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Dynamical inverse problem for the Schrödinger equation (the BC-method)

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Let $\Omega \subset \mathbb{R}^n$ be a bounded domain; $\partial\Omega \in C^2$; $\rho, q \in C^2(\overline{\Omega})$ real functions, $\rho > 0$. We show that for any fixed $T > 0$ the response operator $R^T : f \to u|_{\partial\Omega\times[0,T]}$ of the Schrödinger system $i\rho u_t + \Delta u - qu = 0$ in $\Omega \times (0,T)$, $u|_{t=0} = 0$, $\frac{\partial u}{\partial \nu}|_{\partial\Omega\times[0,T]} = f$ determines the coefficients $\rho = \rho(x), q = q(x)$, $x \in \Omega$ uniquely. The problem is reduced to one of recovering ρ, q through the boundary spectral data. The spectral data are extracted from R^T by the use of a variational principle. A peculiarity of the approach (the Boundary Control method) is that it allows to solve the problem using the data on a finite time interval, avoiding a continuation beyond [0,T].

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0 Introduction

0.1 About the paper

In the article [2], using the Boundary Control (BC) method it was shown that the response operator of the 1-dim dynamical system governed by the Schrödinger equation with the potential determines this potential uniquely. In the present paper this result is generalized to the multidimensional case.

Applicability of the BC method to a specific dynamical system depends to a large extent on our knowledge of controllability of the system. Approximate controllability is sufficient for the unique determination of system parameters. However, exact controllability (when such is the case) allows to make a recovery procedure more efficient. In this paper we use important results by R. Triggiani and P.-F. Yao [12] on the exact boundary controllability of the Schrödinger system.

0.2 Main result

Let us consider the Schrödinger dynamical system in a bounded domain $\Omega \subset \mathbb{R}^n$ with the boundary $\Gamma \in C^2$:

$$i\rho\, u_t + \Delta u - qu = 0 \quad \text{in } Q^T; \tag{0.1}$$

$$u|_{t=0} = 0 \quad \text{in } \Omega\,; \tag{0.2}$$

$$\frac{\partial u}{\partial \nu} = f \quad \text{on } \Sigma^T; \tag{0.3}$$

where $\rho, q \in C^2(\overline{\Omega})$ are real functions, $\rho > 0$ in $\overline{\Omega}$; $Q^T := \Omega \times (0,T)$; $\Sigma^T := \Gamma \times [0,T]$; ν is the outward normal to Γ; $f \in L_2(\Sigma^T)$ is the boundary control; $u = u^f(x,t)$ is the solution.

To the system (0.1)–(0.3) one associates the response operator $R^T : f \mapsto u^f|_{\Sigma^T}$. The dynamical inverse problem is to recover the coefficients ρ and q from the given operator R^T. Our main result is the following.

Theorem 1 *For any fixed $T > 0$ the operator R^T uniquely determines the functions ρ and q in Ω.*

A procedure recovering the coefficients is proposed.

0.3 The approach

Let us consider the boundary value spectral problem

$$-\Delta\varphi + q\varphi = \lambda\rho\varphi \quad \text{in } \Omega; \tag{0.4}$$

$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Gamma; \tag{0.5}$$

let $\{\lambda_k\}_{k=1}^{\infty} : \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ be its spectrum, and let $\{\varphi_k\}_{k=1}^{\infty}$ be (real) eigenfunctions normalized by the condition $\int\limits_{\Omega} \varphi_k\varphi_l\, \rho dx = \delta_{kl}$.

The set of the pairs $\{\lambda_k; \varphi_k|_{\Gamma}\}_{k=1}^{\infty}$ is said to be the (Neumann) spectral data of the problem (0.4)–(0.5). The spectral data, as well as the eigenfunctions corresponding to the given ρ and q, are evidently not unique. Nevertheless, *any* spectral data determine ρ and q in Ω uniquely [11].

The Theorem 1 is proved in two steps:

(i) we state a variational principle which allows to extract the spectral data from the response operator;

(ii) we determine ρ and q by the spectral data using the result by R.G. Novikov [11] mentioned above.

A variational principle for determination of the spectral data was introduced in [3] for the heat equation; here we propose its modified version available for the Schrödinger system. This modification constitutes the dominant bulk of the paper.

A peculiarity of the proposed approach is that all the considerations are restricted to a *finite* time interval $[0, T]$. We avoid continuations to the semiaxis $t > 0$ and the subsequent Fourier-Laplace transforms: from our standpoint, these devices are not quite natural in time-domain inverse problems.

1 Dynamical system

1.1 Spaces and operators

Let us endow the system (0.1)–(0.3) with the standard attributes of control theory, these are spaces and operators. All the spaces are assumed to be complex.

The space of controls $\mathcal{F}^T := L_2(\Sigma^T)$ (with the measure $d\Gamma dt$) is the *outer space* of the system (0.1)–(0.3). It contains a dense linear set of smooth controls $\mathcal{L}^T := \{f \in C^\infty(\Sigma^T) | \operatorname{supp} f \subset \Gamma \times (0, T]\}$; for $f \in \mathcal{L}^T$ the system (0.1)–(0.3) has a unique classical solution (trajectory) u^f.

The space $\mathcal{H} := L_{2,\rho}(\Omega)$ (with the measure ρdx) is said to be *inner*; the states $u^f(\cdot, t), 0 \le t \le T$ are its elements. The "input $\mapsto$ state" correspondence is described by the *control operator* W^T,

$$W^T : \mathcal{F}^T \mapsto \mathcal{H}, \quad \operatorname{Dom} W^T = \mathcal{L}^T, \quad W^T f := u^f(\cdot, T)\,.$$

The "input $\mapsto$ output" correspondence is realized by the *response operator* R^T,

$$R^T : \mathcal{F}^T \mapsto \mathcal{F}^T, \;\; \operatorname{Dom} R^T = \mathcal{L}^T, \;\; R^T f := u^f|_{\Sigma^T}\,.$$

Let $\mathcal{T}^T_{T-\tau} : \mathcal{F}^T \mapsto \mathcal{F}^T$ be the delay operator:

$$(\mathcal{T}^T_{T-\tau} f)(\cdot, t) := \begin{cases} 0, & 0 \le t \le T - \tau; \\ f(\cdot, t - (T - \tau)), & T - \tau < t \le T \end{cases}$$

($\tau \in [0, T]$ is a parameter). Since the coefficients ρ and q do not depend on time, the system (0.1)–(0.3) offers the stationary state property:

$$u^{\mathcal{T}^T_{T-\tau} f}(\cdot, T) = W^T \mathcal{T}^T_{T-\tau} f = u^f(\cdot, \tau)\,, \tag{1.1}$$

which implies the relations

$$u^{if_t} = iu^f_t = < \text{see}\,(0.1) > = \rho^{-1}[-\Delta + q]u^f. \tag{1.2}$$

1.2 Dual system

The system

$$i\rho\, v_t + \Delta v - qv = 0 \quad \text{in } Q^T; \tag{1.3}$$

$$v|_{t=T} = y \quad \text{in } \Omega; \tag{1.4}$$

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on } \Sigma^T \tag{1.5}$$

is called *dual* to the system (0.1)–(0.3). Let us define a linear set $\mathcal{Y} := \{y \in \mathcal{H} |\, [\rho^{-1}(-\Delta + q)]^k y \in \mathcal{H};\; \frac{\partial}{\partial \nu}[\rho^{-1}(-\Delta + q)]^k y\,|_\Gamma = 0,\; k = 0, 1, ...\}$[1] $\subset \mathcal{H}$; for $y \in \mathcal{Y}$ the system (1.3)–1.5) has a unique classical solution $v = v^y(x, t)$. The following result explains what we mean by duality of the two systems.

[1]the conditions defining the set $\mathcal{Y}$ are standard compatibility conditions in Ω and on Γ for $t = T$.

Lemma 1 *For $f \in \mathcal{L}^T$, $y \in \mathcal{Y}$ the relation*

$$(u^f(\cdot, T), y)_{\mathcal{H}} = i\,(f, v^y|_{\Sigma^T})_{\mathcal{F}^T} \tag{1.6}$$

holds.

$\triangleleft$ The equalities

$$0 = \int_{Q^T} dxdt\,[i\rho\, u_t^f + \Delta u^f - qu^f]\bar{v}^y =$$

$$= \int_\Omega \rho dx \int_0^T dt\, i\, u_t^f \bar{v}^y + \int_0^T dt \int_\Omega dx\,[\Delta u^f - qu^f]\,\bar{v}^y =$$

$$= i\int_\Omega \rho dx\, u^f(\cdot, T)\bar{y} + \int_{\Sigma^T} d\Gamma dt\,\frac{\partial u^f}{\partial \nu}\,\bar{v}^y + \int_{Q^T} dxdt\, u^f\,\overline{[iv_t^y + \Delta v^y - qv^y]} =$$

$$= i\,(u^f(\cdot, T),\, y)_{\mathcal{H}} + (f,\, v^y|_{\Sigma^T})_{\mathcal{F}^T}$$

yield (1.6). $\triangleright$

Using (1.1), the duality relation (1.6) can be rewritten in more general form:

$$(u^f(\cdot, \tau), y)_{\mathcal{H}} = i\,(\mathcal{T}_{T-\tau}^T f, v^y|_{\Sigma^T})_{\mathcal{F}^T}. \tag{1.7}$$

In the rest of Subsection 1.2 we derive some implications of the duality relation.

Let us remind that λ_k and φ_k are eigenvalues and eigenfunctions of the problem (0.4),(0.5). Setting $y = \varphi_k$ in (1.4) we easily get $v^{\varphi_k} = e^{i\lambda_k(T-t)}\varphi_k$. Therefore, by the force of (1.6),

$$(u^f(\cdot, T), \varphi_k)_{\mathcal{H}} = (f, e_k^T)_{\mathcal{F}^T}\,, \tag{1.8}$$

where $e_k^T \in \mathcal{F}^T$, $e_k^T(\cdot, t) := -iv^{\varphi_k}|_{\Sigma^T} = -ie^{i\lambda_k(T-t)}\varphi_k|_\Gamma$.

In what follows $H^s(\cdot)$ stands for Sobolev spaces. Let $\mathcal{H}^1$ be $H^1(\Omega)$ endowed with the metric

$$(y, w)_{\mathcal{H}^1} := \int_\Omega dx[\nabla y \nabla \overline{w} + (q - \lambda_1 + \alpha)y\overline{w}]$$

$(\alpha > 0)$, which is equivalent to the standard metric.

Let us consider a linear set of controls $\mathcal{L}_0^T := \{f \in \mathcal{L}^T | f(\cdot, T) = 0\}$; $\mathcal{L}_0^T$ is dense in $\mathcal{F}^T$. For $f \in \mathcal{L}_0^T$ we have: $\frac{\partial}{\partial\nu} u^f(\cdot, T)|_\Gamma = f(\cdot, T) = 0$. Taking $f \in \mathcal{L}_0^T$, $y \in \mathcal{Y}$ and integrating by parts we get

$$(u^f(\cdot, T), y)_{\mathcal{H}^1} = (\rho^{-1}[-\Delta + q - \lambda_1 + \alpha]u^f(\cdot, T), y)_{\mathcal{H}} = <\text{see}\,(1.2)> =$$

$$= (u^{if_t - (\lambda_1 - \alpha)f}(\cdot, T), y)_{\mathcal{H}} = <\text{see}\,(1.6)> =$$

$$= i(if_t - (\lambda_1 - \alpha)f, v^y|_{\Sigma^T})_{\mathcal{F}^T} = i(f, [i\frac{\partial}{\partial t} - (\lambda_1 - \alpha)]v^y|_{\Sigma^T})_{\mathcal{F}^T}. \tag{1.9}$$

Putting $y = \varphi_k$ and taking into account that $v^y = e^{i\lambda_k(T-t)}\varphi_k$, (1.9) implies:

$$(u^f(\cdot, T), \varphi_k)_{\mathcal{H}^1} = (\lambda_k - \lambda_1 + \alpha)(f, e_k^T)_{\mathcal{F}^T}. \tag{1.10}$$

1.3 Spectral representation

Representing the solution of the system (0.1)–(0.3) as a Fourier series with respect to eigenfunctions of the problem (0.4), (0.5), we have:

$$W^T f = u^f(\cdot, T) = \sum_{k=1}^{\infty} (u^f(\cdot, T), \varphi_k)_{\mathcal{H}} \varphi_k =$$

$$= <\text{see}\,(1.8)> = \sum_{k=1}^{\infty} (f, e_k^T)_{\mathcal{F}^T} \varphi_k. \tag{1.11}$$

The following result shows that the response operator is determined by the spectral data.

Lemma 2 *The representation*

$$(R^T f)(\cdot, t) = \sum_{k=1}^{\infty} (\mathcal{T}_{T-t}^T f, e_k^T)_{\mathcal{F}^T} \varphi_k|_\Gamma \tag{1.12}$$

is valid with the series converging in $H^{1/2}(\Gamma)$ for each $t \in [0, T]$.

$\triangleleft$ From (1.7), (1.11) we have:

$$u^f(\cdot,\tau)=\sum_{k=1}^{\infty}(\mathcal{T}^T_{T-\tau}f,e^T_k)_{\mathcal{F}^T}\varphi_k. \tag{1.13}$$

One can easily check that $(\varphi_j,\varphi_k)_{\mathcal{H}^1}=0 \quad (j\neq k)$; it means that the series (1.13) is orthogonal in $\mathcal{H}^1$. Since $u^f(\cdot,\tau)\in\mathcal{H}^1$, the series (1.13) converges to $u^f(\cdot,\tau)$ in $\mathcal{H}^1$ and, therefore, in $H^1(\Omega)$. Passing in (1.13) to the traces on Γ, we complete the proof of the lemma. $\triangleright$

For the classical solution of the problem (1.3)–(1.5) one has the following well-known representation:

$$v^y(\cdot,t)=\sum_{k=1}^{\infty}e^{i\lambda_k(T-t)}(y,\varphi_k)_{\mathcal{H}}\varphi_k. \tag{1.14}$$

1.4 Generalized solutions

It is easy to see that for any $y\in\mathcal{H}$ the right hand side of the equality (1.14) belongs to $C([0,T];\mathcal{H})$; it is said to be a generalized solution of the problem (1.3)–(1.5). One can also check that the map $y\mapsto v^y$ is continuous from $\mathcal{H}^1$ to $C([0,T];\mathcal{H}^1)$.

Let us now define a generalized solution to the problem (0.1)–(0.3). Introduce the rigging $\mathcal{H}^1\subset\mathcal{H}\subset\mathcal{H}^{-1}:=(\mathcal{H}^1)'$. The duality relation (1.7), the standard trace theorem, and the continuity of $y\mapsto v^y$ mentioned above imply:

$$|(u^f(\cdot,\tau),y)_{\mathcal{H}}|\leq\ ||\mathcal{T}^T_{T-\tau}f||_{\mathcal{F}^T}||v^y|_{\Sigma^T}||_{\mathcal{F}^T}\leq$$
$$\leq\text{const}\,||\mathcal{T}^T_{T-\tau}f||_{\mathcal{F}^T}||v^y||_{C([0,T];\mathcal{H}^1)}\leq\text{const}\,||\mathcal{T}^T_{T-\tau}f||_{\mathcal{F}^T}||y||_{\mathcal{H}^1},$$

which yields

$$||u^f(\cdot,\tau)||_{\mathcal{H}^{-1}}\leq\text{const}\,||\mathcal{T}^T_{T-\tau}f||_{\mathcal{F}^T}. \tag{1.15}$$

Using the last estimate one can easily check that the map $f\mapsto u^f$ is continuous from $\mathcal{F}^T$ to $C([0,T];\mathcal{H}^{-1})$. Let us extend this map by continuity from $\mathcal{L}^T$ to all $\mathcal{F}^T$. We take the words solution $u^f\in C([0,T];\mathcal{H}^{-1})$ of the problem (0.1)–(0.3) with $f\in\mathcal{F}^T$ to mean the image of f under the action of this extension.

We shall use below the following regularity result concerning generalized solutions with controls from

$$C_0^1([0,T];L_2(\Gamma)) := \{f \in C^1([0,T];L_2(\Gamma)) \mid f(\cdot,0) = f_t(\cdot,0) = 0\}$$

(with the norm of the space $C^1([0,T];L_2(\Gamma))$).

Lemma 3 *The map $f \mapsto u^f$ is continuous from $C_0^1([0,T];L_2(\Gamma))$ to $C([0,T];H^1(\Omega))$.*

$\triangleleft$ For $f \in \mathcal{L}^T$ the following a priori estimate can be easily obtained from (1.15):

$$||u^f||_{C^1([0,T];H^{-1}(\Omega))} \le \text{const}\, ||f||_{C^1([0,T];L_2(\Gamma))}. \tag{1.16}$$

Let $w = w^f(x,t)$ be the solution to the problem

$$(-\Delta + \varepsilon)w = 0 \quad \text{in } \Omega;$$

$$\frac{\partial w}{\partial \nu} = f(\cdot,t) \quad \text{on } \Gamma$$

($\varepsilon > 0$ is fixed; $t \in [0,T]$ is a parameter). From the standard theory of elliptic problems [10] it follows:

$$||w^f||_{C^1([0,T];H^{3/2}(\Omega))} \le \text{const}\, ||f||_{C^1([0,T];L_2(\Gamma))}. \tag{1.17}$$

One can check that the function $y^f := u^f - w^f$ solves the problem

$$(-\Delta + \varepsilon)\, y = G\,(\cdot,t) \quad \text{in } \Omega;$$

$$\frac{\partial y}{\partial \nu} = 0 \quad \text{on } \Gamma$$

with the right hand side $G = i\rho u_t^f - qu^f + \varepsilon u^f$. From the elliptic theory

$$||y^f||_{C([0,T];H^1(\Omega))} \le \text{const}\, ||G||_{C([0,T];H^{-1}(\Omega))} \le$$

$$\le \text{const}\, ||u^f||_{C^1([0,T];H^{-1}(\Omega))} \le \quad <\text{see}\,(1.16)>$$

$$\le \text{const}\, ||f||_{C^1([0,T];L_2(\Gamma))}. \tag{1.18}$$

Taking into account (1.17), (1.18), we get the estimate for $u^f = w^f + y^f$:

$$||u^f||_{C([0,T];H^1(\Omega))} \le \text{const}\, ||f||_{C^1([0,T];L_2(\Gamma))}.$$

This proves the lemma by virtue of the density of the inclusion $\mathcal{L}^T \subset C_0^1([0,T];L_2(\Gamma))$. $\triangleright$

From this lemma it follows that the response operator R^T acts continuously from $C_0^1([0,T];L_2(\Gamma))$ to $C([0,T];H^{1/2}(\Gamma))$, and the representation (1.12) is valid for $f \in C_0^1([0,T];L_2(\Gamma))$. In what follows we consider R^T to be extended by continuity to $C_0^1([0,T];L_2(\Gamma))$.

1.5 Extensions of the control operator

Since $u^f \in C([0,T];\mathcal{H}^{-1})$, the trace $u^f(\cdot,T) \in \mathcal{H}^{-1}$ is well defined. Along with it, we can define the operator

$$W^T_{\max} : \mathcal{F}^T \mapsto \mathcal{H},\ \ \operatorname{Dom} W^T_{\max} = \{f \in \mathcal{F}^T \,|\, u^f(\cdot,T) \in \mathcal{H}\},\ \ W^T_{\max} f := u^f(\cdot,T),$$

which is the maximal (natural) extension of W^T. One can easily see that the maximal operator is closed: $W^T_{\max} = \overline{W^T_{\max}}$.

In a similar way we define the operator

$$W^T_1 : \mathcal{F}^T \mapsto \mathcal{H}^1,\ \ \operatorname{Dom} W^T_1 = \{f \in \mathcal{F}^T \,|\, u^f(\cdot,T) \in \mathcal{H}^1\},\ \ W^T_1 f := u^f(\cdot,T);$$

note that $W^T_1 = \overline{W^T_1}$.

In the outer space we introduce a linear set of smooth and finite in time controls $\mathcal{M}^T_0 := \{f \in C^\infty([0,T];L_2(\Gamma)) \,|\, \operatorname{supp} f \subset (0,T)\}$. Let us define the operator

$$W^T_0 : \mathcal{F} \mapsto \mathcal{H}^1,\ \ \operatorname{Dom} W^T_0 = \mathcal{M}^T_0,\ \ W^T_0 f := u^f(\cdot,T),$$

so that $W^T_0 \subset W^T_1$. The rest of Section 1 is devoted to the proof of the relation

$$\overline{W^T_0} = W^T_1. \tag{1.19}$$

The proof is given in Subsection 1.6, it is preceded by lemmas about the kernel and image of the operator W^T_0.

Lemma 4 *The following relation is valid:*

$$\operatorname{clos}_{\mathcal{F}^T} \operatorname{Ker} W^T_0 = \operatorname{Ker} W^T_1. \tag{1.20}$$

$\triangleleft$ *Step 1.*

Let $\bigvee$ be the closure of the linear span of a set of elements. Let us prove the equality

$$\mathrm{clos}_{\mathcal{F}^T}\mathrm{Ran}\,(W_1^T)^* = \bigvee_{k=1}^{\infty} e_k^T. \tag{1.21}$$

From (1.10) it easily follows that

$$\varphi_k \in \mathrm{Dom}\,(W_1^T)^*,\ \ (W_1^T)^*\varphi_k = (\lambda_k - \lambda_1 + \alpha)\, e_k^T\,.$$

It means that the right hand side of (1.21) is a subset of the left one.

Let now $f \in \mathrm{clos}_{\mathcal{F}^T}\mathrm{Ran}\,(W_1^T)^*$ and $f \perp \{e_k^T\}_{k=1}^{\infty}$. Then, since $\varphi_k \in \mathcal{H}^1$, we have from (1.8): $\mathcal{H}^{-1} \ni u^f(\cdot, T) = 0 \in \mathcal{H}^1$. So, $f \in \mathrm{Dom}\,W_1^T$ and $W_1^T f = 0$, i.e. $f \in \mathrm{Ker}\,W_1^T$. From this inclusion, using the well-known equality $\mathcal{F}^T = \mathrm{Ker}\,W_1^T \oplus \mathrm{clos}_{\mathcal{F}^T}\mathrm{Ran}\,(W_1^T)^*$ we have $f \perp \mathrm{clos}_{\mathcal{F}^T}\mathrm{Ran}\,(W_1^T)^*$. Then $f = 0$, which proves (1.21).

Step 2.

Let us consider the family of subspaces $\mathcal{F}^T_{T-\tau} := \mathcal{T}^T_{T-\tau}\mathcal{F}^T \subset \mathcal{F}^T,\ \ 0 \le t \le T$ and corresponding operators $W^T_{1,T-\tau} : \mathcal{F}^T_{T-\tau} \to \mathcal{H}^1,\ W^T_{1,T-\tau} := W_1^T|_{\mathcal{F}^T_{T-\tau}}$; note the operators are closed. Put $\mathcal{K}^T_{T-\tau} := \mathrm{Ker}\,W^T_{1,T-\tau};\ \mathcal{K}^T := \bigcup\limits_{0<\tau<T} \mathcal{K}^T_{T-\tau}$. Let us show that

$$\mathrm{clos}_{\mathcal{F}^T}\mathcal{K}^T = \mathrm{Ker}\,W_1^T. \tag{1.22}$$

Let $P^T_{T-\tau}$ be the (cutting) projector in $\mathcal{F}^T$ to $\mathcal{F}^T_{T-\tau}$. The relation

$$P^T_{T-\tau}\mathrm{clos}_{\mathcal{F}^T}\mathrm{Ran}\,(W^T_{1,T-\tau})^* = \bigvee_{k=1}^{\infty} P^T_{T-\tau}e_k^T \tag{1.23}$$

follows from the stationary state property of the system (0.1)–(0.3) and from (1.21); it is really a form of the relation (1.21) with the change of T for τ followed by the shift of time $t \mapsto t - (T-\tau)$.

The inclusion of the left hand side into the right one (1.22) is obvious; let us prove the inverse inclusion. Choose $g \in \mathrm{Ker}\,W_1^T$ such that $g \perp \mathcal{K}^T$. Since g is orthogonal to each $\mathcal{K}^T_{T-\tau}$, then $P^T_{T-\tau}g \in P^T_{T-\tau}\,\mathrm{clos}_{\mathcal{F}^T}\mathrm{Ran}\,(W^T_{1,T-\tau})^*$ and therefore, $P^T_{T-\tau}g \in \bigvee\limits_{k=1}^{\infty} P^T_{T-\tau}e_k^T$ due to (1.23). From the latter relation, taking into account $\lim\limits_{\tau\to T} P^T_{T-\tau} = \mathbf{1}$, it is easy to get that $g \in \bigvee\limits_{k=1}^{\infty} e_k^T$. Returning to (1.21), we have: $g \in \mathrm{clos}_{\mathcal{F}^T} \mathrm{Ran}\,(W_1^T)^* = \mathcal{F}^T \ominus \mathrm{Ker}\, W_1^T$, which means $g \perp \mathrm{Ker}\, W_1^T$. Therefore, $g = 0$, resulting in (1.22).

Step 3.

Let us show that

$$\mathrm{clos}_{\mathcal{F}^T} \mathrm{Ker}\, W_0^T \supset \mathcal{K}^T. \tag{1.24}$$

Fix $\tau \in (0, T)$, and let $f \in \mathcal{K}^T_{T-\tau}$ be such that $u^f(\cdot, T) = 0$. Take a positive $\varepsilon < \frac{T-\tau}{10}$ and denote

$$f_\varepsilon(\cdot, T) := \begin{cases} f(\cdot, t+\varepsilon), & 0 \le t \le T - \varepsilon; \\ 0, & T - \varepsilon < t \le T. \end{cases}$$

Using the stationary state property one can check that $u^{f_\varepsilon}(\cdot, t) = 0$ for $t \in [T - \varepsilon, T]$ and, in particular, $u^{f_\varepsilon}(\cdot, T) = 0$, which means $f_\varepsilon \in \mathrm{Ker}\, W_1^T$.

Take a function $\eta = \eta(t) \in C^\infty(-\infty, \infty)$ such that $\eta \ge 0$, $\mathrm{supp}\, \eta \subset [-1, 1]$, $\int_{-\infty}^{\infty} \eta(t)\, dt = 1$, and put $\eta_\varepsilon(t) := \frac{1}{\varepsilon}\eta(\frac{t}{\varepsilon})$.

Extend f_ε by zero off $0 \le t \le T$, take the convolution $f_\varepsilon * \eta_\varepsilon$ and restrict it to the interval $[0, T]$. Let $f_{\varepsilon\varepsilon}$ be the result these operations. One can check directly that $u^{f_{\varepsilon\varepsilon}}(\cdot, T) = 0$, so $f_{\varepsilon\varepsilon} \in \mathrm{Ker}\, W_1^T$. From the other hand, $\mathrm{supp}\, f_{\varepsilon\varepsilon} \subset \Gamma \times (0, T)$ and $f_{\varepsilon\varepsilon} \in C^\infty([0, T]; L_2(\Gamma))$, and so $f_{\varepsilon\varepsilon} \in \mathcal{M}_0^T$. Finally we have: $f_{\varepsilon\varepsilon} \in \mathrm{Ker}\, W_1^T \bigcap \mathcal{M}_0^T \subset \mathrm{Ker}\, W_0^T$.

As $\varepsilon \to 0$ the convergence $f_{\varepsilon\varepsilon} \to f$ takes place in $\mathcal{F}^T$. Thus, we have approximated the original control $f \in \mathcal{K}^T_{T-\tau}$ by $f_{\varepsilon\varepsilon} \in \mathrm{Ker}\, W_0^T$, which implies (1.24).

The relation (1.20) follows from (1.24) and (1.22). $\triangleright$

Lemma 5 *The relation*

$$\mathrm{clos}_{\mathcal{H}^1} \mathrm{Ran}\, W_0^T = \mathcal{H}^1. \tag{1.25}$$

is valid.

$\triangleleft$ Since $\mathcal{Y}$ is dense in $\mathcal{H}^1$, the equalities (1.9) can be extended to all $y \in \mathcal{H}^1$.

Let y be such that $(u^f(\cdot, T), y)_{\mathcal{H}^1} = 0$ for all $f \in \mathcal{L}_0^T \subset \mathcal{M}_0^T$; we shall prove that $y = 0$.

According to the last equality of (1.9) we have:

$$(if_t - (\lambda_1 - \alpha) f, v^y|_{\Sigma^T})_{\mathcal{F}^T} = 0.$$

Because f is an arbitrary function of $\in \mathcal{L}_0^T$ it follows: $v^y|_{\Sigma^T} = e^{i(\lambda_1-\alpha)(T-t)}a$ with some $a \in L_2(\Gamma)$.

The function $w^y := e^{i(\lambda_1-\alpha)(T-t)}v^y$ satisfies the system

$$i\rho w_t^y + [\Delta - q + \lambda_1 - \alpha]\, w^y = 0 \quad \text{in } Q^T; \tag{1.26}$$

$$w^y = a, \quad \frac{\partial w^y}{\partial \nu} = 0 \quad \text{on } \Sigma^T. \tag{1.27}$$

Choose a (small) $\varepsilon > 0$; let η_ε be the function used in the proof of Lemma 4. The regularization $w_\varepsilon^y := w^y * \eta_\varepsilon$ is well defined in $\Omega \times (\varepsilon, T-\varepsilon)$. According to (1.14), we have the representation

$$w_\varepsilon^y(\cdot, t) = \sum_{k=1}^{\infty} [e^{i(\lambda_k - \lambda_1 + \alpha)(T-t)} * \eta_\varepsilon]\, (y, \varphi_k)_{\mathcal{H}}\, \varphi_k =$$

$$= \sum_{k=1}^{\infty} e^{i(\lambda_k - \lambda_1 + \alpha)(T-t)} (y, \varphi_k)_{\mathcal{H}}\, \kappa_k^\varepsilon\, \varphi_k = \sum_{k=1}^{\infty} e^{i(\lambda_k - \lambda_1 + \alpha)(T-t)} (y_\varepsilon, \varphi_k)_{\mathcal{H}}\, \varphi_k,$$

where

$$\kappa_k^\varepsilon := \int_{-1}^{1} e^{i(\lambda_k - \lambda_1 + \alpha)\, \varepsilon s} \eta(s)\, ds, \qquad y_\varepsilon := \sum_{k=1}^{\infty} (y, \varphi_k)_{\mathcal{H}}\, \kappa_k^\varepsilon\, \varphi_k.$$

As $k \to \infty$ we have $\kappa_k^\varepsilon = o(\lambda_k^{-N})$ with any $N > 0$; therefore $y_\varepsilon \in \mathcal{Y}$ and the solution w_ε^y is the classical one. The conditions (1.26), (1.27) imply

$$i\rho (w_\varepsilon^y)_t + [\Delta - q + \lambda_1 - \alpha]\, w_\varepsilon^y = 0 \quad \text{in } \Omega \times (\varepsilon, T-\varepsilon); \tag{1.28}$$

$$w_\varepsilon^y = a, \quad \frac{\partial w_\varepsilon^y}{\partial \nu} = 0 \quad \text{on } \Gamma \times [\varepsilon, T-\varepsilon]. \tag{1.29}$$

Differentiating with respect to time, we get for $\dot{w}_\varepsilon^y := (w_\varepsilon^y)_t$:

$$i\rho (\dot{w}_\varepsilon^y)_t + [\Delta - q + \lambda_1 - \alpha]\, \dot{w}_\varepsilon^y = 0 \quad \text{in } \Omega \times (\varepsilon, T-\varepsilon); \tag{1.30}$$

$$\dot{w}_\varepsilon^y = 0, \quad \frac{\partial \dot{w}_\varepsilon^y}{\partial \nu} = 0 \quad \text{on } \Gamma \times [\varepsilon, T-\varepsilon]. \tag{1.31}$$

On the strength of the well-known uniqueness theorem (see [9], Theorem 17.2.6), the solution of the equation (1.30) with zero Cauchy data (1.31) is identically zero: $\dot{w}^y_\varepsilon = 0$ in $\Omega \times (\varepsilon, T-\varepsilon)$. Hence, w^y_ε does not depend on time, and from (1.28),(1.29) we get:

$$[\Delta - q + \lambda_1 - \alpha]\, w^y_\varepsilon = 0 \quad \text{in } \Omega; \tag{1.32}$$

$$\frac{\partial w^y_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Gamma. \tag{1.33}$$

Multiplying (1.32) by $\overline{w^y_\varepsilon}$, integrating over Ω, and taking into account (1.33) we get: $||w^y_\varepsilon||^2_{\mathcal{H}^1} = 0$, and then $y_\varepsilon = 0$. Putting $\varepsilon \to 0$ and using the convergence $y_\varepsilon \to y$, we obtain $y = 0$. $\triangleright$

In the course of the proof we have got a stronger relation $\mathrm{clos}_{\mathcal{H}^1} W^T \mathcal{L}^T_0 = \mathcal{H}^1$. In control theory it is called approximate controllability of the system (0.1)–(0.3) in the class $H^1(\Omega)$ (on controls from $\mathcal{L}^T_0$).

1.6 Exact controllability by Triggiani and Yao

As it has been shown in the paper of R. Triggiani and P.-F. Yao [12], for any $T > 0$ the relation is valid:

$$\mathrm{Ran}\, W^T_1 = H^1(\Omega), \tag{1.34}$$

that is called the exact (boundary) controllability of the Schrödinger system in the class $H^1(\Omega)$ on controls from $L_2(\Sigma^T)$[2]. We shall prove that the relation (1.19) is the consequence of this remarkable result. The proof is preceded by a general type lemma.

Lemma 6 *Let F and H be Hilbert spaces, the operator $W : F \mapsto H$ densely defined, closed, and surjective. Let $W_0 \subset W$ be its restriction with the properties:* (α) $\mathrm{clos}_F \mathrm{Ker}\, W_0 = \mathrm{Ker}\, W$, (β) $\mathrm{clos}_H \mathrm{Ran}\, W_0 = H$. *Then* $\overline{W_0} = W$.

$\triangleleft$ From $W_0 \subset W$ we have $\mathrm{Ker}\, W_0 \subset \mathrm{Ker}\, W$; from (α) it follows: $\mathrm{Ker}\, \overline{W_0} = \mathrm{Ker}\, W =: F_0$.

Denote $F_\perp := F \ominus F_0$. The operator $W_\perp : F_\perp \mapsto H$, $W_\perp := W|_{F_\perp}$ is closed, injective, and surjective (since W is surjective). Therefore, the operator $W_\perp^{-1}$ is defined on all H and closed, so it is bounded.

[2]In [12] the boundary conditions of the form $u|_{\Gamma_0 \times [0,T]} = 0$, $\frac{\partial u}{\partial \nu}\big|_{(\Gamma \setminus \Gamma_0) \times [0,T]} = f$, $\Gamma_0 \neq \oslash$ were considered. However, the techniques work also in the case of $\Gamma_0 = \oslash$ (personal communication of R. Triggiani).

The operator $\overline{W}_{0\perp} : F_\perp \to H$, $\overline{W}_{0\perp} := \overline{W_0}|_{F_\perp}$ is closed and injective; along with that $\overline{W}_{0\perp} \subset W_\perp$, and the linear set $\operatorname{Ran} \overline{W}_{0\perp}$ is dense in H by virtue of (β). Therefore, the operator $(\overline{W}_{0\perp})^{-1}$ is densely defined in H, bounded (as a restriction of $(W_\perp)^{-1}$), and closed. It immediately follows that $(\overline{W}_{0\perp})^{-1} = (W_\perp)^{-1}$, so $\overline{W}_{0\perp} = W_\perp$, and at last, $\overline{W_0} = W$ in view of the equality of the kernels of $\overline{W}_{0\perp}$ and W. $\triangleright$

Setting $F = \mathcal{F}^T$, $H = \mathcal{H}$, $W = W_1^T$, $W_0 = W_0^T$ and using the equalities (1.20), (1.25), we come to (1.19).

2 Solving the inverse problem

2.1 The form c^T

Let us introduce the linear set $\mathcal{M}^T := \{f \in C^\infty([0,T]; L_2(\Gamma)) | \operatorname{supp} f \subset (0,T]\} \supset \mathcal{M}_0^T$, and define the Hermite bilinear form $c^T : \mathcal{F}^T \times \mathcal{F}^T \to \mathbb{C}$, $\operatorname{Dom} c^T = \mathcal{M}^T \times \mathcal{M}^T$,

$$c^T[f,g] := (u^f(\cdot,T), u^g(\cdot,T))_{\mathcal{H}} = (W^T_{\max} f, W^T_{\max} g)_{\mathcal{H}}. \tag{2.1}$$

Let $c^T[y] := c^T[y,y]$ be the corresponding quadratic form. We shall prove that the values of c^T are determined by the response operator.

Lemma 7 *The following representation is valid:*

$$c^T[f,g] = i \int\limits_{\Sigma^T} d\Sigma^T \, [R^T f \; \overline{g} - f \; \overline{R^T g}]. \tag{2.2}$$

$\triangleleft$ Take $f \in \mathcal{L}^T$; multiplying (0.1)–(0.3) by $\overline{u}^f$, we obtain:

$$i\rho \, u_t^f \overline{u}^f + \Delta u^f \overline{u}^f - q\,|u^f|^2 = 0.$$

Complex conjugating gives

$$-i\rho \, \overline{u}_t^f u^f + \Delta \overline{u}^f u^f - q\,|u^f|^2 = 0.$$

Combining the last two equalities we get

$$i\rho \, (|u^f|^2)_t = \Delta u^f \overline{u}^f - u^f \Delta \overline{u}^f \,.$$

Integrating over Q^T leads to

$$i\,||u^f(\cdot,T)||^2_{\mathcal{H}} = \int\limits_{\Sigma^T} d\Sigma^T\,[f\;\overline{(R^T f)} - (R^T f)\;\overline{f}].$$

Therefore, we have

$$c^T[f,f] = <\text{see}\,(2.1)> = ||u^f(\cdot,T)||^2_{\mathcal{H}} = -i\int\limits_{\Sigma^T} d\Sigma^T\,[f\;\overline{(R^T f)} - (R^T f)\;\overline{f}],$$

which implies (2.2).

The representation (2.2) can be extended by continuity to $f, g \in \mathcal{M}^T$ by means of the passage to the limit in the norm of the space $C^1_0([0,T]; L_2(\Gamma)) \supset \mathcal{M}^T$ (see the end of Subsection 1.4). $\triangleright$

The form c^T is nonnegative and closable (since $W^T_{\max}$ is closed). Its closure $\overline{c^T}$ determines the (unique) selfadjoint operator $C^T : \mathcal{F}^T \mapsto \mathcal{F}^T$ such that $\operatorname{Dom} C^T \subset \operatorname{Dom}\overline{c^T}[\cdot]$ and $(C^T f, g)_{\mathcal{F}^T} = \overline{c^T}[f,g]$ for $f \in \operatorname{Dom} C^T, g \in \operatorname{Dom}\overline{c^T}[\cdot]$ (see, e.g., [8]).

Lemma 8 *The inclusion* $\operatorname{Dom} W^T_1 \subset \operatorname{Dom} C^T$ *and the representation*

$$C^T g = -iv^y|_{\Sigma^T} \tag{2.3}$$

are valid with $g \in \operatorname{Dom} W^T_1$ *and* $y = W^T_1 g$.

$\triangleleft$ Let $f \in \operatorname{Dom} C^T$ and $y \in \mathcal{H}^1$; let $g \in \operatorname{Dom} W^T_1$ such that $u^g(\cdot,T) = y$ (see (1.34)). On the strength of (1.19), there exists a sequence $\{g_p\} \subset \mathcal{M}^T_0$ such that $g_p \to g$ in $\mathcal{F}^T$ and $u^{g_p}(\cdot,T) \to y$ in $\mathcal{H}^1$. For these sequences we have:

$$(C^T f, g)_{\mathcal{F}^T} = \lim \overline{c^T}[f, g_p] = <\text{see}\,(2.1)> =$$
$$= \lim\,(u^f(\cdot,T), u^{g_p}(\cdot,T))_{\mathcal{H}} = (u^f(\cdot,T), y)_{\mathcal{H}} =$$
$$= <\text{see}\,(1.6)> = (f, -iv^y)_{\mathcal{F}^T}.$$

Using self-adjointness of C^T, we get $g \in \operatorname{Dom} C^T$ and the equality (2.3). $\triangleright$

By virtue of the exact controllability of the system (0.1)– (0.3) in $H^1(\Omega)$, for every eigenfunction φ_k of the problem (0.4), (0.5) there exists a control $f_k \in \operatorname{Dom} W^T_1$ such that $u^{f_k}(\cdot,T) = \varphi_k$. According to (2.3), we have:

$$C^T f_k = -iv^{\varphi_k}|_{\Sigma^T} = <\text{see}\,(1.8)> = e^T_k,$$

and so,

$$i\,(C^T f_k)(\cdot,T) = \varphi_k|_\Gamma. \tag{2.4}$$

2.2 The form l^T

Let us define the form $l^T := \mathcal{F}^T \times \mathcal{F}^T \to \mathbf{C}$, $\operatorname{Dom} l^T = \mathcal{M}_0^T \times \mathcal{M}_0^T$,

$$l^T[f,g] = \int\limits_\Omega dx\,[\nabla u^f(\cdot,T)\,\overline{\nabla u^g(\cdot,T)} + q\,u^f(\cdot,T)\overline{u^g(\cdot,T)}] \tag{2.5}$$

and show that its values are determined by the response operator.

Lemma 9 *The following representations are valid:*

$$l^T[f,g] = c^T[if_t, g] = i\int\limits_{\Sigma^T} d\Sigma^T\,[R^T f\;\;\overline{ig_t} - if_t\;\overline{R^T g}]. \tag{2.6}$$

◁ Substituting $f, g \in \mathcal{L}_0^T$ into (2.5) and integrating by parts we have

$$l^T[f,g] = (\rho^{-1}[-\Delta + q]u^f(\cdot,T), u^g(\cdot,T))_{\mathcal{H}} =$$
$$= <\text{see}\,(1.2)> = (u^{if_t}(\cdot,T), u^g(\cdot,T))_{\mathcal{H}} = <\text{see}\,(2.1)> =$$
$$= c^T[if_t, g] = <\text{see}\,(2.2)> =$$
$$= i\int\limits_{\Sigma^T} d\Sigma^T\,[R^T(if_t)\;\overline{g} - if_t\;\overline{R^T g}]. \tag{2.7}$$

From the stationary state property of the system (0.1)–(0.3) it follows that R^T commutates with $\frac{\partial}{\partial t}$. Using this fact and integrating by parts in (2.7), we obtain (2.6).

The representation (2.6) can be extended to $f, g \in \mathcal{M}_0^T$ using the passage to the limit in $C_0^1([0,T]; L_2(\Gamma)) \supset \mathcal{M}_0^T$. ▷

The definition (2.5) can be written in the form

$$l^T[f,g] = (W_0^T f, W_0^T g)_{\mathcal{H}^1} - (\lambda_1 - \alpha)\,c^T[f,g]\,.$$

Closing the form l^T in $\mathcal{F}^T$ and taking into account the relation (1.19) and inclusion $\operatorname{Dom} W_1^T \subset \operatorname{Dom}\overline{c^T}[\cdot]$, we get the representation

$$\overline{l^T}[f,g] = (W_1^T f, W_1^T g)_{\mathcal{H}^1} - (\lambda_1 - \alpha)\,\overline{c^T}[f,g]$$

on controls $f, g \in \operatorname{Dom}\overline{l^T}[\cdot] = \operatorname{Dom} W_1^T$. Let us note that the controls f_k, which are preimages of the eigenfunctions φ_k, lie in $\operatorname{Dom}\overline{l^T}[\cdot]$.

2.3 Variational principle. Extracting the spectral data

Eigenvalues and eigenfunctions of the problem (0.4),(0.5) can be found with the help of the well-known variational principle (see, e.g., [8]):

$$\lambda_k = \min_{\substack{y\in H^1(\Omega);\,(y,y)_{\mathcal{H}}=1;\\ (y,\varphi_j)_{\mathcal{H}}=0,\,j=1,2,\dots k-1}} \int_\Omega dx\,[|\nabla y|^2 + q\,|y|^2]; \qquad k=1,2\dots \tag{2.8}$$

(for $k=1$ there are no orthogonality conditions); here φ_k are the functions realizing the minima.

By virtue of (1.34) we look for minima in (2.8) over the set $\operatorname{Ran} W_1^T$. This allows to put $y = u^f(\cdot,T)$ and to change the conditions of the search for the following ones:

$$f \in \operatorname{Dom} W_1^T = \operatorname{Dom} \overline{l^T}[\cdot],$$

$$1 = (u^f(\cdot,T), u^f(\cdot,T))_{\mathcal{H}} = <\text{see}\,(2.1)> = \overline{c^T}[f],$$

and

$$0 = (u^f(\cdot,T), u^{f_j}(\cdot,T))_{\mathcal{H}} = <\text{see}\,(2.1)> = \overline{c^T}[f,f_j].$$

Finally we get:

$$\lambda_k = \min_{\substack{f\in \operatorname{Dom}\overline{l}^T;\ \overline{c}^T[f]=1;\\ \overline{c}^T[f,f_j]=0,\ j=1,2,\dots k-1}} \overline{l^T}[f]; \qquad k=1,2\dots \tag{2.9}$$

It is the form in which the variational principle is suitable for the recovery of the spectral data; the procedure can be described as follows.

Step 1. From the response operator R^T find the form c^T according to (2.2); then find $\overline{c^T}$ and the corresponding operator C^T.

Step 2. Find the form l^T by (2.6) and its closure $\overline{l^T}$.

Step 3. Using (2.9), find λ_k and controls f_k, realizing the minima. Then find $\varphi_k|_\Gamma$ according to (2.4).

Let us note in conclusion that the spectral data $\{\lambda_k;\varphi_k|_\Gamma\}$ could be extracted from R^T without the result of Triggiani and Yao, following the scheme of the paper [3] and using the canonical model of the Schrödinger system. However, the exact controllability guarantees the inclusion $\{f_k\}\subset\mathcal{F}^T$, whereas in the general case f_k are the elements of a completion of $\mathcal{F}^T$, which need not be a functional space.

2.4 The result by R.G. Novikov

Let us consider an auxiliary problem

$$-\Delta y + qy = E\rho y \quad \text{in } \Omega; \tag{2.10}$$

$$\frac{\partial y}{\partial \nu} = h \quad \text{on } \Gamma; \tag{2.11}$$

for $E \notin \{\lambda_k\}_{k=1}^{\infty}$ it has the unique solution $y = y^h(x)$. To the problem (2.10), (2.11) we associate the operator $\hat{\Psi}(E) : h \to y^h|_\Gamma$, which transfers the Neumann data into the Dirichlet ones. Recall its well-known representation

$$\hat{\Psi}(E)\, h = \sum_{k=1}^{\infty} \frac{(h, \varphi_k|_\Gamma)_{L_2(\Gamma)}}{\lambda_k - E}\, \varphi_k|_\Gamma \tag{2.12}$$

in the form of the series converging in $H^{1/2}(\Gamma)$. Fix $E < \lambda_1$; the operator $\hat{\Phi}(E) := \hat{\Psi}^{-1}(E)$, transferring the Diriclet data into the Neumann ones is well defined. As was shown in [11], the operator $\hat{\Phi}(E)$ determines the function $q - E\rho$ in Ω uniquely; hence, if $\hat{\Phi}(E)$ is given for two arbitrary fixed E', E'', one can recover ρ and q in Ω.

Let us note that in [11] the explicit formulae determining functions ρ (in the case of $q = 0$) and q (in the case of $\rho = 1$) from the operator $\hat{\Phi}(E)$ are presented.

2.5 Proof of Theorem 1

Let us summarize our considerations with a scheme of recovering the coefficients. This scheme provides the proof of the Theorem 1.

So, let a $T > 0$ be fixed and the operator $R^T : \mathcal{F}^T \to \mathcal{F}^T$ be given;

(i) by the use of the procedure described in Subsection 2.3 find the spectral data $\{\lambda_k; \varphi_k|_\Gamma\}_{k=1}^{\infty}$;

(ii) for $E', E'' : E' < E'' < \lambda_1$ construct the operators $\hat{\Psi}(E'), \hat{\Psi}(E'')$ by the representation (2.12);

(iii) find $\hat{\Phi}(E') = \hat{\Psi}^{-1}(E')$ and $\hat{\Phi}(E'') = \hat{\Psi}^{-1}(E'')$; the operators $\hat{\Phi}(E'), \hat{\Phi}(E'')$ uniquely determine the coefficients ρ, q in Ω (see [11]).

3 Comments

3.1 Remarks. Close problems

In the one-dimensional case the operator W^T is continuous [2]. Probably, the same is valid in the multi-dimensional case. Continuity of W^T (should this be the case) would allow to simplify our procedure.

Considerations quite similar to the given in Subsections 1.5 and 1.6 allow to prove the useful result: the extension of the operator W^T from $\mathcal{L}^T$ to $\mathcal{M}^T$ and subsequent closing lead to the maximal operator $W^T_{\max}$.

Theorem 1 remains valid in the case of controls acting from a part of the boundary. Namely, let $\Gamma_0 \subset \Gamma$ be a (relatively) open subset; denote $\mathcal{L}^T_{\Gamma_0} := \{f \in \mathcal{L}^T \,|\, \operatorname{supp} f \subset \Gamma_0 \times [0,T]\}$. The operator $R^T_{\Gamma_0} : \mathcal{L}^T_{\Gamma_0} \ni f \mapsto u^f|_{\Gamma_0 \times [0,T]}$ determines ρ and q in Ω uniquely.

Consider the Schrödinger system

$$iu_t + \Delta_g u = 0 \quad \text{in } \Omega \times (0,T);$$

$$u|_{t=0} = 0 \quad \text{on } \Omega;$$

$$\frac{\partial u}{\partial \nu} = f \quad \text{on } \Gamma \times [0,T]$$

on a compact Riemann manifold (Ω, g) with the boundary Γ; g is the metric tensor on Ω; Δ_g the Beltrami-Laplace operator; $R^T : f \to \frac{\partial u^f}{\partial \nu}|_{\Gamma \times [0,T]}$ the response operator. The procedure of Subsection 2.3 enables to extract the Neumann spectral data from R^T; these data determine the manifold up to isometry [7], [4]. As a result, we conclude that for any $T > 0$ the operator R^T determines the manifold (Ω, g) up to isometry.

Our techniques can be used for the direct continuation of the response operator from a time interval $0 \le t \le T$ onto the semiaxis $t > 0$ (see [5]). For this end one can extract the spectral data from R^T and, then, use the representation (1.12), replacing T there by a required $T' > T$.

3.2 Numerical realization

In the case of the 1-dim heat equation, a procedure extracting the spectral data from the response operator by means of variational principle has been realized numerically [1]. A computational experiment in the 2-dim problem

[6] has demonstrated a possibility to recover the coefficient ρ from the spectral data. However, a realization of the complete scheme $R^T \to \rho|_\Omega$ still remains problematic. It is not clear whether in the 2-dim case one can extract from R^T a segment of data $\{\lambda_k, \varphi_k|_\Gamma\}_{k=1}^N$ sufficient for satisfactory reconstruction of ρ (in the paper [1] there is $N \sim 20$ whereas in [6] $N \sim 400$ has been used).

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