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FINITE-TEMPERATURE CORRELATIONS IN THE ONE-DIMENSIONAL TRAPPED AND UNTRAPPED BOSE GASES: FUNCTIONAL INTEGRAL APPROACH

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Abstract.

We calculate the dynamic single-particle and many-particle correlation functions at nonzero temperature in one-dimensional trapped repulsive Bose gases. The decay for increasing distance between the points of these correlation functions is governed by a scaling exponent that has a universal expression in terms of observed quantities. This expression is valid in the weak-interaction Gross–Pitaevskii as well as in the strong-interaction Girardeau-Tonks limit, but the observed quantities involved depend on the interaction strength. The confining trap introduces a weak center-of-mass dependence in the scaling exponent. We also conjecture results for the density-density correlation function.

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I. Introduction

Recent advances in experimental techniques have made it possible to study Bose gases in what are effectively two-dimensional and even one-dimensional (1D) traps [1–4]. The confinement of the three-dimensional (3D) gas achieved by making the level spacing of the confining potential in one or two dimensions larger than the energy of the individual atoms. Because of the growing interest in coherent matter-wave interferometry and atom lasers, these recent developments have also revived theoretical interest in the properties of these effectively 1D Bose systems. It is clearly important to understand in more detail the effects of confinement on, *e.g.*, their ground-state and thermal properties including one-particle as well as many-particle correlations. It is well established [5] that a good theoretical framework for ultracold metal vapours is given by bosons with (in the case considered here) repulsive delta-function interactions. We can thus describe the 1D system with a Hamiltonian

$$\hat{H} = \int \left\{ \hat{\psi}^{\dagger}(x) \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu + V(x) \right) \hat{\psi}(x) + \frac{g}{2} \hat{\psi}^{\dagger}(x) \hat{\psi}^{\dagger}(x) \hat{\psi}(x) \hat{\psi}(x) \right\} dx, \quad (1)$$

where $\hat{\psi}(x)$ is a Bose field operator with its adjoint $\hat{\psi}^{\dagger}(x)$ and commutation relations $[\hat{\psi}(x), \hat{\psi}^{\dagger}(x')] = \delta(x - x')$, *etc.*, μ is the chemical potential, $g > 0$ is the interaction strength (the coupling constant), and $V(x) \equiv \frac{m}{2}\Omega^2 x^2$ is the harmonic trap potential. For $V(x) \equiv 0$ this system has been a prototype for exactly solvable models in 1D [6–10], and many of its properties have already been worked out in great detail.

In real physical systems the transition from 3D to 1D behaviour has an interesting aspect in that the resulting effective density of atoms in 1D can be either 'high' or 'low' depending on the parameters of the system. It is a feature of 1D Bose systems that high density means weak interactions and low density strong interactions between the particles. This property opens up the possibility of realising experimentally the 'strong-coupling' limit [11] in which repulsive bosons display fermionic properties [6, 7]. This is the so-called Girardeau–Tonks regime. In this 'true' 1D regime in the sense of [12], the Bose–Einstein condensation in the sense of [12] is not realized, and the system is not coherent.

In contrast, in the limit of high effective 1D densities of atoms, *i.e.*, in the Gross–Pitaevskii, or the weak-interaction regime, short-range coherence builds up between the atoms, and the system forms at low enough temperatures a *quasi condensate* [13–20]. True long-range order is in this regime destroyed by phase fluctuations, while density fluctuations are suppressed. It is thus evident that the local correlations between the atoms are different in these two regimes such that the 'local' two-point correlation function $\Gamma(x_1, x_1)$ tends to zero in the Girardeau-Tonks regime, and to unity in the Gross–Pitaevskii regime with quasi condensate [21]. At high temperatures or in the very small coupling-constant limit it obviously becomes two, the value for uncorrelated bosons. Of course we are not limited to two-point correlations: thus, *e.g.*, the local three-point correlation function $\Gamma(x_1, x_1, x_1)$ is important because it is related to three-body recombination, an inelastic decay process [11]. Notice that in defining the full nonlocal correlation functions we need as many space variables $x_1, x_2, \ldots, x_n$ as are being correlated because in the presence of a confining potential there is no longer translational invariance. Moreover at finite temperatures it becomes natural to correlate also time-like variables τ for which the fields evolve in Matsubara representation $\hat{\psi}(x,\tau) = e^{-\tau\hat{H}}\hat{\psi}(x)e^{\tau\hat{H}}$ with $\hat{H}$ given by (1) and likewise for $\hat{\psi}^{\dagger}(x,\tau)$ [22]. Thus the two-point (single-particle) *thermal* correlation function which will be studied in this paper is given by

$$\Gamma(x_1,\tau_1;x_2,\tau_2)=\left\langle T_\tau\hat{\psi}^\dagger(x_1,\tau_1)\hat{\psi}(x_2,\tau_2)\right\rangle, \tag{2}$$

and the n-point correlation function correspondingly. In these expressions T_τ is a 'time-ordering' operator which places the field operators from right to left in order of increasing τ [22]. Note that these expressions like Eq. (2) are not translationally invariant in x but they are translationally invariant in τ. Variables τ lie in $0\le\tau\le\beta$, $\beta\equiv(k_BT)^{-1}$ and T is temperature, and there is (for bosons) periodicity in τ period β.

Despite the previous results for one or more space dimensions [15, 16, 23–28] the decay of the single-particle and especially many-particle correlations in the presence of a confining potential have not been fully analysed. We address this problem for one space dimension in this paper, and point out that the way these correlations decay for increasing $|x_1-x_2|$ is governed by a *universal* scaling exponent that can be expressed in terms of observed quantities, and, at finite temperature, applies in both of the fundamental regimes described in [12]. This scaling exponent is analogous to the one that appears in the power-law decay of correlations in the homogeneous system of 1D bosons at zero temperature. In the presence of a confining potential, this exponent becomes a function of the center-of-mass coordinate, but this dependence can be expected to be small. This result is valid in the weak-interaction Gross–Pitaevskii as well as the strong-interaction Girardeau–Tonks regime and thus straddles both of the two regimes distinguished in [12]. We also show how asymptotically all multi-particle correlations can be expressed in terms of the two-point correlations, and indicate how the results can be extended to density-density correlations. As all of these properties can be solved exactly by Bethe Ansatz techniques in the translationally invariant case of no confinement, this case provides a good point of reference, and we begin by reviewing it in the next section.

Finally, the existence of the experiments like [1–4], providing essentially one dimensional systems, means that it could become possible to check connected one dimensional theory in great detail at finite temperatures. A purpose of this paper is thus to point out that the methods employed here can be used to provide such a connected theory.

II. Summary of exact results by the Bethe Ansatz method

Let us consider the system described by the Hamiltonian (1). In the absence of the confining potential namely by setting $V(x)\equiv 0$ the Hamiltonian (1) becomes exactly solvable for its eigenstates and eigenvalues by the Bethe Ansatz method [9]. For a gas of N bosons on a ring of circumference L, namely for periodic boundary conditions of period L in the one dimensional space x, eigenfunctions of Hamiltonian (1) are [7, 9]:

$$\begin{aligned}&|\Phi_N(\lambda_1,\dots,\lambda_N)\rangle\\&\quad=\frac{1}{\sqrt{N!}}\int dx_1\dots dx_N f_N(x_1,\dots,x_N\mid\lambda_1,\dots,\lambda_N)\hat{\psi}^\dagger(x_1)\dots\hat{\psi}^\dagger(x_N)\mid 0\rangle,\end{aligned} \tag{3}$$

and these depend on N real parameters $\lambda_1,\dots,\lambda_N$ satisfying the so-called Bethe equations

$$e^{i\lambda_j L}=-\prod_{k=1}^{N}\frac{\lambda_j-\lambda_k+i(\tilde{g}/2)}{\lambda_j-\lambda_k-i(\tilde{g}/2)};\;\; j=1,\dots,N, \tag{4}$$

where $\tilde{g} = (2m/\hbar^2)g$ is the renormalised coupling constant and

$$f_N = C_N \sum_{\mathcal{P}} \exp\left\{ i \sum_{n=1}^{N} x_n \lambda_{\mathcal{P}_n} \right\} \prod_{1\le k<j\le N} \left[1 - \frac{\tilde{g}}{2} \frac{i\epsilon(x_j - x_k)}{(\lambda_{\mathcal{P}_j} - \lambda_{\mathcal{P}_k})} \right]. \tag{5}$$

Here the sum over all the permutations $\mathcal{P}$ of the numbers $1, 2, \ldots, N$ is taken, $\epsilon(x) = sign(x)$ is the sign function, and the factor C_N is equal to

$$C_N = \frac{\prod_{j>k} (\lambda_j - \lambda_k)}{\sqrt{N! \prod_{j>k} \left[(\lambda_j - \lambda_k)^2 + (\tilde{g}/2)^2 \right]}}.$$

The Fock vacuum $|0\rangle$ is defined by the condition $\hat{\psi}(x)|0\rangle = 0$. The corresponding eigenenergies are

$$E_N = \frac{\hbar^2}{2m} \sum_{j=1}^{N} \lambda_j^2. \tag{6}$$

The solutions λ_j of the Bethe equations, Eq. (4), multiplied by $\hbar$ have a natural interpretation as the momenta of the particles each of mass m. The function f_N is a symmetric function of the variables x_j and a continuous function of each x_j. One can also see that f_N is an antisymmetric function of λ_j. Hence, $f_N = 0$ if $\lambda_j = \lambda_k$, $j \neq k$, in the whole coordinate space $-\infty < x_j < \infty$ for each j. This property is the basis of the Pauli principle for one-dimensional interacting bosons. The wave functions (5) form a complete and orthonormal set in configuration space and thus the eigenfunctions (3) are orthogonal for the different sets of the solutions of the Bethe equations. At zero temperature ($T = 0$) the ground state of the system is defined by the solutions lying within the interval $-\lambda_F \le \lambda_j \le \lambda_F$ and these form a "Fermi sphere" in the momentum space. At an arbitrary value of the temperature ($T \neq 0$) the solutions are distributed along the real axis: $-\infty < \lambda_j < \infty$. In the thermodynamic limit in which the number of particles N and the circumference of the ring L both go to infinity, $N, L \to \infty$, at a fixed density $\rho = \frac{N}{L}$, the distribution function of the solutions of the Bethe equations is defined by the solutions of the Lieb-Liniger equations for $T = 0$ [7] and by the solutions of the Yang-Yang equations for $T > 0$ [8]. The free energy density $\mathcal{E}$ of the gas is $\mathcal{E} = \lim_{N,L\to\infty} E_N/L$. The ground state with a particle density $\rho > 0$ corresponds to positive values of the chemical potential in this exactly solvable case.

In [9], it is proved that the correlation function ($\tau_1 < \tau_2$)

$$\langle \hat{\psi}^\dagger(x_1, \tau_1) \hat{\psi}(x_2, \tau_2) \rangle \equiv \frac{\mathrm{tr}\left(e^{-\beta \hat{H}} \hat{\psi}^\dagger(x_1, \tau_1) \hat{\psi}(x_2, \tau_2) \right)}{\mathrm{tr}\left(e^{-\beta \hat{H}} \right)} \tag{7}$$

may be expressed in the form

$$\langle \hat{\psi}^\dagger(x_1, \tau_1) \hat{\psi}(x_2, \tau_2) \rangle = \frac{\langle \Omega_T \mid \hat{\psi}^\dagger(x_1, \tau_1) \hat{\psi}(x_2, \tau_2) \mid \Omega_T \rangle}{\langle \Omega_T \mid \Omega_T \rangle}, \tag{8}$$

where $|\,\Omega_T\rangle$ is any of the eigenfunctions (3) contributing in the thermodynamic limit to the state of thermal equilibrium. It depends on temperature through the distribution function of the solutions of the Bethe equations. For small nonzero temperatures and long distances $|\, x_1 - x_2 \,| \gg l_c$ (here l_c is a healing length $l_c = \hbar/\sqrt{m\rho g}$) the asymptotics of the correlator (7) is

$$\langle\hat{\psi}^\dagger(x_1,\tau_1)\hat{\psi}(x_2,\tau_2)\rangle \simeq \frac{\rho}{\left|\sinh\frac{\pi}{\hbar\beta v}\left(|x_1-x_2| + i\hbar v(\tau_1-\tau_2)\right)\right|^{1/\theta}}, \tag{9}$$

in which θ is the scaling exponent

$$\theta = \frac{2\pi\hbar\rho}{mv}, \tag{10}$$

and v is the sound velocity, $v^2 = (\rho/m)\frac{\partial^2\mathcal{E}}{\partial\rho^2}$. In this form the scaling exponent depends only on observable quantities, namely the density ρ, the sound velocity v, and the atomic mass m. The asymptotic result Eq. (9) with the scaling exponent (10) is valid for arbitrary values of the coupling constant $0 < g \leq \infty$. For the so-called Girardeau-Tonks gas ($g = \infty$) [6, 12, 29] Appendix B shows that

$$\theta = 2 \tag{11}$$

exactly as is well known [9]. For small coupling constants g we may express (10) in the form [31]

$$\theta = 2\pi\hbar\sqrt{\frac{\rho}{mg}}. \tag{12}$$

One may also express θ (12) through the healing length l_c : $\theta = 2\pi\rho l_c$. In Appendix B we also give a simple derivation of these expressions (11), (12).

Equation (9) means that the correlator (7) decays exponentially at long distances $|x_1 - x_2| \gg l_c$,

$$\langle\hat{\psi}^\dagger(x_1,\tau_1)\hat{\psi}(x_2,\tau_2)\rangle \sim \rho\exp\left(-\frac{\pi}{\hbar\beta v\theta}\,|x_1-x_2|\right), \tag{13}$$

with the correlation length

$$\xi = \frac{2\hbar^2\beta\rho}{m} = \frac{\hbar\beta v}{\pi}\theta. \tag{14}$$

The result Eq. (9) also allows us to obtain the asymptotic behaviour of the correlation function at zero temperature. In the zero temperature limit ($\beta \to \infty$) the correlation function (9) transforms into the correlation function for vacuum fluctuations and has a power law decay

$$\langle\hat{\psi}^\dagger(x_1,\tau_1)\hat{\psi}(x_2,\tau_2)\rangle_{T=0} \simeq \frac{\rho}{\left||x_1-x_2| + i\hbar v(\tau_1-\tau_2)\right|^{1/\theta}}. \tag{15}$$

The distinction in the behaviour of the correlators at zero and finite temperatures may be explained by the formation of the Fermi sphere at zero temperature [9]. However this ground state of bosons cannot be considered as a true condensate since $\langle\hat{\psi}^\dagger(x_1,\tau_1)\hat{\psi}(x_2,\tau_2)\rangle$ vanishes when $|x_1 - x_2| \to \infty$ so that there is no long-range order in any general sense. As correlations decay algebraically, *i.e.*, by a power law the system is scale invariant and can be thought to be at a critical point with θ now a critical exponent.

III. The functional integral approach

A. The partition function.

We now evaluate strictly comparable results to those summarized in Sec. II by using methods of functional integration. We also obtain results which are more general than those of Section II in that the effects of the confining potential $V(x) = \frac{m}{2}\Omega^2 x^2$ are taken into account.

The partition function Z for a one-dimensional gas with repulsive δ-function interactions of strength $g > 0$ is given as a functional integral by [13, 30]

$$Z = \int e^{S[\psi,\bar{\psi}]} \mathcal{D}\psi \mathcal{D}\bar{\psi}. \tag{16}$$

The *classical* action S of this system is equal to

$$S[\psi,\bar{\psi}] = \int_0^\beta d\tau \int dx \left(\bar{\psi}(x,\tau)\hat{K}\psi(x,\tau) - \frac{g}{2}\bar{\psi}(x,\tau)\bar{\psi}(x,\tau)\psi(x,\tau)\psi(x,\tau) \right). \tag{17}$$

The differential operator in this action S is $\hat{K} = \frac{\partial}{\partial\tau} - \mathcal{H}$, while the one particle Hamiltonian $\mathcal{H}$ includes the trap potential $V(x) = \frac{m}{2}\Omega^2 x^2$ and the chemical potential μ is also contained in $\mathcal{H}$ which is $\mathcal{H} \equiv -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) - \mu$. We have introduced two c-number complex valued fields $\psi(x,\tau)$, $\bar{\psi}(x,\tau)$ and, consistent with the existence of the potential $V(x)$, the boundary conditions for these fields are chosen to be vanishing at infinity for x (in the sense of quadratic integrability) and periodic, period $\beta = (k_B T)^{-1}$, for τ. These new c-number complex valued fields ψ, $\bar{\psi}$ are two independent fields, and they are introduced in a formal correspondence with the operator fields $\hat{\psi}(x,\tau)$, $\hat{\psi}^\dagger(x,\tau)$ [13]. In (16), $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ is the integration measure.

For small enough temperatures we can expect to put $\psi(x,\tau) = \psi_o(x,\tau) + \psi_e(x,\tau)$ and likewise for $\bar{\psi}$. The condensate variable $\psi_o(x,\tau)$ describes the condensate field and $\psi_e(x,\tau)$ describes highly excited thermal particles. Also we can expect to write the field variable $\psi_o(x,\tau) = \psi_o(x) + \xi(x,\tau)$, where the field $\psi_o(x)$ describes the ground state of the model at zero temperature (for the homogeneous case this field describes the particles in the Fermi sphere), while the field $\xi(x,\tau)$ describes the low lying excited particles (the phonon-like excitations in the vicinity of the Fermi momenta in the homogeneous case). We shall also require that the fields ψ_o, ψ_e are orthogonal in the sense: $\int dx \psi_o(x,\tau)\bar{\psi}_e(x,\tau)$ $= \int dx \bar{\psi}_o(x,\tau)\psi_e(x,\tau) = 0$. Since by construction ψ_o and ψ_e are orthogonal, and likewise for $\bar{\psi}_o$ and $\bar{\psi}_e$, we can write the integration measure as $\mathcal{D}\psi\mathcal{D}\bar{\psi} = \mathcal{D}\psi_o\mathcal{D}\bar{\psi}_o\mathcal{D}\psi_e\mathcal{D}\bar{\psi}_e$.

We shall only consider terms in S up to quadratic (bilinear) in ψ_e, $\bar{\psi}_e$. This means that we are making an approximation in which the noncondensed particles do not interact with each other. By doing this we can then actually integrate out the thermal fluctuations and obtain the effective action functional

$$S_{eff}[\psi_o, \bar{\psi}_o] = \ln \int e^{S[\psi,\bar{\psi}]} \mathcal{D}\psi_e \mathcal{D}\bar{\psi}_e, \tag{18}$$

depending only on the $\psi_o, \bar{\psi}_o$ variables. Then,

$$Z = \int e^{S_{eff}[\psi_o,\bar{\psi}_o]} \mathcal{D}\psi_o \mathcal{D}\bar{\psi}_o. \tag{19}$$

The effective action to lowest order in g is given by (see Eq. (A15) Appendix A):

$$S_{eff}[\psi_o, \bar{\psi}_o] = -\beta F_{nc}(\mu) + \int_0^\beta d\tau \int dx \left\{ \bar{\psi}_o(x,\tau) \right.$$
$$\left. \times \left(\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \Lambda - V(x) \right) \psi_o(x,\tau) - \frac{g}{2} \bar{\psi}_o(x,\tau)\bar{\psi}_o(x,\tau)\psi_o(x,\tau)\psi_o(x,\tau) \right\}, \qquad (20)$$

where $\Lambda = \mu - 2g\rho_{nc}(0)$ is a renormalized chemical potential and $F_{nc}(\mu)$ is the free energy of the nonideal gas of thermal particles (see Appendix A). At this point it is reasonable to pass to new variables, namely the density $\rho(x,\tau)$ and the phase $\varphi(x,\tau)$ of the field $\psi_o(x,\tau)$. These variables are defined through, and compare [13],

$$\psi_o(x,\tau) = \sqrt{\rho(x,\tau)} e^{i\varphi(x,\tau)}, \qquad \bar{\psi}_o(x,\tau) = \sqrt{\rho(x,\tau)} e^{-i\varphi(x,\tau)},$$

where ρ and φ are two independent real fields. Now the integration measure $\mathcal{D}\bar{\psi}_o(x,\tau)\mathcal{D}\psi_o(x,\tau)$ is changed to $\mathcal{D}\rho(x,\tau)\mathcal{D}\varphi(x,\tau)$. In these new variables the effective action (20) will be equal to

$$S_{eff}[\rho,\varphi] = -\beta F_{nc}(\mu) + i \int_0^\beta d\tau \int dx \left\{ \rho \partial_\tau \varphi + \frac{\hbar^2}{2m} \partial_x (\rho \partial_x \varphi) \right\}$$
$$+ \int_0^\beta d\tau \int dx \left\{ \frac{\hbar^2}{2m} \left(\sqrt{\rho} \partial_x^2 \sqrt{\rho} - \rho (\partial_x \varphi)^2 \right) + (\Lambda - V)\rho - \frac{g}{2}\rho^2 \right\}. \qquad (21)$$

Here and below we denote the first order partial derivatives over τ and x as ∂_τ and ∂_x, respectively, and the second order ones as ∂_τ^2 and ∂_x^2.

We consider the stationary phase approximation to Eq. (19). The appropriate extremum condition $\delta(S_{eff}[\rho,\varphi]) = 0$ for the effective action (21) has the form of two thermal, τ-dependent, Gross–Pitaevskii equations, namely

$$i\partial_\tau \varphi + \frac{\hbar^2}{2m} \left(\frac{1}{\sqrt{\rho}} \partial_x^2 \sqrt{\rho} - (\partial_x \varphi)^2 \right) + (\Lambda - V) - g\rho = 0,$$
$$-i\partial_\tau \rho + \frac{\hbar^2}{m} \partial_x (\rho \partial_x \varphi) = 0. \qquad (22)$$

The substitution of the solutions ρ_0, φ_0 of these equations into the effective action gives

$$S_{eff}[\rho_0,\varphi_0] = -\beta F_{nc}(\mu) + \frac{g}{2} \int_0^\beta d\tau \int dx \rho_0^2. \qquad (23)$$

Here $F_{nc}(\mu)$ is the free energy of the nonideal gas of thermal particles. The total free energy of the system is thus [16] $F(\mu) = -\beta^{-1} S_{eff}[\rho_0,\varphi_0]$.

At the Thomas–Fermi approximation [5], which is valid at low enough temperatures the kinetic energy term $(\partial_x^2 \sqrt{\rho})/\sqrt{\rho}$ (so-called quantum pressure) in the first equation of Eqs. (22) may be omitted. The equilibrium solution relative to the stationary ground state with $\partial_\tau \rho = 0 = \partial_\tau \varphi$ is obtained by setting the velocity field $\mathbf{v} = m^{-1}\partial_x \varphi$ equal to zero, $\mathbf{v} = 0$.

The density profile is then given by, and compare with Eq. (A7) where μ is now replaced by Λ,

$$\rho_{TF}(x) \equiv \frac{\Lambda}{g}\tilde{\rho}_{TF}(x) = \frac{\Lambda}{g}\Big(1 - \frac{x^2}{R_c^2}\Big)\Theta\Big(1 - \frac{x^2}{R_c^2}\Big), \tag{24}$$

in which Θ is the Heaviside step function, and (recalling that $V(x) = \frac{m}{2}\Omega^2 x^2$) R_c is the radius of the ground state at zero temperature, $R_c^2 = \frac{2\Lambda}{m\Omega^2}$.

Following the original decomposition $\psi_o(x,\tau) = \psi_o(x) + \xi(x,\tau)$ we suppose that the thermal fluctuations in the vicinity of the stationary state are small so that we may split the density profile as

$$\rho_0(x,\tau) = \rho_{TF}(x) + \pi_0(x,\tau). \tag{25}$$

The Gross–Pitaevskii equations (22) linearized around the equilibrium solution ($\rho_0 = \rho_{TF}, \varphi = const$) will then take the form

$$\begin{aligned} i\partial_\tau\varphi_0 - g\pi_0 + \frac{\hbar^2}{4m\rho_{TF}}\partial_x^2\pi_0 &= 0, \\ i\partial_\tau\pi_0 - \frac{\hbar^2}{m}\partial_x\left(\rho_{TF}\partial_x\varphi_0\right) &= 0. \end{aligned} \tag{26}$$

Eliminating φ_0 and dropping the term proportional to $\hbar^4$, we arrive at the thermal Stringari equation [32] which is

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2\pi_0 + \partial_x\left(\Big(1 - \frac{x^2}{R_c^2}\Big)\partial_x\pi_0\right) = 0, \tag{27}$$

where v is the sound velocity at the center of the trap,

$$v^2 = \frac{\rho_{TF}(0)g}{m} = \frac{\Lambda}{m}. \tag{28}$$

The substitution $\pi_o = e^{i\omega\tau}u(x)$ transforms Eq. (27) into a Legendre-like equation

$$-\frac{\omega^2}{\hbar^2 v^2}u(x) + \frac{d}{dx}\left(\Big(1 - \frac{x^2}{R_c^2}\Big)\frac{d}{dx}u(x)\right) = 0. \tag{29}$$

Since the Thomas–Fermi profile (24) differs from zero only within $|x| < R_c$, we consider this equation for the fluctuating part $u(x)$ also at $|x| < R_c$. After the analytical continuation $\omega \to iE$, the equation (29) possesses polynomial solutions, namely the Legendre polynomials $P_n(x/R_c)$, if and only if

$$\Big(\frac{R_c}{\hbar v}\Big)^2 E^2 \equiv \frac{2}{\hbar^2\Omega^2}E^2 = n(n+1),\ n \geq 0.$$

Then this gives for the excitation spectrum $E_n = \hbar\Omega\sqrt{\frac{n(n+1)}{2}}$ [33].

B. The correlation functions.

In this Subsec. we calculate the two-point correlation function for the nonhomogeneous gas as defined by Eq. (2),

$$\Gamma(x_1,\tau_1;x_2,\tau_2)=\left\langle T_\tau \hat{\psi}^\dagger(x_1,\tau_1)\hat{\psi}(x_2,\tau_2)\right\rangle. \tag{30}$$

The case of multi-point correlators will be studied in Sec. V.

We can express the correlator (30) as the ratio of two functional integrals [13, 34]:

$$\Gamma(x_1,\tau_1;x_2,\tau_2)=Z^{-1}\int e^{S[\psi,\bar\psi]}\bar\psi(x_1,\tau_1)\psi(x_2,\tau_2)\mathcal{D}\psi\mathcal{D}\bar\psi, \tag{31}$$

where the action $S[\psi,\bar\psi]$ is (17) and Z is the partition function Eq. (16).

We are interested in the long distance asymptotics of the correlators when $\mid x_1-x_2\mid\gg l_c$. The main contribution to the asymptotics is given by the low-lying excitations. By integrating out the fields ψ_e, $\bar\psi_e$ included up to quadratic terms in Eq. (31), we find that the leading term of the asymptotics is equal to

$$\Gamma(x_1,\tau_1;x_2,\tau_2)\simeq\frac{\int e^{S_{eff}[\psi_o,\bar\psi_o]}\bar\psi_o(x_1,\tau_1)\psi_o(x_2,\tau_2)\mathcal{D}\psi_o\mathcal{D}\bar\psi_o}{\int e^{S_{eff}[\psi_o,\bar\psi_o]}\mathcal{D}\psi_o\mathcal{D}\bar\psi_o}, \tag{32}$$

where $S_{eff}[\psi_o,\bar\psi_o]$ is the effective action (20). We may rewrite this expression in terms of the density-phase variables

$$\Gamma(x_1,\tau_1;x_2,\tau_2)\simeq$$
$$\frac{\int \exp\Big(S_{eff}[\rho,\varphi]-i\varphi(x_1,\tau_1)+i\varphi(x_2,\tau_2)+\frac{1}{2}\ln\rho(x_1,\tau_1)+\frac{1}{2}\ln\rho(x_2,\tau_2)\Big)\mathcal{D}\rho\mathcal{D}\varphi}{\int\exp\left(S_{eff}[\rho,\varphi]\right)\mathcal{D}\rho\mathcal{D}\varphi}, \tag{33}$$

where $S_{eff}[\rho,\varphi]$ is Eq. (21).

At low enough temperatures we may change $\ln\rho(x_1,\tau_1)$, $\ln\rho(x_2,\tau_2)$ in (33) to $\ln\rho_{TF}(x_1)$, $\ln\rho_{TF}(x_2)$ with ρ_{TF} given by Eq. (24), since in the Thomas–Fermi regime density fluctuations are suppressed [35, 36]. The functional integrals in (33) can be evaluated by steepest descents [34], and the correlation function $\Gamma(x_1,\tau_1;x_2,\tau_2)$ can be expressed in the form

$$\Gamma(x_1,\tau_1;x_2,\tau_2)\simeq\exp\Big(-S_{eff}[\rho_0,\varphi_0]+S_{eff}[\rho_1,\varphi_1]$$
$$-i\varphi_1(x_1,\tau_1)+i\varphi_1(x_2,\tau_2)+\frac{1}{2}\ln\rho_{TF}(x_1)+\frac{1}{2}\ln\rho_{TF}(x_2)\Big), \tag{34}$$

where the term $S_{eff}[\rho_0,\varphi_0]$ is given by Eq. (23), and the fields ρ_0,φ_0 satisfy two stationary Gross–Pitaevskii equations (22). By definition, the fields ρ_1,φ_1 are determined by the extremum condition:

$$\delta\left(S_{eff}[\rho,\varphi]-i\varphi(x_1,\tau_1)+i\varphi(x_2,\tau_2)+\frac{1}{2}\ln\rho_{TF}(x_1)+\frac{1}{2}\ln\rho_{TF}(x_2)\right)=0. \tag{35}$$

This variational equation leads to another pair of the Gross–Pitaevskii type equations. The first equation appears as a coefficient at the variation $\delta\rho(x,\tau)$,

$$i\partial_\tau\varphi + \frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{\rho}}\partial_x^2\sqrt{\rho} - (\partial_x\varphi)^2\right) + (\Lambda - V(x)) - g\rho = 0, \tag{36}$$

while the second one is the coefficient at the variation $\delta\varphi(x,\tau)$,

$$-i\partial_\tau\rho + \frac{\hbar^2}{m}\partial_x(\rho\partial_x\varphi) = i\delta(x-x_1)\delta(\tau-\tau_1) - i\delta(x-x_2)\delta(\tau-\tau_2). \tag{37}$$

Note that $\rho_{TF}(x)$ is a fixed function, and it is not subjected to variation. Note also that the equation Eq. (37) is driven by δ-functions but Eq. (36) is not – corresponding to the suppression of density fluctuations. After substitution of the solutions ρ_1, φ_1 of Eqs. (36), (37) into the effective action Eq. (21), one obtains

$$S_{eff}[\rho_1,\varphi_1] = -\beta F_{nc}(\mu) - 1 + \frac{g}{2}\int_0^\beta d\tau \int dx\rho_1^2. \tag{38}$$

We can furthermore consistently assume that, away from the boundaries, the density fluctuations are small so that $\rho_1(x,\tau) = \rho_{TF}(x) + \pi_1(x,\tau)$, and π_1 is a slowly varying function. Consequently the terms $\sqrt{\pi_1}\partial_x^2\sqrt{\pi_1}$ and $\partial_x\pi_1\partial_x\varphi_1$ are small and can be dropped, and Eqs. (36), (37), when linearized around the Thomas–Fermi solution, can be expressed in the form

$$i\partial_\tau\varphi_1 - g\pi_1 - \frac{\hbar^2}{2m}(\partial_x\varphi_1)^2 = 0, \tag{39}$$

$$-i\partial_\tau\pi_1 + \frac{\hbar^2}{m}\partial_x(\rho_{TF}\partial_x\varphi_1) = i\delta(x-x_1)\delta(\tau-\tau_1) - i\delta(x-x_2)\delta(\tau-\tau_2). \tag{40}$$

By differentiating Eq. (39) with respect to τ, substituting the result into Eq. (40), and dropping the terms of order higher than g or $\hbar^2$, we find that

$$\frac{1}{g}\partial_\tau^2\varphi_1 + \frac{\hbar^2}{m}\partial_x(\rho_{TF}(x)\partial_x\varphi_1) = i\delta(x-x_1)\delta(\tau-\tau_1) - i\delta(x-x_2)\delta(\tau-\tau_2). \tag{41}$$

It is convenient to rewrite this equation in the form

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2\varphi_1 + \partial_x(\tilde{\rho}_{TF}(x)\partial_x\varphi_1) = i\frac{mg}{\hbar^2\Lambda}\Big\{\delta(x-x_1)\delta(\tau-\tau_1) - \delta(x-x_2)\delta(\tau-\tau_2)\Big\}, \tag{42}$$

where v is the sound velocity at the center of the trap, Eq. (24), and $\tilde{\rho}_{TF}$ is defined in Eq. (24). The solution of this equation depends on the coordinates x_1, τ_1, x_2, τ_2 of the δ-sources so that $\varphi_1(x,\tau) \equiv \varphi_1(x,\tau;x_1,\tau_1,x_2,\tau_2)$. Now, with the use of Eqs. (23), (38), (39), and (40), one can calculate the following contribution into the exponent in Eq. (34) which is

$$\begin{aligned}
&- S_{eff}[\rho_0,\varphi_0] + S_{eff}[\rho_1,\varphi_1]\\
&\simeq \frac{g}{2}\int_0^\beta d\tau\int dx\left(\rho_1^2 - \rho_0^2\right) = \frac{g}{2}\int_0^\beta d\tau\int dx\,(\rho_1-\rho_0)(\rho_1+\rho_0)\\
&\simeq g\int_0^\beta d\tau\int dx\pi_1\rho_0 = \int_0^\beta d\tau\int dx\left(i\partial_\tau\varphi_1 - \frac{\hbar^2}{2m}\Big(\partial_x\varphi_1\Big)^2\right)\rho_0\\
&= -\frac{\hbar^2}{2m}\int_0^\beta d\tau\int dx\rho_0\,(\partial_x\varphi_1)^2 = \frac{\hbar^2}{2m}\int_0^\beta d\tau\int dx\varphi_1(x)\partial_x(\rho_0(x)\partial_x\varphi_1(x))\\
&= \frac{i}{2}\left(\varphi_1(x_1,\tau_1) - \varphi_1(x_2,\tau_2)\right).
\end{aligned}$$

By substituting this expression into Eq. (34), we obtain for the correlation function

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\Big(-\frac{i}{2}\left(\varphi_1(x_1,\tau_1)-\varphi_1(x_2,\tau_2)\right)\Big),$$

where $\rho_{TF}(x)$ is given by Eq. (24).

It is convenient to express the solution of Eqs. (41), (42) in terms of the solution $G(x,\tau;x',\tau')$ of the related equation

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2 G(x,\tau;x',\tau') + \partial_x\left(\left(1-\frac{x^2}{R_c^2}\right)\partial_x G(x,\tau;x',\tau')\right) = \frac{g}{\hbar^2 v^2}\delta(x-x')\delta(\tau-\tau'). \quad (43)$$

and we call this equation as the 'δ-function driven Stringari equation'. This gives for the first order correlation function at $x_1 \neq x_2$:

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\Big(-\frac{1}{2}(G(x_1,\tau_1;x_2,\tau_2)$$

$$+G(x_2,\tau_2;x_1,\tau_1)) + \frac{1}{2}G(x_1,\tau_1;x_1,\tau_1) + \frac{1}{2}G(x_2,\tau_2;x_2,\tau_2)\Big). \quad (44)$$

The function $G(x_1,\tau_1;x_2,\tau_2)$ has the sense of the correlation function of phases,

$$G(x_1,\tau_1;x_2,\tau_2) = \big\langle\hat\varphi(x_1,\tau_1)\hat\varphi(x_2,\tau_2)\big\rangle. \quad (45)$$

The substitution of this formula into (44) gives the well known result [13]

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\Big(-\frac{1}{2}\big\langle(\hat\varphi(x_1,\tau_1)-\hat\varphi(x_2,\tau_2))^2\big\rangle\Big). \quad (46)$$

Notice that the terms in the exponent of the formula (44) have different senses. The terms $G(x_1,\tau_1;x_2,\tau_2)$, $G(x_2,\tau_2;x_1,\tau_1)$ in it depend on the relative position of the coordinates and thus define the long distance behaviour of the correlator. The terms $G(x_1,\tau_1;x_1,\tau_1)$, $G(x_2,\tau_2;x_2,\tau_2)$ each depend on a single set of coordinates and thus contribute to the amplitudes only. Thus we may express the correlation function in the form

$$\Gamma(x_1,\tau_1;x_2,\tau_2) \simeq \sqrt{\tilde\rho(x_1,\tau_1)\tilde\rho(x_2,\tau_2)}\exp\Big(-\frac{1}{2}\left(G(x_1,\tau_1;x_2,\tau_2)+G(x_2,\tau_2;x_1,\tau_1)\right)\Big), \quad (47)$$

where $\tilde\rho(x_1,\tau_1), \tilde\rho(x_2,\tau_2)$ are the renormalized densities [16]. The solution $G(x_1,\tau_1;x_2,\tau_2)$ of Eq. (43) is defined up to an imaginary constant which has the sense of a global phase, and, as follows from (44), it does not influence the phase fluctuations.

The results of Section III are thus such that the excitation spectrum is given by the solution of the Stringari equation (27), while the correlation of phases $\big\langle\hat\varphi(x_1,\tau_1)\hat\varphi(x_2,\tau_2)\big\rangle$ is determined by $G(x_1,\tau_1;x_2,\tau_2)$ which is the solution of the δ-function driven Stringari equation (43).

IV. Asymptotic behaviours of the correlation functions

A. Homogeneous gas.

Here we apply the method presented in the previous section, Section III, to the calculation of the asymptotics of the correlation function for the homogeneous gas. To describe the case when the trap is absent, $V(x) \equiv 0$, we may equally send the radius of the condensate R_c to infinity. This way the Eq. (43) will take the form

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2 G(x,\tau;x',\tau') + \partial_x^2 G(x,\tau;x',\tau') = \frac{g}{\hbar^2 v^2}\delta(x-x')\delta(\tau-\tau'). \tag{48}$$

The solution of this equation is the correlation function of phases. We consider Eq. (48) in the ranges of arguments $0 \le x \le L$ and $0 \le \tau \le \beta$. For $|x-x'| \le L$, $|\tau-\tau'| \le \beta$, and $\beta^{-1} \equiv k_B T \gg (\hbar v)/L$, we obtain the solution

$$G(x,\tau;x',\tau') = \frac{g}{2\pi\hbar v}\ln\left\{2\left|\sinh\frac{\pi}{\hbar\beta v}\left(|x-x'| + i\hbar v(\tau-\tau')\right)\right|\right\} - \frac{g}{\hbar^2 v^2}\frac{|x-x'|^2}{2\beta L}. \tag{49}$$

In the limit $L \to \infty$ the second term in this expression disappears, and the substitution of the remaining expression in Eq. (44) gives the expected answer Eq. (9) for the asymptotics with the scaling exponent equal to

$$\theta = \frac{2\pi\hbar v}{g}. \tag{50}$$

Using $v = \sqrt{\Lambda/m}$ and $\rho = \Lambda/g$ for the sound velocity and density, respectively, one obtains the expression Eq. (10) for the scaling exponent. For small coupling constants we may re-express the result (50) in the form of Eq. (12).

These asymptotics for the correlation functions are valid for the ring-shaped Bose–Einstein condensates, and the complete agreement between Eq. (9) found by the exact Bethe Ansatz method and the results of the functional integral methods in this case of homogeneous gas, confirms the validity of the functional integral method itself as described in its actual details above. And this should mean that the results achievable by the functional integral method in the inhomogeneous case $V(x) \neq 0$ should be equally good. The case $V(x) \neq 0$ cannot (so far) be treated by any Bethe Ansatz method as already mentioned.

B. Trapped gas.

Let us consider the nonhomogeneous Stringari equation (43) which determines through its solution $G(x,\tau;x',\tau')$ the long distance behaviour of the correlation functions:

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2 G(x,\tau;x',\tau') + \partial_x\left(\left(1-\frac{x^2}{R_c^2}\right)\partial_x G(x,\tau;x',\tau')\right) = \frac{g}{\hbar^2 v^2}\delta(x-x')\delta(\tau-\tau'). \tag{51}$$

We are looking for the solutions which are finite, periodic in τ, and quadratically integrable at $|x| < R_c$. The substitution of the series

$$G(x,\tau;x',\tau') = \frac{1}{\beta}\sum_\omega e^{i\omega(\tau-\tau')}G_\omega(x,x'), \tag{52}$$

where $\omega = \frac{2\pi n}{\beta}, n = 0, \pm 1, \ldots$, into Eq. (51) leads to an equation for the spectral density,

$$-\frac{\omega^2}{\hbar^2 v^2} G_\omega(x, x') + \frac{d}{dx}\left(\left(1 - \frac{x^2}{R_c^2}\right)\frac{d}{dx} G_\omega(x, x')\right) = \frac{g}{\hbar^2 v^2}\delta(x - x'). \tag{53}$$

The solution of this equation is expressed in terms of the Legendre functions of the first and second kind, $P_\nu(x/R_c)$ and $Q_\nu(x/R_c)$, which are linearly independent solutions of the Legendre equation Eq. (29) with ν taken to be

$$\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} - \left(\frac{R_c}{\hbar v}\right)^2 \omega^2}.$$

We get:

$$G_\omega(x, x') = \frac{g R_c}{2\hbar^2 v^2}\,\epsilon(x - x')\Big\{Q_\nu\big(\frac{x}{R_c}\big)P_\nu\big(\frac{x'}{R_c}\big) - Q_\nu\big(\frac{x'}{R_c}\big)P_\nu\big(\frac{x}{R_c}\big)\Big\}, \tag{54}$$

where $\epsilon(x - x')$ is the sign function: $\epsilon(x) \equiv sign(x)$. The validity of this statement can be checked by substitution of the result (54) into Eq. (53), and with the help of the equality [38]

$$Q'_\nu(y)P_\nu(y) - Q_\nu(y)P'_\nu(y) = (1 - y^2)^{-1}.$$

For a stationary, τ-independent correlator we have $G(x; x') = \frac{1}{\beta}G_0(x, x')$, where

$$\begin{aligned} G_0(x, x') =& \frac{g R_c}{2\hbar^2 v^2}\,\epsilon(x - x')\Big\{Q_0\big(\frac{x}{R_c}\big) - Q_0\big(\frac{x'}{R_c}\big)\Big\} \\ =& \frac{g R_c}{4\hbar^2 v^2}\ln\left[\frac{\left(1 + \frac{|x_1 - x_2|}{2R_c}\right)^2 - \frac{(x_1 + x_2)^2}{4R_c^2}}{\left(1 - \frac{|x_1 - x_2|}{2R_c}\right)^2 - \frac{(x_1 + x_2)^2}{4R_c^2}}\right], \end{aligned} \tag{55}$$

and we have used here the particular properties of the Legendre functions $P_0(x) = 1$ and $Q_0(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$. This result leads to the following correlation function [15, 16, 27, 37]

$$\Gamma(x_1; x_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}\exp\left(-\frac{g R_c}{4\hbar^2 v^2}\ln\left[\frac{\left(1 + \frac{|x_1 - x_2|}{2R_c}\right)^2 - \frac{(x_1 + x_2)^2}{4R_c^2}}{\left(1 - \frac{|x_1 - x_2|}{2R_c}\right)^2 - \frac{(x_1 + x_2)^2}{4R_c^2}}\right]\right), \tag{56}$$

where θ is the scaling exponent as given in Eq. (10).

To study the behaviour of the τ-dependent correlation functions we must first notice that the Green's function of Eq. (53) is defined up to a solution of the homogeneous Legendre equation Eq. (29). To guarantee the convergence of the series (52), we may add such a term to the expression (54), and obtain thereby for the Green's function, with $|\omega| > 0$,

$$G_\omega(x, x') = \frac{g R_c}{2\hbar^2 v^2}\epsilon(x - x')\Big\{Q_\nu\big(\frac{x}{R_c}\big)P_\nu\big(\frac{x'}{R_c}\big) - Q_\nu\big(\frac{x'}{R_c}\big)P_\nu\big(\frac{x}{R_c}\big)\Big\}$$

$$-i\frac{gR_c}{2\hbar^2v^2}\Big\{\frac{2}{\pi}Q_\nu(\frac{x}{R_c})Q_\nu(\frac{x'}{R_c})+\frac{\pi}{2}P_\nu(\frac{x'}{R_c})P_\nu(\frac{x}{R_c})\Big\}. \tag{57}$$

For $|\omega|\gg\hbar v/(2R_c)$, the asymptotics of the Legendre functions are [38]:

$$P_\nu(\cos\theta)\simeq\Big(\frac{2}{\pi\nu\sin\theta}\Big)^{1/2}\sin\left[(\nu+\frac{1}{2})\theta+\frac{\pi}{4}\right],$$
$$Q_\nu(\cos\theta)\simeq\Big(\frac{\pi}{2\nu\sin\theta}\Big)^{1/2}\cos\left[(\nu+\frac{1}{2})\theta+\frac{\pi}{4}\right],$$

where $\cos\theta\equiv x/R_c$, $0<\theta<\pi$, and $|\arg\nu|<\pi/2$. Substituting these expressions into (57) we find the behaviour of the Green's function for large $|\omega|$,

$$G_\omega(x,x')\simeq-\frac{g}{2\hbar v|\omega|}\frac{1}{\sqrt{\sin\theta\sin\theta'}}\exp\Big(-\frac{R_c}{\hbar v}|\omega|\,|\theta-\theta'|\Big). \tag{58}$$

In the quasi homogeneous limit, when $|x_1-x_2|\gg\frac{x_1+x_2}{2}$ and the coordinates x_1 and x_2 are far away from the boundaries defined by R_c, so that $|x_1-x_2|\ll R_c$, the leading term of the Green's function Eq. (55) is

$$G_0(x,x')\simeq\frac{\Lambda}{2\hbar^2v^2\rho_{TF}(S)}\,|x-x'|\,. \tag{59}$$

Here S is the center-of-mass coordinate $S=\frac{x_1+x_2}{2}$, and v is the sound velocity Eq. (28). For the functions with $\omega\neq0$ and at inverse temperatures larger than the lowest energy excitations, $\beta\equiv(k_BT)^{-1}\gg\hbar v/(2R_c)$, we may use the asymptotics Eq. (58) to obtain

$$G_\omega(x,x')\simeq-\frac{\Lambda}{2\hbar v\rho_{TF}(S)}\frac{\exp\Big(-(\hbar v)^{-1}\mid\omega\mid\mid x-x'\mid\Big)}{\mid\omega\mid}. \tag{60}$$

The substitution of Eqs. (59) and (60) in Eq. (52) gives now

$$G(x,\tau;x',\tau')=\frac{\Lambda}{2\pi\hbar v\rho_{TF}(S)}\ln\left\{2\left|\sinh\frac{\pi}{\hbar\beta v}\left(|x-x'|+i\hbar v(\tau-\tau')\right)\right|\right\}. \tag{61}$$

The correlation function for the Bose fields is in this case

$$\Gamma(x_1,\tau_1;x_2,\tau_2)\simeq\frac{\sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}}{\left|\sinh\frac{\pi}{\hbar\beta v}\left(|x_1-x_2|+i\hbar v(\tau_1-\tau_2)\right)\right|^{1/\theta(S)}}, \tag{62}$$

where the scaling exponent $\theta(S)$ depends now on the center-of-mass coordinate $S=\frac{x_1+x_2}{2}$:

$$\theta(S)=\frac{2\pi\hbar\rho_{TF}(S)}{mv}.$$

From the result (62) it follows that the correlation function decreases exponentially at large distances,

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)} \exp\left(-\frac{\Lambda\,|x_1 - x_2|}{2\beta\hbar^2 v^2 \rho_{TF}(S)}\right), \tag{63}$$

in which the correlation length depends on the center-of-mass coordinate and is

$$\xi(S) \equiv \frac{2\hbar^2 \beta \rho_{TF}(S)}{m} = \frac{\hbar\beta v}{\pi}\theta(S). \tag{64}$$

At zero temperature we find correspondingly that

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \frac{\sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)}}{\Big||x_1 - x_2| + i\hbar v(\tau_1 - \tau_2)\Big|^{1/\theta(S)}}. \tag{65}$$

These expressions for the trapped gas are similar to those for the homogeneous gas, Eqs. (9) and (15), but the scaling exponents now depend on the center-of-mass coordinate. We thus find that our results for the trapped Bose gas precisely assume the forms of the exactly known results in the limit of vanishing trap potential.

There is also another way to study the asymptotic behaviour of the τ-dependent correlation function in the quasi-homogeneous limit [16]. We may suppose that the term $(2x/R_c^2)\partial_x G(x, \tau; x', \tau')$ in Eq. (51) can be neglected, and that the factor $\left(v^2 \tilde{\rho}_{TF}(x)\right)^{-1}$ in the first term in Eq. (51) can be substituted by the inverse squared sound velocity at the center of the trap. The corresponding equation for the correlation function of the phases in the quasi homogeneous limit is

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2 G(x, \tau; x', \tau') + \partial_x^2 G(x, \tau; x', \tau') = \frac{g}{\hbar^2 v^2 \tilde{\rho}_{TF}(\frac{x+x'}{2})}\delta(x - x')\delta(\tau - \tau'). \tag{66}$$

It is easy to check that the solution of this equation is Eq. (61).

V. Multi-particle correlation functions

We consider finally the asymptotic behaviour of the multi-particle temperature-dependent correlation functions of the Bose fields $\hat{\psi}^\dagger, \hat{\psi}$,

$$\Gamma_m(x_1, \tau_1; \ldots ; x_{2m}, \tau_{2m}) \equiv \langle T_\tau \hat{\psi}^\dagger(\mathbf{z}_1) \ldots \hat{\psi}^\dagger(\mathbf{z}_m)\hat{\psi}(\mathbf{z}_{m+1}) \ldots \hat{\psi}(\mathbf{z}_{2m})\rangle, \tag{67}$$

where $\mathbf{z} = (x, \tau)$. We show in particular that at low temperatures and for large relative distances $R_{ij} \equiv\mid x_i - x_j \mid$, the m-particle correlation function can be expressed in terms of the two-point ones,

$$\Gamma_m(x_1, \tau_1; \ldots ; x_{2m}, \tau_{2m}) \simeq \prod_{1 \le i < j \le 2m} \Gamma(x_1, \tau_1; x_2, \tau_2)^{-l_i l_j}, \tag{68}$$

where $l_i = 1$ if $i = 1, \ldots, m$, $l_i = -1$ if $i = m+1, \ldots, 2m$, and $\Gamma(x_i, \tau_i; x_j, \tau_j) = \left\langle T_\tau \hat{\psi}^\dagger(x_i, \tau_i) \hat{\psi}(x_j, \tau_j) \right\rangle$ is the two-point correlation function discussed in the previous sections. We suppose as well that the points x_i are lying inside the condensate and sufficiently far from its boundaries. The factorisation of the asymptotic behaviour of the correlation function Eq. (68) is similar to that in the nontrapped case [39]; the lack of the translational invariance only introduces modifications to the two-point correlations.

We can express Eq. (67) as the ratio of two functional integrals over the space of complex-valued functions (c.f. Section III):

$$\Gamma_m(x_1, \tau_1; \ldots; x_{2m}, \tau_{2m})$$
$$= Z^{-1} \int e^{S[\psi, \bar{\psi}]} \bar{\psi}(x_1, \tau_1) \ldots \bar{\psi}(x_m, \tau_m) \psi(x_{m+1}, \tau_{m+1}) \ldots \psi(x_{2m}, \tau_{2m}) \mathcal{D}\psi \mathcal{D}\bar{\psi}, \quad (69)$$

where Z is the partition function Eq. (16). In order to estimate Γ_m, Eq. (69), we proceed as in Section III. The leading term of the asymptotics is as follows:

$$\Gamma_m(x_1, \tau_1; \ldots; x_{2m}, \tau_{2m})$$
$$\simeq \frac{\int e^{S_{eff}[\psi_o, \bar{\psi}_o]} \bar{\psi}_o(\mathbf{z}_1) \ldots \bar{\psi}_o(\mathbf{z}_m) \psi_o(\mathbf{z}_{m+1}) \ldots \psi_o(\mathbf{z}_{2m}) \mathcal{D}\psi_o \mathcal{D}\bar{\psi}_o}{\int e^{S_{eff}} \mathcal{D}\psi_o \mathcal{D}\bar{\psi}_o}, \quad (70)$$

where S_{eff} is the effective action Eq. (20). Repeating the arguments of Section III, we can rewrite the expression for the correlation function in the form

$$\Gamma_m(x_1, \tau_1; \ldots; x_{2m}, \tau_{2m})$$
$$\simeq \frac{\int \exp \left(S_{eff}[\rho, \varphi] - i \sum\limits_{k=1}^{m} \varphi(\mathbf{z}_k) + i \sum\limits_{k=m+1}^{2m} \varphi(\mathbf{z}_k) + \frac{1}{2} \sum\limits_{k=1}^{2m} \ln \rho_{TF}(\mathbf{z}_k) \right) \mathcal{D}\rho \mathcal{D}\varphi}{\int \exp \left(S_{eff}[\rho, \varphi] \right) \mathcal{D}\rho \mathcal{D}\varphi}. \quad (71)$$

At low enough temperatures these remaining functional integrals can be evaluated by steepest descents, and the correlation function takes the form

$$\Gamma_m(x_1, \tau_1; \ldots; x_{2m}, \tau_{2m}) \simeq \exp\Bigg(-S_{eff}[\rho_o, \varphi_o] + S_{eff}[\rho_1, \varphi_1]$$
$$-i \sum_{k=1}^{m} \varphi_1(\mathbf{z}_k) + i \sum_{k=m+1}^{2m} \varphi_1(\mathbf{z}_k) + \frac{1}{2} \sum_{k=1}^{2m} \ln \rho_{TF}(\mathbf{z}_k) \Bigg), \quad (72)$$

where the fields ρ_0, φ_0 satisfy two stationary Gross–Pitaevskii Eqs. (22). By definition, the fields ρ_1, φ_1 are determined by the extremum condition

$$\delta \Big(S_{eff}[\rho, \varphi] - i \sum_{k=1}^{m} \varphi(\mathbf{z}_k) + i \sum_{k=m+1}^{2m} \varphi(\mathbf{z}_k) + \frac{1}{2} \sum_{k=1}^{2m} \ln \rho_{TF}(\mathbf{z}_k) \Big) = 0, \quad (73)$$

and this variational equation also leads to a pair of the Gross–Pitaevskii type equations

$$i\partial_\tau\varphi+\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{\rho}}\partial_x^2\sqrt{\rho}-(\partial_x\varphi)^2\right)+(\Lambda-V(x))-g\rho=0$$

$$-i\partial_\tau\rho+\frac{\hbar^2}{m}\partial_x\left(\rho\partial_x\varphi\right)=i\sum_{k=1}^{m}\delta(x-x_k)\delta(\tau-\tau_k)-i\sum_{k=m+1}^{2m}\delta(x-x_k)\delta(\tau-\tau_k).$$

In the approximation considered in Subsection III.B (see Eqs. (39), (40)), these equations lead us to the following equation:

$$\frac{1}{\hbar^2v^2}\partial_\tau^2\varphi+\partial_x\left(\tilde{\rho}_{TF}(x)\partial_x\varphi\right)=i\frac{mg}{\hbar^2\Lambda}\Big\{\sum_{k=1}^{m}\delta(x-x_k)\delta(\tau-\tau_k)-\sum_{k=m+1}^{2m}\delta(x-x_k)\delta(\tau-\tau_k)\Big\}. \quad (74)$$

One finds now that

$$-S_{eff}[\rho_0,\varphi_0]+S_{eff}[\rho_1,\varphi_1]=\frac{i}{2}\Big(\sum_{k=1}^{m}\varphi_1(\mathbf{z}_k)-\sum_{k=m+1}^{2m}\varphi_1(\mathbf{z}_k)\Big).$$

Substitution of this expression in Eq. (72) leads to the correlation function

$$\Gamma_m(x_1,\ldots,x_{2m})\simeq\prod_{k=1}^{2m}\sqrt{\rho_{TF}(x_k)}\exp\left(-\frac{i}{2}\Big(\sum_{k=1}^{m}\varphi_1(\mathbf{z}_k)-\sum_{k=m+1}^{2m}\varphi_1(\mathbf{z}_k)\Big)\right), \quad (75)$$

where the function $\varphi_1(\mathbf{z})\equiv\varphi_1(\mathbf{z};z_1,\ldots,\mathbf{z}_{2m})$ satisfies Eq. (74). We can then express the solution of Eq. (74) in terms of the function $G(\mathbf{z},\mathbf{z}')$, which is the solution of Eq. (43), and this yields

$$-\frac{i}{2}\Big(\sum_{k=1}^{m}\varphi_1(\mathbf{z}_k)-\sum_{k=m+1}^{2m}\varphi_1(\mathbf{z}_k)\Big)=-\frac{1}{2}\sum_{j=1}^{m}\sum_{k=m+1}^{2m}\left(G(\mathbf{z}_j,\mathbf{z}_k)+G(\mathbf{z}_k,\mathbf{z}_j)\right) \quad (76)$$

$$+\frac{1}{2}\sum_{j=1}^{m-1}\sum_{k=j+1}^{m}\left(G(\mathbf{z}_j,\mathbf{z}_k)+G(\mathbf{z}_k,\mathbf{z}_j)\right)$$

$$+\frac{1}{2}\sum_{j=m+1}^{2m-1}\sum_{k=j+1}^{2m}\left(G(\mathbf{z}_j,\mathbf{z}_k)+G(\mathbf{z}_k,\mathbf{z}_j)\right)+\frac{1}{2}\sum_{j=1}^{2m}G(\mathbf{z}_j,\mathbf{z}_j).$$

The first order correlation function at $\mathbf{z}_1\neq\mathbf{z}_2$ is then equal to that of Eq. (44). Substitution of this expression in Eq. (76) gives the already indicated result Eq. (68):

$$\Gamma_m(\mathbf{z}_1,\ldots,\mathbf{z}_{2m})=\prod_{1\le i<j\le 2m}\Gamma^{-l_il_j}(\mathbf{z}_i,\mathbf{z}_j).$$

VI. Conclusions

It is plain that the functional integral methods used here to calculate correlation functions of 1D bosons are very powerful. They extend the methods introduced by Popov [13],[34] to the case with a confining potential that breaks the translational symmetry of the system. A key new ingredient introduced here is the Green's function G that satisfies the δ-function driven Stringari equation Eq. (43). This Green's function was shown to be precisely the correlation of phases. We thus obtain the results quoted in [15, 16] in a very natural way, for the Green's function could then be used to obtain the two-point correlation function. Note how density fluctuations were suppressed by the step at which δ-function driving terms were deliberately neglected from the Eq. (39). It is rather remarkable that for a vanishing trap potential the two-point correlation function found by the approximated functional-integral method coincides *precisely* with the exact Bethe Ansatz result, especially when one takes into account the changed boundary conditions.

The 'universal' feature in the two-point correlation functions obtained with or without a trap potential is the scaling exponent θ that governs the decay of correlations for increasing separation of the two points. At nonzero temperatures θ is proportional to the correlation length and at zero temperature it becomes a critical exponent in the algebraic decay of correlations. We express this scaling exponent in terms of the observed quantities, sound velocity v and the density ρ of the quasi condensate, which means we need not calculate the T-matrix as, *e.g.*, in [24]. It also means that the expression we find for θ is valid for all coupling strengths: the two physical quantities involved do depend on coupling strength so that $\theta = 2$ in the exact Girardeau–Tonks limit $g \to \infty$, and $\theta > 2$ for any finite g. In the presence of a confining potential the decay of correlations is formally exactly the same, but the scaling exponent will now depend on the center-of-mass coordinate,

$$\theta(S) = \frac{2\pi\hbar\rho_{TF}(S)}{mv}$$

with $S = \frac{x_1+x_2}{2}$. As we expect the S-dependence of θ to be small, it should approach two in the Girardeau–Tonks limit also in this case. In the weak-interaction Gross–Pitaevskii limit θ can also be expressed in the form

$$\theta(S) = 2\pi\hbar\sqrt{\frac{\rho_{TF}^2(S)}{mg\rho_{TF}(0)}}.$$

As all quantities that enter these expressions for θ are themselves observables, a measurement of the decay of correlations in a trapped Bose system would provide a sensitive test of the theoretical framework based on repulsive δ-function interactions between the particles. With a sensitive enough measurement one should also be able to detect the effects of confinement, *i.e.*, the dependence on the center-of-mass coordinate S in the decay of the correlations.

As experiments on trapped Bose condensates have become increasingly more sophisticated, a need for better understanding of also many-particle correlations has become evident also. So far little has been known about the $2m$-point correlation functions for $m > 1$. One of the important results of this paper is thus the demonstration that all $2m$-point correlations in trapped Bose systems can be expressed asymptotically entirely in terms of the two-point correlations. Note that, based on the results given here, and the exact results we have reported earlier on a related Bose system [31], we can also propose that the asymptotic form of the density-density correlator for trapped Bose systems as

$$\left\langle T_\tau \hat{\psi}^\dagger(x_1,\tau_1)\hat{\psi}(x_1,\tau_1)\hat{\psi}^\dagger(x_2,\tau_2)\hat{\psi}(x_2,\tau_2)\right\rangle - \rho_{TF}(x_1)\rho_{TF}(x_2)$$

$$\sim A\left(\frac{1}{w^2}+\frac{1}{\bar{w}^2}\right)+\frac{B}{|w|^{\theta(S)}}\cos\left[2\pi(x_1-x_2)\rho_{TF}(S)\right]$$

with $w = \sinh\{(\pi|x_1-x_2|/\hbar\beta v) + i\hbar v(\tau_1-\tau_2)\}$. Here A and B are real constants. At large separations of points this density-density correlator decays as $Ae^{-\zeta|x_1-x_2|}$ with the correlation length $\zeta = (2\pi)^{-1}\hbar\beta v$. Notice that the scaling exponent θ appears here in the oscillating term which is not the leading term asymptotically. One would nevertheless expect that the oscillating behaviour in the asymptotic density-density correlations should be experimentally observable.

Finally we would like to emphasize that the methods applied here are not restricted to 1D but can equally well be applied in 2D and 3D [16, 42]. These methods are not restricted to the calculation of correlation functions, and can be used to calculate all of the thermodynamic properties of interacting Bose systems, and can be extended beyond the Thomas-Fermi as well as the mean-field approximations.

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Appendix A.

Here we shall give a formal derivation of the effective action $S_{eff}[\psi_o,\bar{\psi}_o]$, Eq. (20). We use the field-theoretical approach of loop expansion [40]. Following the general prescription of Section III, we split the initial fields $\psi,\bar{\psi}$ into $\psi=\psi_o+\psi_e$, $\bar{\psi}=\bar{\psi}_o+\bar{\psi}_e$ so that the action $S[\psi,\bar{\psi}]$, Eq. (17), is divided into three terms:

$$S = S_{cond} + S_{free} + S_{int}, \tag{A1}$$

where S_{cond} is the action of the condensed particles in the so-called tree-level approximation [43]

$$S_{cond}[\psi_o,\bar{\psi}_o]$$
$$\equiv \int_0^\beta d\tau \int dx \left\{\bar{\psi}_o(x,\tau)\hat{K}_+\psi_o(x,\tau) - \frac{g}{2}\bar{\psi}_o(x,\tau)\bar{\psi}_o(x,\tau)\psi_o(x,\tau)\psi_o(x,\tau)\right\}. \tag{A2}$$

The action of the highly excited thermal fluctuations of the noncondensed particles is given by

$$S_{free}[\psi_e,\bar{\psi}_e] \equiv \frac{1}{2}\int_0^\beta d\tau \int dx(\bar{\psi}_e,\psi_e)\hat{G}^{-1}\begin{pmatrix}\psi_e\\ \bar{\psi}_e\end{pmatrix}, \tag{A3}$$

and S_{int} is the part of the action that describes the interaction of condensed particles with the thermal ones,

$$S_{int}[\psi_o, \bar{\psi}_o, \psi_e, \bar{\psi}_e] \equiv \int_0^\beta d\tau \int dx \Big\{ \bar{\psi}_e(x,\tau) \left[\hat{K}_+ - g\bar{\psi}_o\psi_o \right] \psi_o(x,\tau) + \psi_e(x,\tau) \left[\hat{K}_- - g\bar{\psi}_o\psi_o \right] \bar{\psi}_o(x,\tau) \Big\}. \tag{A4}$$

In Eqs. (A2)–(A4) we have defined the differential operators,

$$\hat{K}_\pm \equiv \pm\frac{\partial}{\partial\tau} - \mathcal{H}, \qquad \mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu + V(x),$$

and the matrix operator

$$\hat{G}^{-1} \equiv \hat{G}_0^{-1} - \hat{\Sigma}, \tag{A5}$$

where

$$\hat{G}_0^{-1} \equiv \begin{pmatrix} \hat{K}_+ & 0 \\ 0 & \hat{K}_- \end{pmatrix}, \qquad \hat{\Sigma} \equiv \hat{\Sigma}(\psi_o, \bar{\psi}_o) = g \begin{pmatrix} 2\bar{\psi}_o\psi_o & \psi_o^2 \\ (\bar{\psi}_o)^2 & 2\bar{\psi}_o\psi_o \end{pmatrix}.$$

The approximation used here for S_{int}, Eq. (A4), implies that the noncondensed excitations do not interact with each other.

For our purposes it is enough to consider the functional integral in leading order as given by the method of stationary phase (steepest descents). Therefore we choose $\bar{\psi}_o, \psi_o$ as the stationary points of S_{cond} (A2). They are governed by equations of the Gross–Pitaevskii type,

$$\begin{aligned} \left(\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu - V(x) \right) \psi_o - g(\bar{\psi}_o\psi_o)\psi_o = 0, \\ \left(-\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \mu - V(x) \right) \bar{\psi}_o - g(\bar{\psi}_o\psi_o)\bar{\psi}_o = 0. \end{aligned} \tag{A6}$$

As soon as $\psi_o, \bar{\psi}_o$ are the solutions of these equations, S_{int} drops out of Eq. (A1), and the dynamics of $\psi_e, \bar{\psi}_e$ is just given to lowest order by S_{free}, Eq. (A3). The latter depends nontrivially on $\bar{\psi}_o, \psi_o$ through $\hat{\Sigma}$ in $\hat{G}^{-1}$, Eq. (A5). We denote a τ-independent solution of the Gross–Pitaevskii Eqs. (A6), written in the Thomas–Fermi approximation as

$$\bar{\psi}_o\psi_o = \rho_{TF}(x;\mu) \equiv \frac{1}{g}\left(\mu - V(x)\right)\Theta\left(\mu - V(x)\right), \tag{A7}$$

where one should notice the dependence of the Thomas–Fermi profile on the 'bare' chemical potential μ. The integration over ψ_e, $\bar{\psi}_e$ in Eq. (18) is now Gaussian, and gives a one-loop corrected effective action [13] in terms of the condensate variables ψ_o, $\bar{\psi}_o$ only,

$$S_{eff}[\psi_o, \bar{\psi}_o] \equiv S_{cond}[\psi_o, \bar{\psi}_o] - \frac{1}{2}\ln\,\det(\hat{G}^{-1}). \tag{A8}$$

Here $\hat{G}^{-1}$ is the matrix operator (A5).

To assign a meaning to the resulting expression for the effective action (A8), we have to regularize the determinant $\det(\hat{G}^{-1})$. Our $\hat{G}^{-1}$ is already written in the form of a 2×2

matrix Dyson equation (A5), where the entries of the matrix $\hat{\Sigma}(\psi_o, \bar{\psi}_o)$ play the role of the normal and anomalous self-energy parts. The Dyson equation defines the matrix Green function $\hat{G} \equiv \hat{G}(\psi_o, \bar{\psi}_o)$ (*i.e.*, the propagator matrix) of the fields $\psi_e, \bar{\psi}_e$. The matrix $\hat{G}$ is given by the formal inverse

$$\hat{G} = \left(\hat{G}_0^{-1} - \hat{\Sigma}\right)^{-1}. \tag{A9}$$

The matrix operator $\hat{G}^{-1}$ (A5) can be formally diagonalized by means of the famous (u, v)-transformation of Bogoliubov [41, 35]. Compatibility of the corresponding equations for the unknown fields u, v defines a quasi-classical spectrum of the elementary excitations [32].

For our purposes it is appropriate to represent $\hat{G}^{-1}$ in the form

$$\hat{G}^{-1} = \hat{G}_0^{-1} - \hat{\Sigma} \equiv \hat{\mathcal{G}}^{-1} - \left(\hat{\Sigma} - 2g\rho_{TF}(x;\mu)\hat{I}\right), \tag{A10}$$

where $\hat{I}$ is a 2×2 unit matrix, and $\hat{\mathcal{G}}^{-1}$ is defined as

$$\hat{\mathcal{G}}^{-1} \equiv \begin{pmatrix} \hat{K}_+ - 2g\rho_{TF}(x;\mu) & 0 \\ 0 & \hat{K}_- - 2g\rho_{TF}(x;\mu) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \tag{A11}$$

Here $\rho_{TF}(x;\mu)$ is the solution (A7) of Eq. (A6), and the expression (A10) implies that we have simply added and subtracted $2g\rho_{TF}(x;\mu)$ in the diagonal of the matrix operator $\hat{G}^{-1}$. A formal inverse of $\hat{\mathcal{G}}^{-1}$ can be found from the equation which defines the corresponding Green's functions $\mathcal{G}_\pm$:

$$\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix} \begin{pmatrix} \mathcal{G}_+ & 0 \\ 0 & \mathcal{G}_- \end{pmatrix} = \delta(x - x')\delta(\tau - \tau')\hat{I}\,.$$

Applying the relation $\ln \det = \operatorname{tr} \ln$, which is valid for both matrices and operators, we obtain

$$-\frac{1}{2}\ln\det \hat{G}^{-1} = -\frac{1}{2}\operatorname{tr}\ln\left(\hat{I} - \hat{\mathcal{G}}(\hat{\Sigma} - 2g\rho_{TF}(x;\mu)\hat{I})\right) - \frac{1}{2}\ln\det\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \tag{A12}$$

The first term on the r.h.s. is free of divergences and now we have to assign a meaning to the infinite-order determinant on the r.h.s. of (A12). The operators $\mathcal{K}_\pm$ may be expressed in the form

$$\mathcal{K}_\pm \equiv \pm\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + |V(x) - \mu|. \tag{A13}$$

Notice how minus two times $(\mu - V(x))\,\Theta\,(\mu - V(x))$ in $\mathcal{K}_+, \mathcal{K}_-$ introduces the absolute value $|V(x) - \mu|$ into these differential operators. Let us denote the eigenvalues of $\mathcal{K}_\pm$ as $\pm i\omega - \lambda_n$, where ω is a bosonic Matsubara frequency, and λ_n is an eigenenergy of the operator $-(\hbar^2/2m)\partial^2/\partial x^2 - |V(x) - \mu|$. The regular part of the logarithm of the determinant in Eq. (A12) has the sense of the free energy $\tilde{F}_{nc}$ of an ideal gas of thermal particles

$$\tilde{F}_{nc}(\mu) \equiv \frac{1}{2\beta}\ln\det\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix} = \frac{1}{\beta}\sum_n \ln\left(2\sinh\frac{\beta\lambda_n}{2}\right),$$

where the regularized values of the determinants of the differential operators $\mathcal{K}_{\pm}$ can be obtained, for instance, by means of a zeta-regularization procedure [43]. Then, up to a first order in g, we find that

$$-\frac{1}{2}\ln\,\det\hat{G}^{-1}=-\beta\tilde{F}_{nc}(\mu)+g\int_0^\beta d\tau\int dx\,(\mathcal{G}_-(x,\tau;x,\tau)+\mathcal{G}_+(x,\tau;x,\tau))$$

$$\times(\bar{\psi}_o\psi_o-\rho_{TF}(x;\mu))\equiv-\beta F_{nc}(\mu)+2g\int_0^\beta d\tau\int dx\rho_{nc}(x)\bar{\psi}_o\psi_o. \tag{A14}$$

Here F_{nc} is the free energy of the nonideal gas of thermal particles, and the last term in (A14) describes the interaction of the thermal particles with the condensate. The density of the noncondensed particles is $\rho_{nc}(x)\equiv-\mathcal{G}_{\pm}(x,\tau;x,\tau)$, and it depends, in fact, on the spatial variable only because of the translational invariance in τ. For very low temperatures and far from the boundaries of the condensate we can use $\rho_{nc}(x)\simeq\rho_{nc}(0)$ because $\mathcal{G}_{\pm}(x,\tau;x,\tau)$ is nearly a constant over most parts of the condensate [44].

We now finally obtain the effective action

$$S_{eff}[\psi_o,\bar{\psi}_o]=-\beta F_{nc}(\mu)+\int_0^\beta d\tau\int dx\left\{\bar{\psi}_o(x,\tau)\left(\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\Lambda-V(x)\right)\psi_o(x,\tau)\right.$$

$$\left.-\frac{g}{2}\bar{\psi}_o(x,\tau)\bar{\psi}_o(x,\tau)\psi_o(x,\tau)\psi_o(x,\tau)\right\}, \tag{A15}$$

where $\Lambda=\mu-2g\rho_{nc}(0)$ is a renormalized chemical potential. We consider the S_{eff} (A15) as an effective one-loop action which involves the thermal corrections above the *classical* level (see Eqs. (A6)). Note that this derivation of the effective action does not depend on the dimensionality of the system, and is thus valid in two and three space dimensions as well.

Appendix B.

We derive here the scaling exponents Eqs. (11) and (12) for infinite and small values of the coupling constant g, respectively, from the *universal* result Eq. (10) based on the Bethe Ansatz equations Eq. (4). It is possible to solve these equations exactly in the limiting cases $g\to0$ and $g\to\infty$.

Let us discuss first the weak-interaction limit $g\to0$. In order to study this free boson limit we have to rescale the parameters λ_j in Eq. (4): $\lambda_j=(\tilde{g}/L)^{1/2}\mu_j$, and then take $g\to0$. The Bethe equations for the ground state will now take the form

$$\mu_j=\sum_{k=1,k\neq j}^{N}\frac{1}{\mu_j-\mu_k}\,;\ j=1,\ldots,N.$$

These equations are equations for the zeros of the Hermite polynomials. By applying the equality

$$\sum_{j=1}^{N}\sum_{k\neq j}\frac{x_j}{x_j-x_k}=\frac{1}{2}N(N-1),$$

it becomes easy to calculate the ground-state energy of the weakly interacting gas

$$E_N = \frac{g}{L}\sum_{j=1}^{N}\mu_j^2 = \frac{g}{2L}N(N-1).$$

In the thermodynamic limit $N,\ L \to \infty$ with the density $\rho = \frac{N}{L}$ fixed, the free energy $\mathcal{E}$ of the weakly interacting Bose gas is

$$\mathcal{E} = \lim_{N\to\infty}\frac{E_N}{L} = \frac{g}{2}\rho^2$$

so that for small coupling constants the sound velocity is

$$v = \sqrt{\frac{g\rho}{m}}.$$

The substitution of this result in Eq. (10) gives the result (12).

The case of impenetrable (hard core) bosons, also referred to as the free-fermion limit or the Girardeau–Tonks gas, corresponds to an infinite value of the coupling constant $g = \infty$. The Bethe equations Eq. (4) now become

$$e^{i\lambda_j L} = (-1)^{N+1}.$$

These equations describe noninteracting particles and are easily solved,

$$\lambda_j = \frac{2\pi}{L}\left(j - \frac{N+1}{2}\right),\ j = 1,\dots,N, \tag{B1}$$

and these solutions fill in the interval $\lambda_1 \leq \lambda_j \leq \lambda_N$, where the Fermi radius $\lambda_F \equiv \lambda_N$ corresponds to the solution with $j = N$ $(-\lambda_F \equiv \lambda_1)$:

$$\lambda_F = \frac{\pi}{L}(N-1).$$

By using the solutions of (B1), the ground state energy of the Girardeau–Tonks gas is easily calculated to be

$$E_N = \frac{\pi^2\hbar^2}{6mL^2}N(N^2-1).$$

In the thermodynamic limit the free energy $\mathcal{E}$ of the Girardeau–Tonks gas is

$$\mathcal{E} = \frac{\pi^2\hbar^2}{6m}\rho^3,$$

and the sound velocity at zero temperature is

$$v = \frac{\hbar\pi}{m}\rho.$$

Substitution of this result in Eq. (10) gives the exponent of Eq. (11).

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