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EXPANSIONS OF THE L^2 SPACES OVER GENERAL LÉVY PROCESSES, AND FOCK FACTORIZATIONS

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Abstract

We construct explicitly and study the isometry between the spaces of square integrable functionals of an arbitrary Lévy process and a vector-valued Gaussian white noise. In particular, we obtain explicit formulas for this isometry at the level of multiplicative functionals and orthogonal decompositions, and find its kernel. The key role in our considerations is played by the notion of measure and Hilbert factorizations and related notions of multiplicative and additive functionals and logarithm. The obtained results allow us to introduce a canonical Fock structure (an analogue of the Wiener-Itô decomposition) in the L^2 space over an arbitrary Lévy process.

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Contents

1 Introduction: the problem and main results

In this paper we prove that there is a natural isometry between the Hilbert spaces of square integrable functionals of two arbitrary random processes with independent values (Lévy processes) with the same number of points in the support of the Lévy measure and find explicit formulas for this isometry. More exactly, we describe the isometry between the L^2 space over an arbitrary Lévy process and the L^2 space over the (multidimensional) Gaussian white noise. The isometry is natural in the sense that the image of a functional depending only on the restriction of the first process to a certain subset of the base space also depends only on the restriction of the second process to the same subset; in other words, the isometry preserves the so-called *Hilbert factorization* determined by the process (see Sect. 2.1). The existence of such an isometry means, in particular, that *the factorization determined by any Lévy process is isomorphic to the factorization generated by the (vector-valued) white noise*, or, in other words, that the L^2 space over any Lévy process has a *canonical Fock space structure*, i.e., the orthogonal decomposition into subspaces similar to the Wiener–Itô subspaces ("chaoses").

Though there are numerous (mostly technical) papers devoted to the construction of stochastic integrals and analogues of the Wiener–Itô decomposition for Lévy processes, the important fact of the isomorphism of factorizations remained up to now in the background. It is closely related to the remarkable Araki–Woods theorem [2] (though does not follow from it) which claims that a factorization having sufficiently many multiplicative vectors is isomorphic to a Fock factorization, see [2, 17, 4]. In probabilistic terms this condition means that the existence of the canonical Fock (or Wiener–Itô) structure in the L^2 space over any Lévy process follows from the totality of the set of multiplicative functionals of the process. Recall that a subset of a Hilbert space is called total if its linear span is everywhere dense, and that a functional of a process defined on an interval (set) X is called multiplicative if its value at an arbitrary realization of the process is equal to the product of its values at the restrictions of this realization to subsets forming an arbitrary finite partition of the whole interval (set) X (see Sect. 2.1 for a precise definition). A multiplicative functional of a Gaussian process is the exponential of a linear functional, however for a general Lévy process the set of multiplicative functionals is much wider (see Sect. 3.3).

The simplest example of the isometry under consideration is the isometry between the L^2 spaces over Poisson and Wiener processes. An analogy between the orthogonal structures in these spaces was observed in many papers; however the underlying isometry was not expicitly stated even in this case. The existence of this isometry was originally deduced from the equivalence of two realizations of the canonical representation of the groups of diffeomorphisms [21]; later it was studied in [8] in terms of the so-called holomorphic model of the Fock space (i.e., the Fock space realized as the Hilbert space of holomorphic functionals) which is more popular among physicists. Final explicit formulas for the isometry between the L^2 spaces over Poisson and Gaussian processes are apparently new.

The general case of the isometry between an arbitrary Lévy process and a Gaussian process of an appropriate dimension can be reduced to the above-mentioned Poisson–Gauss case by means of the Lévy–Khintchin decomposition, or Poissonian construction of Lévy processes. In this case we need to consider vector-valued white noises. Namely, the L^2 space over the Lévy process with Lévy measure Λ is naturally isometric to the L^2 space over the $L^2(\Lambda)$-valued white noise. In particular, in the case of a Poisson process this space is one-dimensional, hence the Poissonian factorization is isomorphic to the factorization generated by the ordinary (one-dimensional) Wiener process.

Explicit construction of the Fock–Wiener–Itô structure in the L^2 space over an arbitrary Lévy process, i.e., the decomposition of this space into an orthogonal sum of symmetric tensor powers of the first "chaos", reduces to calculating the *"logarithm" of multiplicative functionals*, which yields additive functionals of the process, i.e., the space of the first chaos. However, the rule for calculating this "logarithm" substantially depends on the factorization, and in general it does not coincide with the ordinary logarithm (the coincidence takes place only for the Gaussian white noise; in this case the space of additive functionals coincides with the space of linear functionals). The existence of this "logarithm" for general Lévy processes is not obvious; the proof of the Araki–Woods theorem consists essentially in construction of this logarithm (in a slightly more general context); a more explicit version of this construction can be found in [17, Appendix A]. Below (Sect. 3.3) we calculate the logarithm for the Gaussian and Poisson cases (which, as we have mentioned above, exhaust the general case of Lévy processes).

To determine the isometry it suffices to establish a correspondence between the sets of multiplicative functionals, or between the linear subspaces of additive functionals ("first chaoses"); if we fix a bijection between the canonical bases in these subspaces, then the isometry is unique. The isometry can be also defined by a kernel, i.e., a generalized function in realizations of the two processes. This kernel for the Poisson–Gauss case is computed in Sect. 3.5; it is defined in purely combinatorial terms. The correspondence of chaoses in this case reduces to a correspondence between Hermite and Charlier functionals. The kernel is not positive, hence the isometry is neither Markovian, nor multiplicative operator.

Thus the only Hilbert invariant (i.e., invariant up to arbitrary isometries of the Hilbert space) of factorizations arising in the theory of Lévy processes is the dimension of the space $L^2(\Lambda)$, i.e., the number of points in the support of the Lévy measure. For example, the stable and gamma processes generate the same factorization as a Wiener process with values in an infinite-dimensional Hilbert space. At the same time, the metric invariants of factorizations generated by Lévy processes, i.e., invariants up to measure-preserving transformations, are much more detailed; namely, two measure factorizations generated by Lévy processes with Lévy measures Λ_1 and Λ_2 are isomorphic if and only if $(\mathbb{R}, \Lambda_1)$ and $(\mathbb{R}, \Lambda_2)$ are isomorphic as measure spaces, see [4]. The metric classification of processes plays an important role in the theory of decreasing families of σ-fields (filtrations).

The established Fock structure in the L^2 spaces over Lévy processes and the isometry between these spaces and the L^2 spaces over Gaussian processes make unnecessary numerous special constructions of orthogonal decompositions (stochastic integrals) in each special case. The key point here is the notion of factorization which was overlooked by the researchers working in this field. Another important detail is that it does not make any difference over which base space the process is considered: it may be an arbitrary measure space, rather than an interval or the line as usual. In particular, the isometry under consideration applies to random fields. The only advantage of the one-dimensional situation is that in this case one may argue in terms of processes with independent increments, rather than independent values, and consider the Wiener process instead of the white noise, which is sometimes more convenient. However, we consider Lévy processes over an arbitrary base space.

One of the most important applications of isometries between the Hilbert spaces of functionals over various Lévy processes is the theory of representations of infinite-dimensional groups, namely, of current and gauge groups, and of Kac–Moody algebras. Unitary representations of these groups naturally generate Hilbert factorizations, and they are usually realized in the Fock space. The realizations of the Fock space as the L^2 space over the white noise or as the space of holomorphic functions in infinitely many variables are only two of possible models. If we fix an arbitrary commutative subgroup (subalgebra) of the infinite-dimensional group (algebra), and construct the representation of this algebra where this subgroup (subalgebra) is diagonalized, then we obtain immediately one of these models. For instance, the commutative model of the canonical representation of the group of $\mathrm{SL}(2, \mathbb{R})$-currents (see [19]) with respect to the subgroup of unipotent matrices yields the isometry of the L^2 spaces over the infinite-dimensional white noise and the gamma process (see [5], and also §5). In particluar, this observation led to discovering new symmetry properties of the gamma process. This example is considered in detail in [15].

It should be mentioned that not only vector-valued Lévy processes, but also processes with independent values in Lie groups and even in infinite-dimensional groups generate Fock-type factorizations. More precisely, it is proved ([16, Theorem 1.6]) that the factorizations generated by Gaussian processes (white noises) with values in the infinite-dimensional unitary group and its continuous subgroups (in particular, in locally compact separable groups) are of Fock type; the question for general groups is still open. At the same time, in [17] there were discovered non-Fock metric factorizations, in dimensions 0 and 1, which have no nonconstant multiplicative and additive functionals. These factorizations are not generated by any group-valued random processes, they are of more complicated nature (the so-called “black noise”).

In conclusion we would like to say some words on the combinatorial and analytical aspects of the problem. The isometry under consideration for a general Lévy process with Lévy measure Λ can be stated in a very explicit form if the moment problem for the measure $t^2 d\Lambda(t)$ is definite. In this case we can consider the basis of orthogonal polynomials in $L^2(\mathbb{R}_+, t^2 d\Lambda(t))$, and obtain interesting relations between orthogonal polynomials with respect to various measures on $\mathbb{R}$ (see, for instance, (14)). The most important role is played by the Hermite and Charlier polynomials. Combinatorial aspects of stochastic integrals for a wide class of processes were considered in [11].

The paper is organized as follows.

§2 contains the necessary background on factorizations and processes with independent values. We also give the definition of the logarithm determined by a factorization. In §3 we consider the fundamental special case — the canonical isometry between the spaces of square integrable functionals over Poisson and Gaussian processes. For both processes we compute all the above-mentioned characteristics (multiplicative and additive functionals, logarithm, orthogonal decomposition). As model cases, we consider the finite-dimensional analogues of the processes which correspond to a finite base space. We would like to draw the reader's attention to formula (14) for the classical Hermite and Charlier orthogonal polynomials, which is apparently new. Formula (27) for the kernel is doubtless new. There are several proofs of this formula including a purely combinatorial one. In §4 we study the isometry for a general Lévy process; this requires no substantially new ideas, since a well-known construction allows one to represent a general Lévy process as a vector-valued Poisson process. Though this representation is well-known, it was not apparently realized that the L^2 space over a general Lévy process with Lévy measure Λ is isometric to the L^2 space over a $L^2(\mathbb{R}_+, \Lambda)$-*valued* Wiener process. Thus all formulas in the general case merely reproduce the corresponding formulas for the Poisson–Gauss case. Finally, in §5 we consider a representation-theoretic application of the isometry between the Fock space and the space of square integrable functionals over the gamma process. In fact, it was this example that gave rise to the series of papers [19, 20, 21, 22, 5, 15] resulting in the present understanding of the whole situation.

2 Basic definitions

2.1 Factorizations

We will consider two types of factorizations: *Hilbert factorizations* and *measure factorizations*.

Given a Hilbert space $\mathcal{H}$, denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$, and by $\mathcal{R}(\mathcal{H})$, the lattice of all von Neumann algebras on $\mathcal{H}$ (recall that the lattice operations in $\mathcal{R}(\mathcal{H})$ are as follows: $R_1 \wedge R_2 = R_1 \cap R_2$, and $R_1 \vee R_2 = (R_1 \cup R_2)''$, where $R' = \{a \in \mathcal{B}(\mathcal{H}) : ar = ra \ \forall r \in R\}$ is the commutant of R). The following definition of a Hilbert factorization goes back to von Neumann.

Definition 1. *A (type I)* Hilbert factorization *of a Hilbert space $\mathcal{H}$ over a Boolean algebra $\mathcal{A}$ is a map $\xi : \mathcal{A} \to \mathcal{R}(\mathcal{H})$ such that each $\xi(A)$ is a type I factor, and*

- $\xi(A_1 \wedge A_2) = \xi(A_1) \wedge \xi(A_2)$;
- $\xi(A_1 \vee A_2) = \xi(A_1) \vee \xi(A_2)$;
- $\xi(A') = \xi(A)'$;
- $\xi(0_{\mathcal{A}}) = \{\alpha \cdot \mathrm{Id}_{\mathcal{H}},\ \alpha \in \mathbb{C}\} = 1_{\mathcal{H}}$[1], *where* $\mathrm{Id}_{\mathcal{H}}$ *is the identity operator in* $\mathcal{H}$*;*

[1]The algebra $\{\alpha \cdot \mathrm{Id}_{\mathcal{H}},\ \alpha \in \mathbb{C}\}$ is traditionally denoted by $1_{\mathcal{H}}$, though it is the *zero* of the lattice of von Neumann algebras in $\mathcal{H}$.

- $\xi(1_{\mathcal{A}}) = \mathcal{B}(\mathcal{H})$

for all $A, A_1, A_2, \ldots \in \mathcal{A}$. A factorized Hilbert space $(\mathcal{H}, \xi)$ *is a Hilbert space $\mathcal{H}$ equipped with a Hilbert factorization ξ of its operator algebra.*

The notion of a measure factorization was introduced by Feldman [4]. We follow the presentation of [17].

Definition 2. *Let $(\Omega, \mathfrak{A}, \mathbb{P})$ be a probability space. Denote by $\Sigma(\mathbb{P})$ the complete lattice of all sub-σ-fields of $\mathfrak{A}$ containing all negligible sets. A* measure factorization *of $(\Omega, \mathfrak{A}, \mathbb{P})$ over a Boolean algebra $\mathcal{A}$ is a map $\zeta : \mathcal{A} \to \Sigma(\mathbb{P})$ such that for all $A, A_1, A_2, \ldots \in \mathcal{A}$ the following conditions (not independent) hold:*

- $\zeta(A_1 \wedge A_2) = \zeta(A_1) \wedge \zeta(A_2)$;
- $\zeta(A_1 \vee A_2) = \zeta(A_1) \vee \zeta(A_2)$;
- *if $A_1 \wedge A_2 = 0$, then $\zeta(A_1)$ and $\zeta(A_2)$ are independent;*
- $\zeta(0_{\mathcal{A}}) = \mathfrak{A}_0$ *(the trivial σ-field),* $\zeta(1_{\mathcal{A}}) = \mathfrak{A}$;
- *if $A_1 \wedge A_2 \wedge \ldots = 0$ and $A_1 \vee A_2 \vee \ldots = A$, then $\zeta(A) = \zeta(A_1) \vee \zeta(A_2) \vee \ldots$.*

A factorized measure space $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ *is a probability space equipped with a measure factorization ζ over some Boolean algebra $\mathcal{A}$.*

The most important examples are factorizations over the Boolean algebra $\mathfrak{B}$ of Borel sets of a standard Borel space $(X, \mathfrak{B})$, and over the algebra of mod0 classes of measurable subsets of a Lebesgue space (X, ν). In this paper we consider only factorizations of these types. Moreover, in all our examples, a factorization over a Borel space can be extended to a factorization over the corresponding Lebesgue space (in fact, this is a consequence of the fact that all factorizations considered in this paper turn out to be Fock factorizations, see below).

Definition 3. *1) Two* factorized Hilbert spaces $(\mathcal{H}_1, \xi_1)$ *and* $(\mathcal{H}_2, \xi_2)$ *over Boolean algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, are called* isomorphic *if there exists an isomorphism of Boolean algebras $S : \mathcal{A}_1 \to \mathcal{A}_2$ and an isometry of the Hilbert spaces $T : \mathcal{H}_1 \to \mathcal{H}_2$ such that the following diagram (where $\bar{T}$ is the operator from $\mathcal{R}(\mathcal{H}_1)$ to $\mathcal{R}(\mathcal{H}_2)$ generated by T) is commutative:*

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{S} & \mathcal{A}_2 \\ \downarrow{\scriptstyle \xi_1} & & \downarrow{\scriptstyle \xi_2} \\ \mathcal{R}(\mathcal{H}_1) & \xrightarrow{\bar{T}} & \mathcal{R}(\mathcal{H}_2). \end{array}$$

2) In a similar way, two factorized measure spaces $(\Omega_1, \mathfrak{A}_1, \mathbb{P}_1, \zeta_1)$ *and* $(\Omega_2, \mathfrak{A}_2, \mathbb{P}_2, \zeta_2)$ *over Boolean algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, are called* isomorphic *if there exists an isomorphism of Boolean algebras $S : \mathcal{A}_1 \to \mathcal{A}_2$ and an isomorphism of measure spaces $T : (\Omega_1, \mathfrak{A}_1, \mathbb{P}_1) \to (\Omega_2, \mathfrak{A}_2, \mathbb{P}_2)$ such that the following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{S} & \mathcal{A}_2 \\ \downarrow{\scriptstyle \zeta_1} & & \downarrow{\scriptstyle \zeta_2} \\ \Sigma(\mathbb{P}_1) & \xrightarrow{T} & \Sigma(\mathbb{P}_2). \end{array}$$

3) If two factorizations (Hilbert or measure) are defined over the same Boolean algebra $\mathcal{A}$, and S is the identity automorphism, then the corresponding isomorphism T is called special.

The following lemma is obvious.

Lemma 1. *Each measure factorization $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ over a Boolean algebra $\mathcal{A}$ generates a Hilbert factorization in $\mathcal{H} = L^2(\Omega, \mathbb{P})$ over the same Boolean algebra $\mathcal{A}$. Namely, for each $A \in \mathcal{A}$, let $H_A = L^2(\Omega, \zeta(A), \mathbb{P}|_{\zeta(A)}) \subset L^2(\Omega, \mathfrak{A}, \mathbb{P})$. Then the map $\xi : \mathcal{A} \to \mathcal{R}(\mathcal{H})$ given by*

$$\xi(A) = \mathcal{B}(H_A) \otimes 1_{H_{A'}} \tag{1}$$

is a Hilbert factorization in $\mathcal{H}$.

In this paper we deal only with Hilbert factorizations of this type. Note that nonisomorphic measure factorizations may generate isomorphic Hilbert factorizations, since not every isometry of L^2 spaces is generated by some isomorphism of the underlying measure spaces.

Remark. In fact, a measure factorization is a triple $(\xi, \mathcal{Z}, \psi)$, where $\xi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a Hilbert factorization, $\mathcal{Z} \subset \mathcal{B}(\mathcal{H})$ is a maximal commutative subalgebra, and $\psi \in \mathcal{H}$ is a factorizable vector (see Definition 4 below) such that $\mathcal{Z} \cap \xi(A)$ is a maximal commutative subalgebra of $\xi(A)$ for each $A \in \mathcal{A}$, and ψ is $\mathcal{Z}$-cyclic.

The key role in the study of factorizations is played by the notion of multiplicative and additive functionals.

Definition 4. *Let $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ be a factorized measure space over a Boolean algebra $\mathcal{A}$. A measurable function $F : \Omega \to \mathbb{C}$ is called an* additive *(resp.,* multiplicative*)* functional *if for any finite partition $A_1, \ldots, A_n$ of the unity element $1_{\mathcal{A}}$ there exist functions $F_{A_1}, \ldots, F_{A_n} : \Omega \to \mathbb{C}$ such that F_{A_k} is $\zeta(A_k)$-measurable, and $F = F_{A_1} + \ldots + F_{A_k}$ (resp., $F = F_{A_1} \cdot \ldots \cdot F_{A_k}$).*

Given a Hilbert factorization $(\mathcal{H}, \xi)$ over $\mathcal{A}$, a vector $h \in \mathcal{H}$ is called factorizable *if for any finite partition $A_1, \ldots, A_n$ of $1_{\mathcal{A}}$ there exist vectors $h_{A_i} \in H_{A_i}$ such that $h = h_{A_1} \otimes \ldots \otimes h_{A_n}$.*

If a factorized Hilbert space $(L^2(\Omega, \mathbb{P}), \xi)$ is generated by a factorized measure space $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ as in Lemma 1, then the set of factorizable vectors in $L^2(\Omega, \mathbb{P})$ coincides with the set of square integrable multiplicative functionals in $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$. Since in this paper we consider only Hilbert factorizations of this type, we will use the term "multiplicative functionals" for factorizable vectors in $L^2(\Omega, \mathbb{P})$.

2.2 The first examples: Fock factorizations, Gaussian and Poisson processes

Fock spaces and Fock factorizations

The (boson) *Fock space* $\operatorname{EXP} H$ over a Hilbert space H is defined as

$$\operatorname{EXP} H = S^0 H \oplus S^1 H \oplus \ldots \oplus S^n H \oplus \ldots,$$

where $S^n H$ is the nth symmetric tensor power of H. Given $h \in H$, let

$$\operatorname{EXP} h = 1 \oplus h \oplus \frac{1}{\sqrt{2!}} h \otimes h \oplus \frac{1}{\sqrt{3!}} h \otimes h \otimes h \oplus \ldots.$$

The vectors $\{\operatorname{EXP} h\}_{h \in H}$ are linearly independent, and

$$(\operatorname{EXP} h_1, \operatorname{EXP} h_2)_{\operatorname{EXP} H} = \exp(h_1, h_2)_H,$$

where $(\cdot,\cdot)_{\mathrm{EXP}\,H}$ stands for the scalar product in $\mathrm{EXP}\,H$. In particular,

$$\|\,\mathrm{EXP}\,h\|^2 = \exp\|h\|^2.$$

The principal example of a Hilbert factorization is given by the following construction.

Definition 5. *Given a Lebesgue space (X,ν), consider the direct integral of Hilbert spaces $\mathcal{K} = \int^{\oplus} K(x)d\nu(x)$, and the corresponding Fock space $\mathcal{H} = \mathrm{EXP}\,\mathcal{K}$. For each measurable $A \subset X$, let $H_A = \mathrm{EXP}\,\mathcal{K}(A)$, where $\mathcal{K}(A) = \int_A^{\oplus} K(x)d\nu(x)$, and set $\xi(A) = \mathcal{B}(H_A)\otimes 1_{H_{A'}}$. The obtained Hilbert factorization $(\mathcal{H},\xi)$ is called a* Fock factorization.

In particular, if $\dim K(x)\equiv 1$, then $\mathcal{H} = \mathrm{EXP}\,L^2(X,\nu)$, and $H_A = \mathrm{EXP}\,L^2(A,\nu_A)$, where ν_A is the restriction of ν to A. More generally, if $\dim K(x)\equiv n$ $(n = 1,2,\dots,\infty)$, then $\mathcal{H}$ can be identified with $\mathrm{EXP}\,L^2((X,\nu);H)$, where H is a Hilbert space with $\dim H = n$, and $L^2((X,\nu);H)$ is the space of square integrable H-valued functions on (X,ν). The corresponding factorization is called a *homogeneous Fock factorization of dimension n.*

Fock factorizations are characterized by the following well-known theorem.

Theorem 1 (Araki–Woods [2]). *1) A Hilbert factorization $(\mathcal{H},\xi)$ over a nonatomic Boolean algebra $\mathcal{A}$ is a Fock factorization if and only if the set of factorizable vectors is total in $\mathcal{H}$.*

2) The complete invariant of a Fock factorization is its spectrum, i.e., $\{n\in\{0,1,\dots;\infty\} : \nu(\{x : \dim K(x) = n\}) > 0\}$. That is, two Fock factorizations are isomorphic if and only if their spectra coincide.

The total set of multiplicative (factorizable) vectors in the Fock space $\mathcal{H} = \mathrm{EXP}\,\mathcal{K}$ is

$$\mathcal{M} = \{c\cdot\mathrm{EXP}\,h,\ h\in\mathcal{K}, c\in\mathbb{C}\}.$$

On the groups of automorphisms of factorizations

According to Definition 3, the group $\mathrm{AUT}(\mathcal{H},\xi)$ of automorphisms of a factorized Hilbert space $(\mathcal{H},\xi)$ over a Boolean algebra $\mathcal{A}$ consists of isometries T of $\mathcal{H}$ such that $\bar{T}\circ\xi = \xi\circ S$ for some automorphism S of $\mathcal{A}$ (recall that $\bar{T}$ stands for the operator in $\mathcal{R}(\mathcal{H})$ generated by T). The subgroup $\mathrm{SAUT}(\mathcal{H},\xi)$ of *special* automorphisms consists of automorphisms that leave the base unchanged: $\bar{T}\xi(A) = \xi(A)$ for all $A\in\mathcal{A}$ (i.e., S is the identity automorphism of $\mathcal{A}$). The subgroup $\mathrm{SAUT}_\psi(\mathcal{H},\xi)\subset\mathrm{SAUT}(\mathcal{H},\xi)$ consists of special automorphisms of $(\mathcal{H},\xi)$ that leave a multiplicative vector $\psi\in\mathcal{H}$ (the vacuum) unchanged.

Remark. If the Hilbert factorization is generated by a measure factorization as in Lemma 1, then we may consider also the group of automorphisms of the underlying measure factorization, which is obviously a subgroup of $\mathrm{AUT}(\mathcal{H},\xi)$. This group may be different for different measure factorizations generating the same Hilbert factorization. For example, the measure factorization generated by the Gaussian white noise has the greatest group of isomorphisms among the measure factorisations generating the same Fock factorization.

For a Fock factorization, the groups $\mathrm{SAUT}(\mathcal{H},\xi)$ and $\mathrm{SAUT}_\psi(\mathcal{H},\xi)$ were computed in [1].

Proposition 1. *For a homogeneous Fock factorization in $\mathcal{H} = \mathrm{EXP}\,L^2((X,\nu);H)$, the group $\mathrm{AUT}(\mathcal{H})$ of automorphisms consists of operators of the form*

$$\mathrm{EXP}\,h\mapsto e^{ib-\frac{\|\psi\|}{2}-(\psi,Uh(S^{-1}\cdot))}\cdot\mathrm{EXP}(Uh(S^{-1}\cdot)+\psi), \tag{2}$$

where $b \in \mathbb{R}$, $\psi \in L^2((X,\nu);H)$, S is an isomorphism of the measure space (X,ν), and U is a unitary operator in $L^2((X,\nu);H)$ that commutes with all projections P_A to subspaces of the form $L^2((A,\nu_A);H)$, where A is a measurable subset of X, and ν_A is the restriction of ν to A.

The subgroup $\mathrm{SAUT}(\mathcal{H})$ *consists of operators (2) with $S = \mathrm{Id}$ (the identity map in X), and* $\mathrm{SAUT}_1(\mathcal{H})$ *is the subgroup of* $\mathrm{SAUT}(\mathcal{H})$ *with $\psi = 0$.*

Remark. In particular, if $\dim H = 1$, that is, we have a Fock space $\mathcal{H} = \mathrm{EXP}\, L^2(X,\nu)$ of dimension 1, then the Spectral Theorem implies that U is a multiplicator $h(\cdot) \mapsto a(\cdot)h(\cdot)$ by a measurable function $a : X \to \mathbb{C}$ with $|a| \equiv 1$. Thus $\mathrm{AUT}(\mathcal{H})$ is isomorphic to the semidirect product $(\mathrm{Aut}(X,\nu) \ltimes \mathbb{T}^X) \ltimes (L^2(X,\nu) \ltimes \mathbb{T})$, where the semidirect product of the additive group of the Hilbert space $L^2(X,\nu)$ with the unit circle $\mathbb{T} = \{e^{ib},\, b \in \mathbb{R}\}$ is determined by the cocycle $c((\psi_1,b_1),(\psi_2,b_2)) = b_1 + b_2 - \mathrm{Im}(\psi_2,\psi_1)$; $\mathrm{Aut}(X,\nu)$ is the group of isomorphisms of the measure space (X,ν) which acts on $\mathbb{T}^X = \{a : X \to \mathbb{C} : |a(x)| \equiv 1\}$ as $Sa(\cdot) = a(S^{-1}\cdot)$, and the pair (S,a) sends (ψ,b) to $(a(\cdot)\psi(S^{-1}\cdot),b)$.

Gaussian white noise

Let (X,ν) be a standard Borel space with a continuous finite positive measure. Consider the *Gaussian white noise* on the space (X,ν), that is, the standard Gaussian measure μ in the nuclear extension $\hat{H}$ of the Hilbert space $L^2(X,\nu)$ with Fourier transform

$$\mathbb{E}e^{i\langle h,\cdot\rangle} d\mu(\cdot) = e^{-\frac{1}{2}\|h\|^2}, \quad h \in L^2(X,\nu), \tag{3}$$

where $\langle h,\cdot\rangle$ is the linear functional on $\hat{H}$ defined by a vector $h \in L^2(X,\nu)$, and $\|h\|$ is the norm of h in $L^2(X,\nu)$.

It is well-known that the space $L^2(\hat{H},\mu)$ of square integrable functionals of the white noise is canonically isomorphic to the Fock space $\mathrm{EXP}\, H$, where $H = L^2(X,\nu)$. This isomorphism is given by the formula

$$\mathrm{EXP}\, h \leftrightarrow e^{-\frac{\|h\|^2}{2} + \langle h,\cdot\rangle}. \tag{4}$$

In particular, the vacuum vector $\mathrm{EXP}\, 0$ corresponds to the unity function $1 \in L^2(\hat{H},\mu)$, and the n-particle subspace $S^n H$ is identified with the subspace of $L^2(\hat{H},\mu)$ spanned by n-multiple stochastic integrals, i.e., by generalized Hermite functionals of order n, see Sect. 3.4. It is worth mentioning that the structure of a unitary ring in $L^2(\hat{H},\mu)$ is described axiomatically in [18].

The following well-known proposition is a consequence of the isomorphism (4) between $L^2(\hat{H},\mu)$ and $\mathrm{EXP}\, L^2(X,\nu)$.

Proposition 2. *The Gaussian white noise on an arbitrary Lebesgue space (X,ν) generates a Fock factorization.*

In particular, the set of multiplicative functionals in the L^2 space over the Gaussian white noise on (X,ν) is $\{c \cdot e^{\langle h,\cdot\rangle},\ h \in L^2(X,\nu),\ c \in \mathbb{C}\}$, and the set of additive functionals is $\{c \cdot \langle h,\cdot\rangle,\ h \in L^2(X,\nu),\ c \in \mathbb{C}\}$ (see Sect. 3.3).

Poisson process

By definition, a point configuration in X is a (non-ordered) empty, finite, or countable set of points of X with positive (integral) multiplicities. Denote by $\mathcal{E} = \mathcal{E}(X)$ the set of all point configurations on X.

The *point Poisson process* on the space X with mean measure ν (in short, the Poisson process on (X,ν)) is a random configuration $\omega \in \mathcal{E}$ such that

- for each measurable $A \subset X$, $\mathrm{Prob}\{\#\{\omega \cap A\} = n\} = \frac{\nu(A)^n}{n!} e^{-\nu(A)}$;
- for any disjoint measurable subsets $A_1, \ldots, A_n \subset X$, the random variables $\#\{\omega \cap A_k\}$, $k = 1, \ldots, n$, are independent.

Let $\mathcal{P}$ be the distribution of this Poisson process (a probability measure in $\mathcal{E}$).

2.3 The logarithm

In this section we describe the operation of "logarithm", introduced in [17], which allows one to construct the space of additive functionals of a factorized measure space $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$, given the set of multiplicative functionals. For example, if the set of multiplicative functionals is total in $L^2(\Omega, \mathbb{P})$, then, by the Araki–Woods theorem, the corresponding Hilbert factorization of $L^2(\Omega, \mathbb{P})$ is a Fock factorization, and Theorem 2 allows one to construct the first chaos, and hence to recover the whole Fock structure in $L^2(\Omega, \mathbb{P})$ by means of the standard orthogonalization process (see Sect. 3.4). The operation of logarithm depends substantially on the factorization. In general it does not coincide with the ordinary logarithm.

Let $(\Omega, \mathfrak{A}, \mathbb{P}, \zeta)$ be a factorized measure space over a Lebesgue space (X,ν). Denote by $\mathcal{A}$ the space of all square integrable additive functionals, and by $\mathcal{M}$ the space of all square integrable multiplicative functionals on this factorized space. Given $F \in \mathcal{M}$, denote by $F_A(\cdot)$ the $\zeta(A)$-measurable function from the definition of a multiplicative functional (it is defined uniquely up to scalar factor). For each measurable $A \subset X$, let $m_F(A) = \log \mathbb{E}|F_A(\cdot)|^2 - \log |\mathbb{E}F_A(\cdot)|^2$. Then m_F is a nonatomic measure on X ([17, Lemma A2]).

Theorem 2 ([17], Theorem A6). *There is a natural one-to-one correspondence* $\mathrm{LOG}_\zeta : \mathcal{M} \to \mathcal{A}$. *Given* $F \in \mathcal{M}$ *with* $\mathbb{E}F = 1$,

$$\mathrm{LOG}_\zeta F(\cdot) = \lim_{\max m_F(A_k) \to 0} \sum_k (F_{A_k}(\cdot) - 1) \qquad \text{in} \quad L^2(\Omega, \mathbb{P}), \tag{5}$$

where $A_1, \ldots, A_k$ *is a measurable partition of* X, *and each* F_{A_k} *is assumed to be normalized so that* $\mathbb{E}F_{A_k}(\cdot) = 1$.

In Sect. 3.3 we use this theorem to compute additive functionals for Gaussian and Poisson processes.

Note that multiplicative and additive functionals can be defined in an obvious way for general processes. However, if the process is not with independent values, then it does not determine a factorization, so that formula (5) does not make sense, and the logarithm is not defined.

Thus if we have two factorized measure spaces with isomorphic Hilbert factorizations of the corresponding L^2 spaces, then this isomorphism can be determined by indicating the correspondence either between multiplicative functionals, or between additive functionals, or between all chaoses. We will construct all these correspondences for the Poisson–Gauss and Lévy–Gauss isomorphisms.

2.4 General Lévy processes

Let (X,ν) be a metric space with a continuous finite measure ν. Denote by $\mathcal{F}$ the space of bounded continuous functions on X, and by $\mathcal{F}'$, the corresponding space of distributions on $\mathcal{F}$.

Let Π be a Borel measure on $\mathbb{R}$ with $\Pi(\{0\}) = 0$ and $\int_{\mathbb{R}} \frac{t^2}{1+t^2} d\Pi(t) < \infty$. The *homogeneous process with independent values* (without Gaussian component) on the space (X, ν) corresponding to the Lévy–Khintchin measure Π is a generalized process on $\mathcal{F}'$ such that for all $a \in \mathcal{F}$

$$\mathbb{E}e^{i\langle a,\cdot\rangle} = \exp\left(\int_X \log \phi(a(x))d\nu(x)\right),$$

where

$$\log \phi(y) = i\alpha y + \int_{\mathbb{R}} \left(e^{ity} - 1 - \frac{ity}{1+t^2}\right) d\Pi(t).$$

The following lemma is well-known.

Lemma 2. *Let η be an arbitrary process with independent values on a standard Borel space $(X, \mathfrak{B})$ with a continuous finite measure ν. For each $A \in \mathfrak{B}$, let $\zeta(A)$ be the σ-field generated by the restriction of η to A. Then ζ is a measure factorization over the Borel algebra $\mathfrak{B}$ which can be extended to a measure factorization over the algebra of* mod 0 *classes of measurable functions (i.e., over the corresponding Lebesgue space).*

According to general construction of Lemma 1, η defines also a Hilbert factorization ξ in the space of square integrable functionals $\mathcal{H} = L^2(\eta)$ of the process; namely, $\xi(A) = \mathcal{B}(H_A) \otimes 1_{H_{X\setminus A}}$, where H_A is the set of square integrable functionals that depend only on the restriction of η to A.

A very important problem is to classify processes according to the corresponding factorizations. It turns out that all processes with independent values generate Fock factorizations, and the complete invariant of this Fock factorization is the number of points in $\operatorname{supp}\Pi$. We will give a constructive proof of this theorem in a particular case of Lévy processes (Theorem 6). In particular, the Poisson and Gaussian processes generate isomorphic factorizations, since in both cases the support of the Lévy-Khintchin measure Π consists of a single point.

Note that the measure factorizations generated by processes with independent values with the same cardinality of $\operatorname{supp}\Pi$ need not be isomorphic. For example, Poisson and Gaussian processes generate nonisomorphic measure factorizations. The result of Feldman [4] claims that two measure factorizations generated by processes with independent values are isomorphic if and only if the measure spaces $(\mathbb{R}, \Pi_1)$ and $(\mathbb{R}, \Pi_2)$, where Π_1 and Π_2 are the corresponding Lévy–Khintchin measures, are isomorphic.

Lévy processes are a particular case of processes with independent values corresponding to *subordinators* for $X = \mathbb{R}_+$. Let Λ be a measure on $\mathbb{R}_+$ such that

$$\int_0^\infty (1 - e^{-s})d\Lambda(s) < \infty.$$

A Lévy process on (X, ν) with Lévy measure Λ is a generalized process on $\mathcal{F}'$ such that

$$\mathbb{E}e^{-\langle a,\eta\rangle} = \exp\left(\int_X \log \psi_\Lambda(a(x))d\nu(x)\right), \qquad a \in \mathcal{F},\ a \geq 0, \tag{6}$$

where ψ_Λ is the Laplace transform of the infinitely divisible distribution F_Λ with Lévy measure Λ:

$$\psi_\Lambda(t) = \exp\left(-\int_0^\infty (1 - e^{-ts})d\Lambda(s)\right).$$

The following explicit Poissonian construction of Lévy processes (see [7, Ch. 8]) plays a key role in our considerations. Consider a Poisson point process on the space $X \times \mathbb{R}_+$ with mean measure $\nu \times \Lambda$. We associate with a realization $\Pi = \{(X_i, Z_i)\}$ of this process the element

$$\eta = \sum_{(X_i,Z_i)\in\Pi} Z_i \delta_{X_i} \in \mathcal{F}'. \tag{7}$$

Then η is the Lévy process with Lévy measure Λ. In particular, it follows that the law P_Λ of the Lévy process is concentrated on the cone D^+, where

$$D = \left\{ \sum z_i \delta_{x_i},\ x_i \in X,\ z_i \in \mathbb{R}, \sum |z_i| < \infty \right\}$$

is the real linear space of all finite real discrete measures on X, and $D^+ = \{\sum z_i \delta_{x_i} \in D :\ z_i > 0\} \subset D$ is the cone in D consisting of all positive measures.

3 Canonical isometry between the factorizations generated by Gaussian and Poisson processes

In this section we study the isometry between the spaces of square integrable functionals of the Gaussian and Poisson processes with the same base space. Many authors observed that these spaces have many common features. However, the isometry was first established in [21] from representation-theoretic considerations, and explicitly described in [8] as an isometry between the L^2 space over the Poisson process and the holomorphic model of the boson Fock space.

From now on we fix the base space (X, ν) which is a standard Borel space with a finite positive continuous measure ν.

Theorem 3. *There exists a unity-preserving isometry (isomorphism of Hilbert factorizations)*

$$\Phi : L^2(\hat{H}, \mu) \to L^2(\mathcal{E}, \mathcal{P}).$$

On the set of multiplicative functionals it is given by the following formula: for each $h \in L^2(X, \nu)$,

$$\Phi :\ e^{<h,\cdot> - \frac{\|h\|^2}{2}} \mapsto \prod_{x\in\omega} (1 + h(x)) \cdot e^{-\int h(x) d\nu(x)}, \qquad \omega \in \mathcal{E}. \tag{8}$$

This is the unique unity-preserving special real (i.e., preserving the real subspace of real-valued functionals) isomorphism of Hilbert factorizations in $L^2(\hat{H}, \mu)$ and $L^2(\mathcal{E}, \mathcal{P})$.

Proof. Formula (8) follows from the description of the boson–Poisson correspondence, which was given in [8] in terms of the holomorphic model of the Fock space, and formula (4) for the isomorphism between $L^2(\hat{H}, \mu)$ and the boson Fock space. It is also easy to verify directly that the map defined by (8) is an isomorphism with desired properties.

Obviously, any two isomorphisms with desired properties differ from each other by a vacuum-preserving special automorphism of the Fock factorization in $\mathrm{EXP}\, H$, where $H = L^2(X, \nu)$. Thus the uniqueness part follows from Proposition 1. $\square$

Remark. In particular, we obtain that the set of multiplicative functionals in $L^2(\mathcal{E}, \mathcal{P})$ is $\{c \cdot \prod_{x\in\omega}(1 + h(x)),\ h \in L^2(X, \nu)\}$.

Our purpose is to study the isometry (8) in more detail. In particular, we would like to find its *kernel*, that is, a (generalized) function $K(\omega, f)$ on $\mathcal{E} \times \hat{H}$ such that for every $F \in L^2(\mathcal{E}, \mathcal{P})$

$$(\Phi^{-1}F)(\cdot) = \int_{\mathcal{E}} K(\omega, \cdot)F(\omega)d\mathcal{P}(\omega). \tag{9}$$

Given a process η with independent values on (X, ν), a functional $F \in L^2(\eta)$ is called *single-generated* if it depends only on the integral of η over the whole space X. Similarly, F is called *finitely generated* if there is a finite measurable partition $X = A_1 \cup \ldots \cup A_n$ such that F depends only on the integrals of η over $A_1, \ldots, A_n$. Clearly, each factorizable map on $L^2(\eta)$ must preserve the set of n-generated functionals. Moreover, $L^2(\eta)$ is the projective limit of the subspaces of finitely-generated functionals with respect to refinement of partitions.

3.1 The restriction of the canonical isometry onto the subspace of single-generated functionals

Since the isometry (8) preserves the subspaces of single-generated functionals, we may consider its restriction to this subspace. If η is the Gaussian process on the space (X, ν) with $\nu(X) = a$, then $\eta(X)$ is the random Gaussian variable with zero mean and variance a, and the subspace of single-generated functionals is naturally isometric to the L^2 space over the Gaussian process on the space consisting of a single point with weight a. Analogously, the subspace of single-generated functionals of the Poisson process ω on (X, ν) is isomertric to the L^2 space over the Poisson process on the same one-point space. Let us consider this situation in detail.

Let $X = \{x\}$ be a one-point space, and $\nu(X) = a$. Then $\hat{H} = \mathbb{R}$, and μ is the ordinary normal law on $\mathbb{R}$ with zero mean and variance a, so that $L^2(\hat{H}, \mu) = L^2(\mathbb{R}, N(0, a))$.

On the other hand, in this case we have $\mathcal{E} = \mathbb{Z}_+$, and $\mathcal{P}$ is the Poisson distribution P_a on $\mathbb{Z}_+$ with parameter a, so that $L^2(\mathcal{E}, \mathcal{P})$ is the space of sequences $b = \{b_n\}_{n \geq 0}$ with scalar product

$$(b, b') = e^{-a} \sum_{n=0}^{\infty} \frac{a^n}{n!} b_n b'_n.$$

Formula (8) takes the following form: for all $t \in \mathbb{R}$

$$e^{tx - \frac{at^2}{2}} \leftrightarrow \left\{ e^{-at}(1+t)^k \right\}_{k=0}^{\infty}. \tag{10}$$

The left-hand side of (10) is the generating function for the Hermite polynomials $H_n^a(x)$, and the right-hand side, the generating function for the Charlier polynomials $C_n^a(k)$ (see Appendix, (50), (54)). These polynomials constitute orthogonal families in $L^2(\mathbb{R}, N(0, a))$ and $L^2(\mathbb{Z}_+, P_a)$, respectively, and $\|H_n^a\|^2_{L^2(\mathbb{R}, N(0,a))} = \|C_n^a\|^2_{L^2(\mathbb{Z}_+, P_a)} = a^n n!$, hence

$$\Phi(H_n^a) = C_n^a. \tag{11}$$

We see that formula (10) is precisely the expression of (11) in terms of generating functions.

Proposition 3. *The kernel* (9) *of the unitary isomorphism between* $L^2(\mathbb{R}, N(0, a))$ *and* $L^2(\mathbb{Z}_+, P_a)$ *is given by the formula*

$$K^a(k, x) = e^{-\frac{a}{2} - x} \frac{H_k^a(x + 2a)}{a^k}. \tag{12}$$

Proof. Observe that if ν_1 and ν_2 are arbitrary measures on $\mathbb{R}$ with finite moments, and $P_n^{(j)}$ are the orthonormalized polynomials with respect to ν_j, $j = 1, 2$, then the kernel of the unitary isomorphism between $L^2(\mathbb{R}, \nu_1)$ and $L^2(\mathbb{R}, \nu_2)$ is given by the formula

$$K(x, y) = \sum_{n=0}^{\infty} P_n^{(1)}(x) P_n^{(2)}(y),$$

provided that the series converges in $L^2(\mathbb{R}, \nu_1)$ for each y.

Thus let us consider the series

$$\sum_{n=0}^{\infty} \frac{H_n^a(x) C_n^a(k)}{a^n n!}.$$

We would like to prove that this series converges in $L^2(\mathbb{R}, N(0, a))$ for each $k \in \mathbb{Z}_+$. Since H_n^a are orthogonal polynomials in $L^2(\mathbb{R}, N(0, a))$, and $\|H_n^a\|^2 = n! a^n$, it suffices to check that

$$\sum_{n=0}^{\infty} \frac{|C_n^a(k)|^2}{a^n n!} < \infty. \tag{13}$$

But in view of (56), $\frac{|C_n^a(k)|^2}{a^n} = |C_k^a(n)|^2 a^{n-2k} \leq \text{const} \cdot n^{2k} a^{n-2k}$, since C_k^a is a polynomial of degree k, and (13) follows immediately.

Thus we have

$$K(k, x) := K^a(k, x) = \sum_{n=0}^{\infty} \frac{H_n^a(x) C_n^a(k)}{a^n n!}.$$

Consider the generating function

$$\mathcal{K}(t, x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} K(k, x).$$

Changing the order of summation yields

$$\mathcal{K}(t, x) = \sum_{n=0}^{\infty} \frac{H_n^a(x)}{n!} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{C_n^a(k)}{a^n}.$$

In view of (56) and (54), the internal sum equals

$$\sum_{k=0}^{\infty} (-1)^{n+k} \frac{t^k}{a^k k!} C_k^a(n) = (-1)^n e^t \left(1 - \frac{t}{a}\right)^n.$$

Thus

$$\mathcal{K}(t, x) = \sum_{n=0}^{\infty} \frac{H_n^a(x) a^n}{n!} \left(\frac{t}{a} - 1\right)^n e^t = \exp\left(2t + \frac{tx}{a} - x - \frac{t^2}{2a} - \frac{a}{2}\right)$$

which coincides in view of (50) with the generating function for $e^{-\frac{a}{2} - x} \frac{H_k^a(x+2a)}{a^k}$. ∎

Remark. Formula (12) can also be checked directly using a known formula (see, e.g., [12, formula 7.377 of the fourth Russian edition])

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} H_n(x + a) H_m(x + b) dx = m! (-b)^{n-m} L_m^{n-m}(-ab), \quad m \leq n,$$

where $L_n^\alpha(x)$ is a Laguerre polynomial with parameter α, and the formula $C_n^a(x) = n! L_n^{x-n}(a)$ relating the Charlier and Laguerre orthogonal polynomials.

Corollary 1. *We have proved the following identity for the Hermite and Cahrlier orthogonal polynomials*

$$\sum_{n=0}^{\infty} \frac{H_n^a(x) C_n^a(k) a^n}{n!} = e^{-\frac{a}{2}-x} \frac{H_k^a(x+2a)}{a^k}. \tag{14}$$

In particular, denoting $C_n(\cdot) = C_n^1(\cdot)$, we obtain

$$\sum_{n=0}^{\infty} \frac{H_n(x) C_n(k)}{n!} = e^{-\frac{1}{2}-x} H_k(x+2). \tag{15}$$

Thus in the case of a one-point space X we have the description of the Gauss–Poisson isomorphism at three levels:

- correspondence of multiplicative functionals: (10);
- correspondence of orthogonal polynomials: (11);
- explicit kernel: (12).

Our purpose is to obtain the formulas of the second and third level for general spaces.

3.2 The restriction of the canonical isometry to the subspace of finitely-generated functionals

Let us consider a measurable partition $X = A_1 \cup \ldots \cup A_m$ of the space X with $\nu(A_j) = a_j$, and the corresponding subspaces of finitely-generated functionals in $L^2(\hat{H}, \mu)$ and $L^2(\mathcal{E}, \mathcal{P})$. Then we arrive at the following situation.

Let the space $X = \{s_1, \ldots, s_m\}$ consist of m points with measures $\nu(s_j) = a_j$, $j = 1, \ldots, m$. Then

$$\begin{aligned} L^2(\hat{H}, \mu) &= \bigotimes_{j=1}^{m} L^2(\mathbb{R}, N(0, a_j)), \\ L^2(\mathcal{E}, \mathcal{P}) &= \bigotimes_{j=1}^{m} L^2(\mathbb{Z}_+, P_{a_j}), \\ \prod_{j=1}^{m} H_{k_j}^{a_j}(\cdot) &\leftrightarrow \prod_{j=1}^{m} S_{k_j}^{a_j}(\cdot), \end{aligned}$$

and for $k = (k_1, \ldots, k_m) \in \mathbb{Z}_+^n$, and $x = (x_1, \ldots, x_m) \in \mathbb{R}^n$,

$$K(k, x) = \prod_{j=1}^{m} e^{-\frac{a_j}{2} - x_j} \frac{H_{k_j}^{a_j}(x_j + 2a_j)}{a_j^{k_j}}. \tag{16}$$

3.3 Logarithm

In this section we apply the logarithm construction of Theorem 2 to compute the sets of additive functionals for Gaussian and Poisson processes.

Poisson process

In this case $(\Omega, \mathbb{P}) = (\mathcal{E}(X), \mathcal{P})$, and normalized multiplicative functionals are given by the formula

$$F_A(\omega) = \prod_{x \in \omega \cap A} (1 + h(x)) e^{-\int_A h(x) d\nu(x)}, \qquad h \in L^2(X, \nu),$$

for each measurable $A \subset X$.

Lemma 3. $\mathrm{LOG}\, F_A(\omega) = \sum_{x \in \omega \cap A} h(x) - \int_A h(x) d\nu(x)$.

Proof. Denote the right-hand side by $G_A(\omega)$. Note that

$$\begin{aligned} \| \sum_k (F_{A_k}(\cdot) - 1) - G(\cdot) \|^2 &= \| \sum_k (F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot)) \|^2 \\ &= \sum_k \| F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot) \|^2, \end{aligned}$$

since the restrictions of the Poisson process to disjoint subsets are independent. Using Campbell's theorem for sums and products over Poisson processes (see, e.g., [7], Sect. 3.2, 3.3), it is not difficult to compute that in our case

$$m_F(A) = \int_A h^2(x) d\nu(x),$$

and

$$\| F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot) \|^2 = e^{m_F(A_k)} - 1 - m_F(A_k).$$

Assume that $m_F(A_k) < \delta$ for all k. Then for δ small enough we have

$$\| F_{A_k}(\cdot) - 1 - G_{A_k}(\cdot) \|^2 < C \cdot m_F(A_k)^2.$$

Hence

$$\begin{aligned} \| \sum_k (F_{A_k}(\cdot) - 1) - G_{A_k}(\cdot) \|^2 &\le C \cdot \sum_k m_F(A_k)^2 \\ &\le C \cdot \sum_k \delta \cdot \int_{A_k} h^2(x) d\nu(x) \le C \cdot \|h\|^2 \cdot \delta, \end{aligned}$$

which is arbitrarily small for sufficiently small δ, and we are done. $\square$

Thus we obtain that the space of additive functionals ("the first chaos") of the Poisson process is

$$\left\{ \sum_{x \in \omega} h(x) - \int_X h(x) d\nu(x), \quad h \in L^2(X, \nu) \right\}. \tag{17}$$

Gaussian process

Let us now compute the logarithm for Gaussian processes. In this case a normalized multiplicative functional is given by the formula

$$F_A(\eta) = e^{\langle h, \eta_A \rangle - \frac{1}{2} \int_A h^2(x) d\nu(x)}, \qquad h \in L^2(X, \nu),$$

where η_A is the restriction of η to $A \subset X$.

Lemma 4. $\mathrm{LOG}\, F(\eta) = \langle h, \eta \rangle$.

Proof. Let $G_A(\eta) = \langle h, \eta_A \rangle$. It is easy to compute that in this case

$$m_F(A) = \log \mathbb{E} \| e^{\langle h, \eta_A \rangle - \frac{1}{2}\int_A h^2(x) d\nu(x)} \|^2 = \int_A h^2(x) d\nu(x),$$

and

$$\begin{aligned} \|F_A(\cdot) - 1 - G_A(\cdot)\|^2 &= \mathbb{E}\| e^{\langle h, \eta_A \rangle - \frac{1}{2}\int_A h^2(x) d\nu(x)} - 1 - \langle h, \eta_A \rangle \|^2 \\ &= e^{m_F(A)} - 1 - m_F(A), \end{aligned}$$

exactly as in the Poissonian case, so the proof just reproduces the proof of Lemma 3. □

3.4 Orthogonal decompositions

In this section we state the canonical isometry (8) at the level of orthogonal decompositions. By orthogonal decomposition we mean the result of the orthogonalization process starting from the subspace of additive functionals, i.e., from the first chaos. The problem of constructing such a decomposition for general processes with independent values can be addressed within the following general scheme.

Let ϕ be an arbitrary process with independent values on X (which can be regarded as a random measure on X). We will construct an orthogonal decomposition in the space $L^2(\phi)$ of square integrable functionals of this process. Set $\mathcal{H}_0 = \mathbb{C}$. Let $\mathcal{H}_1$ the subspace of centralized (i.e., orthogonal to constants) *additive* functionals in the factorization ζ generated by η (if we are given the set $\mathcal{M}$ of square integrable multiplicative functionals in ζ, then the space $\mathcal{H}_1$ of centralized square integrable additive functionals can be obtained by the logarithm construction from Theorem 2). Assuming that we have already constructed $\mathcal{H}_1, \ldots, \mathcal{H}_{n-1}$, the next space $\mathcal{H}_n$ is defined as the orthogonal complement to $\mathcal{H}_1 \oplus \ldots \oplus \mathcal{H}_{n-1}$ in the subspace spanned by the functionals of order n, that is, products of n additive functionals:

$$\mathcal{H}_n = \overline{\{F_1(\phi) \cdot \ldots \cdot F_n(\phi),\ F_k \in \mathcal{H}_1\}} \ominus (\mathcal{H}_0 \oplus \ldots \oplus \mathcal{H}_{n-1}).$$

The space $\mathcal{H}_n$ is called the space of *generalized n-multiple stochastic integrals* of the process ϕ, or the *nth chaos* of ϕ. We have $L^2(\phi) = \oplus_{n=0}^{\infty} \mathcal{H}_n$.

Note that for general Lévy processes the space of additive functionals $\mathcal{H}_1$ is wider than the space of linear functionals L_1 (see §4). If we carry out the above orthogonalization process with $\mathcal{H}_1$ replaced by L_1, then the nth subspace L_n will be the space of ordinary n-multiple stochastic integrals of ϕ, but the sum of this subspaces will not exhaust the whole space $L^2(\phi)$. However, for the Gaussian and Poisson processes the spaces of linear and additive functionals coincide, hence $\mathcal{H}_n = L_n$ for all n.

The spaces L_n can be described explicitly using the general combinatorial approach to stochastic integrals suggested in [11]. Denote by Δ_n the nth diagonal measure of ϕ, that is,

$$\Delta_n(A) = \phi^{\otimes n}\{(x, \ldots, x) : x \in A\}$$

for each measurable A. Then it follows easily from the results of [11] (see also [13]) that L_n is spanned (as a closed subspace) by the n-multiple stochastic integrals of the form

$$I_n(f_1,\dots,f_n)=\int f_1(x_1)\dots f_n(x_n)\,d\phi(x_1,\dots,x_n)$$
$$=\sum_{g\in\mathfrak{S}_n(x_1,\dots,x_n)}(-1)^{n-c(g)}\prod_{(x_{i_1}\dots x_{i_k})\in C(g)}\int f_{i_1}(x)\dots f_{i_n}(x)d\Delta_k(x),\quad(18)$$

where $\mathfrak{S}_n(x_1,\dots,x_n)$ is the symmetric group of degree n realized as the group of permutations of the set $\{x_1,\dots,x_n\}$, $C(g)$ is the set of cycles of a permutation g, and $c(g)=\#C(g)$ is the number of cycles of g.

In particular, in the case when ϕ is the Gaussian white noise on (X,ν), we have $\Delta_2=\nu$, $\Delta_3=\Delta_4=\ldots=0$ ([11], Example G), whence $\mathcal{H}_n=L_n$ is spanned by

$$I_n^g(f_1,\dots,f_n)=\sum_{g\in\mathrm{Inv}_n}\prod_{i\in C_1(g)}\langle f_i,\cdot\rangle\prod_{\{j,k\}\in C_2(g)}\left(-(f_j,f_k)\right),\quad(19)$$

where Inv_n is the set of all involutions in $\mathfrak{S}_n$, $C_k(g)$ is the number of cycles of length k of a permutation $g\in\mathfrak{S}_n$, and $(f_j,f_k)=\int_X f_j(x)f_k(x)d\nu(x)$ is the scalar product in $L^2(X,\nu)$. The functional $I_n^g(f_1,\dots,f_n)$ is called the nth *generalized Hermite functional*. In particular, for $f_1=f_2=\ldots=f_n=f$ we obtain (see Appendix B, (51))

$$I_n^g(f,\dots,f)=\sum_{g\in\mathrm{Inv}_n}\langle f,\cdot\rangle^{c_1(g)}\cdot(-\|f\|^2)^{c_2(g)}=H_n^\sigma(\langle f,\cdot\rangle),\quad(20)$$

the nth ordinary Hermite polynomial in $\langle f,\cdot\rangle$ with parameter $\sigma=\|f\|^2$. The corresponding orthogonal decomposition of $L^2(\hat{H},\mu)$

$$L^2(\hat{H},\mu)=\bigoplus_{n=0}^{\infty}\mathcal{H}_n\quad(21)$$

is the classical Wiener-Itô decomposition. Notice that in terms of the Fock space $\mathrm{EXP}\,H$, the subspace $\mathcal{H}_n$ is precisely the n-particle subspace S^nH.

The corresponding orthogonal decomposition for the Poisson process was first discussed by Itô [6], and explicitly constructed by Ogura [10]. Within the combinatorial approach of [11] it is obtained as follows. In this case ϕ is the compensated Poisson process on (X,ν), that is $\phi=\sum_{x\in\omega}\delta_x-\nu$, where ω is the Poisson process on (X,ν), and the diagonal measures equal $\Delta_2=\Delta_3=\ldots=\omega$ ([11], Example CP). Thus we have

$$L^2(\mathcal{E},\mathcal{P})=\bigoplus_{n=0}^{\infty}V_n,\quad(22)$$

where $V_0=\mathbb{C}$, and V_n is the closed space generated by the n-multiple stochastic integrals given by the formula

$$I_n^p(f_1,\dots,f_n)=\sum_{g\in\mathfrak{S}_n}(-1)^{n-c(g)}\prod_{i\in C_1(g)}\left(\sum_{x\in\omega}f_i(x)-\int_X f_i(x)d\nu(x)\right)$$
$$\cdot\prod_{(x_{i_1},\dots,x_{i_k})\in C(g)}\sum_{x\in\omega}f_{i_1}(x)\dots f_{i_k}(x).\quad(23)$$

$I_n^p(f_1,\ldots,f_n)$ is called the nth *generalized Charlier functional* (see [10]). If $f_1 = \ldots = f_n = \chi_A$, where χ_A is the characteristic function of a measurable set $A \subset X$, then

$$I_n^p(\chi_A,\ldots,\chi_A) = \sum_{g\in\mathfrak{S}_n} (\#(\omega\cap A) - \nu(A))^{c_1(g)} \cdot \#(\omega\cap A)^{-c_2(g)+c_3(g)-\cdots} = C_n^\sigma(\#(\omega\cap A)), \quad (24)$$

the nth ordinary Charlier polynomial with parameter $\sigma = \nu(A)$ (see Appendix B, (55)). (Note that in this case, unlike the Gaussian one, $I_n^p(f,\ldots,f)$ for an arbitrary f is not an ordinary Charlier polynomial. The reason is that all linear functionals of the Gaussian process have Gaussian distributions, while in the Poissonian case only integrals over subsets of X have Poisson distributions.) In particular, the first chaos of the Poisson process is given by (17), and the second chaos is generated by the functionals of the form

$$I_2^p(f,g) = I_1^p(f)I_1^p(g) - \sum_{x\in\omega} f(x)g(x), \quad (25)$$

where $I_1^p(f) = \sum_{x\in\omega} f(x) - \int_X f(x)d\nu(x)$ are the functionals of the first chaos.

Corollary 2. *The canonical isometry Φ sends the generalized Hermite functional to the corresponding generalized Charlier functional:*

$$\Phi(I_n^g(f_1,\ldots,f_n)) = I_n^p(f_1,\ldots,f_n). \quad (26)$$

3.5 Kernel

For the sake of simplicity we will consider the case when $X = [0,1]$ and ν is the Lebesgue measure on $[0,1]$, that is, μ is the distribution of the standard white noise on $[0,1]$, and $\mathcal{P}$ is the distribution of the homogeneous Poisson process on $[0,1]$ with unit rate. The case of an arbitrary continuous Lebesgue space (X,ν) is completely analogous.

Let us introduce the following notation. Given a point configuration $\omega \in \mathcal{E}$ (since the parameter measure ν is continuous, all configurations of the Poisson process are simple, i.e., each point has multiplicity one; thus we may consider only simple (multiplicity-free) configurations), denote by $\Pi_{\le 2}(\omega)$ the set of partitions of the set ω into subsets consisting of at most two points.

For example, if $\omega = \{x,y,z\}$, then

$$\Pi_{\le 2}(\omega) = \{\{\{x\},\{y\},\{z\}\},\ \{\{x,y\},\{z\}\},\ \{\{x\},\{y,z\}\},\ \{\{x,z\},\{y\}\}\}.$$

For each partition $R \in \Pi_{\le 2}$, let $C_k(R)$ be the set of k-point subsets in R, $k = 1,2$, and $|R| = \#C_2(R)$.

Theorem 4. *The kernel (9) of the isomorphism (8) between $L^2(\hat{H},\mu)$ and $L^2(\mathcal{E},\mathcal{P})$ is given by the following formula:*

$$K(\omega,\eta) = e^{-\frac{1}{2}-\langle\eta,1\rangle} \sum_{R\in\Pi_{\le 2}(\omega)} (-1)^{|R|} \prod_{z\in C_1(R)} (\eta+2)(z) \prod_{\{x,y\}\in C_2(R)} \delta(x-y) \quad (27)$$

for almost all ω, η.

(Here $\eta + 2$ is a generalized function: $\langle\eta+2,h\rangle = \langle\eta,h\rangle + 2\int h(t)dt$, and the product is the direct product of generalized functions.)

Remark. The kernel $K(\omega,\eta)$ is a generalized function on $\mathcal{E}\times\hat{H}$. For any measurable $A \subset \mathcal{E}$ and $B \subset \hat{H}$, set $\rho(A,B) = \int_A\int_B K(\omega,\eta)d\mathcal{P}(\omega)d\mu(\eta)$. Then ρ is an additive set function on $\mathcal{E}\times\hat{H}$. Thus K can be regarded as the density of a signed measure (of infinite variation) on $\mathcal{E}\times\hat{H}$.

Proof. It suffices to check that (9) holds for multiplicative functionals F, that is, to show that

$$\mathbb{E}\left(\prod_{x\in\omega}(1+h(x))K(\omega,\eta)\right) = \exp\left(-\frac{\|h\|^2}{2}+\langle\eta+1,h\rangle\right) \tag{28}$$

for all $h\in L^2(X,\nu)$. It is more convenient to rewrite the sum in (27) over *permutations* rather than partitions. Given an n-point configuration $\omega=\{x_1,\dots,x_n\}$ let $\mathfrak{S}_n(\omega)$ be the symmetric group of degree n realized as the group of all permutations of the set $\{x_1,\dots,x_n\}$. Let $C_i(g)$ be the set of all cycles of length i in the permutation $g\in\mathfrak{S}_n$, and $c_i(g)=\#C_i(g)$. Finally, denote by $\mathrm{Inv}(\omega)$ the subset of $\mathfrak{S}_n(\omega)$ consisting of all involutions (i.e., permutations with cycles of length at most two). Recall that the number of points of the homogeneous Poisson process on $[0,1]$ obeys the Poisson distribution with parameter one, and given the number of points equal to n, these points are distributed as n i.i.d. variables with uniform distribution on $[0,1]$. Then the left-hand side of (28) equals

$$e^{-3/2-\langle\eta,1\rangle}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1\dots\int_0^1(1+h(x_1))\dots(1+h(x_n))$$
$$\cdot\sum_{g\in\mathrm{Inv}(x_1,\dots,x_n)}\prod_{\{x_i,x_j\}\in C_2(g)}(-\delta(x_i-x_j))\cdot\prod_{x_k\in C_1(g)}(\eta(x_k)+2)\;dx_1\dots dx_n.$$

The contribution of each pair $\{x,y\}\in C_2(g)$ is equal to

$$-\int_0^1\int_0^1(1+h(x))(1+h(y))\delta(x-y)dxdy = -\int_0^1(1+h(x))^2dx,$$

and the contribution of each element $x\in C_1(g)$ is equal to

$$\int_0^1(1+h(x))(\eta(x)+2)dx=\langle\eta+2,1+h\rangle.$$

Thus the sum under consideration equals

$$e^{-3/2-\langle\eta,1\rangle}\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{g\in\mathrm{Inv}_n}t_1^{c_1(g)}t_2^{c_2(g)}, \tag{29}$$

where

$$\begin{aligned} t_1 &= \langle\eta+2,1+h\rangle,\\ t_2 &= -\int_0^1(1+h(x))^2dx. \end{aligned}$$

But the sum in (29) is just the augmented cycle index $\tilde{Z}(\mathfrak{S}_n)[t_1,t_2,0,0,\dots]$ (see Appendix A, (46)). Thus applying (47) with $z=1$ we obtain that (29) is equal to

$$\exp\left(-3/2-\langle\eta,1\rangle+\langle\eta+2,1+h\rangle-\frac{1}{2}\int(1+h(x))^2dx\right),$$

and (28) follows by trivial computations. ∎

Remark. There is another proof of Theorem 4 which allows to derive formula (27) rather than to check it. The idea of this proof is as follows. Observe that

$$L^2(\hat{H}, \mu) = \varprojlim A_n,$$

where A_n is the subspace consisting of functionals $F(\eta)$ depending only on $\langle \eta, \chi_{[0,\frac{1}{n}]}\rangle, \ldots, \langle \eta, \chi_{[\frac{n-1}{n},1]}\rangle$. Obviously, A_n is isometric to

$$\bigotimes_{j=1}^{m} L^2(\mathbb{R}, N(0, 1/n)).$$

Then one should apply (16) and pass (carefully!) to the limit.

Example 1. Let A_n be the subset in $L^2(\mathcal{E}, \mathcal{P})$ consisting of functions supported by n-point configurations, $n = 1, 2, \ldots$. Then it follows from (27) that the image of A_1 under the canonical isomorphism is the subspace of functions of the form

$$e^{-\frac{3}{2}-\langle \eta,1\rangle}\langle \eta + 2, f\rangle, \quad f \in L^2([0, 1], \nu);$$

the image of A_2 consists of functions of the form

$$e^{-\frac{3}{2}-\langle \eta,1\rangle}\frac{1}{2!}\left[\int_0^1\int_0^1 f(x, y)(\eta + 2)(x)(\eta + 2)(y)dxdy - \int_0^1 f(x, x)dx\right],$$

where $f \in L^2([0, 1] \times [0, 1], \nu \times \nu)$; and the image of L_3 is

$$\begin{aligned} e^{-\frac{3}{2}-\langle \eta,1\rangle} \quad \cdot \quad & \frac{1}{3!}\left[\int_0^1\int_0^1\int_0^1 f(x, y, z)(\eta + 2)(x)(\eta + 2)(y)(\eta + 2)(z)dxdydz\right. \\ - \quad & \int_0^1\int_0^1 f(x, x, z)(\eta + 2)(z)dxdz - \int_0^1\int_0^1 f(x, y, y)(\eta + 2)(x)dxdy \\ - \quad & \left.\int_0^1\int_0^1 f(z, y, z)(\eta + 2)(y)dydz\right], \end{aligned}$$

$f \in L^2([0, 1] \times [0, 1] \times [0, 1], \nu \times \nu \times \nu)$.

Example 2. Each function $h \in L^2(\mathcal{E}, \mathcal{P})$ determines a"linear" functional $F_h(\omega) = \sum_{x\in\omega} h(x)$. Let us compute its image in $L^2(\hat{H}, \mu)$. We have

$$\begin{aligned} \mathbb{E}K(\omega, \eta)F_h(\omega) = e^{-3/2-\langle \eta,1\rangle} & \sum_{n=0}^{\infty}\frac{1}{n!}\int_0^1 \cdots \int_0^1 \sum_{l=1}^{n} h(x_l) \\ & \sum_{g\in \mathrm{Inv}(x_1,\ldots,x_n)} \prod_{\{x_i,x_j\}\in C_2(g)} (-\delta(x_i - x_j)) \cdot \prod_{x_k\in C_1(g)} (\eta + 2)(x_k)dx_1 \ldots dx_n. \end{aligned}$$

It is easy to check that each summand $h(x_l)$ contributes

$$\begin{cases} \langle h, \eta + 2\rangle(\langle 1, \eta\rangle + 2)^{c_1(g)-1}(-1)^{c_2(g)}, & \text{if } x_k \in C_1(g) \\ \langle h\rangle(\langle 1, \eta\rangle + 2)^{c_1(g)}(-1)^{c_2(g)}, & \text{if } x_k \in C_2(g). \end{cases}$$

Thus the sum under consideraton equals $e^{-3/2-\langle\eta,1\rangle}(S_1+S_2)$, where

$$\begin{aligned} S_1 &= \langle h,\eta+2\rangle \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{g\in \mathrm{Inv}(x_1,\dots,x_n)} c_1(g)(\langle 1,\eta\rangle+2)^{c_1(g)-1}(-1)^{c_2(g)},\\ S_2 &= 2\langle h\rangle \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{g\in \mathrm{Inv}(x_1,\dots,x_n)} c_2(g)(\langle 1,\eta\rangle+2)^{c_1(g)}(-1)^{c_2(g)}. \end{aligned}$$

Notice that the sum in S_1 is the derivative of the augmented cycle index $\tilde{Z}(\mathfrak{S}_n)$ in t_1 calculated at $t=(\langle 1,\eta\rangle+2,-1,0,0,\dots)$, hence $S_1=\langle h,\eta+2\rangle e^{\langle 1,\eta\rangle+3/2}$. Similarly, S_2 is the derivative of the same cycle index in t_2, thus $S_2=-\langle h\rangle e^{\langle 1,\eta\rangle+3/2}$, where $\langle h\rangle=\int_X h(x)d\nu(x)$, and simple computations show that the image of the functional F_h in $L^2(\hat{H},\mu)$ equals

$$F_h(\eta)=\langle h,\eta\rangle+\int_X hd\nu(x),$$

in agreement with (17).

4 The isometry for general Lévy processes

The purpose of this section is to apply the results on the Poisson–Gauss isomorphism to general Lévy processes.

In view of the Poissonian representation (7) of Lévy processes, $L^2(D,P_\Lambda)$ is isometric to the space of square integrable functionals of the Poisson process on the product space $X\times\mathbb{R}_+$ with mean measure $\nu\times\Lambda$. Thus let μ^Λ be the law of the Gaussian white noise on the space $(X\times\mathbb{R}_+,\nu\times\Lambda)$, i.e., the standard Gaussian measure in the nuclear extension $\widehat{H^\Lambda}$ of the Hilbert space $L^2(X\times\mathbb{R}_+,\nu\times\Lambda)$. Note that this process may be also regarded as a $L^2(\mathbb{R}_+,\Lambda)$-valued Gaussian process on (X,ν), that is, $L^2(\widehat{H^\Lambda},\mu^\Lambda)$ can be identified with the homogeneous Fock space $\mathrm{EXP}\,L^2((X,\nu);\,L^2(\widehat{H^\Lambda},\mu^\Lambda))$. The spaces $L^2(\widehat{H^\Lambda},\mu^\Lambda)$ and $L^2(D,P_\Lambda)$ are equipped with natural Hilbert factorizations over (X,ν).

Theorem 5. *There exists a unity-preserving isometry (isomorphim of Hilbert factorizations)*

$$\Phi: L^2(\widehat{H^\Lambda},\mu^\Lambda)\to L^2(D,P_\Lambda).$$

On the set of multiplicative functionals it is given by the following formula: for each $h\in L^2(x\times\mathbb{R}_+,\nu\times\Lambda)$,

$$\Phi:\ e^{<h,\cdot>-\frac{\|h\|^2}{2}}\mapsto\prod_i(1+h(x_i,t_i))\cdot e^{-\int\int h(x,t)d\nu(x)d\Lambda(t)},\quad \eta=\sum_i t_i\delta_{x_i}\in D. \tag{30}$$

This is the unique real special vacuum-preserving automorphism of the Hilbert factorizations that acts identically on $L^2(\mathbb{R}_+,\Lambda)$.

Proof. The existence follows immediately from the above observations and Theorem 3.

In order to find the group of special automorphisms of the Hilbert factorization of the Fock space $L^2(\widehat{H^\Lambda},\mu^\Lambda)=\mathrm{EXP}\,L^2((X,\nu);\,L^2(\widehat{H^\Lambda},\mu^\Lambda))$ over (X,ν), we retrace the proof of Theorem 3 and see that the set of operators in $L^2((X,\nu);\,L^2(\widehat{H^\Lambda},\mu^\Lambda))$ that commute with all projections P_A

for $A \subset X$ is $L^\infty(X,\nu)\otimes\mathcal{B}(L^2(\mathbb{R}_+,\Lambda))$. Thus the group of special vacuum-preserving automorphims is generated by operators of the form $\mathrm{EXP}(h_1(\cdot)\otimes h_2(\cdot)) \mapsto \mathrm{EXP}((a(\cdot)h_1(\cdot))\otimes(Uh_2(\cdot)))$, where $h_1 \in L^2(X,\nu)$, $h_2 \in L^2(\mathbb{R}_+,\Lambda)$, a is a complex-valued measurable function on X with $|a| \equiv 1$, and U is an arbitrary unitary operator in $L^2(\mathbb{R}_+,\Lambda)$, and Theorem follows. □

Remark. In order to determine the isomorphism uniquely, one can, for example, fix a basis in $L^2(\mathbb{R}_+,\Lambda)$. We do this below by considering orthogonal polynomials.

Theorem 5 implies immediately

Theorem 6. *Let η be a Lévy process on (X,ν) corresponding to the Lévy measure Λ. Then the Hilbert factorization determined by η is a homogeneous Fock factorization, and its dimension depends only on the number of points in* $\operatorname{supp}\Lambda$.

Remark. This theorem also holds for general processes with independent values. We have proved it only in the particular case of Lévy processes in order to avoid technical difficulties.

Theorem 5 states the Lévy–Gauss isomorphism at the level of multiplicative functionals. Let us now describe it in terms of orthogonal decompositions. By (22) we have

$$L^2(D,P_\Lambda) = \bigoplus_{n=0}^{\infty} V_n,$$

where V_n is the space generated by the nth generalized Charlier functionals (23) in the space $(X\times\mathbb{R}_+,\nu\times\Lambda)$. In particular, additive functionals of the Lèvy process are of the form

$$\sum_i h(x_i,t_i), \qquad h \in L^2(X\times\mathbb{R}_+,\nu\times\Lambda).$$

Note that if $\operatorname{supp}\Lambda$ does not consist of a single point, that is, the Lévy process is neither Gaussian, nor Poisson, *the space of additive functionals does not coincide with the space of linear functionals*

$$\langle a,\eta\rangle = \int_X a(x)d\eta(x) = \sum_i a(x_i)t_i, \qquad a \in L^2(X,\nu). \tag{31}$$

This is exactly the reason of the well-known fact (see, e.g., [3]) that the only Lévy processes with the so-called chaotic representation property (which means that the L^2 space can be decomposed into a direct sum of ordinary multiple stochastic integrals) are the Gaussian and Poisson processes.

In the rest of this section we assume that the measure $t^2\Lambda(t)$ has finite moments of all orders, and the moment problem for this measure is definite. In this case we can obtain a more detailed description of the orthogonal decomposition.

Let $\{P_k(t)\}_{k=0}^\infty$ be the family of orthogonal polynomials on $\mathbb{R}_+$ with respect to the measure $t^2\Lambda(t)$, and let

$$V_{n,k} = \overline{\{I_n^p(a_1(x)tP_{k-1}(t),\dots,a_n(x)tP_{k-1}(t));\ a_1,\dots,a_n \in L^2(X,\nu)\}} \tag{32}$$

be the closed subspace spanned by the nth generalized Charlier functionals corresponding to functions of the form $f(x,t) = a(x)tP_{k-1}(t)$. For example, let $f_a(\eta) = \sum_i a(x_i)t_i - \int a(x)d\nu(x)$. Then $V_{1,1} = \{f_a\}$, $V_{2,1} = \left\{f_af_b - \sum a(x_i)b(x_i)t_i^2\right\}$, $V_{1,2} = \{\sum a(x_i)t_iP_1(t_i)\}$, etc.

It is not difficult to see that

$$V_n = \sum_{\lambda \vdash n} \bigoplus_{\substack{(i_1,\ldots,i_k)\\ \text{distinct}}} \bigotimes_{j=1}^{k} V_{\lambda_j, i_j},$$

where $\lambda = (\lambda_1 \geq \ldots \geq \lambda_k > 0)$ is a partition of n, and the tensor product is symmetric. Thus, rearranging the summands, we obtain the following orthogonal decomposition for the general Lévy process:

$$\begin{aligned} L^2(D, P_\Lambda) &= \bigoplus_n \bigoplus_{\lambda \vdash n} \otimes_s V_{n_k,k} \\ &= \mathbb{C} \oplus V_{1,1} \oplus (V_{1,2} \oplus V_{2,1}) \oplus (V_{3,1} \oplus (V_{1,1} \otimes_s V_{1,2}) \oplus V_{1,3}) \oplus \ldots, \end{aligned}$$

where $\lambda = 1^{n_1} 2^{n_2} \ldots$ is a partition of n with n_k parts equal to k, and $\otimes_s$ is the symmetric tensor product.

The same decomposition can be described in another way (cf. [9]). Given the Lévy process $\eta = \sum t_i \delta_{x_i}$, consider the processes $\eta_k = \sum_i t_i P_{k-1}(t_i) \delta_{x_i}$ (in particular, $\eta_1 = \eta$). Then $V_{n,k}$ is the space of n-multiple stochastic integrals of the process η_k.

The corresponding decomposition for the L^2 space over the vector-valued Gaussian process is obtained in a similar way. Namely,

$$\begin{aligned} L^2(\widehat{H^\Lambda}, \mu^\Lambda) &= \bigoplus_n \bigoplus_{\lambda \vdash n} \otimes_s \mathcal{H}_{n_k,k} \\ &= \mathbb{C} \oplus \mathcal{H}_{1,1} \oplus (\mathcal{H}_{1,2} \oplus \mathcal{H}_{2,1}) \oplus (\mathcal{H}_{3,1} \oplus (\mathcal{H}_{1,1} \otimes_s \mathcal{H}_{1,2}) \oplus \mathcal{H}_{1,3}) \oplus \ldots, \end{aligned}$$

where $\mathcal{H}_{n,k}$ is the closed subspace spanned by the nth generalized Hermite functionals (19) corresponding to functions of the form $f(x,t) = a(x) t P_{k-1}(t)$.

Corollary 3. *At the level of orthogonal decompositions the canonical isometry (30) takes the form*

$$\Phi I_n^g(a_1(x) t P_{k-1}(t), \ldots, a_n(x) t P_{k-1}(t)) = I_n^p(a_1(x) t P_{k-1}(t), \ldots, a_n(x) t P_{k-1}(t)).$$

Example. Gamma processes. The standard gamma process on the space (X, ν) is the Lévy process with Lévy measure

$$\Lambda_\Gamma(t) = \frac{e^{-t}}{t} dt, \qquad t > 0. \tag{33}$$

Denote the law of this process by $\mathcal{G}$. All the considerations of this section can be applied to $L^2(D, \mathcal{G})$, the orthogonal polynomials with respect to $t^2 \Lambda_\Gamma(t) = t e^{-t} dt$ being the Laguerre polynomials with parameter 1: $P_n(t) = L_n^{(1)}(t)$.

5 Representations of the current group $\mathrm{SL}(2,\mathbb{R})^X$

Let (X, ν) be a standard Borel space with a fixed finite measure ν. The *current group* $G^X = \mathrm{SL}(2,\mathbb{R})^X$ on (X, ν) is the group of Borel bounded $\mathrm{SL}(2,\mathbb{R})$-valued functions on X. In other words, G^X consists of 2×2-matrices whose elements are bounded measurable real functions on X.

The *canonical representation* of the current group G^X is a unitary irreducible representation with spherical function given by the formula

$$\Omega(g(\cdot)) = C\exp\left(-\int_X \log\left(2+\mathrm{Tr}(g(x)g^*(x))\right)d\nu(x)\right), \qquad g(\cdot)\in G^X. \tag{34}$$

The restriction of this spherical function to the subgroup of constant functions (isomorphic to $G=\mathrm{SL}(2,\mathbb{R})$) equals $\Omega_0(g)=\frac{C}{2+\mathrm{Tr}\,gg^*}$, the so-called *canonical state* of $\mathrm{SL}(2,\mathbb{R})$, see [19].

5.1 The Fock model of the canonical representation

A general construction of the representation of the current group G^X in a Fock space is described in [1]. This construction requires the existence of a non-trivial cocycle of the group G taking values in an irreducible unitary representation. For $\mathrm{SU}(n,1)$ and $\mathrm{SO}(n,1)$ (in particular for $\mathrm{SL}(2,\mathbb{R})$) such a cocycle was found in [20].

Let us describe the Fock model of the canonical representation of $\mathrm{SL}(2,\mathbb{R})^X$. We start from the so-called *special* representation of the group $G=\mathrm{SL}(2,\mathbb{R})$ in the Hilbert space $H=L^2(\mathbb{R},\frac{dt}{|t|})$ given by the following formulas:

$$\begin{aligned}\left(T\begin{pmatrix}1&0\\b&1\end{pmatrix}\phi\right)(t) &= e^{ibt}\phi(t),\\ \left(T\begin{pmatrix}a^{-1}&0\\0&a\end{pmatrix}\phi\right)(t) &= \phi(a^2t),\\ \left(T\begin{pmatrix}0&1\\-1&0\end{pmatrix}\phi\right)(t) &= \int_{-\infty}^{\infty}K_0(t,s)\phi(s)ds,\end{aligned}\tag{35}$$

where

$$K_0(t,s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|u|^2}e^{-i(tu+su^{-1})}du. \tag{36}$$

This is an irreducible unitary representation of discrete series. (Note that considered over $\mathbb{C}$ it is reducible: it decomposes into the sum of two irreducible subrepresentations.) However, this representation is distinguished as the only representation having a non-trivial cocycle.

Let $\phi_0(t)=e^{-|t|}$, and fix a cocycle $\beta: G\times H\to H$ defined as

$$\beta(g,t)=T_g\phi_0(t)-\phi_0(t). \tag{37}$$

Consider now the Hilbert space

$$H^X=L^2\left(X\times\mathbb{R},\ \nu\times\frac{dt}{|t|}\right),$$

and the corresponding Fock space $\mathrm{EXP}\,H^X$. The realization of the canonical representation of $G^X=\mathrm{SL}(2,\mathbb{R})^X$ in $\mathrm{EXP}\,H^X$ is given by the formula

$$U_{g(\cdot)}\,\mathrm{EXP}\,h(\cdot,\cdot)=\lambda(g(x),h(x,t))\cdot\mathrm{EXP}(T_{g(x)}h(x,t)+\beta(g(x),t)), \tag{38}$$

where

$$\lambda(g,h) = \exp\left(-\frac{1}{2}\|\beta\|^2 - \langle T_g h, \beta\rangle\right)$$
$$= \exp\left(-\frac{1}{2}\int_X\int_{\mathbb{R}} \frac{|\beta(g(x),t)|^2}{|t|} dt\, d\nu(x) - \int_X\int_{\mathbb{R}} \frac{T_{g(x)}h(x,t)\cdot\bar{\beta}(g(x),t)}{|t|} dt\, d\nu(x)\right).$$

The vacuum vector in the Fock realization is EXP 0, and the corresponding spherical function equals (34).

Using the isomorphism (4) between the Fock space and the L^2 space over the Gaussian white noise, one can obtain the Gaussian realization of the canonical representation. We do not reproduce here the corresponding formulas which can be found in [5].

5.2 The Lebesgue model of the canonical representation

A commutative model of the canonical representation of G^X with respect to the unipotent subgroup was given in [5]. Another realization of this model, in the L^2 space over the so-called infinite-dimensional Lebesgue measure, was constructed in [15]. Let us describe this model.

The standard gamma process on (X,ν) is a Lévy process on (X,ν) with Lévy measure $d\Lambda_\Gamma(t) = t^{-1}e^{-t}dt$, $t>0$, that is,

$$\mathbb{E}_{\mathcal{G}}\left[\exp\left(-\int_X a(x)d\eta(x)\right)\right] = \exp\left(-\int_X \log\left(1+a(x)\right)d\nu(x)\right), \tag{39}$$

where $\mathcal{G}$ is the law of the gamma process, and a is an arbitrary non-negative measurable function on the space X with $\int_X \log(a(x)+1)d\nu(x) < \infty$.

The *infinite-dimensional Lebesgue measure* $\mathcal{L}^+$ on the space $D^+(X,\nu)$ is a σ-finite measure (finite on compact sets) equivalent to the law of the gamma process such that

$$\frac{d\mathcal{L}^+}{d\mathcal{G}}(\eta) = \exp(\eta(X)). \tag{40}$$

It follows from (39) and (40) that the Laplace transform of $\mathcal{L}^+$ equals

$$\int_{D^+}\left[\exp\left(-\int_X a(x)d\eta(x)\right)\right]d\mathcal{L}^+(\eta) = \exp\left(-\int_X \log a(x)d\nu(x)\right). \tag{41}$$

The *Lebesgue measure on* $D(X)$ is the convolution $\mathcal{L}^+ * \mathcal{L}^-$, where $\mathcal{L}^-$ is the image of $\mathcal{L}^+$ under the mapping $\eta \to -\eta$.

Consider the triangular subgroup $\mathcal{T}$ of $\mathrm{SL}(2,\mathbb{R})^X$,

$$\mathcal{T} = \left\{T_{a,b} = \begin{pmatrix} a(\cdot)^{-1} & 0 \\ b(\cdot) & a(\cdot)\end{pmatrix}\right\}.$$

Theorem 7 ([15]). *The formula*

$$\mathcal{U}(T_{a,b})F(\eta) = \exp\left(\int_X \log|a(x)|d\nu(x) + i\int_X a(x)b(x)d\eta(x)\right)F(M_{a^2}\eta) \tag{42}$$

defines a unitary irreducible representation of the triangular subgroup $\mathcal{T}$ in the space $L^2(D,\mathcal{L})$ which is extendable to a (unitary irreducible) representation of the whole group $\mathrm{SL}(2,\mathbb{R})^X$.

5.3 An isomorphism of the Fock and Lebesgue models of the canonical representation

The Fock model (38) and the Lebesgue model (42) define isomorphic representations, since their spherical functions coincide. However, now we can construct the isomorphism of these realizations in an explicit form.

Theorem 8. *The isometry of the spaces* $\mathrm{EXP}\, H^X$ *and* $L^2(D,\mathcal{L})$ *which intertwines the Fock realization* U *and the Lebesgue realization* $\mathcal{U}$ *of the canonical representation of the current group* $\mathrm{SL}(2,\mathbb{R})^X$ *is given by*

$$\mathrm{EXP}\, h \leftrightarrow \Psi_h, \qquad h \in L^2(X\times\mathbb{R}, \nu\times\frac{dt}{|t|}),$$

where

$$\Psi_h(\eta) = \prod_i \left(h(x_i,t_i) + e^{-|t_i|/2}\right)\cdot \exp\left(-\int_X\int_{\mathbb{R}} \frac{h(x,t)\cdot e^{-|t|/2}}{|t|} dt\, d\nu(x)\right) \tag{43}$$

for $\eta = \sum t_i\delta_{x_i} \in D$.

Proof. Formula (43) defines an isometry betwen $\mathrm{EXP}\, H^X$ and $L^2(D,\mathcal{L})$, as follows from Theorem 5 in the special case of the gamma process and the obvious isometry between $L^2(D,\mathcal{G})$ and $L^2(D,\mathcal{L})$ given by $F(\eta) \leftrightarrow F(\eta)e^{-\eta(X)/2}$. It is not difficult to verify by direct calculation that this isomorphism intertwines the representations U and $\mathcal{U}$. ☐

Note that in the Lebesgue realization the vacuum vector is $\Psi_0(\eta) = e^{-\frac{|\eta|(X)}{2}}$, where $|\eta| = \sum |t_i|$ is the total charge of the (signed) measure η.

Corollary 4. *The action of the involution* $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *in the Lebesgue realization of the canonical representation of* G^X *is given by the formula*

$$U_s\Psi_{f(\cdot,\cdot)} = \Psi_{T_s f(\cdot,\cdot)}, \tag{44}$$

where

$$T_s f(x,t) = \int_{-\infty}^{\infty} K_0(t,s)f(s)ds, \tag{45}$$

with the kernel K_0 *given by (36).*

Proof. Follows from Theorem 8 and (38), since $\beta(s,t)\equiv 0$. ☐

Appendix

A. Cycle index

Let $\mathfrak{S}_n$ be the symmetric group of degree n. By $c_k(g)$ we denote the number of cycles of $g\in\mathfrak{S}_n$ of length k. Let $t = (t_1, t_2, \dots)$ be a sequence of indeterminates. The (augmented) cycle index of the symmetric group $\mathfrak{S}_n$ is

$$\tilde{Z}(\mathfrak{S}_n) = \tilde{Z}(\mathfrak{S}_n)[t] = \sum_{g\in\mathfrak{S}_n} t_1^{c_1(g)} t_2^{c_2(g)}\cdot\dots. \tag{46}$$

A well-known formula (see, e.g., [14, (5.30)]) claims that

$$\sum_{n=0}^{\infty} \tilde{Z}(\mathfrak{S}_n)[t]\frac{z^n}{n!} = \exp\sum_{i=1}^{\infty} t_i \frac{z^i}{i}. \tag{47}$$

B. Orthogonal polynomials

B.1. Hermite polynomials

Definition:

$$\begin{aligned} H_n(x) &= H_n^1(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}}, \\ H_n^a(x) &= a^{\frac{n}{2}} H_n(x/\sqrt{a}). \end{aligned} \tag{48}$$

Orthogonality: orthogonal with respect to the normal law $N(0,a)$ with zero mean and variance a:

$$\frac{1}{\sqrt{2\pi a}} \int_{-\infty}^{\infty} H_n^a(x) H_m^a(x) e^{-\frac{x^2}{2a}}\, dx = \delta_{nm} a^n n!. \tag{49}$$

Generating function:

$$\sum_{n=0}^{\infty} \frac{H_n^a(x) t^n}{n!} = e^{tx - \frac{t^2 a}{2}}. \tag{50}$$

Combinatorial description:

$$H_n^a(x) = \sum_{g \in \mathrm{Inv}_n} x^{c_1(g)} (-a)^{c_2(g)} = \tilde{Z}_{\mathfrak{S}_n}[x, -a, 0, 0, \ldots], \tag{51}$$

where Inv_n is the set of all involutions in $\mathfrak{S}_n$.

B.2. Charlier polynomials

Definition:

$$C_n^a(x) = a^n \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} a^{-j} \binom{x}{j} j!. \tag{52}$$

Orthogonality: orthogonal on $\mathbb{Z}_+$ with respect to the Poisson law P_a with parameter a:

$$e^{-a} \sum_{k=0}^{\infty} C_n^a(k) C_m^a(k) \frac{a^k}{k!} = \delta_{nm} a^n n!. \tag{53}$$

Generating function:

$$\sum_{n=0}^{\infty} \frac{C_n^a(y) t^n}{n!} = (1+t)^y e^{-ta}. \tag{54}$$

Combinatorial description:

$$C_n^a(x) = \sum_{g \in \mathfrak{S}_n} (x-a)^{c_1(g)} x^{-c_2(g)+c_3(g)-c_4(g)+\cdots} = \tilde{Z}_{\mathfrak{S}_n}[x-a, -x, x, -x, \ldots]. \tag{55}$$

The Charlier polynomials satisfy the following convenient formula

$$\frac{(-1)^n}{a^n} S_n^a(k) = \frac{(-1)^k}{a^k} S_k^a(n). \tag{56}$$

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