sim e^{-\frac{2\pi T}{v_s}(x - v_s t)}. \tag{14}$$

This formula is valid for temperatures smaller than $\Lambda_s = \mu k_F/U \approx k_F v_s$, that is when linear description of spin waves is appropriate. For $|x \pm v_c t| < v_c/T$ the charge part of correlation function is given by (8). For $T > \Lambda_s$ all spin degrees of freedom become saturated and the spin part of the correlation function should become temperature independent. Thus, replacing the temperature T in the right hand side of (14) by the crossover scale Λ_s we get a prediction for the correlation function at $T > \Lambda_s$. Making this substitution and taking the limit of vanishing v_s we get for the fermion correlation function

$$\begin{aligned} G(x,t) = {} & \mathcal{S}(x)\mathcal{C}(x + v_c t, x - v_c t)e^{-ik_F x} \\ & + \mathcal{S}(x)\mathcal{C}(x - v_c t, x + v_c t)e^{ik_F x}, \end{aligned} \tag{15}$$

where function $\mathcal{C}$ is given by (8) and

$$\mathcal{S}(x) = \text{const} \times e^{-\gamma k_F x}. \tag{16}$$

Here γ is a dimensionless constant. The problem of this picture is that it assumes spin-charge separation in the LL sense even though the spin degrees are strongly excited. To see that this is not the case, compare the correlation function of spin up fermions calculated in two different ground states. Let one ground state be the infinite U limit of the ground state of Hamiltonian (1) and another be the fully polarized spin up ground state. Both states have the same statistical weight and equally contribute to the correlation function. In the first state the anomalous dimensions of charge part of fermion correlation function are given by (12). In the second state, due to the Pauli principle fermions do not feel the interaction and the anomalous dimensions are those of non-interacting fermions, that is $\Delta = 1/2$ and $\bar{\Delta} = 0$. It is not, therefore, clear whether after averaging over all spin configurations the anomalous dimensions in (15) should coincide with (12) or, even, whether the correlation function should have the structure (15) at all.

In paper [14] it was shown that the correlation function (2) can be expressed in terms of the Fredholm determinant of a linear integral operator $\hat{V}$ via

$$G(x,t) = \frac{e^{-ik_F^2 t}}{8\pi i} \oint_{|z|=1} \frac{dz}{z} F(z) B_{--}(z) \det(\hat{I} + \hat{V})(z). \tag{17}$$

The kernel

$$V(k,p) = \frac{e_+(k)e_-(p) - e_+(p)e_-(k)}{k-p} \tag{18}$$

of the operator $\hat{V}$ is defined on the square $[-k_F, k_F] \times [-k_F, k_F]$, where $k_F = \sqrt{\mu}$. Functions entering (18) are defined as follows ($z = e^{i\eta}$)

$$e_-(k) = \frac{1}{\sqrt{\pi}} e^{\tau(k)/2}, \tag{19}$$

$$e_+(k) = \frac{e^{-\tau(k)/2}}{2\sqrt{\pi}} [(1 - \cos\eta) e^{\tau(k)} E_0(k) + \sin\eta], \tag{20}$$

$$E_0(k) = \text{p.v.} \int_{-\infty}^{\infty} dp \frac{e^{-\tau(p)}}{\pi(p-k)}, \tag{21}$$

$$\tau(k) = ik^2 t - ikx. \tag{22}$$

Function $B_{--}(z)$ is

$$B_{--}(z) = \int_{-k_F}^{k_F} dk e_-(k) (\hat{I} + \hat{V})^{-1} e_-(k) \tag{23}$$

and

$$F(z) = 1 + \frac{z}{2-z} + \frac{1}{2z-1}. \tag{24}$$

We performed asymptotic analysis of Eq. (17) by asymptotically solving the corresponding Riemann-Hilbert problem [9] by the techniques described in [18]. Detailed calculations are involved and will be presented elsewhere [19]. Our main results for the correlation function (2) are as follows. For $x, t \to +\infty$ and $x/t = \text{const} \neq 0$ we have

$$G(x,t) = \frac{\Xi e^{-k_F x \ln 2/\pi} e^{i(k_F x - \phi_+)}}{(x - 2k_F t)^{2\Delta}(x + 2k_F t)^{2\bar{\Delta}}} - \frac{\Xi e^{-k_F x \ln 2/\pi} e^{-i(k_F x - \phi_-)}}{(x + 2k_F t)^{2\Delta}(x - 2k_F t)^{2\bar{\Delta}}}. \tag{25}$$

Here Ξ is a constant, which is calculated explicitly in [19]. The phases $\phi_\pm$ are given by

$$\phi_\pm = -2\text{Im}\left[\ln\Gamma\left(\frac{i\ln 2}{2\pi}\right)\right] + \frac{\ln 2}{\pi}\ln(2k_F x \pm 4k_F^2 t) \tag{26}$$

and the anomalous dimensions are given by

$$\Delta = \frac{1}{2} - \frac{1}{8}\left(\frac{\ln 2}{\pi}\right)^2, \qquad \bar{\Delta} = -\frac{1}{8}\left(\frac{\ln 2}{\pi}\right)^2. \tag{27}$$

This result (25) is consistent with the previously calculated equal time correlation function [6]. The case $x = 0$ is special. In this case

$$G(0,t) = \frac{\Xi'}{\sqrt{t\ln(k_F^2 t)}} \tag{28}$$

where Ξ' is a constant.

The form of the correlation function (25) is essentially the same as of (15) and shows clear signs of spin-charge separation. The exponentially decaying part can be attributed to the non-propagating spin mode. The algebraically decaying part of the correlation function only depends on combinations $x - 2k_F t$ and $x + 2k_F t$ which correspond to propagation with velocity $2k_F$. This is exactly the velocity of the charge propagation in the model.

Remarkably, the anomalous dimensions given by (27) are completely different from dimensions (12), obtained in the infinite U limit of the LL theory. While the form of the correlation function (25) suggests that the fermion operator can be represented as a product of a spin operator and a charge operator, the emergence of a negative scaling dimension $\bar{\Delta}$ in (27) indicates that the conformal field theory describing the charge sector should be non-unitary. Answering the question of whether such a CFT exists is a subject of further investigation. In particular, it would be interesting to calculate the three point correlation functions involving two fermion operators and the charge density or the spin density operator.

Note that the phases $\phi_\pm$ (26) instead of being constant are logarithmic functions of the light cone coordinates $x \pm v_c t$. A similar phenomenon occurs in the theory of classical non-linear wave equations[20].

While the exponential decay of the correlation function in x direction smears the power law scaling of the momentum distribution function near the Fermi points, the power law scaling can be observed in the tunnelling density of states, which is given by

$$A(\omega) = \mathrm{Re}\tilde{G}(0, \omega) \tag{29}$$

where tilde denotes the Fourier transform. From our result (28) we find

$$A(\omega) \sim \omega^{-\frac{1}{2}}. \tag{30}$$

The divergence of the tunnelling density of states at zero energy is similar to the one found in [15] but the numerical value of the tunnelling exponent 1/2 is different from 3/8 calculated in [15].

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[1] A.O. Gogolin, A.A. Nersesyan, and A.M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, England, 1998).

[2] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge University Press, Cambridge, England, 1993).

[3] F.D.M. Haldane, J. Phys C **14**, 2585 (1981); S. Tomonaga, Progr. Teor. Phys. **5**, 544 (1950); J.M. Luttinger, J. Math. Phys. (N.Y.) **4**, 1154 (1963); D.C. Mattis and E.H. Lieb, J. Math.Phys. (N.Y.) **6**, 304 (1965).

[4] J. Sólyom, Adv. Phys. **28**, 201 (1979).

[5] F.D.M. Haldane, Phys. Lett. A **81**, 153 (1981).

[6] A. Berkovich, J. Phys. A: Math. Gen. **24**, 1543 (1991).

[7] M. Ogata and H. Shiba, Phys. Rev. B **41**, 2326 (1990).

[8] A. Parola and S. Sorella, Phys. Rev. Lett. **64**, 1831 (1990).

[9] F. Göhmann, A.G. Izergin, V.E. Korepin, and A.G. Pronko, Int. J. Mod. Phys. B, **12**, 2409 (1998).

[10] Z.N.C. Ha, Phys. Rev. Lett. **73**, 1574 (1994).

[11] C.N. Yang, Phys. Rev. Lett. **19**, 1312 (1967).

[12] M. Takahashi, Prog. Theor. Phys. **46**, 1388 (1971); C.K. Lai, Phys. Rev. A, **8**, 2567 (1973).

[13] H. Frahm and V.E. Korepin, Phys. Rev. B, **42**, 10553 (1990); N. Kawakami and S.K. Yang, Phys. Lett. A **148**, 359 (1990).

[14] A.G. Izergin and A.G. Pronko, Nucl. Phys. B **520**, 594 (1998).

[15] K. Penc, F. Mila, and H. Shiba, Phys. Rev. Lett. **75**, 894 (1995); K. Penc, *et al.*, Phys. Rev. B **55**, 15475 (1997).

[16] Y. Ren and P.W. Anderson, Phys. Rev. B, **48**, 16662 (1993).

[17] H.J. Schulz, Phys. Rev. Lett. **64**, 2831 (1990).

[18] P. Deift, T. Kriecherbauer, K. T-R McLaughlin, S. Venakides, and X. Zhou, Commun. Pure Appl. Math. **52**, 1491 (1999); P. Deift, *Orthogonal Polynomials and Random Matrices:A Riemann-Hilbert Approach* (Courant Institute of Mathematical Sciences, New York, NY; American Mathematical Society, Providence, RI, 1999).

[19] Vadim V. Cheianov and M.B. Zvonarev (unpublished).

[20] V.E. Zakharov and S.V. Manakov, Sov. Phys. JETP **44**, 106 (1976).