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NEW COUNTEREXAMPLES TO AN OLD UNIQUENESS HYPOTHESIS FOR CONVEX BODIES

G. PANINA

Institute for Informatics and Automation,
V.O. 14 line 39,
St.Petersburg 199178, Russia

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ABSTRACT

The paper presents a series of principally different C^∞ -smooth counterexamples to the hypothesis on characterization of the sphere:

If for a smooth convex body $K \subset \mathbb{R}^3$ and a constant C , in each point of ∂K we have $R_1 \le C \le R_2$, then K is a ball. (R_1 and R_2 are the principal curvature radii of ∂K).

The hypothesis was proved by A. D. Alexandrov and H. F. Münzner for analitic bodies. For the general smooth case it remained an open problem for years. Recently, Y. Martinez–Maure presented a C^2-smooth counterexample to the hypothesis.

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1. Introduction

The paper aims at a construction of a series of counterexamples to the following old hypothesis.

Hypothesis.

Suppose that for a smooth convex body $K \subset \mathbb{R}^3$ and a constant C , in each point of ∂K the inequality $R_1 \leq C \leq R_2$ is valid. Then K is a ball of radius C. ($R_1 \leq R_2$ are the principal curvature radii of ∂K).

This hypothesis was proved by A.D.Alexandrov and H. F. Münzner for analitic bodies [A1], [M1]. There exist also some partial results when the hypothesis is valid (see [K] and [M2]). For the general case of smoothness the hypothesis has remained an open problem for years. For a long time, mathematicians did beleive that it is also true for all smooth convex bodies, but gradually abandoned their attempts to prove it. Only recently, nearly at the same time (and absolutely independently) there appeared two mutually contradicting papers: one written by A. V. Pogorelov containing a proof of the hypothesis [P] and one written by Y. Martinez–Maure [M-M2] containing a C^2-smooth counter example to the hypothesis. Despite of their independence, both authors used the same trick: they reduced the problem to consideration of smooth hyperbolic hérissons, that is, a special type of saddle surfaces. (The theory of hérissons is widely developed in [LLR], [M-M1, 2, 3], [RR], and other papers. Although Pogorelov used a different terminology, his constructions are equivalent to those of this theory.)

This trick reveals a simple but unknown before relationship between the hypothesis and the extrinsic geometry of saddle surfaces. It is remarkable that the latter subject was investigated rather actively in the 60s of the last century by some Russian mathematicians, but without any connection with the hypothesis (see [B] and [Roz] for detailed rewiews).

The Pogorelov's paper contains an uncorrectable mistake. The example of Matrinez–Maure is correct.

On the other hand, the investigation of hérissons (which are, roughly speaking, Minkowski differences of smooth convex bodies) has an obvious interplay with the theory of virtual polytopes (which are, roughly speaking, Minkowski differences of convex polytopes). Virtual polytopes were introduced originaly by A. Pukhlikov and A. Khovanskij for some reasons of algebraic geometry. They also appear in the polytope algebras of P. McMullen. Besides, Martinez–Maure also presented recently a general theory of virtual polytopes (he calls them "discrete hérissons") which leads to the same notion [M-M3].

The counterexample of Martinez–Maure is based on a construction of a C^2-smooth saddle surface containing 4 cross-caps (see fig. 2.4.) The surface is given by a formula. Some later, Martinez–Maure presented an example of a polytopal analogue of his counterexample, a discrete hérissons (i.e., a virtual polytope) which has 4 discrete cross-caps (see fig. 5.4).

In the present paper in attempt to get new counter-examples, we move in the opposite direction: we start (section 5) by a construction of a hyperbolic virtual polytope with N cross-caps ($N \geq 4$ is any even number) and after that, using a smoothening technique, we obtain (theorem 6.1) a C^∞-smooth saddle hérisson with N smooth cross-caps. The latter give us (due to the trick of Martinez–Maure and Pogorelov) the desired series of counterexamples .

The paper is organized as follows.

Section 2 contains all necessary notions concerning smooth hérissons and explains the mentioned above trick of Martinez–Maure and Pogorelov.

Section 3 recalls briefly the notion of virtual polytopes.

Section 4 introduces the notion of hyperbolic hérissons and hyperbolic virtual polytopes.

Section 5 presents a construction of a hyperbolic virtual polytope with N cross-caps ($N = 4, 6, 8...$).

Section 6 describes a smooth hyperbolic approximation of such a polytope. This implies the desired series of counterexamples.

2. Smooth hérissons and the hypothesis

The section reviews the necessary for the completeness of the paper methods and constructions of [LLR], [M-M1, 2, 3], [P], and some others.

As is known, C^2-smooth convex bodies form a semigroup $\mathcal{H}$ with respect to the Minkowski addition $\otimes$. Since the usual cancellation law is valid in this semigroup, its Grothendieck group $\mathcal{H}^*$ coincides with the group of formal expressions of type $B_1 \otimes B_2^{-1}$, where $B_{1,2} \in \mathcal{H}$.

Remark 2.1. In the polytope algebra ([KP], [McM1,2]), the Minkowski addition plays the role of multiplication. Besides, it reflects the multiplication in Picard group of toric varieties.That's why we use $\otimes$ instead of more usual $+$. By $(.)^{-1}$ we mean the inversion with respect to $\otimes$.

Due to linearity, there exist reasonable definitions of support function, support hyperplanes and principal curvature radii for elements of $\mathcal{H}^*$.

Definition 2.2. Let $B \in \mathcal{H}^*, B = B_1 \otimes B_2^{-1}$, where B_1 and B_2 are convex smooth bodies. The *support function h of B* is defined as the (pointwise) difference of the support functions of B_1 and B_2:

$$h = h_{B_1} - h_{B_2}.$$

We say that a hérisson B is C^2-smooth (C^∞-smooth) if its support function is C^2-smooth (C^∞-smooth).

For each point ξ lying on the unit sphere S^2 centered at the origin O (which is identified with the set of unit vectors in $\mathbb{R}^3$), we define the *oriented support hyperplane $e_B(\xi)$ with the normal vector ξ* of B by the equation

$$(x, \xi) = h(\xi).$$

By the *hérisson* B we mean the envelope of the collection $\{e_B(\xi)\}_{\xi \in S^2}$. It is an oriented sphere homeomorphic surface with possible self-intersections and self-overlapings.

As a set of points, a hérisson B coincides with the image of the mapping

$$\phi : S^2 \longrightarrow \mathbb{R}^3,$$

$$(x, y, z) \longrightarrow (h'_x(x, y, z), h'_y(x, y, z), h'_z(x, y, z)).$$

Definition 2.3. Analogously to the classical convex case, we define the *principal curvature radii* R_1 and R_2 of a hérisson B at the point $\xi \in S^2$ (or at the point $\phi(\xi) \in B$) as the eigenvalues of the matrix

$$\begin{pmatrix} h''_{xx}(\xi) & h''_{xy}(\xi) \\ h''_{yx}(\xi) & h''_{yy}(\xi) \end{pmatrix}.$$

(ξ is codirected with the axes (z).)

Although the support function of a hérisson is smooth, the hérisson itself can have singular points.

For singular points of B, we have $R_1R_2 = 0$.

If $R_1R_2 \neq 0$, the hérisson B is a smooth surface in a neighborhood of $\phi(\xi)$ with the normal vector ξ in the point $\phi(\xi)$ and the radii R_1 and R_2 coincide with the principal curvature radii (in the classical sense) of the surface B.

Suppose that a convex body B with a smooth support function is a counter example to the hypothesis, that is, we have everywhere

$$R_1 \leq C \leq R_2.$$

Consider now the hérisson $B^0 = B \otimes D^{-1}$ where D is the ball of radius C. Since the support function behaves additively with respect to the Minkowski addition, the principal radii R_1^0 and R_2^0 of the hérisson B^0 satisfy the inequality

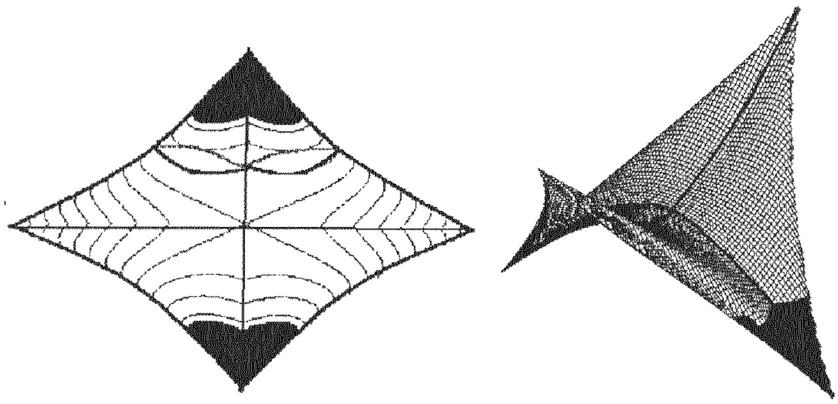
$$R_1^0 \leq 0 \leq R_2^0.$$

It means that the hérisson B^0 is a saddle surface.

Backwards, let a hérisson B^0 be a saddle surface (except for its singular points). Due to C^2-smoothness, its principal curvature radii R_1^0 and R_2^0 are restricted from below by a constant C. Then the convex body $B = B^0 \otimes D$ presents a counter example to the hypothesis. (As above, D is the ball of radius C.)

Thus the hypothesis is reduced to the problem of existing of a saddle hérisson. A. V. Pogorelov tried to prove that such a surface doesn't exist.

The required hérisson (figure 2.4.) was constructed by Martinez–Maure [M-M2]. It is a self-intersecting surface with a C^2- smooth support function which is obtained by glueing together graphs of two functions.



View from above   Sidewiew

The counterexample

Virtual polytopes

The section reviews the necessary for the completeness of the paper methods and constructions of [KP], [Pa1], [Pa2] and [Pa3] for dimension 3.

Denote by $\mathcal{P}$ the set of all compact convex polytopes in the Euclidean space E^3 with a fixed origin O (degenerate polytopes are also included). It is a semigroup with respect to the Minkowski addition $\otimes$.

Denote by $\mathcal{P}^*$ the Grothendieck group of $\mathcal{P}$. The inverse to K element of $\mathcal{P}^*$ is denoted by K^{-1}.

A function $F : \mathbb{R}^3 \to \mathbb{Z}$ is *polytopal* if it admits a representation of the form

$$F = \sum_i a_i I_{K_i},$$

where $a_i \in \mathbb{Z}$, $K_i \in \mathcal{P}$, and I_{K_i} is the indicator function of the polytope K_i:

$$I_{K_i}(x) = \begin{cases} 1 & \text{if } x \in K_i, \\ 0 & \text{otherwise.} \end{cases}$$

The set of all polytopal functions is denoted by $\mathcal{M}$. It is endowed with two ring operations. The role of addition is played by the pointwise addition, denoted by $+$. The multiplication is generated by $\otimes$ and is denoted by the same symbol.

The unit element of the ring $\mathcal{M}$ is obviously the function $E = I_{\{O\}}$.

Identifying convex compact polytopes with their indicator functions, we get an inclusion $\pi : \mathcal{P} \subset \mathcal{M}$. Keeping this identification in mind, we write for convenience K instead of I_K.

All elements of the semigroup $\pi(\mathcal{P})$ are invertible in $\mathcal{M}$.

Hence the inclusion $\mathcal{P} \subset \mathcal{M}$ induces an inclusion $\mathcal{P}^* \subset \mathcal{M}$. **Definition 3.1.** The image of the latter inclusion is called the group of *virtual polytopes*. For convenience, we denote it by the same letter $\mathcal{P}^*$.

Definition 3.2. Let K be a virtual polytope. Then there exist convex polytopes L and M such that $K = L \otimes M^{-1}$. The *support function* h_K of the virtual polytope K is defined to be the difference of support functions of L and M:

$$h_K = h_L - h_M.$$

This definition is consistent with other definitions of support function [KP], [Pa2]. Since we have defined the support function, we have the notion of *support oriented planes* of a virtual polytope as well.

Definition 3.3.[Pa3] Let $K \in \mathcal{M}, K = \sum_i a_i K_i$ with $K_i \in \mathcal{P}$. Let $l_i(\xi)$ be the support hyperplane to K_i with the outer normal vector ξ. The polytope $K_i^\xi = K_i \cap l_i(\xi)$ is called *the face of the polytope K_i with the normal vector ξ*, while the polytopal function $K^\xi = \sum_i a_i K_i^\xi$ is called *the face of the polytopal function K with the normal vector ξ*.

A face of a virtual polytope is a virtual polytope as well. The 0-dimensional, 1-dimensional and 2-dimensional faces are called vertices, edges and facets respectively.

Definition 3.4. A *fan* Σ is a finite collection of compact spherical polytopes on the unit sphere S^2 (possibly nonconvex and disconnected ones) such that

- $U, V \in \Sigma \Rightarrow U \cap V \in \Sigma$;
- $\cup\Sigma = S^2$;
- $U \neq V \in \Sigma \Rightarrow RelintU \cap RelintV = \emptyset$.

The *fan* of a virtual polytope is defined below analogously to the classical definition of the outer normal fan.

For a virtual polytope $K \in \mathcal{P}$, its *fan* Σ_K is the collection of spherically polytopal sets $\{\Sigma_K(\nu)\}$, where ν ranges over the set of faces of K , and

$$\Sigma_K(\nu) = cl\{\xi | K^\xi = \nu\}$$

(cl denotes the closure.)

These polytopal sets are called *cells* of the fan. Similarly to the convex case, the support function of K is linear on each cell of Σ_K. And similarly to convex polytopes, the fan of a virtual polytope K can be defined as the minimal fan for which h_K is linear on each cell.

The 0 -dimensional cells are called the *vertices* of the fan. The set of all vertices of a fan is denoted by Σ^0. It equals the set of normal vectors of all facets of K.

The collection of all 1-dimensional cells (= edges) of Σ is denoted by Σ^1 and is called the *skeleton* of Σ.

The support of the skeleton *supp* Σ^1 is the union of all edges of Σ.

We say that K *fits* a fan Σ if Σ is a refinement of Σ_K. It means that h_K is linear on each cell of Σ.

For a unit vector ξ and a real number h, denote by $e(\xi, h)$ the plane whose equation is $(x, \xi) = h$. Note that a convex polytope $K \in \mathcal{P}$ is uniquely defined by the set $\Sigma^0 = \{\xi_i\}$ and the values $h_K(\xi_i) = h_i$. Indeed, the collection $\{e(\xi_i, h_i)\}$ is the collection of affine hulls of its facets.

However, this assertion fails when passing to virtual polytopes. In this case we have more freedom: we are free to choose any fan with vertices in $\{\xi_i\}$. The only thing we have to worry about is the consistency condition which is motivated by the following remark.

Remark 3.5. Let K be a virtual polytope. For any cell α of Σ_K with vertices $\{\xi_1, ..., \xi_k\}$, the planes $e(\xi_1, h_1), ..., e(\xi_k, h_k)$ have a common point. This point is the vertex of K corresponding to the cell α.

Consistency condition. We say that a fan Σ with a function $h : \Sigma^0 \to \mathbb{R}$ satisfy the *consistency condition* if for any cell α of Σ with vertices $\{\xi_1, \ldots, \xi_k\}$, the hyperplanes $e(\xi_1, h_1), ..., e(\xi_k, h_k)$ have a common point.

The following theorem demonstrates that virtual polytopes are not uniquely restored by the set $\{\xi_i\}$ and values h_i. It also shows that the cells in a virtual case may be non-convex, disconnected or of a complicated topological form.

Theorem 3.6. [Pa3] *Let a fan Σ and a function $h : \Sigma^0 \to \mathbb{R}$ satisfy the consistency condition. Then there exists a unique virtual polytope K such that Σ_K fits Σ and $h_K(\xi_i) = h_i$.* $\square$

This theorem yields a simple way of constructing examples of virtual polytopes.

Theorem 3.7. Construction of a virtual polytope related to an embedded triangle complex.

Let B in $\mathbb{R}^3$ be a closed sphere homeomorphic embedded (with possible self-intersections) polytopal complex generated by a set of triangles $\{T_i\}$. Suppose there exist a collection of normal vectors ξ_i of the triangles T_i and a fan Σ with vertices in $\{\xi_i\}$ such that the combinatorics of Σ is dual to that of B. (In particular, ξ_i and ξ_j are connected by an edge of Σ if and only if T_i and T_j are adjacent in the complex B.)

Then there exists a virtual polytope K such that

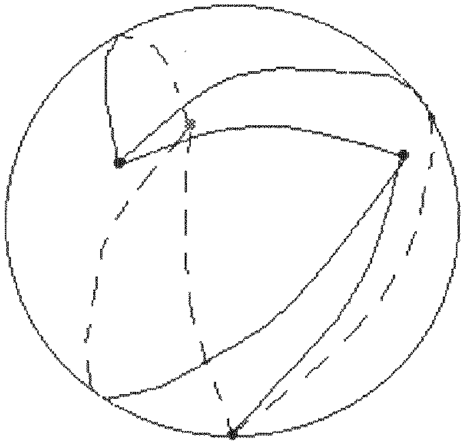
- *the closures of supports of facets of K coincide with the set of triangles $\{T_i\}$, and*

• $\Sigma_K = \Sigma$.

Here is an important example of this construction.

Example 3.8. [Pa2] **Hyperbolic tetrahedron.**

Let B be the complex of faces of a tetrahedron $\Delta \subset \mathbb{R}^3$. Choose the "usual" normal vectors of the triangles, i.e. looking outwards and connect them as shown in figure 3.9. The sphere is divided into four equal non-convex parts. Due to the above construction, we obtain a virtual polytope $\Delta' \neq \Delta$ which corresponds to this fan and this triangle complex. It is a polytopal function which equals -1 everywhere on the tetrahedron except two fixed edges. On these edges the values of the function are equal to 0.



The fan of the hyperbolic tetrahedron

Hyperbolic virtual polytopes and hyperbolic hérissons

Let K be a virtual polytope or a smooth hérisson, whose support function is $h = h_K$. For $\xi \in S^2$, let $e(\xi)$ be the plane defined by the equation $(x, \xi) = 1$. Consider the restriction of h on the plane $e(\xi)$ and denote by $F = F_K(\xi)$ the graph of the restriction. For a virtual polytope K, F is a piecewise linear surface whose vertices and edges correspond to those of Σ_K intersected with the open semisphere with the pole ξ. For a smooth hérisson K, F is a smooth surface.

Let $\Xi = \{\xi_i\}$ be a collection of points on S^2 such that each open semisphere contains at least one point from the collection.

Proposition 4.1. *A hérisson K is hyperbolic if and only if $F_K(\xi)$ is a saddle surface for any $\xi \in \Xi$. Then $F_K(\xi)$ is saddle for any other ξ as well.* $\square$

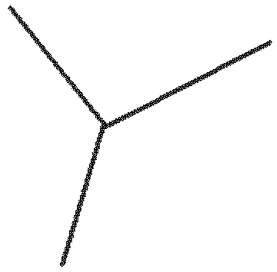
This motivates the following natural definition.

Definition 4.2. A virtual polytope K is called *hyperbolic* if $F_K(\xi)$ is a saddle surface for any $\xi \in S^2$. (As above, it is sufficient to take $\xi \in \Xi$.)

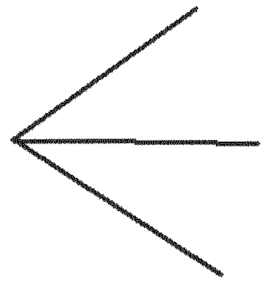
Recall that a piecewise linear (or any other non-smooth surface) F is called *saddle* if there is no plane cutting a bounded connected component off F [B].

Remark 4.3. Martinez–Maure has already given two definitions of weak hyperbolicity and strong hyperbolicity of virtual polytopes [M-M3]. The above definition is equivalent to that of weak hyperbolicity.

Definition 4.4. We say that a vertex ξ of a fan Σ is *non-convex* if for a neighbourhood $U(\xi)$ of ξ, $\Sigma^1 \cap U(\xi)$ lies on one side of a big circle containing ξ.



A convex vertex



A non-convex vertex

Proposition 4.6. *Let K be a virtual polytope. Suppose that each vertex of its fan Σ_K has exactly 3 adjacent edges. Then the two assertions are equivalent:*
- *K is hyperbolic.*
- *Each vertex of Σ_K is non-convex.*

Proof. A vertex with exactly 3 adjacent edges can be cut off the surface F if and only if the vertex is convex.

It remains to observe that if F is non-hyperbolic, there exists a plane cutting off F a bounded component containing exactly one vertex. □

Corollary 4.7. *The hyperbolic tetrahedron (example 3.8) is a hyperbolic virtual polytope.* □

Example of a hyperbolic virtual polytope with N cross-caps

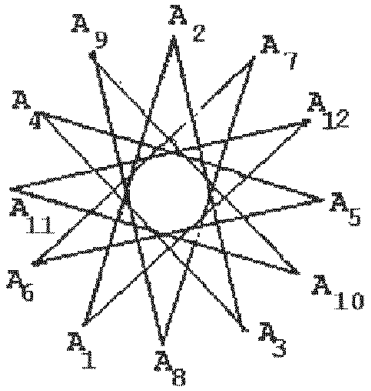
The desired virtual polytope $\mathcal{K}$ is constructed as follows. First, we construct an embedded triangle complex. It consists of its upper and lower central parts (which look like two folded stars) and of N cross-caps (fig. 5.9). The triangles are oriented, and the endpoints of their normal vectors can serve as vertices of a fan (fig. 5.8).

According to theorem 3.7, there exists a virtual polytope related to the complex.

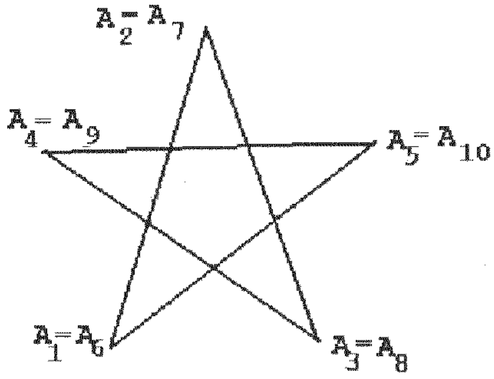
Planar stars. Let N be even, $N \geq 4$.

Case 1. $N/2$ is odd. Consider a (planar) proper $N/2$ -gon P. By the *star with N rays* (based on P) we mean the broken line consisting of all longest diagonals of P. Each of the diagonals is taken twice, so the broken line has a double self-overlaping. The vertices $\{A_i\}$ of the star (i.e., the vertices of P taken twice) are enumerated clockwise.

Case 2. $N/2$ is even. Consider a (planar) proper N -gon P. By the *star with N rays* (based on P) we mean the broken line consisting of all diagonals of P connecting a vertex of P with a vertex lying next to the opposit one. Dislike the previous case, the broken line is not self-overlaping. The vertices $\{A_i\}$ of the star (i.e., the vertices of P) are enumerated clockwise.



A star with even N/2



A star with odd N/2

Central collections of triangles.

Fix a Cartesian coordinate system (O, x, y, z). All the vectors in $\mathbb{R}^3$ forming an angle with the axes (z) less than $\pi/2$ will be refered as *"vectors looking upwards"*. If the angle is more than $\pi/2$, we say that the vector *looks downwards.* If the angle equals 0, the vector is called *horizontal.*

Consider a star with N $(N = 4, 6, 8, \ldots)$ rays lying in the plane $z = 0$ and centered at O. Denote its vertices by $\{A'_i\}_{i=1}^{N}$. Let $A'_i = (x_i, y_i, 0)$.

Assuming that ϵ is small (its precise value is discussed later), introduce a new collection of points:

For odd $N/2$, put $A_i = (x_i, y_i, (-1)^i\epsilon)$, $i = 1, \ldots, N$.

For even $N/2$, put $A_i = (x_i, y_i, (-1)^i\epsilon), \overline{A}_i = (x_i, y_i, (-1)^{i+1}\epsilon)$, $i = 1, \ldots, N.$.

Remark 5.2. Here and in the sequel, given an ordered set of any objects $\{X_i\}_{i=1}^{M}$, we assume that for any $k \in \mathbb{Z}$, we have $X_k = X_i$ if $k \equiv i(mod M)$.

Consider now two collections of oriented triangles which will form the central part of the desired virtual polytope.

Case 1. $N/2$ is odd. Put

$$J^+(\epsilon) = \{T_i^+\}_{i=1}^{N} = \{OA_iA_{i+1}\}_{i=1}^{N}$$

and

$$J^-(\epsilon) = \{T_i^-\}_{i=1}^{N} = \{A_iOA_{i+1}\}_{i=1}^{N}.$$

These collections differ only by orientation: the normal vectors of J^+ look upwards while the normal vectors of J^- look downwards.

Remark 5.3. We indicate an orientation of a triangle in two ways:

by indicating its normal vector ξ or by the order of its vertices.

Case 2. $N/2$ is even. Put

$$J^+(\epsilon) = \{T_i^+\}_{i=1}^{N} = \{OA_iA_{i+1}\}_{i=1}^{N}$$

and

$$J^-(\epsilon) = \{T_i^-\}_{i=1}^{N} = \{\overline{A}_iO\overline{A}_{i+1}\}_{i=1}^{N}.$$

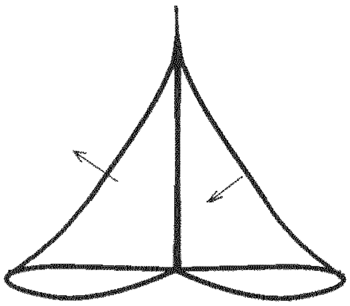
As in the previous case, the normal vectors of the first (respectively, second) collection look upwards (respectively, downwards).

Discrete cross-cap. A general construction.

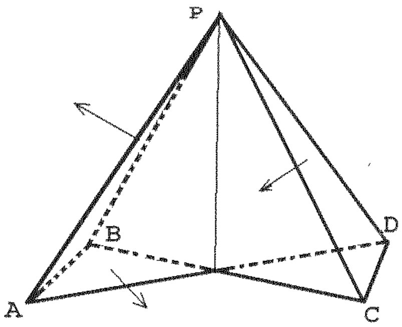
A *smooth cross-cap* is a smooth oriented self-intersecting saddle surface with one singular point which looks like the one in fig. 5.4.

Smooth cross-caps with *infinite horn* (in terminology of [B], when the singular point is on infinity) were investigated by A.Verner [V]. Cross-caps with finite hornswere used by Martinez-Maure as a crucial tool for constructing his counterexample.

A *discrete cross-cap* consists of four oriented triangles $\{APD, PBC, DPC, BPA\}$ (figure 5.4.).We say that the cross-cap is *based* on the rectangle $ABCD$.



A smooth cross-cap



A discrete cross-cap

Construction of N cross-caps glued to the central parts.

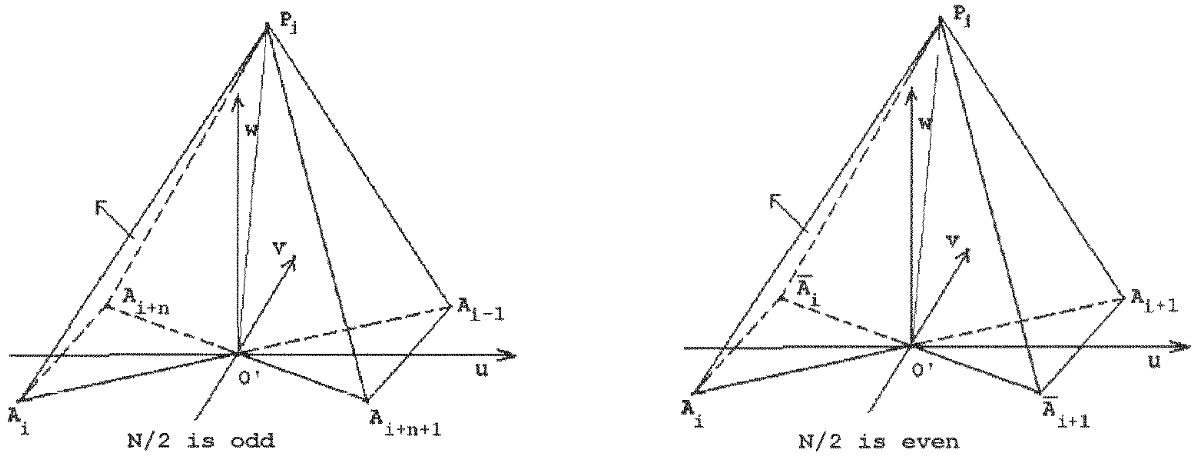
Here we construct N cross-caps based on the rectangles $A_iA_{i+N/2}A_{i+1}A_{i+N/2+1}$ for odd $N/2$ and on the rectangles $A_i\overline{A}_iA_{i+1}\overline{A}_{i+1}$ for even $N/2$.

Case 1. $N/2$ is odd.

Denote $n = N/2$. Choose another Cartesian coordinate system (O', u, v, w) related to the rectangle $A_iA_{i+n}A_{i+1}A_{i+n+1}$. That is, $O' = [A_iA_{i+1}] \cap [A_{i+n}A_{i+n+1}]$. The plane $w = 0$ coincides with the $aff(A_iA_{i+n}A_{i+1}A_{i+n+1})$. The direction of the axes (w) is choosen to make $w(O) \leq 0$. The axes (v) is codirected with the vector $\overrightarrow{A_iA_{i+n}}$. The direction of (u) is codirected with the vector $(-1)^i\overrightarrow{A_iA_{i+n+1}}$.

Let $P_i = P_i(l, \delta) = (o, \delta, l)$. ($l$ is assumed to be big, but the precise values of l and δ will be specified later.) The required cross-cap $C_i = C_i(\delta, l)$ is based on the rectangle $A_iA_{i+1}A_{i+n}A_{i+n+1}$. Namely,

$$C_i = \{A_iA_{i+n}P_i,\ A_{i+1}A_iP_i,\ A_{i+n+1}P_iA_{i+n},\ A_{i+n+1}A_{i+1}P_i\} = \{S_i,\ R_i^+,\ R_i^-,\ S_i'\}.$$



Case 2. $N/2$ **is even.**

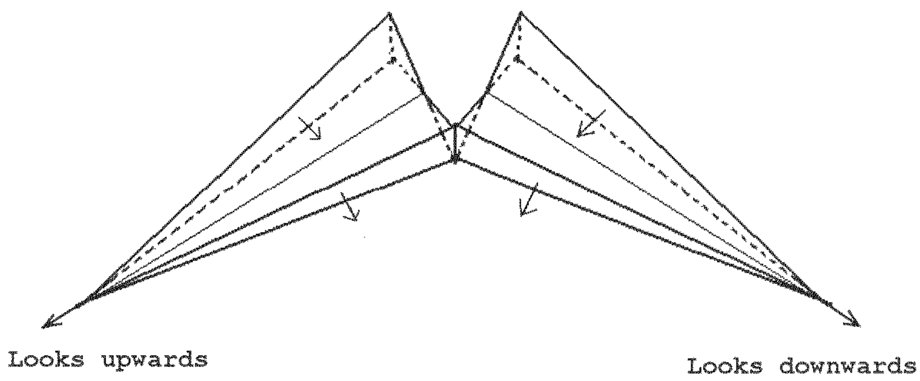
Choose a Cartesian coordinate system $(O'u, v, w)$ related to the rectangle $A_iA_{i+1}\overline{A}_i\overline{A}_{i+1}$. That is, $O' = [A_iA_{i+1}] \cap [\overline{A}_i\overline{A}_{i+1}]$. The plane $w = 0$ coincides with the $aff(A_iA_{i+1}\overline{A}_i\overline{A}_{i+1})$. The direction of the axes (w) is choosen to make $w(O) \leq 0$. The axes (v) is codirected with the vector $\overrightarrow{A_i\overline{A}_i}$. The direction of (u) is codirected with the vector $(-1)^i\overrightarrow{A_i\overline{A}_{i+1}}$.

Let $P_i = P_i(l, \delta) = (o, \delta, l)$. (Again, l is assumed to be big.) Consider the cross-cap $C_i = C_i(\delta, l)$, based on the rectangle $A_iA_{i+1}\overline{A}_i\overline{A}_{i+1}$.

Namely,

$$C_i = \{A_i\overline{A}_iP_i,\ A_{i+1}A_iP_i,\ \overline{A}_{i+1}P_i\overline{A}_i,\ \overline{A}_{i+1}A_{i+1}P_i\} = \{S_i,\ R_i^+,\ R_i^-,\ S_i'\}.$$

The figure presents two sequential cross-caps. The vectors $\overrightarrow{O'P_i}$ look in turn upwards and downwards.



Note that the normal vectors of R_i^+ look upwards (with respect to the old coordinate system (x, y, z)), while the normal vectors of R_i^- look downwards. The normal vectors of triangles S_i and S_i' are horizontal. The triangles S_i and S_{i+1}' have a mutual edge.

Now consider the union of all constructed above collections of triangles

$$\{T_i^+\}_{i=1}^N \cup \{T_i^-\}_{i=1}^N \cup \bigcup_{i=1}^N C_i.$$

After specifying below the pairs of adjacent triangles, we obtain an embedded triangle complex which is denoted by $\mathcal{C}$.

T_i^+ is adjacent to T_{i+1}^+, T_{i-1}^+ and R_i^+;

T_i^- is adjacent to T_{i+1}^-, T_{i-1}^- and R_i^-;

R_i^+ is adjacent to S_i, S_i' and T_i^+;

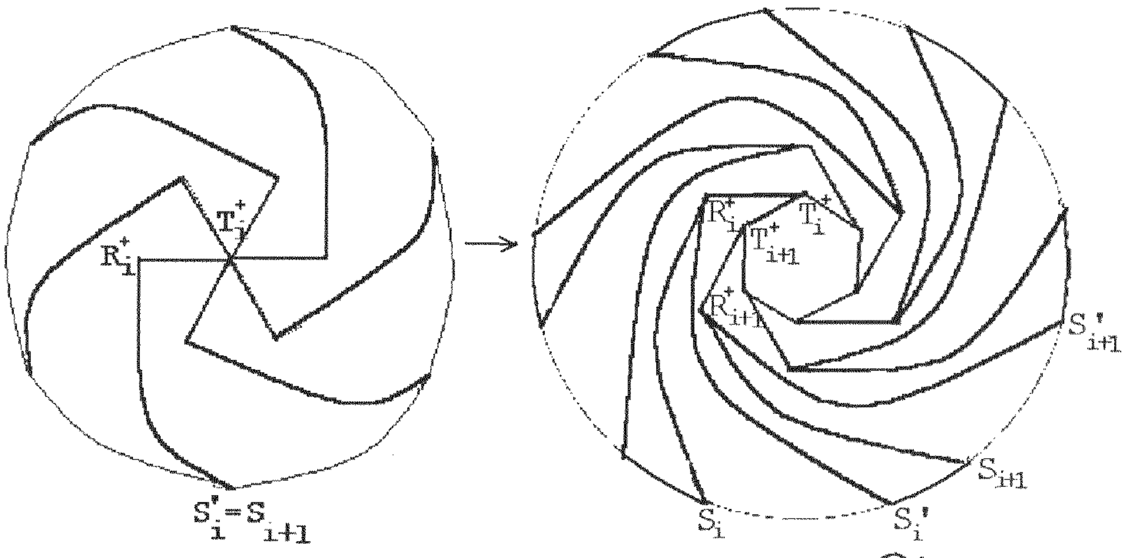
R_i^- is adjacent to S_i, S_i' and T_i^-;

S_i is adjacent to R_i^+, S_{i-1}' and R_i^-;

S_i' is adjacent to R_i^+, S_{i+1} and R_i^-.

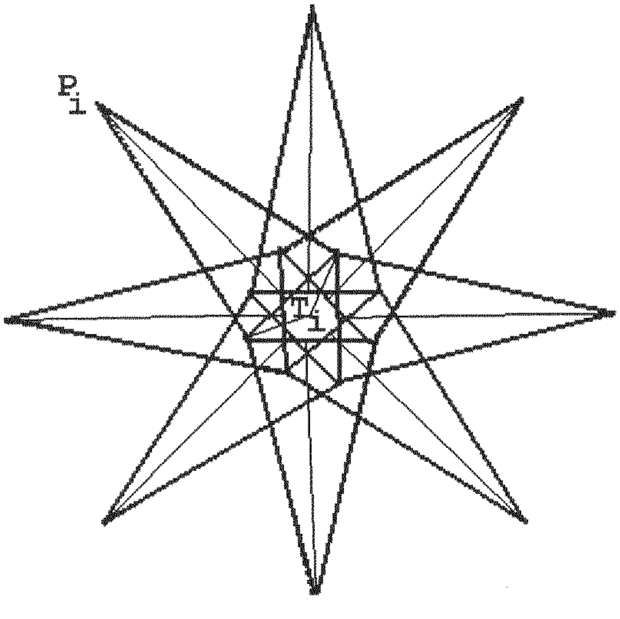
Theorem 5.7. *For an appropriate choice of ϵ, δ and l, the complex $\mathcal{C}$ generates (according to theorem 3.7) a hyperbolic virtual polytope $\mathcal{K}$ with N discrete cross-caps. For $N = 4$, the polytope coincides with the example constructed by Martinez-Maure in [M-M3].*

Proof. We have already specified the orientations of the triangles. Their spherical images (which are denoted by the same letters as the triangles) can be connected to make a fan (see figure 5.8, right part). The opposite side of the fan is symmetric to this one with respect to the origin O. The obtained virtual polytope is hyperbolic due to the proposition 4.6, since each its vertex is obviously non-convex. The only thing which remains to prove is that for a proper choice of ϵ, δ and l, the geodesic lines connecting the vertices of the fan do not intersect (except for the vertices). It is demonsrated through the pair of fans (figure 5.8). The first fan reflects the limit situation when $\epsilon \to 0$ and $l \to \infty$ (all the points T_i^+ coincide, and $S_i = S_i'$. Besides, the angles between T_iR_i and R_iS_i equal $\pi/2$.) Its edges obviouly don't intersect. Therefore, it is possible to move a little the vertices of the first fan and get a fan of the required type (the second one). The values of ϵ, δ and l can be easily restored by the fan. $\square$



The fan of $\mathcal{K}$

From above, the polytope $\mathcal{K}$ looks like in figure 5.9.



HYPERBOLIC SMOOTHING

In the section, we construct a C^∞-smooth hyperbolic approximation of the support function $h = h_{\mathcal{K}}$ of the virtual polytope $\mathcal{K}$.

We construct mutually consistent approximations of surfaces $F_K(\xi)$ for some vectors ξ (steps 1,2) and bring all these approximations together (step 3).

Thus we prove the central theorem of the paper.

Theorem 6.1. *For any even $N = 4, 6, 8...$, there exists a hyperbolic hérisson such that*

- *its support function is C^∞-smooth;*
- *it contains N cross-caps. Due to the trick of Martinez–Maure and Pogorelov, each such hérisson generates a counterexample to the hypothesis.*

Proof. When approximating the below polyhedral surfaces, we don't change them in the points lying far from the edges. Along the edges (but far from the vertices) we replace the surface either by a cylinder or by a cone.

A *cylinder* is a set of points which is invariant under all translations parallel to some line l.

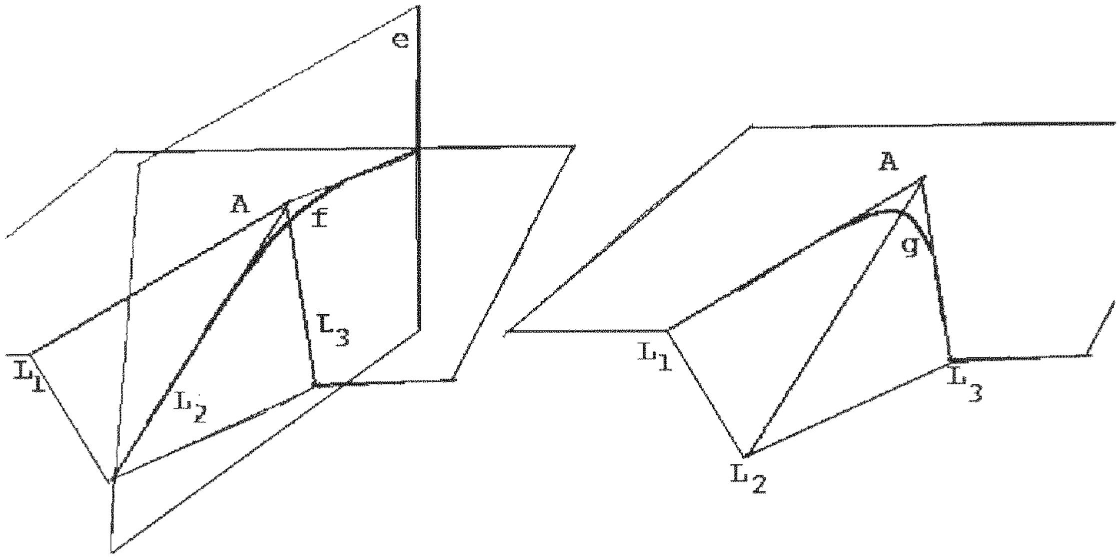
A *cone* with a vertex A is a set of points which is invariant under homotheties with the center in A.

Step 1.

Let $F = F_\xi$ for $\xi = (0, 0, \pm 1)$. Consider a big neigborhood of a vertex A of the surface F. Denote the adjacent edges by L_1, L_2, and L_3, as is shown in figure 6.2. Let e be the plane containing L_2 and orthogonal to $aff(L_1 \cup L_3)$. Introduce two planar C^∞-smooth convex curves $f = f(A)$ and $g = g(A)$ such that:

f is inscribed in $e \cap F$ and coincides with $e \cap F$ outside a small neighbourhood of A;

g is inscribed in $(L_1 \cup L_2)$ and coincides with $(L_1 \cup L_2)$ outside a small neighbourhood of A (fig. 6.2).



In the considered big neighbourhood, we have

$F = \{u(e \cap F)|\ u$ is a translation such that $u(A) \in L_1 \cup L_2\}$.

Now replace F (in this big neighbourhood) by the smooth surface

$\overline{F} = \{u(f)|\ u$ is a translation such that $u(A) \in g\}$.

The edges L_1, L_2 and L_3 are replaced by cylinders c_1, c_2 and c_3. The cylinders c_1 and c_3 are determined by the choice of f. The cylinder c_2 is determined by the choice of g.

When constructing a global approximation of F, we must take in account the dependence between g and f for different vertices: for adjacent vertices $A_{1,2}$ we must get the same cylinders approximating the edge A_1A_2.

The following simple scheme shows that there exists a global choice of g and f for all vertices which yields a global approximation of F and demonstrates the freedom of choice.

1.We may choose independently g and f for all R_i^+.

2. Now, $f(T_i)$ is determined by $f(R_i^+)$ and $g(T_i)$ is determined by $f(R_{i+1}^+)$.

This gives us an approximation of h everywhere on the sphere except for a neighborhood of the horizontal circle. Moreover, we are free to choose independently any (sufficiently narrow) cylinders which replace $S_iR_i^\pm$ and $S_i'R_i^\pm$.

Step 2.

Consider a horizontal vector ξ such that the semisphere with the pole ξ contains the edge $S_i'S_{i+1}$ for some i. Consider the surface $F = F(\xi)$ in a big neighborhood of the point S_i' . Denote the adjacent edges by L_1, L_2 and L_3. We have already a smooth approximation of F along the edges L_1 and L_2, which comes from step 1, but dislike the previous case, the edges are approximated by cones with vertices in the point S_i'. Thus the parallel sections of the approximating surface change. (Being the sections of a cone, they all are homothetic.)

Analogously to the previous step, consider the plane e containing L_2 and orthogonal to $aff(L_1 \cup L_3)$. In the previous step, the parallel to e sections of F were mutually equal. This time they will be mutually homothetic. The homothety coefficient will be ruled by the below function k which is linear beyond a small neighborhood of S_i'. We replace F (in the big neighborhood) by the smooth surface

$\overline{F} = \{\Lambda_k(u_X(f))|\ u \text{ is a translation such that } u(S_i') \in g\}$.

Λ_k is a homothety with the origin $u(S_i')$ with coefficient k.

g is a smooth curve inscribed in $L_1 \cup L_2$ which coincides with the graph of the function k.

Here is the precise construction.

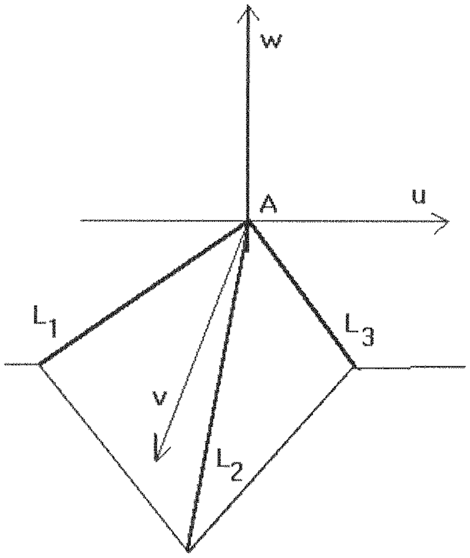
Assume for simplicity that in a coordinate system (u, v, w) (figure 6.3), we have

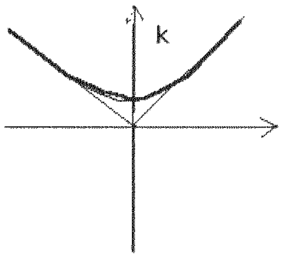
$$aff(L_1, L_3) \text{ is the plane } w = 0;$$

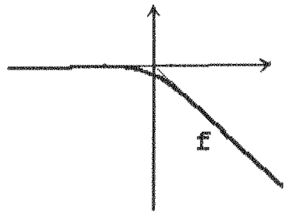
$$S_i' = (0, 0, 0);$$

$$L_1 \cup L_3 = \{(u, v, 0)|\ |u| = v\};$$

$$L_2 = \{(0, v, w)|\ 0 \le v = -w\}.$$







Introduce two C^∞-smooth functions f and k such that the convex graph of k is inscribed in the graph of $v = |u|$.

The convex graph of f is inscribed in the graph of the function $v = -\frac{u+|u|}{2}$.

Note that for any constant C, f can be choosen such that $f'' \neq 0$ implies $f \geq Cf'$.

Put

$$\mathcal{F}(u, v) = k(u) f\left(\frac{v}{k(u)} - 1\right).$$

The graph of the function $\mathcal{F}$ is the desired smooth hyperbolic local approximation. The edges L_1 and L_3 are approximated by cones (determined by the choice of f), while L_2 is approximated by a cylinder (determined by the choice of k).

Ordinary calculations that check hyperbolicity show that

$$det \begin{pmatrix} \mathcal{F}''_{uu}(u, v) & \mathcal{F}''_{uv}(u, v) \\ \mathcal{F}''_{vu}(u, v) & \mathcal{F}''_{vv}(u, v) \end{pmatrix} =$$

$$= \frac{k''(u)}{k^2(u)} f''\left(\frac{v}{k(u)} - 1\right) \left[k(u) f\left(\frac{v}{k(u)} - 1\right) - f'\left(\frac{v}{k(u)} - 1\right) v\right] \leq 0.$$

Indeed, if $f''\left(\frac{v}{k(u)} - 1\right) = 0$, the inequality is obviously valid.

If $f''\left(\frac{v}{k(u)} - 1\right) \neq 0$, and $k'' \neq 0$, then k and v are restricted from above and below by some positive numbers. Therefore, for a proper choice of f, we have

$$k(u) f\left(\frac{v}{k(u)} - 1\right) - f'\left(\frac{v}{k(u)} - 1\right) v \geq 0,$$

which together with $k''(u) f''\left(\frac{v}{k(u)} - 1\right) \leq 0$ implies the required inequality.

Analogously to the previous step, the function k determines the cylinder which approximates the edge L_2. Construct now an approximation of the surface in a neighborhood of S_{i+1} with the same cylinder for L_2.

Step 3. Consider a narrow belt about the horizontal big circle $S^1 \subset S^2$. According step 2, construct a smooth hyperbolic approximation of $h_{\mathcal{K}}$ in the belt such that the edges $R_i^{\pm} S_i$ and $R_i^{\pm} S_i'$ are approximated by cones. Passing to $F = F_{\mathcal{K}}((0,0,\pm 1))$, we get cylindric approximations of the edges $R_i^{\pm} S_i$ and $R_i^{\pm} S_i'$. According to step 1, this can be extended to a global approximation of F, which yields a global smooth hyperbolic approximation of $h_{\mathcal{K}}$. $\square$

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