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The Calderon problem for two-dimensional manifolds by the BC-method (a corrected version)

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As was shown by M.Lassas and G.Uhlmann (2001), the smooth two-dimensional compact orientable Riemann manifold with the boundary is uniquely determined by its Dirichlet-to-Neumann map up to conformal equivalence. We give a new proof of this fact based on relations between the Calderon problem and Function Algebras: the manifold is identified with the spectrum of the algebra of holomorphic functions determined by the DN-map up to isometry; as such, the manifold is recovered from the DN-map by the use of the Gelfand transform. A simple formula linking the DN-map to the Euler characteristic of the manifold is derived.

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0 Introduction.

0.1 General statement.

Let (Ω, g) be a smooth [1] n-dimensional Riemann manifold with the boundary Γ; g the metric tensor; Δ_g the Beltrami–Laplace operator. Consider the Dirichlet problem

$$\Delta_g u = 0 \quad in \ \mathrm{int}\,\Omega \tag{0.1}$$

$$u = f \quad on \ \ \Gamma; \tag{0.2}$$

let $u = u^f(x)$ be the solution for a smooth$f$. With the problem (0.1),(0.2) one associates the DN (Dirichlet-to-Neumann)–map $\Lambda_g : f \to \frac{\partial_u f}{\partial \nu}|_\Gamma$ (ν is the outward normal).

Assume that the boundary Γ is given and the operator $\Lambda_g : C^\infty(\Gamma) \mapsto C^\infty(\Gamma)$ is known; the *Calderon problem* is to recover (Ω, g). In other formulation, one needs to construct a manifold with a prescribed DN-map.

0.2 The case of $n = 2$. The results.

The well known pecularity of the 2–dim case is the following. If g' and g'' are two metrics on Ω such that $g'' = \rho g'$ with a positive function ρ (i.e. g'' is a conformal deformation of g') then the DN-maps of the manifolds (Ω, g') and (Ω, g'') are connected through the relation $\Lambda_{g''} = \frac{1}{\sqrt{\rho}}\Lambda_{g'}$, whereas the additional condition $\rho|_\Gamma = 1$ implies $\Lambda_{g''} = \Lambda_{g'}$. This motivates the following definition.

Let (Ω', g') and (Ω'', g'') be two smooth manifolds with the common boundary $\partial\Omega' = \partial\Omega'' = \Gamma$; we call them *conformally equivalent* if there exist a diffeomorphism $\beta : \Omega' \mapsto \Omega''$, $\beta(\Omega') = \Omega''$, $\beta|_\Gamma = \mathrm{id}$ and a positive function $\rho \in C^\infty(\Omega')$, $\rho|_\Gamma = 1$ such that β is an isometry of $(\Omega', \rho g')$ onto (Ω'', g'').

The conformal equivalence of manifolds implies the coincidence of their DN-maps. The remarkable fact is that the converse is also true:

[1] Everywhere in the paper 'smooth' means C^∞ -smooth.

Theorem 1 *Two 2-dim compact orientable manifolds with the common boundary are conformally equivalent iff their DN-maps coincide.*

In other words, in dimension 2 the class of conformally equivalent manifolds is determined by its DN-map. This fact was first established by M.Lassas and G.Uhlmann in [7] by means of technique based on the analytic continuation.

This paper is a corrected and extended version of the preprint [3]. Our results are the following:

- we show a relationship between the Calderon problem and Function Algebras and give a new proof of Theorem 1 exploiting this relationship;
- a simple formula (see (1.6)) linking the DN-map to the Euler characteristic of the manifold is derived.

The paper is addressed to the specialists in Inverse Problems. Perhaps, it is of some interest as an application of the Commutative Banach Algebras.

0.3 A bit of philosophy.

Setting the goal to construct the manifold with a prescribed DN-map one encounters the naive question: what the 'material' can it be constructed from?

A traditional situation in the boundary value inverse problems is the following: a domain $\Omega \subset \mathbf{R}^n$ *is given* whereas a function $c(x)$ on Ω (density, conductivity etc) has to be found through inverse data. Thus, we possess an initial reserve of points Ω, can mark an $x_0 \in \Omega$ and then discuss a possibility to determine $c(x_0)$.

Another situation occurs in the Calderon problem for manifolds: Ω is unknown in itself and has to be recovered, so that the question of 'material' turns out to be fundamental one. In similar dynamic inverse problem this role is played by a subset of the space-time cylinder $\Gamma \times [0, T]$ (the so-called 'pattern' - see [1]). In [7] the manifold is constructed as a Riemann surface of a (real) analytic function found by the classical procedure of analytic continuation along paths from the boundary.

We propose to identify Ω to the spectrum (the set of multiplicative functionals or, the same, the space of maximal ideals) of some function algebra

determined by inverse data. Roughly speaking, the 'material' proposed is the set of the Dirac measures $\{\delta_{x_0} \,|\, x_0 \in \Omega\}$. As we show, it is the set which may be recovered through the DN-map.

0.4 The plan.

The following basic facts are in use:

(i) (Ω, g) has the complex structure of a Riemann surface; this structure determines the class of metrics conformally equivalent to g;

(ii) the algebra $\mathcal{A}(\Omega)$ of functions continuous in Ω and holomorphic in $\operatorname{int}\Omega$ is nontrivial; functions $w \in \mathcal{A}(\Omega)$ (as local homeomorphisms $\Omega \mapsto \mathbf{C}$) determine the complex structure;

(iii) algebra $\mathcal{A}(\Omega)$ is generic: its (topologized) spectrum is homeomorphic to the manifold: $\operatorname{sp}\mathcal{A}(\Omega) \asymp \Omega$ whereas the algebra itself is identical to its Gelfand transform: $\hat{\mathcal{A}}(\Omega) \equiv \mathcal{A}(\Omega)$;

(iv) the algebra of traces $\mathcal{A}(\Gamma) := \{w|_\Gamma \,|\, w \in \mathcal{A}(\Omega)\}$ is isometrically isomorphic to $\mathcal{A}(\Omega)$ (through the map $\mathrm{tr} : w \mapsto w|_\Gamma$); the isometry yields $\operatorname{sp}\mathcal{A}(\Gamma) \asymp \operatorname{sp}\mathcal{A}(\Omega)$, $\hat{\mathcal{A}}(\Gamma) \equiv \hat{\mathcal{A}}(\Omega)$ that leads to $\operatorname{sp}\mathcal{A}(\Gamma) \asymp \Omega$ and $\hat{\mathcal{A}}(\Gamma) \equiv \mathcal{A}(\Omega)$;

(v) the algebra $\mathcal{A}(\Gamma)$ is determined by the DN-map Λ_g.

To solve the Calderon problem we use these facts in the reversed order:

(α) from the operator Λ_g one recovers the trace algebra $\mathcal{A}(\Gamma)$;

(β) finding its spectrum and Gelfand transform we get $\Omega \asymp \operatorname{sp}\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Omega) \equiv \hat{\mathcal{A}}(\Gamma)$;

(γ) using functions of algebra $\mathcal{A}(\Omega)$ we endow Ω with the complex structure;

(δ) introducing a metric g on Ω conformal to the complex structure we get the manifold (Ω, g) whose DN-map coincides with Λ_g by construction.

The procedure $(\alpha) - (\delta)$ gives a canonical representative of the class of conformally equivalent manifolds which has the given DN-map. The assertion of Theorem 1 is a simple corollary of determinacy of this procedure.

0.5 A little bit more of philosophy.

Let us explain why we look at our approach as a variant of the BC–method. In dynamic problems the BC-method

constructs a model (isometric copy) of the initial system through inverse data;

proposes a mechanism (the amplitude formulas of Geometric Optics) recovering the initial system through the model

(see [1], [2]). The same is done in this paper: DN-map determines trace algebra $\mathcal{A}(\Gamma)$ which is a model of algebra $\mathcal{A}(\Omega)$; the Gelfand transform recovers $\mathcal{A}(\Omega)$ (together with manifold Ω). Moreover, there are some reasons to believe that the mechanism based on Geometric Optics formulas is in fact a kind of the Gelfand transform.

1 Harmonic functions and fields.

1.1 Harmonic functions.

So, we have a 2-dim smooth orientable compact (Ω, g) with the boundary Γ. Just for simplicity we assume that Γ is homeomorphic to the circle (consists of one connected component).

Let $\mathrm{Harm}_g\,\Omega := \{u \,|\, \Delta_g\, u = 0 \quad in \ \mathrm{int}\,\Omega\}$ be the set of functions harmonic in Ω; in local coordinates one has

$$\frac{\partial}{\partial x^i}\det{}^{\frac{1}{2}} g(x)\, g^{ij}(x)\frac{\partial u(x)}{\partial x^j} = 0. \tag{1.1}$$

Solutions of the problem (0.1),(0.2) are said to be *potentials*; the family of potentials $\mathcal{U} := \{u^f \,|\, f \in C^\infty(\Gamma)\}$ coincides with $\mathrm{Harm}_g\,\Omega \bigcap C^\infty(\Omega)$.

1.2 Rotation operator.

Introduce near Γ the semigeodesic coordinates γ, τ such that $ds^2 = Ed\gamma^2 + d\tau^2$ and $E|_\Gamma = 1$, so that γ is the natural parameter on Γ and $\tau(x) = \mathrm{dist}\,(x, \Gamma)$. The field $\frac{\partial}{\partial\gamma}$ fixes an orientation of Γ and an orientation of Ω; note that $\frac{\partial}{\partial\tau} = -\frac{\partial}{\partial\nu}$.

The orientation determines the continuous family of rotations $\Phi(x) \in \mathrm{End}\, T_x\Omega$ such that $\langle \Phi(x)\, a, \Phi(x)\, b\rangle = \langle a, b\rangle$, $\langle \Phi(x)\, a, a\rangle = 0$ for $a, b \in T_x\Omega$ and $\Phi(\gamma)\frac{\partial}{\partial\nu} = \frac{\partial}{\partial\gamma}$ on Γ. Note that $\Phi^2(x) = -\mathrm{id}$, $x \in \Omega$.

Let $\vec{\mathcal{L}}$ be the (real) space of the square integrable vector fields on (Ω, g) with the inner product $(a,b) = \int\limits_{\Omega} d\Omega_x \, \langle a(x), b(x)\rangle$. The *rotation operator* $\Phi : \vec{\mathcal{L}} \mapsto \vec{\mathcal{L}}$, $(\Phi y)(x) := \Phi(x) y(x)$, $x \in \Omega$ is a unitary operator in $\vec{\mathcal{L}}$.

1.3 Harmonic fields.

Let $\vec{\mathcal{H}} := \{h \in \vec{\mathcal{L}} \,|\, \operatorname{div} h = \operatorname{div} \Phi h = 0 \quad in \ \operatorname{int}\Omega\}$ be the (sub) space of *harmonic fields*. Harmonic fields are smooth in $\operatorname{int}\Omega$; the set $\vec{\mathcal{H}}^{\infty} := \vec{\mathcal{H}} \bigcap \vec{C}^{\infty}(\Omega)$ is dense in $\vec{\mathcal{H}}$.

In the sequel we denote by the common symbol 'tr' the reduction of fields and functions given in Ω on the boundary Γ. The well-known and important fact is that harmonic field is determined by its trace: $h \in \vec{\mathcal{H}}$ and $\operatorname{tr} h = 0$ implies $h = 0$ in Ω.

The space $\vec{\mathcal{H}}$ contains the subspace of *potential fields* $\vec{\mathcal{E}} := \{\nabla u \in \vec{\mathcal{H}} \,|\, u \in \operatorname{Harm}_g \Omega \bigcap H^1(\Omega)\}$ ($H^1(\Omega)$ is the Sobolev class). The set $\vec{\mathcal{E}}^{\infty} := \vec{\mathcal{E}} \bigcap \vec{C}^{\infty}(\Omega) = \nabla \mathcal{U}$ is dense in $\vec{\mathcal{E}}$.

Recall in addition that every $h \in \vec{\mathcal{H}}$ is 'locally potential': for any $x_0 \in \Omega$ there exist a neghborhood U_{x_0} and a function $v \in \operatorname{Harm}_g U_{x_0}$ such that $h = \nabla v$ in U_{x_0}.

Introduce the subspace $\vec{\mathcal{D}} := \vec{\mathcal{H}} \ominus \vec{\mathcal{E}}$ and list some of its known properties (see e.g. [10]). This subspace has a finite dimension determined by topology of Ω:

$$\dim \vec{\mathcal{D}} = \dim \operatorname{Hom}_1(\Omega) = 1 - \chi(\Omega) \tag{1.2}$$

where $\operatorname{Hom}_1(\Omega)$ is the 1-st homology group of the manifold, $\chi(\Omega)$ is its Euler characteristic. Elements of the subspace are smooth: $\vec{\mathcal{D}} \subset \vec{C}^{\infty}(\Omega)$. The fields $b \in \vec{\mathcal{D}}$ are tangent on Γ; indeed, for any $f \in C^{\infty}(\Gamma)$ we have

$$0 = \int\limits_{\Omega} d\Omega \, \langle \nabla u^f, b\rangle = \int\limits_{\Gamma} d\gamma \, f \, \langle b, \frac{\partial}{\partial \nu}\rangle$$

yielding $\langle b, \frac{\partial}{\partial \nu}\rangle = 0$ or, equivalently, $\operatorname{tr} b = \kappa \frac{\partial}{\partial \gamma}$ with a smooth $\kappa = \langle b, \frac{\partial}{\partial \gamma}\rangle$.

1.4 Topology from DN-map.

Let $\dot{C}^\infty(\Gamma) := \{f \in C^\infty(\Gamma) \,|\, \int\limits_\Gamma d\gamma\, f(\gamma) = 0\}$ be the subset of smooth functions with zero mean value. For $f \in \dot{C}^\infty(\Gamma)$ we denote by Jf the primitive function with zero mean value: $Jf \in \dot{C}^\infty(\Gamma)$, $\frac{d}{d\gamma} Jf = f$.

Recall the well-known properties of the DN-map: $\operatorname{Ker} \Lambda_g = \{\text{const}\}$, $\operatorname{Ran} \Lambda_g \subset \dot{C}^\infty(\Gamma)$. Note also that by standart elliptic theory the operator $\Lambda_g J : L_2(\Gamma) \mapsto L_2(\Gamma)$, $\operatorname{Dom} \Lambda_g J = \dot{C}^\infty(\Gamma)$ is continuous.

Fix $f \in C^\infty(\Gamma)$; in accordance with the decomposition $\vec{\mathcal{H}} = \vec{\mathcal{E}} \oplus \vec{\mathcal{D}}$ the field $\Phi \nabla u^f \in \vec{\mathcal{H}}$ may be represented in the form

$$\Phi \nabla u^f = \nabla u^p + b^f \tag{1.3}$$

with $\nabla u^p \in \vec{\mathcal{E}}^\infty$ (p being determined by f) and $b^f \in \vec{\mathcal{D}}$; let $\operatorname{tr} b^f = \kappa^f \frac{\partial}{\partial \gamma}$.

Lemma 1 *(i) The equality*

$$[\mathbf{1} + (\mathbf{\Lambda_g J})^2] \frac{\mathbf{df}}{\mathbf{d}\gamma} = \mathbf{\Lambda_g J} \kappa^{\mathbf{f}} \tag{1.4}$$

holds.

(ii) The inclusion $\Phi \nabla u^f \in \vec{\mathcal{E}}^\infty$ is equivalent to the relation

$$[\mathbf{1} + (\mathbf{\Lambda_g J})^2] \frac{\mathbf{df}}{\mathbf{d}\gamma} = \mathbf{0}. \tag{1.5}$$

(iii) The equality

$$\dim [\mathbf{1} + (\mathbf{\Lambda_g J})^2]\, \dot{\mathbf{C}}^\infty(\mathbf{\Gamma}) = \mathbf{1} - \chi(\mathbf{\Omega}) \tag{1.6}$$

is valid.

Proof. (i) The traces of the left and right hand sides of (1.3) may be written as follows:

$$\operatorname{tr} \Phi \nabla u^f = \Phi \operatorname{tr} \nabla u^f = \Phi \,[\frac{df}{d\gamma} \frac{\partial}{\partial \gamma} + \Lambda_g f \frac{\partial}{\partial \nu}] =$$

$$= \frac{df}{d\gamma}\, \Phi \frac{\partial}{\partial \gamma} + \Lambda_g f\, \Phi \frac{\partial}{\partial \nu} = \Lambda_g f\, \frac{\partial}{\partial \gamma} - \frac{df}{d\gamma} \frac{\partial}{\partial \nu} \tag{1.7}$$

and

$$\mathrm{tr}\,[\nabla u^p + b^f] = (\frac{dp}{d\gamma} + \kappa^f)\,\frac{\partial}{\partial\gamma} + \Lambda_g p\,\frac{\partial}{\partial\nu}. \tag{1.8}$$

Using (1.7) and (1.8) we get

$$\Lambda_g f = \frac{dp}{d\gamma} + \kappa^f, \qquad -\frac{df}{d\gamma} = \Lambda_g J\,\frac{dp}{d\gamma}$$

(note that the first equality yields $\kappa^f \in \dot{C}^\infty(\Gamma)$). Eliminating $\frac{dp}{d\gamma}$ one easily gets (1.4).

(ii) In accordance with (1.3) the inclusion $\Phi\nabla u^f \in \vec{\mathcal{E}}^\infty$ is equivalent to $b^f = 0$ that is equivalent to $\kappa^f = 0$ leading to (1.5).

(iii) Let P be the (orthogonal) projection in $\vec{\mathcal{H}}$ onto $\vec{\mathcal{D}}$; we will show that

$$P\Phi\vec{\mathcal{E}}^\infty = \vec{\mathcal{D}}. \tag{1.9}$$

Indeed, if $b \in \vec{\mathcal{D}},\ b \perp P\Phi\vec{\mathcal{E}}^\infty$ then for any smooth f one has

$$0 = \int\limits_\Omega d\Omega\,\langle\Phi\nabla u^f, b\rangle = \int\limits_\Omega d\Omega\,\langle\nabla u^f, \Phi^* b\rangle =$$

$$= \int\limits_\Gamma d\gamma\,u^f\,\langle\frac{\partial}{\partial\nu}, \Phi^* b\rangle = \int\limits_\Gamma d\gamma\,f\,\langle\Phi\frac{\partial}{\partial\nu}, b\rangle = \int\limits_\Gamma d\gamma\,f\,\langle\frac{\partial}{\partial\gamma}, b\rangle.$$

This leads to $\langle\frac{\partial}{\partial\gamma}, b\rangle = 0$, then $\mathrm{tr}\,b = \langle b, \frac{\partial}{\partial\gamma}\rangle\frac{\partial}{\partial\gamma} = 0$ (recall that b's are tangent to Γ) and, finally, to $b = 0$. Thus, the set $P\Phi\vec{\mathcal{E}}^\infty$ is dense in $\vec{\mathcal{D}}$ that implies (1.9) in view of $\dim\vec{\mathcal{D}} < \infty$.

Let f in (1.3) run over $C^\infty(\Gamma)$; due to (1.9) the corresponding projections b^f cover $\vec{\mathcal{D}}$. Since the map $b^f \mapsto \kappa^f$ is injective, one has $\dim\{\kappa^f \,|\, f \in C^\infty(\Gamma)\} = \dim\vec{\mathcal{D}}$. As it is easy to see, the operator $\Lambda_g J$ is injective on $\dot{C}^\infty(\Gamma)$; therefore

$$\dim\vec{\mathcal{D}} = \dim\{\kappa^f \,|\, f \in C^\infty(\Gamma)\} = \dim\{\Lambda_g J\kappa^f \,|\, f \in C^\infty(\Gamma)\}.$$

Taking into account (1.2) and (1.4) we get

$$\dim\,[\mathbf{1} + (\mathbf{\Lambda_g J})^2]\,\frac{\mathbf{d}}{\mathbf{d}\gamma}\mathbf{C}^\infty(\mathbf{\Gamma}) = \dim\vec{\mathcal{D}} = \mathbf{1} - \chi(\mathbf{\Omega})$$

that is equivalent to (1.6). The lemma is proved.

In the situation of the Calderon problem Lemma 1 may be exploited as follows. Assume that Ω is a fortiori known to be homeomorphic to the sphere with handles and one removed disk. In this case $\chi(\Omega) = 1 - 2m$ where m is the number of handles, and one can find m by the formula

$$m = \frac{1}{2} \dim \left[\mathbf{1} + (\mathbf{\Lambda_g J})^{\mathbf{2}} \right] \dot{\mathbf{C}}^{\infty}(\mathbf{\Gamma}).$$

In particular, Ω is homeomorphic to the disk (so that $m = 0$) iff $(\Lambda_g J)^2 = -\mathbf{1}$ on $\dot{C}^{\infty}(\Gamma)$.

Given the DN-map and checking (1.5) one can select a subset of f's in $C^{\infty}(\Gamma)$ such that the 'conjugate' fields $\Phi \nabla u^f$ are also potential. Later we'll use this opportunity.

2 Algebras.

2.1 CBA-dictionary.

We begin with a minimal information concerning the Commutative Banach Algebras (CBA's); for details see [6], [9].

A. The CBA is a (complex or real) Banach space $\mathcal{A}$ equipped with the multiplication operation ab satisfying $ab = ba$; $||ab|| \leq ||a||\,||b||$, $a, b \in \mathcal{A}$. Example: the algebra $C(X)$ of functions continuons on a (topological) compact X with the norm $||a|| = \max_{\Omega} |a(\cdot)|$; the subalgebras of $C(X)$ are called *function algebras*.

CBA is said to be *uniform* if it has the unit $e : ea = a$ and the relation $||a^2|| = ||a||^2$ holds. All the function algebras are uniform.

B. A subspace $I \neq \mathcal{A}$ is called *ideal*, if $ja \in I$ for any $j \in I$, $a \in \mathcal{A}$. Ideal I is *maximal* if for any ideal $\tilde{I} \subset \mathcal{A}$ the relation $I \subset \tilde{I}$ implies $I = \tilde{I}$.

Let $\mathcal{I}$ be the set of maximal ideals of algebra $\mathcal{A}$. Every $I \in \mathcal{I}$ is closed; $\operatorname{codim} I = 1$; the quotient space $\mathcal{A}/I$ is isometrically isomorphic to the field of complex numbers $\mathbf{C}$.

C. Let $\mathcal{A}'$ be the space of continuons functionals on $\mathcal{A}$. A functional $\delta \in \mathcal{A}'$ is called *multiplicative* if $\delta(ab) = \delta(a)\delta(b)$. Example: the Dirac measure $\delta_{x_0} \in C'(X) : \delta_{x_0}(a) = a(x_0)$. The set of multiplicative functionals is denoted by $\mathcal{M}$.

D. There exists a canonical bijection between the sets $\mathcal{M}$ and $\mathcal{I}$: if $\delta \in \mathcal{M}$ then $I_\delta := \operatorname{Ker}\delta \in \mathcal{I}$; if $I \in \mathcal{I}$ then the projection $\delta_I : \mathcal{A} \mapsto \mathcal{A}/I = \mathbf{C}$ is an element of $\mathcal{M}$. In what follows we identify $\mathcal{M}$ to $\mathcal{I}$ through this bijection.

E. The *Gelfand transform* maps element $a \in \mathcal{A}$ into function $\hat{a}$ on $\mathcal{M}$ by the rule: $\hat{a}(\delta) := \delta(a)$, $\delta \in \mathcal{M}$. The *Gelfand topology* is defined as the weakest topology on $\mathcal{M}$ in which all of $\hat{a}$ are continuous. The set $\mathcal{M}$ equipped with this topology is a compact; this compact is called *spectrum* (or the maximal ideal space) of the algebra $\mathcal{A}$ and denoted by $\operatorname{sp}\mathcal{A}$.

The Gelfand transform $\hat{\mathcal{A}} := \{\hat{a} \,|\, a \in \mathcal{A}\}$ of algebra $\mathcal{A}$ is a subalgebra of the algebra $C(\operatorname{sp}\mathcal{A})$. If $\mathcal{A}$ is uniform, the map $a \mapsto \hat{a}$ turns out to be an isometric isomorphism (on its image): $(\alpha a + \beta b + cd)\hat{} = \alpha\hat{a} + \beta\hat{b} + \hat{c}\hat{d}$, $||\hat{a}|| = ||a||$ for all $\alpha, \beta \in \mathbf{C}$ and $a, b, c, d, \in \mathcal{A}$; in this case $\hat{\mathcal{A}}$ is a canonical realization of $\mathcal{A}$ in the form of function algebra.

F. An isomorphism $t : A(X) \mapsto B(Y)$ of two function algebras is called *spatial* if there exists a bijection $\beta : X \mapsto Y$ such that $tw = w \circ \beta^{-1}$.

For a function algebra $\mathcal{A} \subset C(X)$, to each point $x_0 \in X$ one associates the Dirac measure $\delta_{x_0} \in \operatorname{sp}\mathcal{A}$, due to that the natural embedding $\varepsilon : X \mapsto \operatorname{sp}\mathcal{A}$, $\varepsilon(x_0) = \delta_{x_0}$ occurs.

A function algebra $\mathcal{A} \subset C(X)$ is said to be *generic* if ε is a homeomorphism from X onto $\operatorname{sp}\mathcal{A}$. Generic algebra is spatially isomorphic to its Gelfand transform $\hat{\mathcal{A}}$: $\hat{w} = w \circ \varepsilon^{-1}$.

2.2 Algebra $\mathcal{A}(\Omega)$.

The functions $u, u^* \in \operatorname{Harm}_g \Omega$ are said to be *conjugate* if they satisfy the Cauchy-Riemann conditions: $\nabla u^*(x) = \Phi(x)\nabla u(x), \quad x \in \operatorname{int}\Omega$. Note the relation $(u^*)^* = -u \,(\operatorname{mod const})$. If Ω is homeomorphic to the disk each harmonic u possesses the conjugate u^*; it is not the case if topology of Ω is nontrivial [5].

We assign a harmonic u to the set $\mathcal{P}$ if it has the conjugate u^* and $u, u^* \in \operatorname{Harm}_g \Omega \bigcap C(\Omega)$.

let us define the family of complex functions $\mathcal{A}(\Omega) := \{w = u + iu^* \,|\, u \in \mathcal{P}\}$ and mention some of its properties [5]:

(i) in proper (isothermal) local coordinates x^1, x^2 each $w \in \mathcal{A}(\Omega)$ is a holomorphic function of $z = x^1 + ix^2$; functions of $\mathcal{A}(\Omega)$, as local homeomorphisms from Ω into $\mathbf{C}$, determine on Ω a complex structure of the Riemann surface;

(ii) for each $w \in \mathcal{A}(\Omega)$ $|w|$ attains maximum on Γ,
(iii) $\mathcal{A}(\Omega)$ is a closed subalgebra of the (complex) algebra $C(\Omega)$;

Lemma 2 *The algebra $\mathcal{A}(\Omega)$ is generic.*

Proof. (1) The embedding $\varepsilon(\Omega) \subset \operatorname{sp}\mathcal{A}(\Omega)$ is a general fact (see F, sec.2.1) and one needs to prove the converse embedding only.

(2) Choose an $I \in \operatorname{sp}\mathcal{A}(\Omega)$; we are going to show that there exists a point $x_I \in \Omega$ such that $j(x_I) = 0$ for all $j \in I$. Assume the opposite: for any $x \in \Omega$ one can find $j_x \in I$ such that $j_x(x) \neq 0$. Due to continuity of j_x there exists a neighborhood $U_x \ni x$ such that j_x doesn't vanish in U_x. Since $\bigcup\limits_x U_x$ covers Ω, we can select a finite subcovering $U_{x_1}, ..., U_{x_p}$; in this case the corresponding $j_{x_1}, ..., j_{x_p}$ have no common zeros in Ω.

(3) Due to smoothness of the boundary Γ the manifold (Ω, g) may be embedded into a larger manifold $(\Omega', g') : \Omega \subset \Omega', \operatorname{dist}_{g'}(\partial\Omega', \Omega) = r_0 > 0, g'|_\Omega = g$. One way to realize such the embedding is to use the semigeodesic coordinates, extending E on $\tau < 0$.

Denote $\Omega_r := \{x \in \Omega' \,|\, \operatorname{dist}_{g'}(x, \Omega) < r\}, 0 < r < r_0$. Assign a function $w \in \mathcal{A}(\Omega)$ to the family $\mathcal{O}(\Omega)$ if it has a holomorphic continuation in Ω_r (with r depending on w). The family $\mathcal{O}(\Omega)$ is dense in $\mathcal{A}(\Omega)$ (see [4]; p.48, Corollary 2).

(4) Since $\operatorname{codim} I = 1$, the set $I \bigcap \mathcal{O}(\Omega)$ is dense in I. Due to this one can approximate $j_{x_1}, ..., j_{x_p}$ by functions $\tilde{j}_{x_1}, ..., \tilde{j}_{x_p} \in I \bigcap \mathcal{O}(\Omega)$ so that $\tilde{j}_{x_k} \neq 0$ in U_{x_k} and $\tilde{j}_{x_1}, ..., \tilde{j}_{x_p}$ have no common zeros in Ω.

Since the set $\{\tilde{j}_{x_k}\}$ is finite, there exists $\Omega_r \supset \Omega$ such that $\tilde{j}_{x_1}, ..., \tilde{j}_{x_p}$ are defined and have no common zeros in Ω_r. Considering Ω_r as a non-compact Riemann surface one can find functions $g_1, ..., g_p$ holomorphic in Ω_r and satisfying the condition $g_1\tilde{j}_{x_1} + ... + g_p\tilde{j}_{x_p} = 1$ everywhere in Ω_r (see [5], p.205). Reducing functions on Ω we get $\tilde{g}_1\tilde{j}_{x_1} + ... + \tilde{g}_p\tilde{j}_{x_p} = 1$ in Ω for $\tilde{g}_k := g_k|_\Omega \in \mathcal{A}(\Omega)$ and $\tilde{j}_{x_k} \in I$ that yields the inclusion $1 \in I$. The last leads evidently to $I = \mathcal{A}(\Omega)$ that contradicts I to be a proper ideal.

Thus, all the functions belonging to I vanish in a point $x_I \in \Omega$ and the embedding $\varepsilon(\Omega) \supset \operatorname{sp}\mathcal{A}(\Omega)$ is established. So, $\varepsilon(\Omega) = \operatorname{sp}\mathcal{A}(\Omega)$ and it remains to show that ε is a homeomorphism.

(5) Identifying maximal ideals of $\mathcal{A}(\Omega)$ to multiplicative functionals we set $I_{x_0} \equiv \delta_{x_0}$ where δ_{x_0} is the Dirac measure. In terms of the bijection

$\varepsilon : \Omega \mapsto \operatorname{sp}\mathcal{A}(\Omega)$, existence of which has been proved above, the Gelfand transform takes the form $\hat{w}(\delta) := \delta(w) = w(x_\delta) = (w \circ \varepsilon^{-1})(\delta)$ (is a spatial isomorphism).

Topologizing $\operatorname{sp}\mathcal{A}(\Omega)$ by Gelfand we endow it with the weakest topology in which all $\hat{w} \in \hat{\mathcal{A}}(\Omega)$ are continuous. Since the corresponding topology on Ω determined by functions $w \in \mathcal{A}(\Omega)$ coincides with the initial (metric) topology of (Ω, g), the bijection ε turns out to be a homeomorphism. The lemma is proved.

The density of $\mathcal{O}(\Omega)$ in $\mathcal{A}(\Omega)$ used in the proof provides one more important property of the algebra $\mathcal{A}(\Omega)$:

(iv) the smooth subalgebra $\mathcal{A}^\infty(\Omega) := \mathcal{A}(\Omega) \bigcap C^\infty(\Omega)$ is dense in $\mathcal{A}(\Omega)$.

2.3 Conforming metrics.

The algebra $\mathcal{A}(\Omega)$ as well as the corresponding complex structure on Ω (see (i), sec.2.2) are determined by the metric g, the structures corresponding to metrics g and ρg being identical since these metrics define one and the same reserve of harmonic functions [2].

The converse is true in the following sense. We call a metric g *conforming* to the complex structure determined by the algebra $\mathcal{A}(\Omega)$ if $\operatorname{Re}\mathcal{A}(\Omega) \subset \operatorname{Harm}_g \Omega$. It can be shown, that any two metrics conforming to the given structure are distinguished by a functional multiplier.

To construct a metric conforming to a given structure one can use the following standart trick. Let us choose an atlas $\{U_k, \varphi_k\} : \bigcup\limits_k U_k \supset \Omega$; $\varphi_k : U_k \mapsto \mathbf{R}^2$, $\varphi_k(x) = \{\operatorname{Re} w_k(x), \operatorname{Im} w_k(x)\}, w_k \in \mathcal{A}(\Omega)$; let η_k be a subordinated partition of unity: $\eta_k \geq 0$, $\operatorname{supp}\eta_k \subset U_k$, $\sum\limits_k \eta_k = 1$. Define on Ω the tensors $g_{ij}^{(k)} := \eta_k \delta_{ij}$. The tensor $g := \sum\limits_k g^{(k)}$ defines a required metric.

Another way to find a conforming metric is to solve the equation (1.1) with respect to the entries $\det^{\frac{1}{2}} g\, g^{ij}$ for sufficiently many $u \in \operatorname{Re}\mathcal{A}(\Omega)$, i.e. to recover the metric g up to a function multiplier.

[2] In dimension 2 the equalities $\Delta_g u = 0$ and $\Delta_{\rho g} u = 0$ are equivalent.

2.4 Algebra $\mathcal{A}(\Gamma)$.

The set of traces $\mathcal{A}(\Gamma) := \mathrm{tr}\,\mathcal{A}(\Omega) = \{\mathrm{tr}\,w \,|\, w \in \mathcal{A}(\Omega)\}$ is a closed subalgebra of the (complex) algebra $C(\Gamma)$. We list some of its properties.

(i) Since $\max_\Omega |w|$ for $w \in \mathcal{A}(\Omega)$ is attained at Γ, the equality $||w||_{\mathcal{A}(\Omega)} = ||\mathrm{tr}\,w||_{\mathcal{A}(\Gamma)}$ holds. Therefore, the mapping tr is an isometric isomorphism of algebras.

(ii) Since $\mathcal{A}(\Omega)$ is generic, whereas $\mathcal{A}(\Gamma)$ is isometrically isomorphic to $\mathcal{A}(\Omega)$, the Gelfand transform $\hat{\mathcal{A}}(\Gamma) \subset C(\mathrm{sp}\,\mathcal{A}(\Gamma))$ turns out to be spatially isomorphic to $\mathcal{A}(\Omega)$. The corresponding spatial isomorphism from $\mathcal{A}(\Omega)$ onto $\hat{\mathcal{A}}(\Gamma)$ is $\wedge\,\mathrm{tr}$ [3], whereas the corresponding space bijection is $\beta : \Omega \mapsto \mathrm{sp}\,\mathcal{A}(\Gamma)$, $\beta(x_0) = \mathrm{tr}\,I_{x_0}$.

(iii) Define the smooth subalgebra $\mathcal{A}^\infty(\Gamma) := \mathcal{A}(\Gamma) \bigcap C^\infty(\Gamma) = \mathrm{tr}\,\mathcal{A}^\infty(\Omega)$. Since $\mathcal{A}^\infty(\Omega)$ is dense in $\mathcal{A}(\Omega)$ (see (iv), sec.2.2) $\mathcal{A}^\infty(\Gamma)$ turns out to be dense in $\mathcal{A}(\Gamma)$.

The following lemma plays a key role: it shows that the trace algebra $\mathcal{A}(\Gamma)$ is determined by the DN-map. Introduce the set

$$\mathcal{F} := \{f \in C^\infty(\Gamma) \,|\, [\mathbf{1} + (\mathbf{\Lambda_g J})^2]\frac{\mathbf{df}}{\mathbf{d\gamma}} = \mathbf{0}\}$$

of (real) functions satisfying (1.5). To each $f \in \mathcal{F}$ we associate the *conjugate function* f^* such that $\frac{df^*}{d\gamma} = \Lambda_g J \frac{df}{d\gamma}$ holds. As is easy to see, f^* is determined by f up to constant, $f^* \in \mathcal{F}$ and $f^{**} = -f \ (\mathrm{mod\,const})$.

Lemma 3 *The representation*

$$\mathcal{A}(\Gamma) = \mathrm{clos}_{C(\Gamma)}\ \{f + if^* \,|\, f \in \mathcal{F}\} \tag{2.1}$$

is valid.

Proof. If $w = u + iu^* \in \mathcal{A}^\infty(\Omega)$ then $u = u^f$ with $f = \mathrm{tr}\,u$, $u^* = u^p$ with $p = \mathrm{tr}\,u^*$ and $\Phi\nabla u^f = \nabla u^p$. According to (ii) of Lemma 1 the last equality implies that $f, p \in \mathcal{F}$. Taking the traces in the same equality one easily gets $\frac{dp}{d\gamma} = \Lambda_g f = \Lambda_g J \frac{df}{d\gamma}$, so $p = f^*$. Hence $\mathrm{tr}\,w = f + ip = f + if^* \in \mathcal{A}^\infty(\Gamma)$ that establishes the inclusion $\mathcal{A}^\infty(\Gamma) \subset \{f + if^* \,|\, f \in \mathcal{F}\}$.

[3] Here and below (sec.3.2) $\wedge$ denotes the Gelfand transform: $\wedge\mathcal{A} := \hat{\mathcal{A}}$.

Take $f \in \mathcal{F}$ and its conjugate $f^* \in \mathcal{F}$. By virtue of Lemma 1, (ii) the fields ∇u^f and ∇u^{f^*} are potential and satisfy

$$\operatorname{tr} \Phi \nabla u^f = \Lambda_g f \frac{\partial}{\partial \gamma} - \frac{df}{d\gamma}\frac{\partial}{\partial \nu} = \frac{df^*}{d\gamma}\frac{\partial}{\partial \gamma} + \Lambda_g f^* \frac{\partial}{\partial \nu} = \operatorname{tr} \nabla u^{f^*}.$$

Equality of the traces leads to equality of the fields: $\Phi \nabla u^f = \nabla u^{f^*}$ in Ω. Hence, the function $w = u^f + i u^{f^*}$ belongs to $\mathcal{A}^\infty(\Omega)$ whereas its trace $f + i f^*$ belongs to $\mathcal{A}^\infty(\Gamma)$. Thus, we have $\{f + i f^* \,|\, f \in \mathcal{F}\} \subset \mathcal{A}^\infty(\Gamma)$ that establishes the equality $\mathcal{A}^\infty(\Gamma) = \{f + i f^* \,|\, f \in \mathcal{F}\}$.

Since $\mathcal{A}^\infty(\Gamma)$ is dense in $\mathcal{A}(\Gamma)$ one gets (2.1). The lemma is proved.

3 Solving the problem.

3.1 Construction of the manifold.

So, let an operator Λ_g acting in (real) $C^\infty(\Gamma)$ be given and known to be the DN-map of a manifold. By results of J.Lee and G.Uhlmann [8] Λ_g determines the metric tensor at the boundary $g|_\Gamma$; therefore one can also assume the length element $d\gamma$ to be known. Due to the latter the class $\dot{C}^\infty(\Gamma)$ as well as the integration operator J acting in this class are at our disposal.

The following procedure gives a canonical representative of the class of manifolds whose DN-maps coincide with Λ_g.

Step 1. Select the set $\mathcal{F}$ in $C^\infty(\Gamma)$. Through $\mathcal{F}$ recover the algebra $\mathcal{A}(\Gamma)$ by the representation (2.1).

Step 2. Find $\operatorname{sp} \mathcal{A}(\Gamma) =: \Omega$ (see (ii), sec.2.4). Identify the subset of ideals $\{I_{\gamma_0} \,|\, \gamma_0 \in \Gamma\} \subset \Omega$ to Γ by the rule $I_{\gamma_0} \equiv \gamma_0$.

Step 3. Find the Gelfand transform $\hat{\mathcal{A}}(\Gamma) =: \mathcal{A}(\Omega)$. Note that the 'boundary' $\{I_{\gamma_0} \,|\, \gamma_0 \in \Gamma\}$ constructed in the previous step may be also identified as the Shilov boundary of $\mathcal{A}(\Omega)$.

Step 4. Using functions $w \in \mathcal{A}(\Omega)$ as local homeomorphisms $\Omega \mapsto \mathbf{C}$ recover on Ω the complex structure. Its realifying determines on Ω a structure of the smooth 2-dim manifold with the boundary Γ.

Step 5. Equip Ω with a metric $\tilde{g}$ conforming to the complex structure (see sec.2.3). Choose a positive function $\rho \in C^\infty(\Omega)$ such that the metric $g := \rho \tilde{g}$ satisfies $(\rho \tilde{g})_\Gamma = g|_\Gamma$.

The manifold (Ω, g) solves the Calderon problem : its DN-map coincides with Λ_g by construction.

3.2 Proof of Theorem 1.

We recall that $\wedge$ denotes the Gelfand transform: $\wedge\mathcal{A} = \hat{\mathcal{A}}$. The statement of the theorem is just a consequence of the well-determinacy of the procedure described above [4]. Indeed,

if (Ω', g') and (Ω'', g'') have the common DN-map, the corresponding sets $\mathcal{F}$ determined by Λ_g coincide: $\mathcal{F}' = \mathcal{F}''$. Whence, by Lemma 3 the trace algebras coincide: $\mathcal{A}'(\Gamma) = \mathcal{A}''(\Gamma)$;

by the property (ii),sec.2.4 the algebras $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$ are spatially isomorphic to the Gelfand transform of the common trace algebra:

$$\wedge'[\mathrm{tr}'\mathcal{A}(\Omega')] = \hat{\mathcal{A}}'(\Gamma) = \hat{\mathcal{A}}''(\Gamma) = \wedge''[\mathrm{tr}''\mathcal{A}(\Omega'')];$$

for the corresponding space bijections one has

$$\beta'(\Omega') = \mathrm{sp}\,\mathcal{A}'(\Gamma) = \mathrm{sp}\,\mathcal{A}''(\Gamma) = \beta''(\Omega''). \tag{3.1}$$

Therefore the algebras $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$ are spatially isomorphic (through the map $(\mathrm{tr}'')^{-1}\,(\wedge'')^{-1}\,\wedge'\,\mathrm{tr}'$) whereas the manifolds Ω' and Ω'' are connected through the bijection $\beta : \Omega' \mapsto \Omega''$, $\beta = \beta''^{-1}\beta'$;

since the differentiable structures on Ω' and Ω'' are determined by the algebras $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$, the bijection β is a diffeomorphism. Due to (3.1) one has $\beta'(\gamma_0) = I'_{\gamma_0} = I''_{\gamma_0} = \beta''(\gamma_0)$, $\gamma_0 \in \Gamma$ that follows to $\beta(\gamma_0) = \beta''^{-1}(\beta'(\gamma_0)) = \gamma_0$, i.e.$\beta|_\Gamma = \mathrm{id}$;

introduce on Ω' the (induced) metric $\tilde{g}''$ providing β to be isometry from $(\Omega', \tilde{g}'')$ onto (Ω'', g''). Since $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$ are isomorphic, the metric $\tilde{g}''$ (as well as g') turns out to be conforming to the structure determined by $\mathcal{A}(\Omega')$. Due to the last one has $\tilde{g}'' = \rho g'$ with a $\rho \in C^\infty(\Omega')$. By $\beta|_\Gamma = \mathrm{id}$ one has $\tilde{g}''|_\Gamma = g''|_\Gamma$; taking into account $g'|_\Gamma = g''|_\Gamma = g|_\Gamma$ we obtain $\tilde{g}''|_\Gamma = g'|_\Gamma$ implying $\rho|_\Gamma = 1$.

So, β is an isometry from $(\Omega', \rho g')$ onto (Ω'', g'') i.e. (Ω', g') and (Ω'', g'') belong to one and the same conformal class. The theorem is proved.

3.3 Comments and acknowledgements.

- The first variant of this paper published as preprint in June'02 [5] doesn't contain references on M.Lassas and G.Uhlmann results: unfortunatelly,

[4] Such the situation is quite usual for the BC-method.

[5] See www.pdmi.ras.ru/activities/publishing/ PDMI PREPRINT 10/2002).

we didn't know them. We would like to apologize for this confusion and to thank G.Alessandrini and V.Isakov for informing about the paper [7].

- To the best of our knowlege, the relation $\mathrm{sp}\,\mathcal{A}(\Omega) \asymp \Omega$ was first established by J.Wermer [11] in the case of the Riemann surface with analytic Γ homeomorphic to the circle. If $\Omega \subset \mathbf{C}$ the algebra $\mathcal{A}(\Omega)$ is generic in much more general situation (without assumptions on smoothness and connectedness of Γ: see e.g. [6]). Perhaps, the same is valid for Riemann surfaces, so that, most probably, Lemma 2 is well known for specialists in Function Algebras. However, we didn't succeed to find exact references.

- Perhaps, the operator $\Lambda_g J$ has an abstract analog in a class of the uniform algebras with nontrivial Shilov boundary.

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