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INVERSE PROBLEMS ON A CLASS OF GRAPHS (TREES) BY THE BC-METHOD

M. I. Belishev

Russian Academy of Sciences
Steklov Mathematical Institute

191011, St.Petersburg, Fontanka, 27
belishev@pdmi.ras.ru
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**Dedicated to A. S. Blagovestchenskii
on occasion of his 65th birthday**

ABSTRACT

Planar graph consisting of strings of variable densities is considered. The spectrum of the Dirichlet problem on the graph and the values of derivatives of the (normalized) eigenfunctions at the boundary vertices form the spectral data. We show that the graph without cycles (tree) and the densities of its edges are determined by the spectral data uniquely up a natural isometry in the plane. In the framework of the appoach (the BC-method) we establish a boundary controllability of the dynamical system gouverned by the wave equation on the tree and exploit this property for recovering the tree from its spectral data.

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0. Introduction

0.1. About the paper.

Mechanical object inspiring the problem is a planar graph consisting of a finite set of strings of variable densities. Graph can be displaced in normal (with respect to the plane) direction whereas its boundary vertices are rigidly fixed on the plane. Such the system can oscillate; it possesses a spectrum of eigenfrequences and the corresponding eigenmodes.

Assume that we know the spectrum of the graph and the values of derivatives of the eigenmodes at the boundary vertices. What the information about graph may be extracted from these data? We show that, if the graph does not contain cycles (is a tree), such the data determine the graph (its structure, lenghts of strings and their densities) uniquely, up to a natural isometry in the plane. Moreover, a procedure recovering the tree is proposed.

As each variant of the BC-method, our approach invokes results of the boundary control theory. We show that the dynamical system associated with the tree and gouverned by the wave equation is controllable from the boundary in appropriate sense and use this property in the recovering procedure.

0.2. Statement and main result.

Let $\Omega \subset \mathbf{R}^2$ be a finite connected planar graph with edges $\{e\} = E$ (intervals of straight lines) and boundary vertices $\{\gamma_1, \dots, \gamma_n\} = \Gamma$. The graph is equipped with density ρ which is a function on Ω strictly positive and C^2-smooth on the edges.

To the graph Ω one associates the (real) Hilbert space $\mathcal{H} = L_2(\Omega; \rho|dx|)$ ($|dx|^2$ is the length element on Ω induced by the $\mathbf{R}^2$-metric) the Sobolev class $\mathcal{H}_0^1 = \{y \in \mathcal{H} \mid y \in C(\Omega);\quad y|_e \in H^1(e),\ e \in E;\ y|_\Gamma = 0\}$ and the positive definite bilinear form $l : \mathcal{H} \times \mathcal{H} \mapsto \mathbf{R}$, $\operatorname{Dom} l = \mathcal{H}_0^1 \times \mathcal{H}_0^1$,

$$l\,[y, v] = \sum_{e \in E} \int\limits_e \frac{dy}{de}\frac{dv}{de}\,|dx|.$$

Let L be the selfadjoint operator in $\mathcal{H}$ corresponding to the form l. The operator L has a discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}$: $0 < \lambda_1 < \lambda_2 \le \dots$; let $\varphi_1, \varphi_2, \dots$ be the corresponding eigenfunctions orthonormalized in $\mathcal{H}$. We denote

$$\frac{d\varphi_k}{de}\Big|_\Gamma = \operatorname{col}\left\{\frac{d\varphi_k}{de}(\gamma_1), \dots, \frac{d\varphi_k}{de}(\gamma_n)\right\};$$

the set of pairs $\{\lambda_k; \frac{d\varphi_k}{de}|_\Gamma\}_{k=1}^{\infty}$ is said to be the (Dirichlet) *spectral data* of the (equipped) graph Ω.

We say that two graphs Ω' and Ω'' equipped with densities ρ' and ρ'' are *spatially isometric* if there exists an isometry I connecting the metric spaces $(\Omega', |dx|^2)$ and $(\Omega'', |dx|^2)$ such that $I(\Omega') = \Omega''$ and $\rho'' \circ I = \rho'$. As is easy to see, spatially isometric graphs have one and the same spectral data. The main result of the paper is that in the case of trees the opposite turns out to be true.

Theorem 1. *If the spectral data of two trees coincide the trees are spatially isometric.*

In other words, tree is determined by its spectral data up to spatial isometry.

0.3. Remarks.

To the best of our knowledge, an ivestigation of iverse problems on graphs was originated by N. I. Gerasimenko in [9]. A review of results in this area the reader can find in [12]. Most of them concerns to recovering the unknown parameters (densities, potentials) of a *known* graph. The examples of nonuniqueness to the problem of reconstruction of *unknown* graph are present in [12]; all of them concern to graphs with cycles. A pecularity of our paper is that we deal with *unknown* graph equipped with *unknown* density.

At the end of the paper we discuss some related inverse problems with another kinds of data.

In order to restrict the volume of the paper, we omit some of technical details (mainly in Secs. 2.2–2.4). In particular, we do not prove results placed in Propositions: they are either known or their proofs should present no problems.

1. The graph

1.1. Terminology.

The *edge* is a finite interval of a straight line on the plane $\mathbf{R}^2$:

$$e = \left\{ \operatorname{col}\{x^1, x^2\} \;\middle|\; x^i = a^i + b^i s, \;\; a^i, b^i = \text{const}, \;\; s \in (0,1) \right\};$$

the *vertex* is a point $v \in \mathbf{R}^2$. The *graph* $\Omega \subset \mathbf{R}^2$ is the disjoint sum $\Omega = E \cup V$ of a finite set of edges $E = \{e\}$ and a finite set of vertices $V = \{v\}$ such that Ω is a closed connected set on the plane.

We say that $v \in V$ and $e \in E$ are *incident* (and write $e \sim v$, $v \sim e$) if $v \in \overline{e}$ (the closure in $\mathbf{R}^2$) and denote $r(v) := \sharp\{e \in E \,|\, e \sim v\}$. Just for simplicity we consider only the graphs satisfying $r(\,\cdot\,) \neq 2$.

The set $\Gamma := \{v \in V \,|\, r(v) = 1\}$ is the *boundary* of Ω; the set $\operatorname{int}\Omega := \Omega\backslash\Gamma$ is *interior* points of the graph.

The *path* $\pi[a,b] \subset \Omega$ connecting points $a, b \in \Omega$ is the image of the segment [0,1] through a homeomorphism $\kappa : [0,1] \mapsto \Omega$ such that $\kappa(0) = a$, $\kappa(1) = b$; the notations $\pi\,[a,b)$, $\pi(a,b]$, $\pi(a,b)$ are also of the clear meaning.

Graph Ω is the *tree* if any $a, b \in \Omega$, $a \neq b$ are connected by the unique path (so, there are no cycles in Ω). In this paper we deal mainly with trees.

Tree may be semiordered in a standard way. Namely, let $\xi \in \Omega$ be fixed and declared to be the *root*; for $a, b \in \Omega$ we write $a \preceq b$ ($a \prec b$) if $a \in \pi\,[\xi, b]$ ($a \in \pi\,[\xi, b)$). Of course, this semiorder is determined by the choice of ξ.

1.2. Optical metric.

A function ρ defined on $\Omega\backslash V$ is said to be the *density* if

(i) ρ is strictly positive: $\rho(\,\cdot\,) \geq \rho_0 > 0$;

(ii) for each e, the function $\rho|_e$ may be extended on $\overline{e}$ up to a function of the class $C^2(\overline{e})$.

Thus, the boundary values $\rho(\gamma)$, $\gamma \in \Gamma$ are well–defined whereas at interior vertices $v \in V\backslash\Gamma$ the density may be considered as an $r(v)$-valued function.

Let $|dx|^2$ be the Euclidean length element induced on Ω by the $\mathbf{R}^2$-metric. The *optical metric* τ on the graph is determined by the element

$$d\tau^2 := \rho(x)|dx|^2 \qquad x \in \Omega\backslash V; \tag{1.1}$$

by the properties of ρ this metric is equivalent to the Euclidean one. If Ω is a tree one easily has

$$\tau(a,b) = \begin{cases} \int\limits_{\pi[a,b]} \rho^{1/2}\,|dx|\,, & a \neq b; \\ 0, \quad a = b. \end{cases}$$

We denote

$$B_r[a] := \{x \in \Omega \big|\ \tau(a,x) < r\}\,;$$

for a subset $A \subset \Omega$ we define its (optical) diameter

$$d\,(A) := \sup_{a,b\,\in\,A} \tau(a,b)\,;$$

let

$$T_* := \frac{d\,(\Omega)}{2}$$

be the 'radius' of the graph. The point $x_* \in \Omega$ satisfying $\tau(x_*, \Gamma) = T_*$ is called by the (optical) *center* of the graph.

Proposition 1. *The center of the tree is unique.*

1.3. Optical image. Spatial isometry.

Denote $n := \sharp\Gamma$, numerate the boundary vertices and introduce the map $i : \Omega \mapsto \mathbf{R}^n$,

$$i(x) := \mathrm{col}\left\{\tau(x,\gamma_1), \dots, \tau(x,\gamma_n)\right\}.$$

Proposition 2. *For any tree Ω the map i is injective.*

Thus, the functions $\tau(\,\cdot\,,\gamma_j)$ form a coordinate system on the tree. The set

$$\Omega_{\rm opt} := i\,(\Omega)$$

is a graph in $\mathbf{R}^n$ called by the *optical image* of Ω.

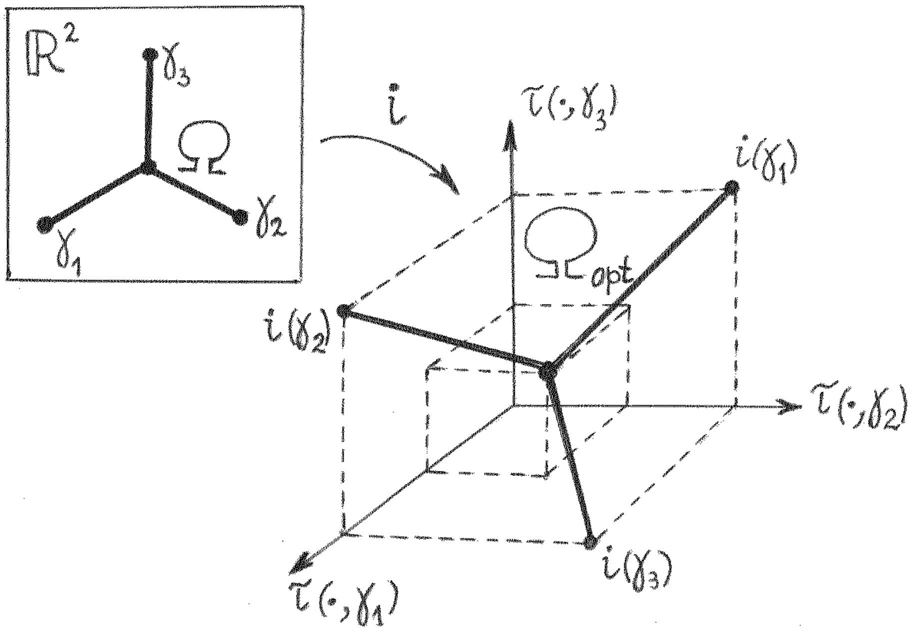


Fig. 1. Optical image

Let $|dT|^2$ be the length element on $\Omega_{\rm opt}$ induced by the surrounding $\mathbf{R}^n$–metric; the properties of the density easily lead to the following result.

3. Proposition. *The map i is an isometry of the metric space (Ω, τ) onto $(\Omega_{\rm opt}, \frac{1}{n}|dT|^2)$. The spaces $(\Omega_{\rm opt}, |dT|^2)$ and $(\Omega, |dx|^2)$ are homeomorphic.*

Thus, the optical image of the tree determines its topology.

Define the function

$$\rho_{\rm opt} := \rho \circ i^{-1}$$

on $\Omega_{\rm opt}$. As we'll see, it is the pair $\Omega_{\rm opt}$, $\rho_{\rm opt}$ which may be recovered from inverse data. This pair determines the original Ω, ρ (which is the subject of the inverse problem) up to a natural nonuniqueness in the following sense.

We say that two graphs Ω' and Ω'' equipped with densities ρ' and ρ'' are *spatially isometric* if there exists an isometry I between the metric spaces $(\Omega', |dx|^2)$ and $(\Omega'', |dx|^2)$ such that $I(\Omega') = \Omega''$ and $\rho' = \rho'' \circ I$.

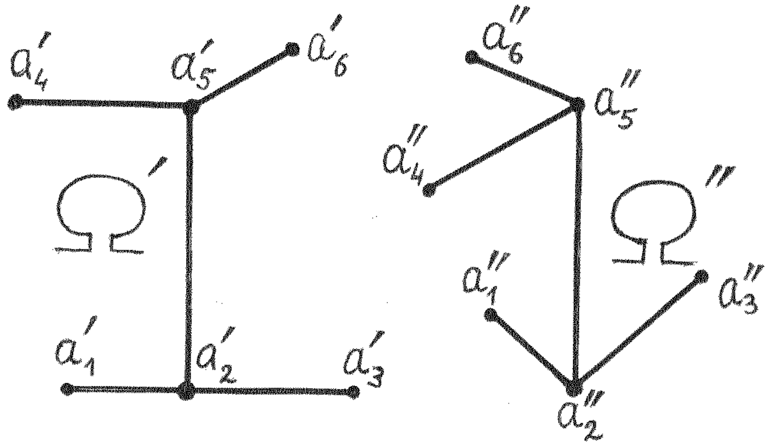


FIG. 2. SPATIALLY ISOMETRIC GRAPHS: $|a'_i a'_j| = |a''_i a''_j|$; $\rho' \equiv 1$, $\rho'' \equiv 1$.

A simple fact is that spatial isometry of Ω' and Ω'' implies $\Omega'_{\rm opt} = \Omega''_{\rm opt}$ and $\rho'_{\rm opt} = \rho''_{\rm opt}$. The converse also turns out to be true.

Lemma 1. *Let Ω', Ω'' be the graphs equipped with densities ρ' ρ'' and bijective to their optical images. If $\Omega'_{\rm opt} = \Omega''_{\rm opt}$ and $\rho'_{\rm opt} = \rho''_{\rm opt}$ then Ω' and Ω'' are spatially isometric.*

$\triangleleft$ Since

$$i'(\Omega') = \Omega'_{\rm opt} = \Omega''_{\rm opt} = i''(\Omega''),$$

the map $I := (i'')^{-1}\, i'$ is a bijection from Ω' onto Ω''.

Let $|dx'|^2$ and $|dx''|^2$ be the Euclidean length elements on Ω' and Ω''. In accordance with (1.1), the equality $\rho'_{\rm opt} = \rho''_{\rm opt}$ implies

$$|dx'|^2 = \frac{d\tau^2}{\rho'_{\rm opt}} = \frac{d\tau^2}{\rho''_{\rm opt}} = |dx''|^2;$$

hence, I is an isometry of the metric spaces $(\Omega', |dx'|^2)$ and $(\Omega'', |dx''|^2)$. The relation $\rho' = \rho'' \circ I$ is evident. $\triangleright$

So, the pair $\Omega_{\rm opt}$, $\rho_{\rm opt}$ determines the tree Ω up to spatial isometry.

1.4. Spaces. Dirac measures.

Introduce the (real) Hilbert space $\mathcal{H}=L_{2,\rho}(\Omega)$,

$$(y,w)_{\mathcal{H}} := \int\limits_{\Omega} yw\,\rho|dx| = \sum_{e\in E}\int\limits_{e} yw\,\rho|dx|.$$

We assigne a function y on Ω to the class $\mathcal{H}_0^1$ if

(i) $y\in C(\Omega)$;

(ii) $y\,|_{\overline{e}}$ belongs to the Sobolev class $H^1(\overline{e})$ on every edge e;

(iii) $y=0$ on Γ,

and denote

$$(y,w)_{\mathcal{H}_0^1} := \int\limits_{\Omega} \nabla y\cdot\nabla w\,|dx| = \sum_{e\in E}\int\limits_{e} \frac{dy}{de}\frac{dw}{de}\,|dx|,$$

where ∇ is the Euclidean gradient on Ω, $\frac{d}{de}$ is the differentiation with respect to Euclidean length in any (but one and the same for y and w) direction along e.

We also introduce the tripple (the rigged Hilbert space)

$$\mathcal{H}_0^1\ \subset\ \mathcal{H}\ \subset \mathcal{H}^{-1}$$

where $\mathcal{H}^{-1}$ is dual to $\mathcal{H}_0^1$ with respect to the product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}}$. As is customary, we denote by $(h,y)_{\mathcal{H}}$ the value of the functional $h\in\mathcal{H}^{-1}$ on the element $y\in\mathcal{H}_0^1$.

Let

$$\mathcal{D} := \{c\delta_\xi\,|\,\xi\in\operatorname{int}\Omega,\ 0\neq c\in\mathbf{R}\}\ \subset\mathcal{H}^{-1}$$

be the subset of functionals proportional to Dirac measures normalized by $(\delta_\xi,y)_{\mathcal{H}}=y(\xi)$; recall one a simple fact. For each functional $h\in\mathcal{H}^{-1}$ we define its (optical) *diameter* by

$$d\,[h]\ := d\,(\operatorname{supp} h)$$

and denote

$$\mathcal{D}_1 := \{h\in\mathcal{D}\,\big|\ ||h||_{\mathcal{H}^{-1}}=1\}.$$

Proposition 4. *The inclusion $h\in\mathcal{D}$ is equivalent to the relation $d\,[h]\ =0$, so that the characterization*

$$\mathcal{D}_1=\left\{h\in\mathcal{H}^{-1}\,\Big|\ ||h||_{\mathcal{H}^{-1}}=1,\ d\,[h]\ =0\right\} \tag{1.2}$$

holds.

Later, solving the inverse problem, we'll use the following evident fact: the set $\mathcal{D}_1$ is bijective to $\operatorname{int}\Omega$.

1.5. Operator L and spectral data.

Let us begin with the well-known facts. The product $(\,\cdot\,,\,\cdot\,)_{\mathcal{H}_0^1}$ is a closed positive definite bilinear form in $\mathcal{H}$ defined on $\mathcal{H}_0^1\times\mathcal{H}_0^1$. As such, this form determines the (unique) selfadjoint operator $L:\mathcal{H}\mapsto\mathcal{H}$ satisfying

$$(Ly,w)_{\mathcal{H}}=(y,w)_{\mathcal{H}_0^1}$$

for $y \in \operatorname{Dom} L,\ w \in \mathcal{H}_0^1$ (see e.g., [6]). This operator may be described in more detail as follows.

A function $y \in \mathcal{H}$ belongs to $\operatorname{Dom} L$ iff

(i) $y \in \mathcal{H}_0^1$;

(ii) $y\,|_{\overline{e}} \in H^2(\overline{e})$ on every $e \in E$;

(iii) for each interior vertex $v \in V \setminus \Gamma$ the equality (the Kirchhoff law)

$$\sum_{e \sim v} \frac{dy}{de}(v) = 0 \tag{1.3}$$

holds, where

$$\frac{dy}{de}(v) := \lim_{x \in e,\, x \to v} \frac{y(x) - y(v)}{|x - v|}.$$

Then, for $y \in \operatorname{Dom} L$ and $e \in E$ one has

$$(L\, y)\Big|_e = -\,\rho^{-1}\,\frac{d^2 y}{de^2} \tag{1.4}$$

(see, e.g., [8]).

In what follows we use also the operator $L_{\max} \supset L$ defined on continuous functions satisfying (ii), (iii) and acting by the same rule (see the right hand side of (1.4)).

Operator L has a discrete spectrum $\{\lambda_k\}_{k=1}^{\infty}\ :\ 0 < \lambda_1 < \lambda_2 \leq \ldots$ (see, e.g., [8, 10]); let $\{\varphi_k\}_{k=1}^{\infty}$ be the corresponding eigenfunctions normalized by $(\varphi_k, \varphi_l)_{\mathcal{H}} = \delta_{kl}$. For each boundary vertex $\gamma \in \Gamma$ the value

$$\frac{d\varphi_k}{de}(\gamma) = \lim_{x \to \gamma} \frac{\varphi_k(x) - \varphi_k(\gamma)}{|x - \gamma|}$$

is well-defined, and we denote

$$\frac{d\varphi_k}{de}\Big|_{\Gamma} := \operatorname{col}\Big\{\frac{d\varphi_k}{de}(\gamma_1), \ldots, \frac{d\varphi_k}{de}(\gamma_n)\Big\}.$$

The set of the pairs $\{\lambda_k\,;\ \frac{d\varphi_k}{de}\,|_{\Gamma}\ \}_{k=1}^{\infty}$ is called by the (Dirichlet) *spectral data* of the (equipped) graph Ω. It is the set which will play the role of data in the inverse problem.

As was already mentioned in Introduction, spatially isometric graphs have one and the same spectral data. Indeed, if $I : \Omega' \mapsto \Omega''$ is a spatial isometry one can esily check that the map $\hat{I} : \mathcal{H}' \mapsto \mathcal{H}'',\ \hat{I}y := y \circ I^{-1}$ is a unitary operator which satisfies $\hat{I}L' = L''\hat{I}$ (yielding $\lambda'_k = \lambda''_k$) and $\hat{I}\varphi'_k = \varphi''_k$ (implying $\frac{d\varphi'_k}{de'}(\gamma'_j) = \frac{d\varphi''_k}{de''}(\gamma''_j)$ where $\gamma''_j = I\gamma'_j$). Therefore, setting the goal to recover a graph from its spectral data, what the boundary spectral inverse problem is, one can hope for determination only up to spatial isometry. Theorem 1 claims that in the case of trees such the determination is possible. This fact is the main result of our paper; its proof is postponed until Sec. 4.

2. Waves on the graph

2.1. Dynamical system.

Speaking about dynamical system associated to the graph, we mean the initial boundary value problem of the form

$$u_{tt} + L_{\max} u = 0 \ in \ \operatorname{int}\Omega \times (0,T)\,; \tag{2.1}$$

$$u\,|_{t=0} \ = u_t\,|_{t=0} \ = 0 \ in \ \Omega\,; \tag{2.2}$$

$$u \ = f \ on \ \Gamma \times [0,T]\,, \tag{2.3}$$

with the (Dirichlet) boundary control $f = f(\gamma,t)$; the solution $u = u^f(x,t)$ describes the wave initiated at Γ and propagating in Ω. If f is C^2-smooth with respect to t and satisfies $f(\,\cdot\,,0) = f_t(\,\cdot\,,0)$ the problem has the unique classical solution satisfying the wave equation

$$\rho\, u_{tt}(\,\cdot\,,t) - u_{ee}(\,\cdot\,,t) = 0$$

on edges and the Kirchhoff law at interior vertices for all times.

The space of controls $\mathcal{F}^T := L_2\,(\Gamma \times [0,T])$ with the inner product

$$(f,g)_{\mathcal{F}^T} = \sum_{\gamma\in\Gamma} \int\limits_0^T f(\gamma,t)g(\gamma,t)\,dt$$

is called by the *outer space* of the system (2.1)–(2.3). The space $\mathcal{H}$ is the *inner space*; the waves (states) $u^f(\,\cdot\,,t)$ are time-dependent elements of $\mathcal{H}$.

2.2 Fundamental solution.

Fix $\gamma \in \Gamma$ and denote

$$\delta_\gamma(\gamma') := \begin{cases} 0, & \gamma' \neq \gamma; \\ 1, & \gamma' = \gamma. \end{cases}$$

Take $\chi \in C^2[0,T]$ such that $\chi \geq 0$, $\operatorname{supp}\chi \subset (0,T)$, and $\int\limits_0^T \chi(t)dt = 1$. Choose (small) $\varepsilon > 0$ and denote

$$\chi_\varepsilon(t) := \begin{cases} \dfrac{1}{\varepsilon}\chi(\dfrac{t}{\varepsilon}), & 0 \leq t \leq \varepsilon T; \\ 0, & \varepsilon T < t < T; \end{cases}$$

let $u^{\delta_\gamma \chi_\varepsilon}$ be the classical solution to (2.1)–(2.3) corresponding to the control $f(\gamma',t) = \delta_\gamma(\gamma')\chi_\varepsilon(t)$. As may be shown, tending $\varepsilon \to 0$ (so that χ_ε tends to the Dirac δ- function) the limit passage gives a function

$$u^{\delta_\gamma \delta} := \lim_{\varepsilon\to 0} u^{\delta_\gamma \chi_\varepsilon} \in C\left([0,T]\,;\ \mathcal{H}^{-1}\right)$$

which is the *fundamental solution* to the problem (2.1)–(2.3). This solution describes the wave initiated by the instanteneous source supported at γ. In the next section we describe the properties of $u^{\delta_\gamma\delta}$ in more detail.

2.3. Propagation of singularities.

All the results of Secs. 2.3 and 2.4 are valid for any graph (not only for trees). Recall that $\tau(a,b)$ is the optical distance between $a, b \in \Omega$.

First edge. Let e be the edge incident to γ, $v \in V$ the second vertex incident to e. For times $0 < t \le \tau(\gamma, v)$ the well-known representation (time-domain WKB expansion)

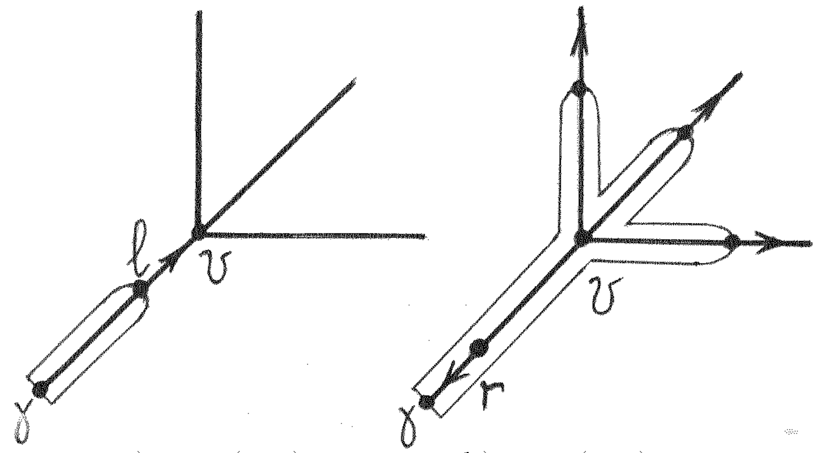
$$u^{\delta_\gamma \delta}(x,t) = \left[\frac{\rho(x)}{\rho(\gamma)}\right]^{-\frac{1}{4}} \delta\Big(t - \tau(x,\gamma)\Big) + u_\gamma(x,t)\,, \;\; x \in \Omega \tag{2.4}$$

holds, where u_γ is a bounded function satisfying $\operatorname{supp} u_\gamma(\,\cdot\,,t) \subseteq \overline{B_t[\gamma]} \subseteq \overline{e}$ and C^2-smooth on $[\operatorname{supp} u_\gamma] \setminus \{(\gamma, 0)\}$ (see e.g. [Blag]). The first term in (2.4) is interpreted as the leading singularity which moves from γ towards v along e with variable speed $\rho^{-1/2}$: see the point l on Fig. 3a (the ball $B_t[\gamma]$ is contoured).

Passing through the first vertex. At the moment $t = \tau(\gamma, v)$ the singularity reaches v and, then, passes through v. A simple analysis using (2.4) and (1.3) gives the following representation for times $\tau(\gamma,v) < t \le \tau(\gamma,v) + \varepsilon$ (with small $\varepsilon > 0$):

$$u^{\delta_\gamma \delta}(x,t) = \alpha(x)\, \delta\Big(t - \tau(\gamma,v) - \tau(v,x)\Big) + u_\gamma(x,t)\,, \;\; x \in \Omega \tag{2.5}$$

with a bounded u_γ satisfying $\operatorname{supp} u_\gamma(\,\cdot\,,t) \subseteq \overline{B_t[\gamma]}$ and piecewise smooth on $\operatorname{supp} u_\gamma$. The *amplitude* α is a piecewise smooth function strictly positive on edges $e' \sim v$, $e' \neq e$ incident to v.



a) $t = \tau(\gamma, v) - \varepsilon$ b) $t = \tau(\gamma, v) + \varepsilon$

FIG. 3. PASSING THROUGH v

Note in addition that $\alpha\big|_e$ can be negative. From the physical point of view this means that the reflected part of the singularity moving back to γ (point r on Fig. 3b) can have the phase opposite to one of the incident singularity (point l on Fig. 3a; in both cases the ball $B_t[\gamma]$ is contoured).

First reflection from the boundary. Let $\gamma' \in \Gamma$ be a boundary vertex nearest to v:

$$\tau(\gamma', v) = \min_{\gamma'' \in \Gamma} \tau(\gamma'', v)$$

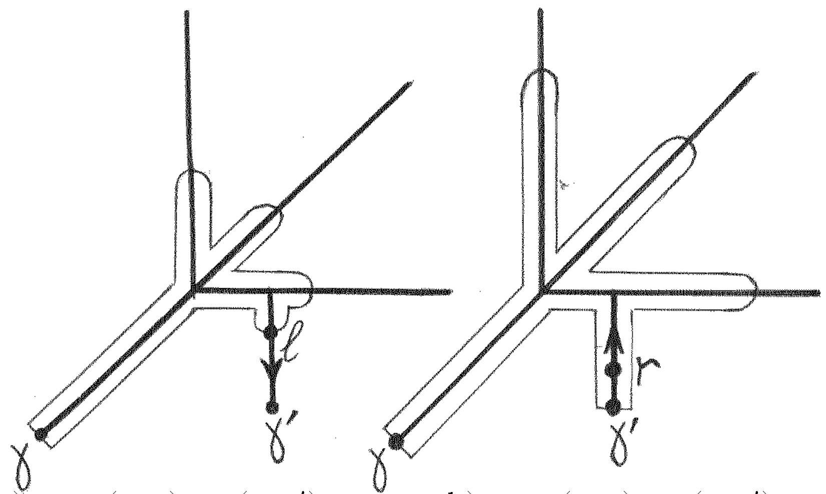
(may be $\gamma = \gamma'$). As $t \to \tau(\gamma, v) + \tau(v, \gamma') - 0$, the singularity passed through v (and, perhaps, through another vertices or reflected from v back to γ) approaches to γ'. Then the singularity is reflected by γ': see the point r on Fig. 3b. A simple analysis using the condition

$$u^{\delta_\gamma \delta}(\,\cdot\,, t)\,|_\Gamma \;= 0\,, \quad t > 0$$

gives the representation

$$u^{\delta_\gamma \delta}(x,t) = \alpha(x)\,\delta\Big(t - \tau(\gamma, v) - \tau(v, \gamma') - \tau(\gamma', x)\Big) + u_\gamma(x,t) \tag{2.6}$$

with a bounded piecewise smooth u_γ which holds for $t \sim \tau(\gamma, v) + \tau(v, \gamma') + \varepsilon$ and x close to γ'. The amplitude α is negative if $\gamma' \neq \gamma$ and positive if $\gamma' = \gamma$ which means that reflection from boundary leads to inverting the phase from boundary incident singularity.



a) $t = \tau(\gamma, v) + \tau(v, \gamma') - \varepsilon$ b) $t = \tau(\gamma, v) + \tau(v, \gamma') + \varepsilon$

FIG. 4. REFLECTING FROM $\gamma' \neq \gamma$

Arbitrary times. Representations (2.4)–(2.6) are sifficient for further analysis of propagation of singularities in Ω. Omitting technical details, let us summarize the results.

In the space-time domain $\Omega \times \overline{\mathbf{R}}_+$, for a fixed (x_0, t_0) let us define the *characteristic cone*

$$\mathcal{K}_{(x_0,t_0)} := \Big\{(x,t)\,|\, t - t_0 = \tau(x, x_0)\Big\}$$

and introduce the *characteristic set* $\mathcal{C}_{(x_0,t_0)}$ which is defined as the minimal subset in $\Omega \times \overline{\mathbf{R}}_+$ sutisfying the conditions:

(i) $\mathcal{K}_{(x_0,t_0)} \subset \mathcal{C}_{(x_0,t_0)}$;

(ii) if $v \in V$ and $t_v \in \mathbf{R}_+$ are such that $(v, t_v) \in \mathcal{C}_{(x_0,t_0)}$ then $\mathcal{K}_{(v,t_v)} \subset \mathcal{C}_{(x_0,t_0)}$.

This set is of the form of a space-time graph; Fig. 5 illustrates the case of $x_0 = \gamma$, $t_0 = 0$. We also denote by

$$\mathcal{K}^s_{(x_0,t_0)} := \{x \in \Omega\,|\,(x,s) \in \mathcal{K}_{(x_0,t_0)}\}\,; \quad \mathcal{C}^s_{(x_0,t_0)} := \{x \in \Omega\,|\,(x,s) \in \mathcal{C}_{(x_0,t_0)}\}$$

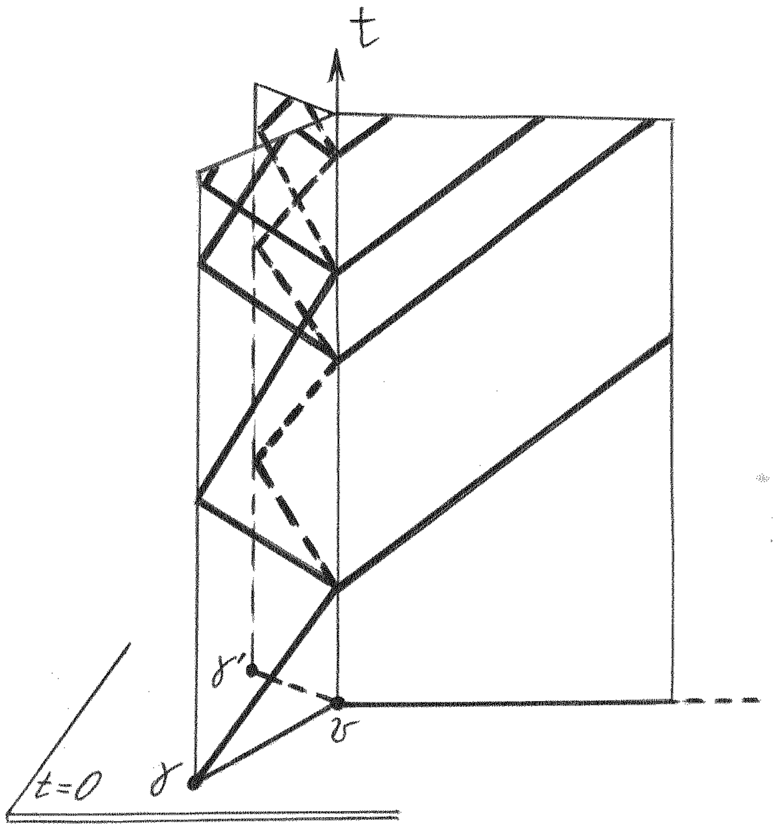


FIG. 5. THE SET $C_{(\gamma,0)}$

the crossections by the plane $t=s$; as is easy to see, one has

$$\mathcal{K}^s_{(x_0,t_0)}=\{x\in\Omega\,|\,\tau(x,x_0)=s-t_0\}\,.$$

Proposition 5. *The leading singularities of $u^{\delta_\gamma\delta}$ (as a space-time distribution) are located on the characteristic set $\mathcal{C}_{(\gamma,0)}$. For any $t>0$ the representation*

$$u^{\delta_\gamma\delta}(\,\cdot\,,t)=\sum_{\xi\in\mathcal{K}^t_{(\gamma,0)}}\alpha_\gamma(\xi)\,\delta_\xi(\,\cdot\,)+\sum_{\xi\in\mathcal{C}^t_{(\gamma,0)}\setminus\mathcal{K}^t_{(\gamma,0)}}\beta_\gamma(\xi,t)\,\delta_\xi(\,\cdot\,)+u_\gamma(\,\cdot\,,t)\tag{2.7}$$

holds in Ω where:

$\delta_\xi\in\mathcal{H}^{-1}$ is the Dirac measure;

the amplitude function α_γ is strictly positive on Ω piecewise smooth and C^2-smooth out of the interior vertices; the value $\alpha_\gamma(\xi)$ is determined by the density $\rho\,|_{\pi[\gamma,\xi]}$;

β_γ is a piecewise smooth function defined on $\mathcal{C}_{(\gamma,0)}\setminus\mathcal{K}_{(\gamma,0)}$;

$u_\gamma(\,\cdot\,,t)$ is a bounded piecewise smooth function (with possible jumps on $\mathcal{C}^t_{(\gamma,0)}$) satisfying $\operatorname{supp}u_\gamma(\,\cdot\,,t)\subset\overline{B_t[\gamma]}$.

The first sum in (2.7) selects the so-called *forward singularities* which lie on the forward front of the wave $u^{\delta_\gamma\delta}(\,\cdot\,,t)$ moving from γ. These singularities are present only if $t<\max\limits_{x\in\Omega}\tau(x,\gamma)$ whereas $\alpha_\gamma(\xi)$ is the amplitude of the *first* singularity arriving at ξ from γ at the moment $t=\tau(\xi,\gamma)$.

The second sum is the *secondary singularities* appearing due to reflection from vertices and supported into $B_t[\gamma]$.

A typical picture is shown on Fig.6 where the points $\xi_1, \dots, \xi_5$ support the forward singularities whereas $\xi_6, \dots, \xi_9$ support the secondary ones (the ball $B_t[\gamma]$ is contoured).

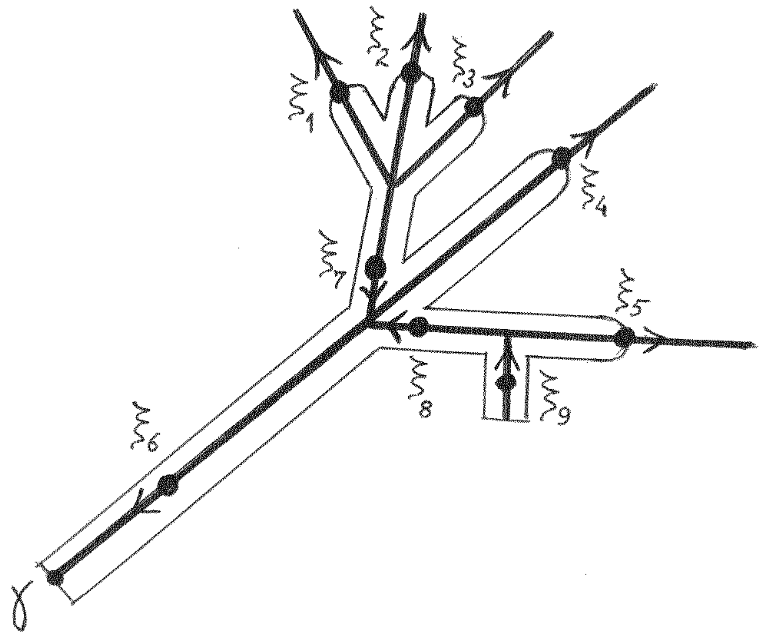


FIG. 6. SINGULARITIES OF $u^{\delta_\gamma \delta}(\cdot, t)$

2.4. Generalized solutions.

Here we describe some of the properties of solutions to the problem (2.1)–(2.3) which may be easily derived from the representation (2.7).

Let

$$\mathcal{F}^T_\gamma := \left\{ f \in \mathcal{F}^T \mid \operatorname{supp} f \subset \gamma \times [0, T] \right\}$$

be the subspace of controls acting from $\gamma \in \Gamma$; introduce the classes

$$(\mathcal{F}^T)^1_0 := \left\{ f \in \mathcal{F}^T \mid f(\,\cdot\,, t) = \sum_{\gamma\,' \in \Gamma} \delta_{\gamma\,'}(\,\cdot\,) g_{\gamma\,'}(t), \quad g_{\gamma\,'} \in H^1_0[0, T] \right\}$$

(so that controls belonging to this class satisfy $f(\,\cdot\,, 0) = f(\,\cdot\,, T) = 0$) and

$$(\mathcal{F}^T_\gamma)^1_0 := (\mathcal{F}^T)^1_0 \bigcap \mathcal{F}^T_\gamma\,.$$

Let $\mathcal{H}^1 := H^1(\Omega) \supset \mathcal{H}^1_0$ be the Sobolev class of functions on Ω satisfying (i),(ii), Subsec. 1.4.

(i) (*The Duhamel formula*). If f is such that u^f is classical, the representation

$$u^f = \sum_{\gamma \in \Gamma} u^{\delta_\gamma \delta} * f(\gamma, \,\cdot\,) \tag{2.8}$$

(the convolution with respect to time) holds.

(ii) (*Generalized solutions*). For any $f \in \mathcal{F}^T$ the right hand side of (2.8) is well-defined and belongs to the class $C([0,T]\,;\,\mathcal{H})$; it defines the (generalized) solution u^f to (2.1)–(2.3) of this class.

(iii) (*Localization of solutions*). For controls $f \in \mathcal{F}^T_\gamma$ the inclusion

$$\operatorname{supp} u^f(\,\cdot\,,t) \subset \overline{B_t[\gamma]}\,, \quad t \geq 0 \tag{2.9}$$

holds.

(iv) (*Delayed solutions*). For $f \in \mathcal{F}^T$ denote

$$f_{t_0}(\,\cdot\,,t) := \begin{cases} 0, & \leq t < t_0; \\ f(\,\cdot\,,t-t_0), & t_0 \leq t \leq T. \end{cases}$$

Since the graph and its density do not depend on time, one has

$$u^{f_{t_0}}(\,\cdot\,,t) = u^f(\,\cdot\,,t-t_0).$$

As result, if $f(\,\cdot\,,t) = 0$ for $0 \leq t \leq t_0$ then one has

$$\operatorname{supp} u^f(\,\cdot\,,t) \subset \overline{B_{t-t_0}[\gamma]}, \quad t \geq 0. \tag{2.10}$$

(v) (*Smoothness*). In the case of $f \in (\mathcal{F}^T)^1_0$ one has $u^f(\,\cdot\,,t) \in \mathcal{H}^1$ for all $t \in [0,T]$ and $u^f(\,\cdot\,,T) \in \mathcal{H}^1_0$.

2.5. Propagators p_γ.

Here we introduce the solutions (waves) of a special kind. Playing the role of device prospecting the graph from its boundary, these waves will be invoked for solving the inverse problem.

Let

$$\theta(t) := \begin{cases} 0, & t < 0; \\ 1, & t \geq 0; \end{cases}$$

be the Heavyside function; taking (2.1)–(2.3) $f(\gamma',t) = \delta_\gamma(\gamma')\theta(t)$ we call the solution

$$u^{\delta_\gamma \theta} = u^{\delta_\gamma \delta} * \theta =: p_\gamma$$

by the *propagator* corresponding to the vertex $\gamma \in \Gamma$. Propagator is a function of the class $C([0,T]\,;\,\mathcal{H})$ satisfying (2.9).

Assume that the final moment T is large enough: $T > d(\Omega)$ and take $t < \max_{x \in \Omega} \tau(x,\gamma)$, so that $\mathcal{K}^t_{(\gamma,0)} \neq \{\emptyset\}$. Fix a $\xi \in \mathcal{K}^t_{(\gamma,0)}$ (see points $\xi_1, \ldots, \xi_5$ on Fig. 6) and note the relation $t = \tau(\gamma,\xi)$. Using (2.7), one can easily derive the representation

$$p_\gamma(\,\cdot\,,t) = \alpha_\gamma(\xi)\,\theta(t - \tau(\,\cdot\,,\gamma)) + w_\gamma(\,\cdot\,,t) \tag{2.11}$$

which is valid in $B_\varepsilon[\xi]$ with a small enough $\varepsilon > 0$ and continuons w_γ vanishing on $\{x \in B_\varepsilon[\xi] \mid \tau(x,\gamma) \geq t\,\}$ (in particular, $w(\xi,t) = 0$). Thus, the propagator p_γ carries a jump on its forward front.

Choose $\gamma', \gamma'' \in \Gamma\,,\ \gamma' \neq \gamma''$; the value

$$t_{\gamma'\gamma''} := \inf\Big\{\, t > 0 \;\Big|\; \Big(p_{\gamma'}(\,\cdot\,,t)\,,\,p_{\gamma''}(\,\cdot\,,t)\Big)_{\mathcal{H}} \neq 0 \,\Big\}$$

may be interpreted as the time at which the propagators $p_{\gamma'}$ and $p_{\gamma''}$ begin to interact.

Lemma 2. *The equality*

$$t_{\gamma'\gamma''} = \frac{\tau(\gamma', \gamma'')}{2} \tag{2.12}$$

is valid.

$\triangleleft$ Let $m \in \Omega$ be the middle point of the path $\pi\,[\gamma', \gamma'']$; denote

$$T_m := \tau(\gamma', m) = \tau(m, \gamma'') = \frac{\tau(\gamma', \gamma'')}{2}.$$

Consider the generic case $m \notin V$; the case of $m \in V$ may be treated quite analogously. So, let m lie on an edge e.

If $t < T_m$, by virtue of (2.9) one has

$$\Big(p_{\gamma'}(\,\cdot\,, t), p_{\gamma''}(\,\cdot\,, t)\Big)_{\mathcal{H}} = \int\limits_{B_t[\gamma'] \cap B_t[\gamma'']} p_{\gamma'}(\,\cdot\,, t)\, p_{\gamma''}(\,\cdot\,, t)\, \rho\, |dx| = 0$$

because the balls do not intersect. Hence, we have $t_{\gamma'\gamma''} \geq T_m$.

Take $\varepsilon > 0$ so small that

$$B_{T_m+\varepsilon}\,[\gamma'] \bigcap B_{T_m+\varepsilon}\,[\gamma''] \subset e\,;$$

let $m'(\varepsilon)$ and $m''(\varepsilon)$ be the endpoints of this intersection nearest to γ' and γ'' correspondingly (see Fig. 7; the balls $B_{T_m+\varepsilon}[\gamma']$ and $B_{T_m+\varepsilon}[\gamma'']$ are contoured).

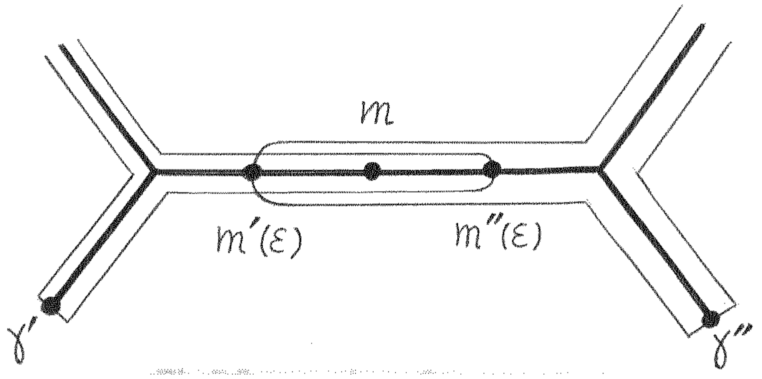


Fig. 7. Interaction of propagators

In accordance with (2.11) (for $t = T_m + \varepsilon$), on the segment $\pi[m'(\varepsilon),\ m''(\varepsilon)]$ one has the representations

$$p_{\gamma'}(\,\cdot\,, T_m + \varepsilon) = \alpha_{\gamma'}(m''(\varepsilon))\, \theta(T_m + \varepsilon - \tau(\,\cdot\,, \gamma')) + w_{\gamma'}(\,\cdot\,, T_m + \varepsilon)$$

with continuons bounded $w_{\gamma'}$ satisfying $w_{\gamma'}(m''(\varepsilon),\ T_m + \varepsilon) = 0$ and

$$p_{\gamma''}(\,\cdot\,, T_m + \varepsilon) = \alpha_{\gamma''}(m'(\varepsilon))\, \theta\,(T_m + \varepsilon - \tau(\,\cdot\,, \gamma'')) + w_{\gamma''}(\,\cdot\,, T_m + \varepsilon)$$

with continuons bounded $w_{\gamma''}$ satisfying $w_{\gamma''}(m'(\varepsilon),\ T_m+\varepsilon)=0$. Tending $\varepsilon\to 0$, these representations easily lead to the asymptotic

$$\Big(p_{\gamma'}(\,\cdot\,,T_m+\varepsilon)\,,\ p_{\gamma''}(\,\cdot\,,T_m+\varepsilon)\Big)_{\mathcal{H}}=\alpha_{\gamma'}(m)\,\alpha_{\gamma''}(m)\,\Big(\int\limits_{B_\varepsilon[m]}\rho\,|dx|\Big)\,[1+o(1)]$$

which implies $\Big(p_{\gamma'}(\,\cdot\,,T_m+\varepsilon)\,,\ p_{\gamma''}(\,\cdot\,,T_m+\varepsilon)\Big)_{\mathcal{H}}\neq 0$ for $t\sim T_m+0$. Whence, $t_{\gamma'\gamma''}\le T_m+\varepsilon$ and, finaly, $t_{\gamma'\gamma''}\le T_m$ by arbitrariness of ε. As result, we obtain $t_{\gamma'\gamma''}=T_m$, i.e., (2.12) holds. $\triangleright$

2.6. Propagators p_ξ.

Fix $\xi\in\operatorname{int}\Omega$ and denote $c_\xi:=\|\,\delta_\xi\,\|^{-1}_{\mathcal{H}^{-1}}$; the solution $p=p_\xi(x,t)$ to the problem

$$p_{tt}+L\,p=0\ \ in\ \operatorname{int}\Omega\times\mathbf{R}_+\,; \tag{2.13}$$

$$p\,|_{t=0}\ =0\,,\quad p_t\,|_{t=0}\ =c_\xi\delta_\xi\ \ in\ \Omega\,; \tag{2.14}$$

$$p\ =0\ \ on\ \Gamma\times\overline{\mathbf{R}}_+ \tag{2.15}$$

is the *propagator* corresponding to the interior point ξ. The following is a list of its properties:

(i) $p_\xi\in C\,(\overline{\mathbf{R}}_+\,;\mathcal{H})$; $\operatorname{supp}p_\xi(\,\cdot\,,t)\subset\overline{B_t[\xi]}$;

(ii) in the case of $\xi\notin V$, for times $0<t<\tau(\xi,V)$ the representation

$$p_\xi(x,t)=\frac{c_\xi}{2}\left[\frac{\rho(x)}{\rho^3(\xi)}\right]^{-\frac14}\theta\,\Big(t-\tau(x,\xi)\Big)+w_\xi(x,t)\,,\quad x\in\Omega \tag{2.16}$$

holds with $w_\xi(\,\cdot\,,t)\in C(\Omega)$ (so that $w_\xi(\,\cdot\,,t)=0$ at the endpoints of $\overline{B_t[\xi]}$);

(iii) for any $\xi\in\operatorname{int}\Omega$ and times $0<t<\max\limits_{x\in\Omega}\tau(x,\xi)$ the representation

$$p_\xi(\,\cdot\,,t)=c_\xi\alpha_\xi(\,\cdot\,)\,\theta\,\Big(t-\tau(\,\cdot\,,\xi)\Big)+w_\xi(\,\cdot\,,t) \tag{2.17}$$

is valid in Ω with a piecewise C^2-smooth strictly positive α_ξ and $w_\xi(\,\cdot\,,t)$ continuons near $\mathcal{K}^t_{(\xi,0)}$ (so that $w_\xi(\,\cdot\,,t)=0$ on $\mathcal{K}^t_{(\xi,0)}$).

Note in addition that the discontinuities (jumps) of p_ξ are located on the characteristic set $\mathcal{C}_{(\xi,0)}$.

Let $1(\,\cdot\,)$ be the function equal identically to 1 and $\xi\notin V$; using (2.16), simple calculations give the asymptotics as $t\to+0$:

$$\Big(p_\xi(\,\cdot\,,t),1\Big)_{\mathcal{H}}=c_\xi\,\rho(\xi)\,t+o(t)\,;\qquad \|\,p_\xi(\,\cdot\,,t)\,\|^2_{\mathcal{H}}=\frac{c_\xi^2}{2}\,\rho^{3/2}(\xi)\,t+o(t)\,,$$

leading to the useful relation

$$\left\{\lim_{t\to0}\ \frac{(\,p_\xi(\,\cdot\,,t),1\,)^2_{\mathcal{H}}}{2t\,\|\,p_\xi(\,\cdot\,,t)\,\|^2_{\mathcal{H}}}\right\}^2=\rho\,(\xi) \tag{2.18}$$

which will work in the inverse problem. In the case of $\xi=v\in V$ the left hand side of (2.18) gives a weighted meanvalue of the limit values of densities of edges incident to v.

For $\xi',\xi''\in\operatorname{int}\Omega$ introduce the interaction time

$$t_{\xi'\xi''}:=\inf\Big\{\,t>0\ \Big|\ \Big(p_{\xi'}(\,\cdot\,,t),\ p_{\xi''}(\,\cdot\,,t)\Big)_{\mathcal{H}}\neq 0\Big\}.$$

The representation (2.17) together with arguments quite analogous to ones used in the proof of Lemma 2 lead to the following result.

Proposition 6. *For any $\xi', \xi'' \in \operatorname{int}\Omega$ the equality*

$$t_{\xi'\xi''} = \frac{\tau(\xi', \xi'')}{2} \tag{2.19}$$

holds.

One could also introduce the interaction time $t_{\gamma\xi}$ for the propagators p_γ and p_ξ. The result would be the same: $t_{\gamma\xi} = \frac{\tau(\gamma,\xi)}{2}$.

3. Controllability

3.1. Wave shaping.

Here we consider the question: Managing the boundary control, can one create the waves of prescribed shape ? In rigorous formulation, this is a variant of the boundary control problem for the system (2.1)–(2.3) (see, e.g. [2]): given function $y \in \mathcal{H}$ to find a control $f \in \mathcal{F}^T$ satisfying

$$u^f(\,\cdot\,, T) \;=\; y\,. \tag{3.1}$$

The following theorem gives a positive answer. Being of some independent character this result is planned to be a subject of a separate publication; here we present just a scetch of the proof. Note that in solving the inverse problem Theorem 2 is not used. Recall that radius T_* and center x_* of the tree were introduced in Subsec. 1.2.

Theorem 2. *Let Ω be a tree and $T \geq T_*$. For any $y \in \mathcal{H}$ there exists $f \in \mathcal{F}^T$ such that* (3.1) *holds.*

$\triangleleft$ *(Scetch)*

(i) Let Ω be semiordered with respect to the root x_*. To each $a \in \operatorname{int}\Omega$, $a \neq x_*$ we associate the subtree

$$\Omega_a := \{x \in \Omega \,|\, x \succ a\}$$

and endow it with semiorder with respect to a. We denote $\Gamma_a := \Omega_a \bigcap \Gamma$.

Fix a $\xi \in \Omega\backslash V$, $\xi \neq x_*$; let Ω_ξ be the corresponding subtree and $\gamma_\xi \in \Gamma_\xi$ be such that

$$\tau(\xi, \gamma_\xi) = \max_{\gamma \in \Gamma_\xi} \tau(\xi, \gamma)\,;$$

we call γ_ξ by the *controlling vertex*. Note the simple relations $\tau(\xi, \gamma_\xi) < T_*$ and $\overline{B_{\tau(\xi,\gamma_\xi)}[\gamma_\xi]} \subseteq \Omega_\xi$.

Return to the system (2.1)–(2.3) and take $T = T_*$. In the part (i) of the proof we are going to show that there exists a (singular) control f_ξ such that

$$u^{f_\xi}(\,\cdot\,, T_*) \;=\; \alpha(\xi)\,\delta_\xi(\,\cdot\,) + u_\xi(\,\cdot\,) \quad in\;\; \Omega \tag{3.2}$$

with a piecewise smooth strictly positive α and a piecewise smooth u_ξ satisfying supp $u_\xi \subset \Omega^\xi$. The control f_ξ will be found in the form

$$f_\xi(\gamma, t) = \delta_{\gamma_\xi}(\gamma)\,\delta\left(t - [T_* - \tau(\xi,\gamma_\xi)]\right) + \sum_{j=1}^{N(\gamma)} \kappa^j_\gamma\,\delta\left(t - [T_* - t^j_\gamma\,]\right) \tag{3.3}$$

with some constants κ^j_γ and $t^j_\gamma < \tau(\xi, \gamma_\xi)$, i.e., in the form of a combination of delayed δ-functions.

Step 1 (creating the singularity at ξ). Take

$$f^{(0)}_\xi(\gamma, t) := \delta_{\gamma_\xi}(\gamma)\,\delta\Big(t - [T_* - \tau(\xi, \gamma_\xi)]\Big).$$

In accordance with the delay relation (see (iv), Subsec. 2.4) one has

$$u^{f^0_\xi}(\,\cdot\,, T_*) = u^{\delta_{\gamma_\xi}\delta}(\,\cdot\,, \tau(\xi, \gamma_\xi))$$

whereas (2.7) and (2.7) imply

$$u^{f^{(0)}_\xi}(\,\cdot\,, T_*) \;=\; \alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,) + u^{(0)}_\xi(\,\cdot\,) \tag{3.4}$$

where the term $u^{(0)}_\xi \in \mathcal{H}^{-1}$ satisfies $\operatorname{supp} u^{(0)}_\xi \subset \overline{B_{\tau(\xi,\gamma_\xi)}[\gamma_\xi]} \subset \Omega_\xi$ and is regular (doesn't contain Dirac measures) in a neighbourhood of ξ. The control $f^{(0)}_\xi$ creates the wanted singularity at ξ and is considered as a zero iteration of the required f_ξ.

Step 2 (removing the nearest singularity). In general case, the term $\alpha_{\gamma_\xi}\delta_\xi$ in (3.4) is not the unique singularity produced by $f^{(0)}_\xi$: detalizing $u^{(0)}_\xi$, one can write

$$u^{f^{(0)}_\xi}(\,\cdot\,, T_*) \;=\; \alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,) + \beta\,\delta_\eta(\,\cdot\,) + u^{(00)}_\xi(\,\cdot\,) \tag{3.5}$$

where $\eta \in \Omega_\xi$ is a point which supports the singularity of $u^{f^{(0)}_\xi}(\,\cdot\,, T_*)$ nearest (in optical metric) to ξ whereas $u^{(00)}_\xi$ is regular on $\pi\,[\xi, \eta]$. This singularity cann't be too close to ξ: a simple analysis of structure of the set $\mathcal{C}_{(\gamma_\xi, 0)}$ gives the estimate

$$\tau(\eta, \xi) \;\geq\; 2\min\Big\{\tau(\xi, v_\xi),\; \min_{e\in E_\xi} d\,(e)\Big\} \tag{3.6}$$

where $v_\xi \in \Omega_\xi \cap V$ is the vertex nearest to ξ, E_ξ is the set of edges of Ω_ξ.

Consider the subtree $\Omega_\eta \subset \Omega_\xi$ (with the root η), choose its controlling vertex γ_η, and define

$$g^{(0)}_\eta(\gamma, t) := -\frac{\beta}{\alpha_{\gamma_\eta}(\eta)}\,\delta_{\gamma_\eta}(\gamma)\,\delta\Big(t - [T_* - \tau(\eta, \gamma_\eta)]\Big).$$

By virtue of the evident inequality $\tau(\eta, \gamma_\eta) < \tau(\xi, \gamma_\xi)$ the control $g^{(0)}_\eta$ is delayed with respect to $f^{(0)}_\xi$. In accordance with (2.7) and (2.10) such the control produces the wave

$$u^{g^{(0)}_\eta}(\,\cdot\,, T_*) \;=\; -\beta\,\delta_\eta(\,\cdot\,) + u^{(0)}_\eta(\,\cdot\,) \tag{3.7}$$

with $u^{(0)}_\eta$ supported in Ω_η and regular near η.

Denoting

$$f^{(1)}_\xi := f^{(0)}_\xi + g^{(0)}_\eta$$

and comparing (3.5) with (3.7) one easily obtains

$$u^{f_\xi^{(1)}}(\,\cdot\,,T_*)\;=\;\alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,)+u_\xi^{(1)}(\,\cdot\,) \tag{3.8}$$

where $u_\xi^{(1)}:=u_\xi^{(00)}+u_\eta^{(0)}$ is supported in Ω_ξ and is regular on $\pi\,[\xi,\eta]$. Thus, the first iteration $f_\xi^{(1)}$ preserves the singularity wanted at ξ and removes the singularity at η. In other words, we succeded to clean of singularities a part of $\Omega_\xi\backslash\{\xi\}$ containing the segment $\pi\,(\xi,\eta]$.

Step 3 (extending the part cleaned). Detalizing (3.8), one has

$$u^{f_\xi^{(1)}}(\,\cdot\,,T_*)\;=\;\alpha_{\gamma_\xi}(\xi)\,\delta_\xi(\,\cdot\,)+\beta'\,\delta_{\eta'}(\,\cdot\,)+u_\xi^{(11)}(\,\cdot\,)$$

where $\eta'\in\Omega_\xi$ is a point which supports a singularity of $u^{f_\xi^{(1)}}(\,\cdot\,,T_*)$ nearest to ξ and, evidently, satisfies $\tau(\eta',\xi)\geq\tau(\eta,\xi)$ whereas $u_\xi^{(11)}$ is regular on $\pi\,[\xi,\eta']$. Applying the trick of Step 2 to the subtree $\Omega_{\eta'}$, one can remove $\beta'\,\delta_{\eta'}$ and clean $\pi\,(\xi,\eta']$ of singularities.

Continuing this procedure, on each step we preserve $\alpha_{\gamma_\xi}\,\delta_\xi$ and extend the part of $\Omega_\xi\setminus\{\xi\}$ free of singularities. Using the separation property (3.6) it is not difficult to show that the whole of $\Omega_\xi\setminus\{\xi\}$ will be exhausted after a finite number of steps.

The cases $\xi\in V$ and $\xi=x_*$ may be analyzed in the same way. So, the control f_ξ of the form (3.3) producing the wave of the form (3.2) exists.

As may be shown, f_ξ is well-defined as element of the space $(\mathcal{F}^{T_*})^{-1}$ dual to $(\mathcal{F}^{T_*})^1_0$ with respect to $\mathcal{F}^{T_*}$. For any $\varphi\in(\mathcal{F}^{T_*})^1_0$ the value $(f_\xi,\varphi)_{\mathcal{F}^{T_*}}$ is a bounded piecewise continuous function of $\xi\in\operatorname{int}\Omega$ smooth in $\Omega\backslash V$. Due to the last property the superposition

$$f^{\,\psi}=\int\limits_\Omega f_\xi\,\psi(\xi)\,|d\xi|$$

defined by

$$(f^{\,\psi},\varphi)_{\mathcal{F}^{T_*}}=\int\limits_\Omega (f_\xi,\varphi)_{\mathcal{F}^{T_*}}\,\psi(\xi)\,|d\xi|$$

turns out to be well-defined for any $\psi\in\mathcal{H}$ and determines an element of $\mathcal{F}^{T_*}$.

Note in addition that a more delicate analysis shows that, adding regular terms to the right–hand side of (3.3), one can construct f_ξ such that (3.2) holds with $u_\xi\in C^k(\Omega)$ with any prescribed $k=0,1,\ldots$.

(ii) In the inner space $\mathcal{H}$ consider the increasing family of subspaces

$$\mathcal{H}^\mu:=\Big\{y\in\mathcal{H}\;\Big|\;\operatorname{supp}y\subset\Omega\setminus B_{T_*-\mu}\,[x_*]\Big\}\,,\quad 0\leq\mu\leq T_*\,,$$

so that $\mathcal{H}^0=\{0\}\,,\ \mathcal{H}^{T_*}=\mathcal{H}$. As is easy to check, this family is continuons:

$$P^\mu\;=\;s-\lim_{\nu\to\mu}\,P^\nu\,,$$

where P^μ is the (orthogonal) projection in $\mathcal{H}$ onto $\mathcal{H}^\mu$.

Introduce the operator $B : \mathcal{H} \mapsto \mathcal{H}$,

$$(By)(x) = \int\limits_\Omega u_\xi(x)\, y(\xi)\, |d\xi|\,, \quad x \in \Omega$$

where $u_\xi(x)$ is the regular term in (3.2). Since $\operatorname{supp} u_\xi \subset \Omega_\xi$, the integral in fact is taken over the path connecting x with the center of the tree:

$$(By)(x) = \int\limits_{\pi\,[x,x_*]} u_\xi(x)\, y(\xi)\, |d\xi|$$

which implies the *triangularity* of B with respect to the family $\{\mathcal{H}^\mu\}$:

$$B\,\mathcal{H}^\mu \subset \mathcal{H}^\mu\,, \quad 0 \le \mu \le T_*.$$

Due to the piecewise smoothness of the kernel $u_\xi(x)$, the operator B acts continuonsly from $\mathcal{H}$ into $C(\Omega)$ and, hence, is a compact operator in $\mathcal{H}$. Triangularity plus compactness provide B to be a *Volterra operator*: its spectrum consists of the unique point $\lambda = 0$ (see [11]).

Return again to (3.2) and introduce the operator $C : \mathcal{H} \mapsto \mathcal{H}$,

$$(Cy)(x) := \int\limits_\Omega u^{f_\xi}\,(x, T_*)\, y(\xi)\, |d\xi| =$$

$$= \alpha(x)\, y(x) + \int\limits_\Omega u_\xi(x)\, y(\xi)\, |d\xi| = ([A+B]\, y)\,(x)\,, \quad x \in \Omega\,,$$

where A multiplies functions by the amplitude α separated from zero. Since A is boundedly invertible and B is Volterra operator, the operator C turns out to be boundedly invertible.

(iii) Take $y \in \mathcal{H}$ and denote

$$\psi := [A+B]^{-1}\, y\,;$$

the function $\psi \in \mathcal{H}$ determines the control

$$f = \int\limits_\Omega f_\xi\, \psi(\xi)\, |d\xi| \quad \in \mathcal{F}^{T_*}.$$

Since the map $f \mapsto u^f$ is linear, the control f produces the wave

$$u^f(\,\cdot\,, T_*) = \int\limits_\Omega u^{f_\xi}(\,\cdot\,, T_*)\, \psi(\xi)\, |d\xi| \;=\; [A+B]\, \psi = y\,.$$

Thus, f solves the boundary control problem at the time $T = T_*$.

Since the reachable set $\{u^f(\,\cdot\,, T) \mid f \in \mathcal{F}^T\}$ doesn't decrease as T grows, the problem is solvable for any $T > T_*$. $\triangleright$

In terms of Control Theory, Theorem 2 establishes the *exact boundary controllability* of the system (2.1)–(2.3). As we guess, this fact may be also derived from results of [2] based on a spectral (frequency-domain) analysis. However, our time-domain technique looks more natural and gives the sharp estimate for the time of controllability: $T_{\rm contr} \le \frac{d\,(\Omega)}{2}$. Perhaps, Theorem 1 is known for specialists in Control Theory but we didn't succeed in finding it in literature.

3.2. Local controllability near forward front.

The result of this section detalizes the character of controllability of the system (2.1)–(2.3). Note in advance that it is valid for any graphs, not only trees.

Fix $\gamma \in \Gamma$; let $T > 0$ be such that $\Omega \setminus \overline{B_T[\gamma]} \neq \{\emptyset\}$, so that the ball $\overline{B_T[\gamma]}$ is a proper subgraph of Ω. In this case the set

$$\mathcal{K}^T_{(\gamma,0)} = \{x \in \Omega \mid \tau(x,\gamma) = T\}$$

is nonempty and, in accordance with (2.7), supports the forward singularities of the fundamental solution $u^{\delta_\gamma\delta}(\,\cdot\,,T)$.

Choose any $a \in \mathcal{K}^T_{(\gamma,0)}$; let e be an edge of $\overline{B_T[\gamma]}$ incident to a (on Fig.8 one has $e = \pi\,(a,v)$).

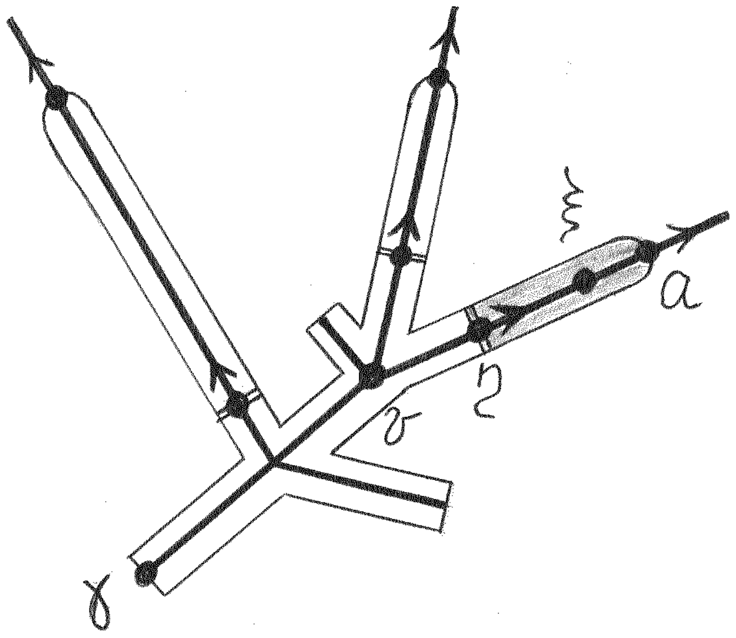


FIG. 8. THE NEIGHBORHOOD OF THE FORWARD FRONT

The point a supports the forward singularity $\alpha_\gamma(a)\,\delta_a(\,\cdot\,)$ of $u^{\delta_\gamma\delta}(\,\cdot\,,T)$; at the same time there exists an interval $e_a \subseteq e$ (see $\pi\,(a,\eta)$ on Fig.8) which is free of secondary singularities. The following result shows that e_a is a part of neighborhood of the forward front available for controlling from the vertex γ. Recall that $d(\,\cdot\,)$ is the optical diameter.

Lemma 3. *For any $y \in \mathcal{H}$ such that $\operatorname{supp} y \subset \overline{e_a}$ there exists the unique control $f \in \mathcal{F}^T_\gamma$ satisfying $\operatorname{supp} f \subset \gamma \times [0, d\,(e_a)]$ and providing the equality*

$$u^f\,(\,\cdot\,,T)\,|_{e_a} = y\,. \tag{3.9}$$

If, additionally, $y \in \mathcal{H}^1_0$ then one can find $f \in (\mathcal{F}^T_\gamma)^1_0$ such that (3.9) holds.

$\triangleleft$ Fix $\xi \in e_a$ and denote

$$f^\xi(\,\cdot\,,t) := \delta_\gamma(\,\cdot\,)\,\delta(t - \tau(\xi,a)).$$

Taking into account the delay $\tau(\xi,a)$ and using (2.7), one can easily represent

$$u^{f^\xi}(\,\cdot\,,T) = \alpha_\gamma(\xi)\,\delta_\xi(\,\cdot\,) + u^\xi(\,\cdot\,) \qquad on\ e_a \tag{3.10}$$

with C^2-smooth strictly positive α_γ and u^ξ satisfying

$$\operatorname{supp} u^\xi\ \subset\ \{x \in e_a \mid \tau(x,a) \geq \tau(x,\xi)\}$$

and C^2-smooth on its support.

Choose $y \in \mathcal{H}$ with $\operatorname{supp} y\ \subset \overline{e_a}$. Solving the control problem (3.9) with respect to f, let us seek for f in the form of a composition

$$f = \int\limits_{e_a} f^\xi\,\psi(\xi)\,|d\xi| \tag{3.11}$$

which is evidently well-defined for any $\psi \in L_2(e_a)$ and determines an element of $\mathcal{F}^T$. By linearity of the map $f \mapsto u^f$, for $x \in e_a$ one has

$$y(x) = u^f(x,T) = \int\limits_{e_a} u^{f^\xi}(x,T)\,\psi(\xi)\,|d\xi| = \ <\text{see (3.10)}>\ =$$

$$= \alpha_\gamma(x)\,\psi(x) + \int\limits_{\pi\,[x,a]} u^\xi(x)\,\psi(\xi)\,|d\xi|.$$

Parametrizing $x \in e_a$ by the optical distance $\tau = \tau(x,a)$ we obtain the Volterra equation

$$\alpha_\gamma(\tau)\,\psi(\tau) + \int\limits_0^\tau u^s(\tau)\,\psi(s)\,\frac{ds}{\rho^{1/2}(s)} = y(\tau)\,, \quad 0 \leq \tau \leq d\,(e_a) \tag{3.12}$$

which is uniquely solvable in $L_2(0,d\,(e))$. Determining ψ and substituting in (3.11) we find

$$f(\,\cdot\,,t) = \delta_\gamma(\,\cdot\,) \int\limits_0^{d\,(e_a)} \delta(t-\tau)\,\psi(\tau)\,\frac{d\tau}{\rho^{1/2}(\tau)} = \delta_\gamma(\,\cdot\,)\,\frac{\psi(t)}{\rho^{1/2}(t)}\,, \quad 0 \leq t \leq d\,(e_a)\,. \tag{3.13}$$

At last, extending f onto $0 \leq t \leq T$ by zero, we obtain a control of the class $\mathcal{F}^T_\gamma$ mentioned in the statement of the lemma.

Due to the smoothness of $\alpha_\gamma\,, u^\xi$ and ρ, in the case of $y \in \mathcal{H}^1_0$ the equation (3.13) gives $\psi \in H^1\,[0,d\,(e_a)]\,, \quad \psi(0) = y(0) = 0$. Consequently, the control (3.13) satisfyies $f \in H^1\,[0,d\,(e)]$ and $f(\gamma,0) = 0$. Such the control may be extended (not uniquely) from $0 \leq t \leq d\,(e)$ onto $0 \leq t \leq T$ up to a function of the class $(\mathcal{F}^T_\gamma)^1_0$. This extension doesn't violate (3.9). Indeed, any of two such the extensions f' and f'' differ on $d\,(e_a) \leq t \leq T$; hence, by (2.10) the corresponding solutions $u^{f'}(\,\cdot\,,T)$ and $u^{f''}(\,\cdot\,,T)$ can differ only in $B_{T-d\,(e_a)}\,[\gamma]$ and must coincide on e_a. $\triangleright$

Note in addition that in the framework of our approach an analysis of controllability is inavoidable: it is a basic property which motivates the term 'Boundary Control method' (see [1, 3, 4]).

3.3. Interaction 'wave-functional'.

The formula derived in this section plays the key role in the inverse problem. It expresses the optical diameter of the functional in 'wave' terms. The derivation uses the local controllability established in Lemma 2.

Fix a boundary vertex $\gamma \in \Gamma$; due to the smoothness property (see (iv), Subsec. 2.4) for functionals $h \in \mathcal{H}^{-1}$ the value

$$t_\gamma(h) := \inf \{ T > 0 \mid \exists f \in (\mathcal{F}_\gamma^T)_0^1 : \quad (h, u^f(\,\cdot\,, T))_{\mathcal{H}} \neq 0 \}$$

is well-defined and may be interpreted as the time at which the waves generated at γ begin to interact with the functional.

Taking $T < \tau(\gamma, \operatorname{supp} h)$, by virtue of the localization property (2.9) one has

$$\{ \operatorname{supp} u^f(\,\cdot\,, T) \cap \operatorname{supp} h \} \subset \{ \overline{B_T[\gamma]} \cap \operatorname{supp} h \} = \{\emptyset\}$$

which implies $(h, u^f(\,\cdot\,, T))_{\mathcal{H}} = 0$ and easily leads to

$$t_\gamma(h) \geq \tau(\gamma, \operatorname{supp} h). \tag{3.14}$$

At the same time it is not difficult to mention the examples with the rigorous inequality in (3.14).

Recall that $d[h] := d(\operatorname{supp} h)$; the following result is of crucial character for the inverse problem.

Theorem 3. *If Ω is a tree, for any nonzero $h \in \mathcal{H}^{-1}$ the equality*

$$d[h] = \max_{\gamma', \gamma'' \in \Gamma} \left\{ 2t_{\gamma'\gamma''} - \left[t_{\gamma'}(h) + t_{\gamma''}(h) \right] \right\} \tag{3.15}$$

is valid.

$\triangleleft$

(i) Take a nonzero $h \in \mathcal{H}^{-1}$; the set

$$M := \bigcup_{a, b \in \operatorname{supp} h} \pi[a, b]$$

is the minimal subtree in Ω containing $\operatorname{supp} h$. One can easily see that $d(M) = d[h]$.

Choose two different vertices $\gamma', \gamma'' \in \Gamma$; define

$$M_{\gamma'\gamma''} := M \cap \pi[\gamma', \gamma''] \supset \operatorname{supp} h \cap \pi[\gamma', \gamma'']$$

and note the simple fact:

$$d(M_{\gamma'\gamma''}) \leq d[h].$$

In the case of $M_{\gamma'\gamma''} \neq \{\emptyset\}$ one has the evident inequalities

$$\tau(\gamma', M_{\gamma'\gamma''}) \leq \tau(\gamma', \operatorname{supp} h), \quad \tau(\gamma'', M_{\gamma'\gamma''}) \leq \tau(\gamma'', \operatorname{supp} h)$$

which yield

$$\tau(\gamma',\gamma'') = \tau(\gamma', M_{\gamma'\gamma''}) + d\,(M_{\gamma'\gamma''}) + \tau(M_{\gamma'\gamma''},\gamma'') \le$$
$$\le \tau(\gamma', \operatorname{supp} h) + d\,(M_{\gamma'\gamma''}) + \tau(\gamma'', \operatorname{supp} h)\,. \tag{3.16}$$

Consider the case $M_{\gamma'\gamma''} = \{\emptyset\}$ (so that $d\,(M_{\gamma'\gamma''}) := 0$); let $m \in M$ be a point nearest to $\pi\,[\gamma',\gamma'']$. Since Ω doesn't contain cycles, such the point is unique. By the triangular inequality one has

$$\tau(\gamma',\gamma'') \le \tau(\gamma', m) + \tau(m,\gamma'') = \tau(\gamma', M) + \tau(\gamma'', M) \le$$
$$\le \tau(\gamma', \operatorname{supp} h) + \tau(\gamma'', \operatorname{supp} h)\,;$$

therefore the inequality (3.16) holds in any case.

Now we can estimate the diameter of h as follows:

$$d\,[h] \ge d\,(M_{\gamma'\gamma''}) \ge \ < \text{see } (3.16) > \ \ge \tau(\gamma',\gamma'') - \Big[\tau(\gamma', \operatorname{supp} h) + \tau(\gamma'', \operatorname{supp} h)\Big] \ge$$
$$< \text{see } (3.14) > \ \ge \tau(\gamma',\gamma'') - \Big[t_{\gamma'}(h) + t_{\gamma''}(h)\Big]\,,$$

whereas (2.12) leads to the inequality

$$d\,[h] \ge 2\,t_{\gamma'\gamma''} - \Big[t_{\gamma'}(h) + t_{\gamma''}(h)\Big]$$

which also is valid for $\gamma' = \gamma''$. Taking the maximum in the right hand side, we obtain

$$d\,[h] \ge \max_{\gamma',\gamma''\in\Gamma} \Big\{ 2\,t_{\gamma'\gamma''} - \Big[t_{\gamma'}(h) + t_{\gamma''}(h)\Big] \Big\}\,. \tag{3.17}$$

(ii) Let us show that the equality in (3.17) is realized. Let $b_1, b_2 \in \operatorname{supp} h$ be a pair of extreme points:

$$\tau(b_1, b_2) = d\,[h] = d\,(M)\,,$$

so that $\pi\,[b_1, b_2]$ is a path in M of the maximal optical length (in the case of $d\,[h] = 0$, $\operatorname{supp} h = \{b\}$ we put $\pi\,[b, b] := \{b\}$). Choose $\gamma_1, \gamma_2 \in \Gamma$ such that $\pi\,[\gamma_1,\gamma_2] \supset \pi\,[b_1, b_2]$ (on Fig.9 the set M is contoured with the solid line).

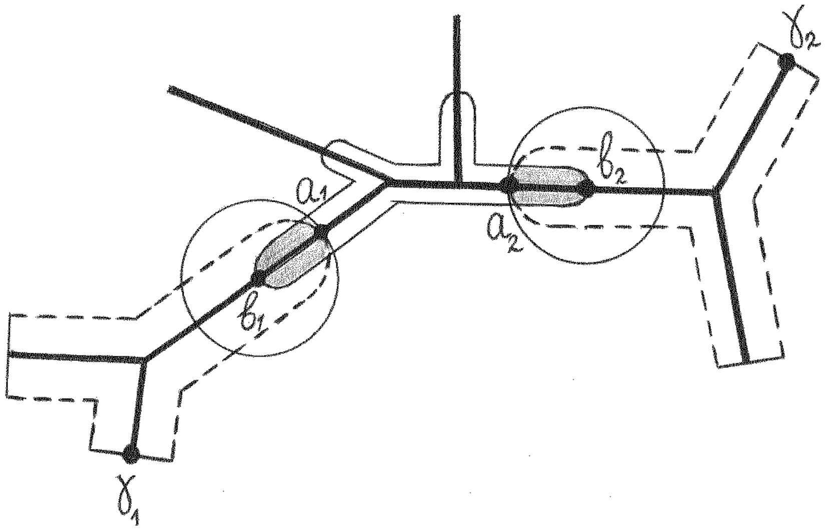


FIG. 9. THE PATHS $\pi[b_1, b_2]$ AND $\pi[\gamma_1, \gamma_2]$

By the choice, the equalities

$$\tau\,(\gamma_i\,,\,b_i) = \tau\,(\gamma_i\,,\,\operatorname{supp} h\,)$$

are valid that yields

$$\tau\,(\gamma_1, \gamma_2)\ =\ \tau\,(\gamma_1, b_1)\ +\ \tau\,(b_1, b_2)\ +\ \tau\,(b_2, \gamma_2)\ =$$

$$=\ \tau\,(\gamma_1,\,\operatorname{supp} h)\ +\ d\,[h]\ +\ \tau\,(\gamma_2,\,\operatorname{supp} h)\,. \tag{3.18}$$

Fix a small positive ε and take $y_i\ \in\ \mathcal{H}^1_0$ such that $\operatorname{supp} y_i\ \subset\ B_\varepsilon\,[b_i]$ and $(h\,, y_i)_{\mathcal{H}}\ \neq 0\,,\quad i = 1, 2$ (on Fig.9 the balls $B_\varepsilon\,[b_i]$ are marked with circles). Denote $T_i := \tau(\gamma_i, b_i) + \varepsilon$; following Lemma 3, find controls $f_i\ \in\ (\mathcal{F}^{T_i}_{\gamma_i})^1_0\,,\quad T_i\ =\ \tau\,(\gamma_i, b_i)\ +\ \varepsilon$ such that the waves $u^{f_i}\,(\,\cdot\,, T_i)$ are supported in the balls $\overline{B_{T_i}[\gamma_i]}$ (contoured with dotted lines) and satisfy

$$u^{f_i}\,(\,\cdot\,, T_i)\ |_{e_{a_i}}\ =\ y_i\,,$$

where the sets $e_{a_i}\ :=\ M\ \cap\ B_{T_i}\,[\gamma_i]$ (on Fig. 9 shadowed) are of the same meaning as in Lemma 3. By the choice made one has

$$(h\,,\,u^{f_i}\,(\,\cdot\,, T_i))_{\mathcal{H}}\ =\ (h\,,\,y_i)_{\mathcal{H}}\ \neq\ 0\,;$$

hence,

$$t_{\gamma_i}\,(h)\ \leq\ T_i\ =\ \tau\,(\gamma_i, b_i)\ +\ \varepsilon\ =\ \tau\,(\gamma_i,\,\operatorname{supp} h)\ +\ \varepsilon\,.$$

Comparing this with (3.14), by arbitrariness of ε we get

$$t_{\gamma_i}\,(h)\ =\ \tau\,(\gamma_i,\,\operatorname{supp} h)\,. \tag{3.19}$$

Returning to (3.18), one has

$$d\,[h]\ =\ \tau(\gamma_1, \gamma_2)\ -\ [\tau(\gamma_1,\,\operatorname{supp} h)\ +\ \tau(\gamma_2,\,\operatorname{supp} h)]$$

$$=\ <\text{see (2.12), (3.19)}>\ =\ 2\,t_{\gamma_1,\gamma_2}\ -\ [t_{\gamma_1}\,(h)\ +\ t_{\gamma_2}\,(h)]\,.$$

So, the maximum in (3.17) is attained and we get (3.15). $\triangleright$

The technique used in the proof allows to obtain the following useful result conserning the Dirac measures. It remains valid for any graph, not only for trees.

Proposition 7. *For any $\xi\ \in \operatorname{int}\Omega$ and $c = \text{const}\ \neq 0$ the equality*

$$t_\gamma\,(c\,\delta_\xi) = \tau\,(\gamma,\,\xi) \tag{3.20}$$

holds.

Note in addition, that (3.15) is invalid on graphs with cycles: to construct an example with

$$d\,[h]\ >\ \max_{\gamma', \gamma''\,\in\,\Gamma}\Big\{2\,t_{\gamma'\gamma''}\ -\ [t_{\gamma'}\,(h)\ +\ t_{\gamma''}\,(h)]\Big\}$$

is not difficult.

4. The inverse problem

4.1. Statement.

The boundary spectral inverse problem is to recover the equipped grapf (the pair Ω, ρ) from its spectral data. This statement goes back to the classical works by I. M. Gelfand and B. M. Levitan (on the Sturm–Liouville operator), M. G. Krein (inhomogeneous string), and Yu. M. Berezanskii (multidimensional version). However, the direct analog of the classical statement would be to recover ρ on *given* Ω (see [9, 12, 13]). A pecularity of our version is that one needs to recover *unknown* graph together with *unknown* density supported on it.

So, we are going to extract a 'material object' (equipped graph) from the spectral data which are just a set of ordered numbers. The rest of the paper is devoted to a procedure constructing such the object. The role of 'raw material' is played by Dirac measures δ_ξ or, more precisely, by the set of their spectral images. As we'll see, it is the set which is determined by the spectral data.

4.2. Spectral representation.

Recall that $\{\lambda_k\}_{k=1}^\infty$ and $\{\varphi_k\}_{k=1}^\infty$ are the spectrum and the (orthonormalized) eigenbase of the operator L in $\mathcal{H}$.

Let us introduce the unitary operator (Fourier transform) $U\,:\,\mathcal{H} \mapsto \tilde{\mathcal{H}} := l_2$,

$$U\,y \;=\; \{\,(y, \varphi_k)_{\mathcal{H}}\,\}_{k=1}^\infty \;=:\; \tilde{y}\;;$$

we call $\tilde{y}$ by the spectral image of y.

Operator U maps the Sobolev class $\mathcal{H}^1_0$ onto the set

$$\tilde{\mathcal{H}}^1_0 \;:=\; U\,\mathcal{H}^1_0 \;=\; \Big\{\{c_k\}_{k=1}^\infty \,\in\, l_2 \;\big|\; \sum_k \lambda_k\, c_k^2 \;<\; \infty\Big\} \tag{4.1}$$

(summation over $k = 1, 2, \dots$), the relation

$$(y, w)_{\mathcal{H}^1_0} \;=\; \sum_k \lambda_k\,(y, \varphi_k)_{\mathcal{H}}\,(w, \varphi_k)_{\mathcal{H}} \;=:\; (\tilde{y}, \tilde{w})_{\tilde{\mathcal{H}}^1_0}$$

being valid. Correspondingly, the negative space $\tilde{\mathcal{H}}^{-1}$ of the tripple $\tilde{\mathcal{H}}^1_0 \subset \tilde{\mathcal{H}} \subset \tilde{\mathcal{H}}^{-1}$ is

$$\tilde{\mathcal{H}}^{-1} \;:=\; U\,\mathcal{H}^{-1} \;=\; \Big\{\{c_k\}_{k=1}^\infty \;\big|\; \sum_k \lambda_k^{-1}\, c_k^2 \;<\; \infty\Big\} \tag{4.2}$$

(here we extend U on $\mathcal{H}^{-1}$ defining Uh by

$$(Uh\,,\,q)_{\tilde{\mathcal{H}}} \;:=\; (h\,,\,U^{-1}q)_{\mathcal{H}}$$

for $q \in \tilde{\mathcal{H}}^1_0$). A noticible fact is that the classes $\tilde{\mathcal{H}}^1_0$ and $\tilde{\mathcal{H}}^{-1}$ are determined by the spectrum of L and, thus, by the spectral data. The transform U maps $\operatorname{Dom} L$ onto the set

$$\operatorname{Dom}\tilde{L} \;=\; U\operatorname{Dom} L \;=\; \Big\{\{c_k\}_{k=1}^\infty \,\in\, l_2 \;\big|\; \sum_k \lambda_k^2\, c_k^2 \;<\; \infty\Big\}\,,$$

where

$$\tilde{L} \,:=\, U\,L\,U^{-1} \;=\; \operatorname{diag}\{\lambda_1\}_{k=1}^{\infty}\,.$$

Recall that $1(\,\cdot\,) \equiv 1$ on Ω; integration by parts easily gives

$$\tilde{1} \,=\, U\,1 \,=\, \Big\{\frac{1}{\lambda_k}\sum_{\gamma\in\Gamma}\frac{d\varphi_k}{de}\,(\gamma)\Big\}_{k=1}^{\infty}. \tag{4.3}$$

Let us return to the dynamical system (2.1)–(2.3). Representing

$$u^f\,(\,\cdot\,,t) \;=\; \sum_k c_k^f\,(t)\,\varphi_k\,(\,\cdot\,)$$

and integrating by parts one can easily find $c_k^f\,(t)$ and get the spectral image of the wave:

$$\tilde{u}^f\,(t) \;=\; U\,u^f\,(\,\cdot\,,t) \;=\; \Big\{\sum_{\gamma\in\Gamma}\frac{d\varphi_k}{de}\,(\gamma)\int_0^t\frac{\sin\sqrt{\lambda_k}\,(t-s)}{\sqrt{\lambda_k}}\,f\,(\gamma,s)\,ds\Big\}_{k=1}^{\infty} \tag{4.4}$$

(see, e.g., [2]). An important fact is that for given f this image is determined by the spectral data.

Fix $\gamma\in\Gamma$ and recall that the propagator corresponding to γ is defined by $p_\gamma \;=\; u^{\delta_\gamma\theta}$. Substituting $f=\delta_\gamma\theta$ in (4.4) we have

$$\tilde{p}_\gamma\,(t) \;=\; U\,p_\gamma\,(\,\cdot\,,t) \;=\; \Big\{\frac{d\varphi_k}{de}\,(\gamma)\,\frac{1-\cos\sqrt{\lambda_k}t}{\lambda_k}\Big\}_{k=1}^{\infty}. \tag{4.5}$$

Return to the system (2.13)–(2.15) determining the propagator p_ξ. The Fourier method easily gives

$$p_\xi\,(\,\cdot\,,t) \;=\; \sum_k\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}(c_\xi\,\delta_\xi\,,\,\varphi_k)_{\mathcal{H}}\,\varphi_k\,(\,\cdot\,),$$

so that

$$\tilde{p}_\xi\,(t) \;=\; U\,p_\xi\,(\,\cdot\,,t) \;=\; \Big\{\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\,(c_\xi\,\delta_\xi\,,\varphi_k\,)_{\mathcal{H}}\Big\}_{k=1}^{\infty}. \tag{4.6}$$

4.3. Interaction times from spectral data.

Fix a pair of boundary points $\gamma',\gamma''\in\Gamma$. Calculating the inner product of the corresponding propagators by isometry, one obtains

$$\Big(p_{\gamma'}\,(\,\cdot\,,t),\,p_{\gamma''}\,(\,\cdot\,,t)\Big)_{\mathcal{H}} \;=\; \Big(\tilde{p}_{\gamma'}\,(t),\,\tilde{p}_{\gamma''}\,(t)\Big)_{\tilde{\mathcal{H}}} \;=< \text{see (4.5)} >$$

$$=\;\sum_k\frac{d\varphi_k}{de}\,(\gamma')\,\frac{d\varphi_k}{de}\,(\gamma'')\,\Big[\frac{1-\cos\sqrt{\lambda_k}t}{\lambda_k}\Big]^2,$$

and we get a possibility to determine the interaction time $t_{\gamma'\gamma''}$ (see Subsec. 2.5) from the spectral data:

$$t_{\gamma'\gamma''} \;=\; \inf\Big\{t>0 \mid \sum_k \frac{d\varphi_k}{de}(\gamma')\,\frac{d\varphi_k}{de}(\gamma'')\,\Big[\frac{1-\cos\sqrt{\lambda_k}t}{\lambda_k}\Big]^2 \neq 0\Big\}. \tag{4.7}$$

Additionally, due to (2.12) the optical distance $\tau(\gamma',\gamma'') = 2t_{\gamma'\gamma''}$ also turns out to be determined.

Recall that the time $t_\gamma(h)$ was defined in the beginning of Subsec. 3.3. To determine it in spectral terms let us note that any $f \in (\mathcal{F}^T_\gamma)^1_0$ has the form

$$f(\gamma',t) \;=\; \delta_\gamma(\gamma')\,\chi(t)$$

with a $\chi \in H^1_0[0,T]$. Hence, for such f and $h \in \mathcal{H}^{-1}$ we have by isometry:

$$(h, u^f(\,\cdot\,,T))_{\mathcal{H}} \;=\; <\text{see (4.4)}> \;=\sum_k h_k\,\frac{d\varphi_k}{de}(\gamma)\int_0^T \frac{\sin\sqrt{\lambda_k}(T-s)}{\sqrt{\lambda_k}}\,\chi(s)\,ds$$

where $h_k = (h,\varphi_k)_{\mathcal{H}}$ are the Fourier coefficients of h. Therefore, one can represent the interaction time as follows:

$$t_\gamma(h) \;=\; \inf\Big\{T>0 \mid \exists\,\chi\in H^1_0[0,T]\;:$$

$$\sum_k h_k\,\frac{d\varphi_k}{de}(\gamma)\int_0^T \frac{\sin\sqrt{\lambda_k}(T-s)}{\sqrt{\lambda_k}}\,\chi(s)\,ds \;\neq\; 0\Big\}.$$

4.4. Function $\tilde{d}$, set $\tilde{\mathcal{D}}_1$, and metric $\tilde{\tau}$.

The last representation may be interpreted in the following way. The function $\tilde{t}_\gamma\,:\,\tilde{\mathcal{H}}^{-1} \mapsto \overline{\mathbf{R}}_+$,

$$\tilde{t}_\gamma(g) \;:=\; t_\gamma(U^{-1}g)$$

is determined by the spectral data: its value on an element $g = \{g_k\}_{k=1}^\infty \in \tilde{\mathcal{H}}^{-1}$ may be calculated by

$$\tilde{t}_\gamma(g) \;=\; \inf\Big\{T>0 \mid \exists\,\chi\in H^1_0[0,T]\;:$$

$$\sum_k g_k\,\frac{d\varphi_k}{de}(\gamma)\int_0^T \frac{\sin\sqrt{\lambda_k}(T-s)}{\sqrt{\lambda_k}}\,\chi(s)\,ds \;\neq\; 0\Big\}. \tag{4.8}$$

As result, the function $\tilde{d}\,:\,\tilde{\mathcal{H}} \mapsto \overline{\mathbf{R}}_+$,

$$\tilde{d}\,[g] \;:=\; d\,[U^{-1}\,g]$$

also is determined by the spectral data: in accordance with (3.15), (4.7), and (4.8) one can find its values by

$$\tilde{d}\,[g] \;=\; \max_{\gamma',\gamma''\in\Gamma}\Big\{\,2\,t_{\gamma'\gamma''} \;-\; \Big[\,\tilde{t}_{\gamma'}\,(g) \,+\, \tilde{t}_{\gamma''}\,(g)\Big]\,\Big\}. \tag{4.9}$$

If the function $\tilde{d}$ is at our disposal, we can characterize the set $\tilde{\mathcal{D}}_1 \;=\; U\,\mathcal{D}_1 \;\subset\; \tilde{\mathcal{H}}^{-1}$ of spectral images of the normalized Dirac measures: in accordance with (1.2) one has

$$\tilde{\mathcal{D}}_1 \;=\; \Big\{\,g\in\tilde{\mathcal{H}}^{-1}\;|\;||g||_{\tilde{\mathcal{H}}^{-1}} \;=\; 1\,,\quad \tilde{d}\,[g] \;=\; 0\,\Big\}. \tag{4.10}$$

For any points $\xi',\xi'' \,\in\, \mathrm{int}\,\Omega$ the interaction time of the corresponding propagators (see Subsec. 2.6) may be expressed in terms of the spectral data: by virtue of (4.6) one has

$$\begin{aligned} t_{\xi'\xi''} \;&=\; \inf\Big\{t>0\;|\;\sum_k\Big(\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\Big)^2\,c_\xi{}^2\,(\delta_{\xi'})_k\,(\delta_{\xi''})_k \;\neq\; 0\,\Big\} \\ &=\; <\text{see (2.19)}>=\;\frac{\tau(\xi',\xi'')}{2}\,, \end{aligned} \tag{4.11}$$

where $(\delta_\xi)_k = (\delta_\xi,\varphi_k)_{\mathcal{H}}$ are the Fourier coefficients.

For $g\in\tilde{\mathcal{D}}_1$ denote by

$$\xi\,(g) \;:=\; \mathrm{supp}\,U^{-1}\,g \;\in\; \mathrm{int}\,\Omega$$

the point supporting the preimage of g and introduce the function (metric) $\tilde{\tau}\;:\;\tilde{\mathcal{D}}_1\,\times\,\tilde{\mathcal{D}}_1 \;\mapsto\; \overline{\mathbf{R}}_+$,

$$\tilde{\tau}\,(g',\,g'') \;:=\; \tau\,(\,\xi(g')\,,\,\xi(g'')\,)\,. \tag{4.12}$$

In accordance, with (4.11) one has

$$\tilde{\tau}\,(g',\,g'') \;=\; 2\,\inf\Big\{\,t>0\;|\;\sum_k\Big(\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\Big)^2\,g'_k\;g''_k \;\neq\; 0\,\Big\}\,, \tag{4.13}$$

i.e., the values of $\tilde{\tau}$ are determined by the spectrum of L.

In summary, we arrive to the facts of principal character for the inverse problem:

(i) the set $\tilde{\mathcal{D}}_1$ and the function $\tilde{\tau}$ are determined by the spectral data;

(ii) the metric space $(\tilde{\mathcal{D}}_1,\tilde{\tau})$ is isometric to $(\mathrm{int}\,\Omega,\tau)$ (through the map $g\mapsto\xi(g)$).

4.5. Solving the inverse problem.

So, we are going to recover the tree Ω and the density ρ on Ω from the spectral data $\Big\{\,\lambda_k\,;\;\frac{d\varphi_k}{de}\,|_\Gamma\,\Big\}_{k=1}^{\infty}$. Note that solving the inverse problem we prove Theorem 1.

Step 1 (preliminaries). Take an example of $l_2 =: \tilde{\mathcal{H}}$ and determine the classes $\tilde{\mathcal{H}}^1_0$ and $\tilde{\mathcal{H}}^{-1}$ by (4.1) and (4.2).

Find the element $\tilde{1}$ by (4.3).

Find $t_{\gamma'\gamma''}$ for all $\gamma',\gamma''\in\Gamma$ by (4.7). Determine the functions $\tilde{t}_\gamma(\,\cdot\,),\gamma\in\Gamma$ by (4.8) and, then, the function $\tilde{d}\,[\,\cdot\,]$ by (4.9).

Step 2 (constructing the isometric copy of (Ω, τ)*).* In $\tilde{\mathcal{H}}^{-1}$ select the set $\tilde{\mathcal{D}}_1$ by representation (4.10) and determine the function $\tilde{\tau}$ on $\tilde{\mathcal{D}}_1 \times \tilde{\mathcal{D}}_1$ by (4.13). The pair $(\tilde{\mathcal{D}}_1\,,\,\tilde{\tau})$ is a metric space isometric to $(\operatorname{int}\Omega\,,\,\tau)$. Completing with respect to metric $\tilde{\tau}$ we get a compact $(\overline{\tilde{\mathcal{D}}_1}\,,\,\tilde{\tau})$ isometric to (Ω, τ).

Step 3 (recovering the density on the copy). For $g = \{g_k\}_{k=1}^{\infty} \in \tilde{\mathcal{D}}_1$ find

$$(p_{\xi(g)}\,(\,\cdot\,,t)\,,\,1)_{\mathcal{H}} \;=\; (\tilde{p}_{\xi(g)}\,(t)\,,\,\tilde{1})_{\tilde{\mathcal{H}}} \;= <\text{see (4.3), (4.6)}>$$

$$= \sum_k \Big(\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\;g_k\Big)\;\Big(\frac{1}{\lambda_k}\sum_{\gamma\in\Gamma}\frac{d\varphi_k}{de}(\gamma)\Big)$$

and

$$||\,p_{\xi(g)}\,(\,\cdot\,,t)\,||^2_{\mathcal{H}} \;=\; ||(\tilde{p}_{\xi(g)}\,(t)\,||^2_{\tilde{\mathcal{H}}} \;= <\text{see (4.6)}>$$

$$= \sum_k \Big(\frac{\sin\sqrt{\lambda_k}t}{\sqrt{\lambda_k}}\;g_k\Big)^2\;;$$

then determine the function $\tilde{\rho} := \rho\circ\xi$ on $\overline{\tilde{\mathcal{D}}_1}$ by

$$\tilde{\rho}\,(g) \;=\; \rho\,(\xi\,(g)\,) \;=\; <\text{see (2.18)}> \;=$$

$$= \left\{\lim_{t\to 0}\frac{\Big[\sum_k\Big(\frac{\sin\sqrt{\lambda_k}\,t}{\sqrt{\lambda_k}}\,g_k\Big)\;\Big(\frac{1}{\lambda_k}\sum_\gamma\frac{d\varphi_k}{de}\,(\gamma)\Big)\Big]^2}{2\,t\,\sum_k\Big(\frac{\sin\sqrt{\lambda_k}\,t}{\sqrt{\lambda_k}}\,g_k\Big)^2}\right\}^2.$$

Step 4 (recovering the optical image). In accordance with Proposition 7, for $g \in \tilde{\mathcal{D}}_1$ one has

$$\tilde{t}_\gamma\,(g) \;=\; t_\gamma\,(\,c_{\xi(g)}\,\delta_{\xi(g)}\,) \;= <\text{see (3.20)}>=\; \tau\,(\gamma\,,\,\xi\,(g)\,)\,.$$

Therefore, the map $\tilde{i}\,:\,\tilde{\mathcal{D}}_1 \;\mapsto\; \mathbf{R}^n$,

$$\tilde{i}\,(g) \;=\; \operatorname{col}\{\,\tilde{t}_{\gamma_1}\,(g), \ldots\,,\,\tilde{t}_{\gamma_n}\,(g)\,\} \;=\; \operatorname{col}\{\,\tau\,(\gamma_1\,,\,\xi(g))\,, \ldots\,,\,\tau\,(\gamma_n\,,\,\xi(g))\,\} \;=\; i\,(\xi\,(g)\,)$$

determines the set

$$\tilde{i}\,(\tilde{\mathcal{D}}_1) \;=\; \operatorname{int}\Omega_{\text{opt}}\,,$$

closing which (in $\mathbf{R}^n$) we get Ω_{opt}.

Step 5 (recovering ρ_{opt}*).* Expressing

$$\rho_{\text{opt}} \;=\; \rho\circ i^{\,-1} \;=\; \rho\circ[\,\tilde{i}\circ\xi^{-1}\,]^{-1} \;=\; \rho\circ\xi\circ\tilde{i}^{\,-1} \;=\; \tilde{\rho}\circ\tilde{i}^{\,-1}$$

we recover the density ρ_{opt} on Ω_{opt}.

In accordance with Lemma 1, the pair $\Omega_{\text{opt}}\,,\,\rho_{\text{opt}}$ determines Ω, ρ up to a spatial isometry.

The inverse problem is solved. Theorem 1 is proved.

Note in addition an evident fact: if the boundary vertices $\gamma_1, \ldots, \gamma_n$ are given on the plane and the tree is determined by their positions then the spectral data determine such the tree uniquely.

5. Comments

5.1. Wave equation.

To the dynamical system (2.1)–(2.3) one associates the (Dirichlet) response operator $R^T : \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$R^T_w f := \frac{du^f}{de}\Big|_{\Gamma \times [0,T]}.$$

Using the result of Theorem 2 and repeating the scheme of [5], one can show that for any fixed $T \geq d(\Omega)$ operator R^T_w determines the (Dirichlet) spectral data uniquely. As result, for $T \geq d(\Omega)$ operator R^T_w determines Ω, ρ up to spatial isometry.

The same is evidently valid if the Neumann response operator $R^T_N = (R^T_w)^{-1}$ is given in capacity of the inverse data.

It presents no problems to adapt our scheme for the *Neumann* spectral data.

5.2. Heat equation.

The dynamical system describing the heat process on the graph is

$$\begin{aligned} & u_t + L_{\max} u = 0 \quad in \ \operatorname{int} \Omega \times (0,T); \\ & u\,|_{t=0} = 0 \quad in \ \Omega; \\ & u = f \quad on \ \Gamma \times [0,T]; \end{aligned}$$

function ρ^{-1} has the physical meaning of the heat conductivity, solution $u = u^f(x,t)$ is the temperatuve. The 'input-output' correspondence is described by the response operator $R^T_h : \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$R^T_h f := \frac{du^f}{de}\Big|_{\Gamma \times [0,T]}.$$

The well-known fact is that for any fixed $T > 0$ operator R^T_h determines the Dirichlet spectral data. Therefore, for any $T > 0$ the operator R^T_h determines the tree up to spatial isometry.

5.3. Schrödinger equation.

To the system

$$\begin{aligned} & i\,u_t - L_{\max} u = 0 \quad in \ \operatorname{int} \Omega \times (0,T); \\ & u\,|_{t=0} = 0 \quad in \ \Omega; \\ & u = f \quad on \ \Gamma \times [0,T]; \end{aligned}$$

one associates the response operator $R^T_s : \mathcal{F}^T \mapsto \mathcal{F}^T$,

$$R^T_s f := \frac{du^f}{de}\Big|_{\Gamma \times [0,T]}.$$

Following the scheme of [1], one can prove that for any fixed $T > 0$ operator R^T_s determines the Dirichlet spectral data uniquely. Hence, for any $T > 0$ operator R^T_s determines the tree up to spatial isometry.

5.4. Inverse scattering.

Assume that all of the edges of the graph Ω incident to the boundary vertices can be extended (as straight lines) to infinity and the density of such the noncompact graph satisfies $\rho \equiv 1$ for $|x| \geq R$ with an R large enough. In this case one can associate to the graph the time-domain (see, e.g., [7]) or frequency-domain (see [9, 10, 12]) scattering data.

If Ω is a tree and the scattering data are such that they determine the response operator R^T_w for $T > 2R$ of the unknown compact part $\Omega \cap B_R[0]$, then these data determine the compact part (together with density on it) up to spatial isometry.

5.5. Another operators.

Our scheme may be easily adapted to the case of the operators $L = -\frac{d}{de}\,\rho\,\frac{d}{de}$ and $L = -\frac{d^2}{de^2} + q$.

5.6. Open problems.

Sure, the first challenging problem is to extend our approach on the graphs with cycles. The principal dificulties are the following:

(i) cycles can imply a lack of controllability i.e., Theorem 2 may be invalid (see e.g. [2]);

(ii) the basic formula (3.15) doesn't remain hold and, therefore, our trick of selecting the set $\tilde{\mathcal{D}}_1$ in $\tilde{\mathcal{H}}^{-1}$ by means of characterization (4.10) does not work.

In the case of trees, an important problem is to find an efficient method of determining the zeros of function $\tilde{d}$ realizable numerically. A difficulty is that $\tilde{d}$ is doscontinuous at every element of $\mathcal{H}^{-1}$.

Another problem is to recover a part of the tree from the response operator R^T_w given for a fixed $T < d(\Omega)$. The case of inverse data given on a part of Γ is also of interest.

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