PREPRINT PDMI 17 / 2002

Push-forwards in oriented cohomology theories of algebraic varieties: II

I. Panin

(after I.Panin and A.Smirnov)

06.10.2002

Abstract

This preprint contains proofs of the results anounced in [PS] in the part concerning the construction of push-forwards for an oriented cohomology theory. One-to-one correspondences are constructed between orientations, Chern structures, Thom structures and integrations on a given ring cohomology theory. The theory is illustrated by motivic cohomology, algebraic K-theory, algebraic cobordism theory and by other examples. Other terms often used in the literature for push-forwards are Gysin homomorphisms, trace homomorphisms, corestriction homomorphisms, transfers.

CONTENTS

1 Introduction

The concept of oriented cohomology theory is well-known in topology [Ad]. An algebraic version of this concept was introduced in [PS] and is considered here. More precisely, this preprint concerns those part of the preprint [PS] which is devoted to the construction of integrations on a ring cohomology theory. So in this preprint we consider a field k and the category of pairs (X, U) with a smooth variety X over k and its open subset U. By a cohomology theory we mean a contrvariant functor A from this category to the category of abelian groups endowed with a functor transformation $\partial : A(U) \to A(X, U)$ and satisfying the localization, Nisnevich excision and homotopy invariance properties (2.0.1).

For a ring cohomology theory A *an integration on* A is a rule assigning to each projective morphism $f : Y \to X$ of smooth varieties a two-sided $A(X)$-module operator $A(Y) \to A(X)$ satisfying certain natural properties (4.1.2) and called *push-forwards*. Other terms often used for push-forwards in literature are trace homomorphisms, Gysin homomorphisms, Thom-Gysin homomorphisms, corestriction homomorphisms, transfers and some others. It seems that there is no canonical terminology for this. One of the main aim of the preprint is to construct an integration on a ring cohomology theory begining with small additional data.

For this we consider three structures a ring cohomology theory A can be equipped with: *an orientation on* A, *a Thom structure on* A and *a Chern structure on* A. *An orientation on* A is a rule ω assigning to each variety X and to each vector bundle E/X a two-sided $A(X)$-module isomorphism $A(X) \to A(E, E - X)$ satisfying certain natural properties (3.1.1) and called *Thom isomorphisms*. *A Thom structure on* A is a rule assigning to each smooth variety X and each line bundle L over X a class $th(L) \in A(L, L - X)$ satisfying certain natural properties (3.2.2) and called *Thom class*. *A Chern structure on* A is a rule assigning to each smooth variety X and each line bundle L over X a class $c(L) \in A(X)$ satisfying certain natural properties (3.2.1) and called *first Chern class* (or some times called *Euler class*).

It is proved in the preprint that for a given A these structures are in natural bijections with each other. More precisely we construct the following diagram

$$\begin{tikzcd} \textit{Orientations on } A \arrow[r, "\alpha"] & \textit{Integrations on } A \arrow[d, "\beta"] \\ \textit{Chern structures on } A \arrow[u, "\delta"] & \textit{Thom structures on } A \arrow[l, "\gamma"'] \end{tikzcd} \qquad (1)$$

in which each arrow is a bijection and each round trip coincides with the identity (Theorems 4.1.4 3.7.4 3.7.5). The constructions of these arrows are described briefly below in this introduction. One of the consequence of the Theorem is this: *the existence at least one of these structures on* A implies *the existence of an integration on* A; an integration on A is *never defined by* the ring cohomology theory itself (even on usual singular cohomology there are plenty different integrations (see an example below in the Introduction)).

However in practice certain ring cohomology theories are equipped either with a Chern structure or with a Thom structure *on the nose.* Thus they are equipped with distinguished integrations. Say, usual singular cohomology with integral coefficients (on complex algebraic varieties) are equipped with the known Chern structure, algebraic K-theory is equipped with a Chern structure as well ($L \mapsto [\mathbf{1}] - [L^\vee]$). The corresponding integrations coincide with the well-known one. The motivic cohomology $H^*(-, \mathbb{Z}(*))$ is equipped with a Chern structure. The algebraic cobordism theory $MGL^{*,*}$ is equipped with a natural Thom structure. Thus these two theories are equipped with the corresponding integrations. The algebraic cobordism theory $MGL^{*,*}$ is one of *the main motivating example* for this preprint, but it is expected to be written in details later.

Using Theorems 4.1.4 3.7.4 3.7.5 we describe rather explicitely all integrations on A (Theorem 4.9.1). This description may be considered as one of *the main result of the preprint.*

To explain this description suppose there is given a ring cohomology theory A which can be equipped with at least one of the mentioned structures. In this case the ring $A(\mathbf{P}^\infty)$ is isomorphic to the formal power series $A(pt)[[t]]$ in one variable over the coefficient ring $A(pt)$ of the theory A (this follows from the projective bundle theorem 3.3.1). Consider an assignment which takes an integration on A to the corresponding by $\gamma \circ \beta$ Chern class $\xi := c(\mathcal{O}(-1))$ of the tautological line bundle over $\mathbf{P}^\infty$

$$\textit{Integrations on } A \xrightarrow{\ \xi\ } \textit{Local parameters of } A(\mathbf{P}^\infty)$$

Theorem 4.9.1 states that this assignment is bijective. In particular, this implies that there are plenty integrations on a ring cohomology theory A which can be equipped with a Chern structure or with a Thom structure. The injectivity of the assignment ξ says that two integrations $f \mapsto f_A^{(1)}$ and $f \mapsto f_A^{(1)}$ on A coincide if the corresponding two local parameters $\xi^{(1)}$ and $\xi^{(2)}$ coincide in $A(\mathbf{P}^\infty)$.

In the case of singular cohomology $H^*(-, \mathbb{Z})$ on complex algebraic varieties and the standart integration the mentioned local parameter coincides with the minus generator of the group $H^2(\mathbb{C}\mathbf{P}^\infty, \mathbb{Z})$. In the case of algebraic K-theory and Grothendieck's integration defined via higher direct images the local parameter coincides with the class

$[\mathbf{1}] - [\mathcal{O}(1)] \in K_0(\mathbf{P}^\infty)$. In the case of the de Rham cohomology and the usual integration of the differential forms the mentioned local parameter coincides with the class in $H^2(\mathbb{C}\mathbf{P}^n, \mathbb{C})$ of the $(1,1)$-differential form $-(dz_0 \wedge d\bar{z}_0 + dz_1 \wedge d\bar{z}_1 + \cdots + dz_n \wedge d\bar{z}_n)$ on the complex projective space $\mathbb{C}\mathbf{P}^n$ (more precisely this form is the restriction of the local parameter to $\mathbb{C}\mathbf{P}^n$).

For well-known theories (singular cohomology on complex algebraic varieties, etale cohomology theory, motivic cohomology theory) one can use the usual Chern structure in order to get an integration (via the assignment $\alpha \circ \delta$) coinciding with the classical ones. In the case of algebraic K-theory the assignment $L \mapsto [\mathbf{1}] - [L^\vee]$ is a Chern structure which gives via $\alpha \circ \delta$ the well-known Grothendieck's integration on K-theory defined via the higher direct images of coherent sheaves. If $k = \mathbb{C}$ and the theory is the complex cobordism theory restricted to the category of pairs of algebraic varieties and equipped with the Chern structure given by the Conner-Floyed classes [CF] then the corresponding integration coincide with the family of Gysin maps in the complex cobordism theory described in [Qu2, 1.2 and 1.4]. The algebraic cobordism theory **MGL** of Voevodsky [V1] is equipped with an orientation ω by choosing a tautological class as the Thom structure 3.8.7 and then applying the assignment $\delta \circ \gamma$ to this Thom structure. Applying the assignment α to the orientation ω we get an integration on the theory represented by **MGL**. This example is supposed to be considered in details in a next preprint. Here it is just briefly indicated. An example of *unstandart Chern structure* on the usual singular cohomology with rational coeffitients is given by the assignment $L \mapsto 1 - exp(-c_1(L))$. This Chern structure gives an integration on $H^*(-, \mathbb{Q})$ which respects via the Chern character to Grothendieck's integration on the algebraic K-theory (see [Pa, Corollary 1.1.10]).

An oriented ring cohomology theory is a cohomology theory equipped with an orientation in the sense above. As it was already mentioned to orient a ring theory A is the same as to fix an integration on A or to fix a Thom structure on A or to fix a Chern structure on A. An orientation is usually denoted ω. The integration corresponding to ω via α is written usually as $f \mapsto f_\omega$. The Thom structure corresponding to ω via $\beta \circ \alpha$ is written often as $L \mapsto th_\omega(L)$. The Chern structure corresponding to ω via $\gamma \circ \beta \circ \alpha$ is written often as $L \mapsto c_\omega(L)$. An oriented cohomology theory is as well an oriented cohomology pretheory in the sence of [Pa] because the mentioned integration is perfect in the sence of [Pa]. *An orientable ring cohomology theory* is a ring cohomology theory which can be equipped with an orientation.

If an oriented cohomology theory A is $\mathbb{Z}$-graded and the first Chern class is homogeneous of degree 1, then the integration on A defines a Borel-Moore theory in the sence of [LM].

Now describe briefly the relation of orientable ring cohomology theories to *the commutative formal groups* and the relation of the oriented theories to *the commutative formal group laws*. The last relation gives one of *the key tool* in order to construct the assignment α. Here are the relations: by the projective bundle theorem one has the Kunnet formular $A(\mathbf{P}^\infty \times \mathbf{P}^\infty) = A(\mathbf{P}^\infty) \otimes_{A(pt)} A(\mathbf{P}^\infty)$. The Segre morphism $\mu : \mathbf{P}^\infty \times \mathbf{P}^\infty \to \mathbf{P}^\infty$

(corresponding to the line bundle $p_1^*\mathcal{O}(1) \otimes p_2^*\mathcal{O}(1)$) induces the pull-back operator

$$\mu^* : A(\mathbf{P}^\infty) \to A(\mathbf{P}^\infty \times \mathbf{P}^\infty) = A(\mathbf{P}^\infty) \otimes_{A(pt)} A(\mathbf{P}^\infty)$$

which in turns defines a Hopf-algebra structure on $A(\mathbf{P}^\infty)$. This is *the formal group* F_A associated with the orientable ring cohomology theory A.

An orientation ω on A identifies the ring $A(pt)[[u]]$ with $A(\mathbf{P}^\infty)$ sending the variable u to the local parameter $\xi_\omega := c_\omega(\mathcal{O}(-1))$. In the same way the orientation ω identifies the ring $A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ with the ring $A(pt)[[u_1, u_2]]$. Under the last identification the element $\mu^*(\xi_\omega)$ becomes a formal power series $F_\omega(u_1, u_2)$ in two variables over the ring $A(pt)$. It is *the formal group law* F_ω corresponding to the orientation ω of the theory A. The series $F_\omega(u_1, u_2)$ is a unique series satisfying the following property: for each smooth variety X and every line bundles L_1 and L_2 over X one has the relation in $A(X)$

$$c_\omega(L_1 \otimes L_2) = F_\omega(c_\omega(L_1), c_\omega(L_2)).$$

Now sketch the structure of the text. In the Section 1.1 certain general notation are introduced. In the Section 2 a notion of cohomology theory introduced in [PS] is recalled and general properties of a cohomology theory are proved. The deformation to the normal cone construction is recalled in 2.2.7 as well. An analog of the purity theorem from [MV] is proved (Theorem 2.2.8). The canonical isomorphism $A(X, X - Z) \cong A(N, N - Z)$ from Theorem 2.2.8 one should consider as a replacement of the excision isomorphism for a tubular neighbourhood (well-known in the topology). In the end of the Section we recall the notion of a ring cohomology theory [PS].

In Section 3 we construct a triangle of correspondences ρ, γ, δ

$$\begin{tikzcd}
\textit{Orientations on } A \arrow[drr, "\rho"] & & \\
\textit{Chern structures on } A \arrow[u, "\delta"] & & \textit{Thom structures on } A \arrow[ll, "\gamma"']
\end{tikzcd}$$

in which each arrow is a bijection and each round trip coincides with the identity (see Theorems 3.2.4, 3.7.4 and 3.7.5). Proofs of these three theorems use Projective Bundle Theorem 3.3.1, Splitting Principle, 3.5.1 and higher Chern classes 3.6.2. In the very end of the Section it is shown how an orientation ω on the theory A gives rise to the formal group law F_ω over the coefficient ring of A.

In Section 4 the notion of an integration on A is defined (here the requirement 5 concerning a short Gysin sequence is added to the corresponding notion introduced in [PS]). We construct a triangle of correspondences α, ϵ, δ

$$\begin{tikzcd}
\textit{Orientations on } A \arrow[r, "\alpha"] & \textit{Integrations on } A \arrow[dl, "\epsilon"] \\
\textit{Chern structures on } A \arrow[u, "\delta"] &
\end{tikzcd}$$

in which each arrow is a bijection and each round trip coincides with the identity (Theorem 4.1.4). Combining the two triangles we get the square diagram (1) except of the arrow β.

The arrow β is described just below Theorem 4.1.4. It follows from the description of β that $\rho = \beta \circ \alpha$ and $\epsilon = \gamma \circ \beta$.

The proof of the Theorem is rather long. It ocupies all the Section 4 and is organized as follows. The uniqueness assertion of the item 1 and the item 2 of the Theorem are proved in Subsection 4.1. The proof of the uniqueness assertion of the item 1 is based on the first variant of Riemann-Roch type theorem proved in [Pa, Theorem 1.1.9]. The proof of the existence assertion is given below in Subsection 4.8 and it requires a construction of an integration.

The construction of an integration subjecting to an orientation is rather long itself and it takes several subsections. Namely, in subsection 4.2 push-forwards are constructed for closed imbeddings of smooth varieties (they are called Gysin operators). In subsection 4.3 push-forwards are constructed for the case of projections $X \times \mathbf{P}^n \to X$ (they are called Quillen's operators). In the subsection 4.4 properties of Gysin operators are proved and in the subsection 4.5 properties of Quillen's operators are proved except of the Key property, which is proved in the subsection 4.6. Push-forwards for any projective morphism are constructed in subsection 4.7. These push-forwards form an integration on A as it is stated in Theorem 4.7.1 from the same subsection. The proof of Theorem 4.1.4 is completed in Subsection 4.8.

The preprint is completed by Theorem 4.9.1. In this theorem an orientable ring cohomology theory A is considered and the set of all integrations is identified with the set of all universally central local parameters of the ring $A(\mathbf{P}^\infty)$. Moreover this identifications is given in one direction very explicetly. Namely it sends an integration $f \mapsto f_A$ on A to the corresponding by Theorem 4.1.4 first Chern class $c(\mathcal{O}(-1)) \in A(\mathbf{P}^\infty)$ of the tautological line bundle $\mathcal{O}(-1)$ over $\mathbf{P}^\infty$. Theorem 4.9.1 might be considered as *the main result* of the preprint.

Since the text is rather long it is reasonable to sketch here our constructions of the assignments α, ϵ and δ (the assignments β and γ are described in Theorems 4.1.4 and 3.7.5). We first describe the arrow ϵ, then sketch the description of δ and finally sketch the description of α.

Suppose we are given with an integration $f \mapsto f_A$ on A. For the zero section $z : X \to L$ of a line bundle L over X set $c(L) = z^A(z_A(1)) \in A(X)$ (the pull-back of the push-forward of the element 1). The assignment $L \mapsto c(L)$ is the Chern structure on A corresponding via ϵ to the integration (see 4.1.4).

Suppose we are given with a Chern structure $L \mapsto c(L)$ on A. In this case the projective bundle theorem holds (see 3.3.1) and there is a Chern class theory $E \mapsto c_n(E)$ with values in A. To produce an orientation ω on A we associate to each vector bundle E/X its Thom class $th(E) \in A(E, E-X)$. Firstly for a rank n vector bundle E we define the element $\bar{th}(E) := c_n(\mathcal{O}_E(1) \otimes p^*(E)) \in A(\mathbf{P}(\mathbf{1} \oplus E))$. It turns out that this element belongs to the subgroup $A(\mathbf{P}(\mathbf{1} \oplus E), \mathbf{P}(\mathbf{1} \oplus E) - \mathbf{P}(\mathbf{1}))$ of the group $A(\mathbf{P}(\mathbf{1} \oplus E))$. The class $th(E)$ is defined as the image of the element $\bar{th}(E)$ under the excision isomorphism identifying $A(\mathbf{P}(\mathbf{1} \oplus E), \mathbf{P}(\mathbf{1} \oplus E) - \mathbf{P}(\mathbf{1}))$ with $A(E, E-X)$. The required orientation ω on A is given by the assignment which associate to a vector bundle $p : E \to X$ the map $(\cup th(E)) \circ p^A : A(X) \to A(E, E-X)$. Details are given in the proof of Theorem 3.7.4.

Now describe the construction of the assignment α. This is *the very heart* of the preprint. So suppose we are given with an orientation ω on A and sketch our construction of the corresponding via α integration $f \mapsto f_\omega$. In this sketch we skip all the details concerning commutativity of certain rings and centrality of certain elements because in the main body of the text it is done in a proper manner. Now begin the mentioned sketch. First of all each projective morphism $f : Y \to X$ can be presented as a composition of a closed imbedding $i : Y \hookrightarrow \mathbf{P}^n \times X$ and the projecton $p : \mathbf{P}^n \times X \to X$. So it suffices to define push-forwards for closed imbeddings, for projections, to set $f_\omega = p_\omega \circ i_\omega$ and to verify that the resulting operator f_ω does not depend on the choice of the decomposition of f. For a closed imbedding $i : S \hookrightarrow T$ of smooth varieties we define push-forward $i_\omega : A(S) \to A(T)$ as the composition

$$A(S) \xrightarrow{th^N} A(N, N-S) \cong A(T, T-S) \to A(T)$$

where $N = N_{T/S}$ is the normal bundle to S in T, th^N is the Thom isomorphism (given by the orientationn ω), the isomorphism $A(N, N-S) \cong A(T, T-S)$ is the "excision isomorphism" above. The homomorphism $A(T, T-S) \to A(T)$ is just the pull-back homomorphism induced by the inclusion of pairs $(T, \emptyset) \subset (T, T-S)$.

The push-forward operators for the projections $p : \mathbf{P}^n \times X \to X$ are defined as follows. We first define a Chern structure on A corresponding to the orientation: given a line bundle L over a smooth X consider the zero section $z : X \to L$, the composition map $A(X) \xrightarrow{th^E} A(E, E-X) \to A(E) \xrightarrow{z^A} A(X)$ and set $c_\omega(L)$ to be equal the evaluation of this composition on the element $1 \in A(X)$. Then we define certain elements $[\mathbf{P}^m]_\omega$ in $A(pt)$ called the classes of projective spaces. By the projective bundle theorem the ring $A(\mathbf{P}^n \times X)$ as the two-sided $A(X)$-module is a free module with the basis $1, \zeta, \dots, \zeta^n$, where $\zeta = c_\omega(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. So we define the push-forward p_ω as a unique two-sided $A(X)$-module operator $p_\omega : A(\mathbf{P}^n \times X) \to A(X)$ which takes the element ζ^r to the class $[\mathbf{P}^{n-r}]_\omega$ for $r = 0, 1, \dots, n$. So to finish the definition of the operators p_ω it remains to define the classes $[\mathbf{P}^m]_\omega$.

To do this we need in the formal group law corresponding to the orientation ω of A. This formal group law is the formal power series $F_\omega(u_1, u_2) \in A(pt)[[u_1, u_2]]$ described above. Write down the unique normalized F_ω-invariant differential 1-form ω_F on the formal group (law) in terms of the local parameter u: $\omega_F = (P_0 + P_1 u + P_2 u^2 + \dots)du$. Now set

$$[\mathbf{P}^m]_\omega = P_m \in A(pt).$$

These completes the definition of the classes $[\mathbf{P}^m]_\omega$ and thus completes the definition of the operators $p_\omega : A(\mathbf{P}^n \times X) \to A(X)$. Finally for the morphism $f : Y \to X$ and the presentation $f = p \circ i$ set $f_\omega = p_\omega \circ i_\omega$. Theorem 4.7.1 states that the operator f_ω is well-defined, respects the composition of projective morphisms and moreover the assignment $f \mapsto f_\omega$ is an integration required by item 1 of Theorem 4.1.4.

Another definition (coinciding with the given one) of the classes $[\mathbf{P}^m]_\omega$ uses the complex cobordism theory $\mathrm{MU}(*)$ and the formal group law F_{MU} associated with this theory and its canonical Chern class for line bundles (the Conner-Floyd class) [CF]. This law is defined over the ring $\mathrm{MU} = \mathrm{MU}(pt)$. According to a theorem of Quillen ([Qu1, Th.2]) F_{MU} is a universal commutative formal group low of dimension 1. This implies that there exists a unique ring homomorphism $l_\omega : \mathrm{MU} \to A(pt)$ such that the coefficients of F_ω coincide with the l_ω-images of the corresponding coefficients of F_{MU}. Now set

$$[\mathbf{P}^m]_\omega = l_\omega([\mathbb{C}P^m]),$$

where $[\mathbb{C}P^m]$ is the class of $\mathbb{C}P^m$ in MU. These completes the second definition of the classes $[\mathbf{P}^m]_\omega$. The two definitions coincides because by Mischenko's theorem (see [**?**]) the normalized invariant differential form ω_{MU} coincides with the form

$$([\mathbb{C}\mathbf{P}^0] + [\mathbb{C}\mathbf{P}^1]u + [\mathbb{C}\mathbf{P}^2]u^2 + \dots)du$$

and clearly a scalor extension takes the normalized invariant differential form to the normalized invariant differential form.

The normalized invariant form ω of a formal group law $F = F(u_1, u_2)$ is computed as follows (see [Qu1]): $\omega_F = du/F_2(u, 0)$, where F_2 is the derivative of F with respect to the variable u_2. Now let us make a test checking that for certain examples of cohomology theories and certain choice of the Chern structures the classes $[\mathbf{P}^m]_\omega$ do coincide with the known ones. Namely, if the cohomology theory is the usual singular cohomology or de Rham cohomology or motivic cohomology and if the Chern structure $L \mapsto c(L)$ is the usual one, then the associated formal group law is additive: $F = u_1 + u_2$. In this case one has $\omega = du$ and the classes are given by $[\mathbf{P}^0]_\omega = 1$ and $[\mathbf{P}^r]_\omega = 0$ for all $r > 0$. This agrees with what is well-known. If the cohomology theory is the algebraic K-theory and the Chern structure is given by $L \mapsto [\mathbf{1}] - [L^\vee]$, then the associated formal group law is multiplicative: $F = u_1 + u_2 - u_1u_2$. In this case one has $\omega = du/(1-u) = (1 + u + u^2 + \dots)du$ and the classes are given by $[\mathbf{P}^r]_\omega = 1$ for all $r > 0$. This agrees with Grothendieck's integration on the algebraic K-theory given via the higher direct images. In the case of the complex K-theory topologists like to take as the Chern structure the assignment $L \mapsto [L] - [\mathbf{1}]$. In this case the associated formal group law is again multiplicative: $F = u_1 + u_2 + u_1u_2$, $\omega = du/(1+u)$ and the classes are given by $[\mathbf{P}^r]_\omega = (-1)^r$ for all $r > 0$. However in [BFM] the Chern structure for the complex K-theory is given by $L \mapsto [\mathbf{1}] - [L^\vee]$ (this is why the morphism $K^{alg} \to K^{top}$ commutes with the push-forwards corresponding to the chosen Chern structures).

The author thanks A. Suslin for his very stimulating interest to the subject. His remark on possible existence of push-forwards in general situation was the origin of our work on the subject. The subject of (3.3) and (3.9) was inspired by Morel's lectures (Muenster, June, 1999). The author thanks as well A. Merkurjev E. Friedlander and S. Yagunov for their stimulating interest to the subject.

The author thanks very much for the support the the TMR Network ERB FMRX CT-97-0107, the grants of the years 2001 and 2002 of the " Support Fund of National Science" at the Russian Academy of Science, the grant INTAS-99-00817, and the RFFI-grant 00-01-00116.

1.1 Terminology and notation

Let k be a field. The term "variety" is used in this text for an arbitrary quasi-projective variety over k (in particular, for a singular variety). We fix the following notation:

- $\mathcal{A}b$ - the category of abelian groups;
- $\mathcal{S}m$ - the category of smooth varieties;

  $\mathcal{S}m\mathcal{O}p$ - the category of pairs (X, U) with smooth X and open U in X. Morphisms are morphisms of pairs.

  We identify the category $\mathcal{S}m$ with a full subcategory of $\mathcal{S}m\mathcal{O}p$ assigning to a variety X the pair $(X, \emptyset)$;
- $pt = \mathrm{Spec}(k)$;

  For a smooth X and an effective divisor $D \subset X$ we write $L(D)$ for a line bundle over X whose sheaf of sections is the sheaf $\mathcal{L}_X(D)$ (see [Har, Ch.II,§6, 6.13]).

  $\mathbf{P}(V) = Proj(S^*(V^\vee))$-the space of lines in a finite dimensional k-vector space V;

  $L_V = \mathcal{O}_V(-1)$-the tautological line bundle over $\mathbf{P}(V)$;
- $\mathbf{P}(E)$ - the space of lines in a vector bundle E;

  $L_E = \mathcal{O}_E(-1)$ - the tautological line bundle on $\mathbf{P}(E)$;

  E^0 - the complement to the zero section of E;

  $E^\vee$ - the vector bundle dual to E;

  $z : X \to E$ - the zero section of a vector bundle E;
- For a contravariant functor A on $\mathcal{S}m$ set

  $$A(\mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V)), \tag{2}$$

  where the projective system is induced by all the finite dimensional vector subspaces $V \hookrightarrow k^\infty$.

  Similarly set

  $$A(\mathbf{P}^\infty \times \mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V) \times \mathbf{P}(W))$$

  where the projective system is induced by all the finite dimensional subspaces $V, W \subset k^\infty$.

2 Cohomology theories

2.0.1. Definition. *A cohomology theory is a contravariant functor* $\mathcal{S}m\mathcal{O}p \xrightarrow{A} \mathcal{A}b$ *together with a functor morphism* $\partial : A(U) \to A(X, U)$ *satisfying the following properties*

1. *Localization: the sequence* $A(X) \xrightarrow{j^A} A(U) \xrightarrow{\partial_P} A(X,U) \xrightarrow{i^A} A(X) \xrightarrow{j^A} A(U)$ *is exact for each pair* $P = (X,U) \in \mathcal{S}m\mathcal{O}p$, *where* $j : U \hookrightarrow X$ *and* $i : (X, \emptyset) \hookrightarrow (X,U)$ *are the natural inclusions;*

2. *Excision: the operator* $A(X,U) \to A(X',U')$ *induced by a morphism* $e : (X',U') \to (X,U)$ *is an isomorphism, if the morphism* e *is etale and for* $Z = X - U$, $Z' = X' - U'$ *one has* $e^{-1}(Z) = Z'$ *and* $e : Z' \to Z$ *is an isomorphism;*

3. *Homotopy invariance: the operator* $A(X) \to A(X \times \mathbf{A}^1)$ *induced by the projection* $X \times \mathbf{A}^1 \to X$ *is an isomorphism.*

The operator ∂_P *is called the boundary operator and is written usually as* ∂.

A morphism of cohomology theories $\varphi : (A, \partial^A) \to (B, \partial^B)$ *is a functor transformation* $\varphi : A \to B$ *commuting with the boundary morphisms in the sence that for every pair* $P = (X,U) \in \mathcal{S}m\mathcal{O}p$ *one has* $\partial^B_P \circ \varphi_U = \varphi_P \circ \partial^A_P$.

We write also $A_Z(X)$ *for* $A(X,U)$, *where* $Z = X - U$, *and call the group* $A_Z(X)$ *cohomology of* X *with the support on* Z. *The operator*

$$A_Z(X) \xrightarrow{i^A} A(X) \tag{3}$$

is called the support extension operator for the pair (X,U).

We do NOT assume at all in this text that cohomology theories are graded and the boundary operator is of degree $+1$. We do not assume this in particular because it is never used below and it is even unconvinient to assume this for certain points.

One could replace in this definition the category of abelian group by any abelian category or even by additive one which is equipped with Kernels and Cokernels for projectors. We left such a replacement to a reader to avoid technicalities as much as it is possible.

2.1 Examples

Consider a number of examples.

2.1.1. Classical singular cohomology. Let $k = \mathbb{C}$ and let A be an abelian group $(X,U) \mapsto \oplus_{-\infty}^{\infty} H^p(X(\mathbb{C}), U(\mathbb{C}); A$ be the usual singular cohomology (with coeffitients in A) of the pair of the complex point sets with respect to the complex topology. Take as a boundary ∂ the usual boundary map ∂ (see for instance [**?**, ??]).

2.1.2. A generalized cohomology theory. Let $k = \mathbb{C}$ and let $(X,U) \mapsto \oplus_{p=-\infty}^{\infty} E^p(X(\mathbb{C}), U(\mathbb{C}))$ be a generalized cohomology theory say represented by a spectra E with the usual boundary map ∂ (see for instance [Ad, ??] or [Sw, 8.33]).

2.1.3. Singular cohomology of the real point sets. Let $k = \mathbb{R}$ and let A be an abelian group $(X,U) \mapsto \oplus_0^{\infty} H^p((X(\mathbb{R}), U(\mathbb{R}); A)$ be the usual singular cohomology (with coefficients in A) of the pair of real points set considered with respect to the strong topology. Take as a boundary ∂ the usual boundary map for the pair $(X(\mathbb{R}), U(\mathbb{R}))$.

2.1.4. Witt theory of Balmer. Let $(X, X - Z) \mapsto \oplus_{p=-\infty}^{\infty} W_Z^p(X)$ be the Witt functor defined in [Ba]. Clearly it is a cohomology theory in the sence of Definition 2.0.1.

2.1.5. Bloch's Higher Chow groups Let $(X, X - Z) \mapsto \oplus_{p=-\infty}^{\infty} \oplus_{q=-\infty}^{\infty} CH_Z^p(X, q)$ be the higher Chow groups defined in [Bl].

2.1.6. Motivic cohomology of M.Levine. .... [L].

2.1.7. Etale cohomology. Let F be a locally constant torsion sheaf on the etale k-situs and assume that char(k) is prime to the torsion of F. In this example $A^n(X, U) = H_Z^n(X_{et}, F)$ [Mi, 3.1] and ∂ is defined in [Mi, 3.1.25]. The localization property for the pair (A, ∂) is proved in [Mi, 3.1.25], the excision property is proved in [Mi, 3.1.27] and the homotopy invariance is proved in [Mi, 6.4.20].

2.1.8. K-theory. Algebraic K-theory also can be fitted to the definition 2.0.1. To do this use, for instance, K-groups with support $K_n(XonZ)$ $(n \geq 0)$ of [TT]. So set $A^n(X, U) = K_{-n}(XonZ)$, where $Z = X - U$. Further set $A(X, U) = \oplus_{n=0}^{\infty} A^n(X, U)$. The definition of ∂ and the exactness of the localization sequence are contained in [TT, Theorem 5.1] (except the surjectivity of the restriction $A^0(X) \to A^0(U)$). If X is quasi-projective then $K(XonX)$ coincides with the Quillen's K-groups $K_n^Q(X)$ by [TT, 3.9, 3.10]. This proves in particular the homotopy invariance $A^n(X)$ for smooth X. The excision property for A follows from [TT, 3.19]. It remains now to check the surjectivity of the restriction $A^0(X) \to A^0(U)$. Clearly $A^0(X) = K_0^Q(X)$ coincides with the Grothendieck group of the vector bundles on X. Since X is smooth the desired surjectivity follows from [BS, §8, Prop.7]. Thus (A, ∂) satisfies the definition 2.0.1. Details of this example are considered in an Appendix below.

2.1.9. Motivic cohomology. Here $A_Z^{\mathcal{C},p}(X) = H_Z^p(X, \mathcal{C}) := Hom_{DM^-(k)}(M_Z(X), \mathcal{C}[p])$ is the motivic cohomology with coefficients in a motivic complex $\mathcal{C} \in DM^-(k)$ [SV], where the motive $M_Z(X)$ with supports on Z is defined in [SV, the text just below the proof of Th.4.8]. The motive $M_Z(X)$ is identified with the complex $C^*(\mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X - Z))$ in the proof of Lemma 4.11 in [SV]. Set $A_Z^{\mathcal{C}}(X) = \oplus_{-\infty}^{\infty} A_Z^{\mathcal{C},p}(X)$. The boundary operator ∂ which we denote in this example $\partial^{\mathcal{C}}$ is defined in [SV][??].

The homotopy invariance property holds by [SV, Prop.4.2]. The excision property is proved in [SV, the proof of Lemma 4.11]. The localization property follows from the exactness of the complex

$$0 \to \mathbb{Z}_{tr}(X - Z) \to \mathbb{Z}_{tr}(X) \to \mathbb{Z}_{tr}(X)/\mathbb{Z}_{tr}(X - Z) \to 0$$

because the functor C^* takes short exact sequences to exact triangles [SV, Th.1.12].

2.1.10. Semi-topological complex and real K-theories [FW]. If the ground field k is the field $\mathbb{R}$ of reals then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{semi}$ defined in[FW] is a cohomology theory as it is proved in [FW].

2.1.11. Representable theories. Here $A^p(X,U) = \oplus_q E^{p,q}(X/U)$ where E is a T-spectra [V1]. Set $A(X,U) = \oplus_{-\infty}^{\infty} A^p(X,U)$. The boundary operator is described in [V1][??] and is defined via the triangulated structure on the stable homotopy category ?? [V1][??]. In particular, in the case $E = MGL$ [V1, Sect.6.3] we obtain the algebraic cobordism theory.

2.2 General properties of cohomology theories

We specify here certain properties of an arbitrary cohomology theory A which are useful below in the text.

2.2.1. The localization property implies that $A_\emptyset(X) = A(X,X) = 0$. Therefore $A(\emptyset) = A_\emptyset(\emptyset) = 0$.

2.2.2. If any two of morphisms $(X,U) \to (Y,V)$, $X \to Y$, $U \to V$, defined by a morphism $f : (X,U) \to (Y,V)$, induce isomorphisms on the level of A then the third of these morphisms induces an isomorphism on the level of A.

2.2.3. Localization sequence for a triple. Let $T \subset Y \subset X$ be closed subsets of a smooth variety X. Let $\partial : A(X-T) \to A_T(X)$ be the boundary map for the pair $(X, X-T)$. Consider the support extension map $e^A : A_{Y-T}(X-T) \to A(X-T)$ and set $\partial_{Y,T} = \partial \circ e^A : A_{Y-Z}(X-Z) \to A_T(X)$.

We claim that the sequence

$$\ldots \to A_T(X) \xrightarrow{\alpha} A_Y(X) \xrightarrow{\beta} A_{Y-T}(X-T) \xrightarrow{\partial_{Y,T}} A_T(X) \xrightarrow{\gamma} A_Y(X) \to \ldots$$

with the obvious mappings α, β and γ is a complex and moreover it is exact. We call this sequence the localization sequence for the triple $(X, X-T, X-Y)$. If $Y = X$, then this sequence coincides with the localization sequence for the pair $(X, X-T)$.

2.2.4. Mayer-Vietoris sequence. If $X = U_1 \cup U_2$ is a union of two open subsets U_1 and U_2 and if Y is a closed subset in X, then set $T_i = Y - U_i$, $Y_i = Y \cap U_i$, $U_{12} = U_1 \cap U_2$ $Y_{12} = U_{12} \cap Y$. Consider the morphism of the localization sequences for the triples $(X, X-T_1, T-Y)$ and $(U_2, U_2 - T - 1, T - Y)$ induced by the inclusion of the triples $(U_2, U_2 - T_1, U_2 - Y_2) \subset (X, X - T_1, T - Y)$

$$\begin{array}{ccccccc}
A_Y(X) & \xrightarrow{\alpha_1} & A_{Y_1}(U_1) & \longrightarrow & A_{T_1}(X) & \xrightarrow{e^A} & A_Y(X) \\
{\scriptstyle\alpha_2}\downarrow & & {\scriptstyle\beta_1}\downarrow & & {\scriptstyle\gamma}\downarrow & & \downarrow \\
A_{Y_2}(U_2) & \xrightarrow{\beta_2} & A_{Y_{12}}(U_{12}) & \xrightarrow{\partial} & A_{T_1}(U_2) & \longrightarrow & A_{Y_2}(U_2).
\end{array}$$

The map γ is an isomorphism by the excision property. Set $d = e^A \circ \gamma^{-1} \circ \partial : A_{Y_{12}}(U_{12}) \to A_Y(X)$. We claim that the sequence

$$\ldots \to A^n_Y(X) \xrightarrow{\alpha_1+\alpha_2} A^n_{Y_1}(U_1) \oplus A^n_{Y_2}(U_2) \xrightarrow{(\beta_1,-\beta_2)} A^n_{Y_{12}}(U_{12}) \xrightarrow{d} A^{n+1}_Y(X) \to \ldots$$

is exact and call this sequence *the Mayer-Vietoris sequence* of the open covering $X = U_1 \cap U_2$. The proof of the exactness is straightforward and we skip it.

The Mayer-Vietoris sequence is natural in the following sence.If $f : X' \to X$ is a morphism and $X' = U_1' \cup U_2'$ is a Zariski covering of X' such that $f(U_i') \subset U_i$ and if Y' is a closed subset in X' containing $f^{-1}(Y)$, then the pull-back mappings $f^A : A_Y(X) \to A_{Y'}(X')$, $f^A : A_{Y_i}(U_i) \to A_{Y_i'}(U_i')$, $f^A : A_{Y_{12}}(U_{12}) \to A_{Y_{12}'}(U_{12}')$ form a morphism of the corresponding Mayer-Vietoris sequences.

2.2.5. Let $i_r : X_r \hookrightarrow X_1 \amalg X_2$ be the natural inclusion $(r = 1, 2)$. Let $Y_r \subset X_i$ be a closed subset for $(r = 1, 2)$ Then the induced map $A_{X_1 \amalg X_2}(X_1 \amalg X_2) \to A_{Y_1}(X_1) \oplus A_{Y_2}(X_2)$ is an isomorphism.

Proof. This follows from the Mayer-Vietoris property and the fact that $A_\emptyset(\emptyset) = 0$.

2.2.6. Strong homotopy invarance. Let $p : T \to X$ be an affine bundle (i.e., a torsor under a vector bundle). Let $Z \subset X$ be a closed subset and let $S = p^{-1}(Z)$. Then the pull-back map $p^A : A_Z(X) \to A_S(T)$ is an isomorphism. If $s : X \to T$ is a section then the induced operator $s^A : A_S(T) \to A_Z(X)$ is an isomorphism as well.

Proof. First consider the case $Z = X$. Then $S = T$ and we have to check that the pull-back map $p^A : A(X) \to A(T)$ is an isomorphism. Choose a finite Zariski open covering $X = \cup U_i$ such that $T_i = p^{-1}(U_i)$ is isomorphic to the trivial vector bundle over each U_i and then use the morphism of the Mayer-Vietoris sequences and the homotopy invariance property of the cohomology theory A.

To prove the general case consider the localization sequences for the pairs $(X, X - Z)$ and $(T, T - S)$. The pull-back mappings form a morphism of these two long exact sequences. The 5-Lemma completes the proof.

2.2.7. Deformation to the normal cone. The deformation to the normal cone is a well-known construction (for example, see [Fu]). Since the construction and its property (6) play an important role in what follows we give here some details.

Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N. There exists a smooth variety X_t together with a smooth morphism $p_t : X_t \to \mathbf{A}^1$ and a closed imbedding $i_t : Y \times \mathbf{A}^1 \hookrightarrow X_t$ such that the map $p_t \circ i_t$ coincides with the projection $Y \times \mathbf{A}^1 \to \mathbf{A}^1$ and

- the fiber of p_t over $1 \in \mathbf{A}^1$ is canonically isomorphic to X and the base change of i_t by means of the imbedding $1 \hookrightarrow \mathbf{A}^1$ coincides with the imbedding $i : Y \hookrightarrow X$;
- the fiber of p_t over $0 \in \mathbf{A}^1$ is canonically isomorphic to N and the base change of i_t by means of the imbedding $0 \hookrightarrow \mathbf{A}^1$ coincides with the zero section $Y \hookrightarrow N$.

Thus we have the diagram

$$(N, N - Y) \xrightarrow{i_0} (X_t, X_t - Y \times \mathbf{A}^1) \xleftarrow{i_1} (X, X - Y) \tag{4}$$

Here and further we identify a variety with its image under the zero section of any vector bundle over this variety.

Let us recall a construction of X_t, p_t and i_t. For that take X'_t to be the blow-up of $X \times \mathbf{A}^1$ with the center $Y \times \{0\}$. Set $X_t = X'_t - \tilde{X}$ where $\tilde{X}$ is the the proper preimage of $X \times \{0\}$ under the blow-up map. Let $\sigma : X_t \to X \times \mathbf{A}^1$ be the restriction of the blow-up map $\sigma' : X'_t \to X \times \mathbf{A}^1$ to X_t and set p_t to be the composition of σ and the projection $X \times \mathbf{A}^1 \to \mathbf{A}^1$.

The proper preimage of $Y \times \mathbf{A}^1$ under the blow-up map is mapped isomorphically to $Y \times \mathbf{A}^1$ under the blow-up map. Thus the inverse isomorphism gives the desired imbedding $i_t : Y \times \mathbf{A}^1 \hookrightarrow X_t$ (observe that $i_t(Y \times \mathbf{A}^1)$ does not cross $\tilde{X}$).

It's not difficult to check that the imbedding i_t satisfies the mentioned two properties (the preimage of $X \times 0$ under the map σ' consists of two irreducible components: the proper preimage of X and the exceptional divizor $\mathbb{P}(N \oplus 1)$. Their intersection is $\mathbb{P}(N)$ and $i_t(Y \times \mathbb{A}^1)$ crosses $\mathbb{P}(N \oplus 1)$ along $\mathbb{P}(1) =$ the zero section of the normal bundle N).

We claim that the diagram (4) consists of isomorphisms on the level of A.

2.2.8. Theorem. *The following diagram consists of isomorphisms*

$$A_Y(N) \xleftarrow{i_0^A} A_{Y \times \mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(X). \tag{5}$$

Moreover for each closed subset $Z \subset Y$ the following diagram consists of isomorphisms as well

$$A_Z(N) \xleftarrow{i_0^A} A_{Z \times \mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Z(X). \tag{6}$$

2.2.9. Corollary. *Let $j_0 : \mathbf{P}(1 \oplus N) \hookrightarrow X'_t$ be the imbedding of the exceptional divisor into X'_t and let $j_1 = e_t \circ i_1 : X \hookrightarrow X'_t$, where $e_t : X_t \hookrightarrow X'_t$ is the open inclusion. Then the diagram*

$$A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N) \xleftarrow{j_0^A} A_{Y \times \mathbf{A}^1}(X'_t) \xrightarrow{j_1^A} A_Y(X). \tag{7}$$

consists of isomorphisms.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N)) \arrow[d, "e^A"'] & A_{Y \times \mathbf{A}^1}(X'_t) \arrow[l, "j_0^A"'] \arrow[d, "e_t^A"] \\
A_Y(N) & A_{Y \times \mathbf{A}^1}(X_t). \arrow[l, "i_0^A"']
\end{tikzcd}$$

where the vertical arrows are the obvious pull-backs. These vertical arrows are isomorphisms by the excision property. The operator i_0^A is an ismomorphism by the first item of Theorem 2.2.8. Thus the operator j_0^A is an isomorphism. $\square$

2.2.10. Let X be a smooth variety and let L be a line bundle over X. Let $E = 1 \oplus L$ and let $\bar{i}_L : X = \mathbf{P}(L) \hookrightarrow \mathbf{P}(E)$ be the closed imbedding induced by the direct summund

L of E. Let $A_{\mathbf{P(1)}}(\mathbf{P}(E)) \xrightarrow{i^A} A(\mathbf{P}(E))$ be the support extension operator and let $\bar{i}_L^A : A(\mathbf{P}(E)) \to A(\mathbf{P}(L))$ be the pull-back operator. We claim that the following sequence

$$0 \to A_{\mathbf{P(1)}}(\mathbf{P}(E)) \xrightarrow{i^A} A(\mathbf{P}(E)) \xrightarrow{\bar{i}_L^A} A(\mathbf{P}(L)) \to 0. \tag{8}$$

is exact.

To prove this consider $U = \mathbf{P}(E) - \mathbf{P}(1)$ with the open inclusion $j : U \hookrightarrow \mathbf{P}(E)$ and observe that U becomes a line bundle over X by means of the linear projection $q : U \to \mathbf{P}(L) = X$ (the line bundle is isomorphic to $L^\vee$) The obvious inclusion $i_L : \mathbf{P}(L) \hookrightarrow U$ is just the zero section of this line bundle, $\bar{i}_L = j \circ i_L$ and the pull-back operator $i_L^A : A(U) \to A(\mathbf{P}(L))$ is an isomorphism (the inverse to the one q^A).

Now consider the pair $(\mathbf{P}(E), U)$. By the localization property 2.0.1 the following sequence

$$\ldots \to A_{\mathbf{P(1)}}(\mathbf{P}(E)) \xrightarrow{i^A} A(\mathbf{P}(E)) \xrightarrow{j^A} A(U) \to \ldots$$

is exact. If $\mathbf{P}(E) \xrightarrow{p} X$ is the natural projection then the operator $p^A \circ i_L^A : A(U) \to A(\mathbf{P}(E))$ splits j^A. This implies the surjectivity of j^A and the injectivity of i^A. To proof that the sequence (8) is short exact it remains to recall that the operator i_L^A is an isomorpism and $\bar{i}_L = j \circ i_L$.

2.2.11. We use here the notation from 2.2.7. Let $e_t : X_t \hookrightarrow X_t'$ be the open inclusion and let $p : \mathbf{P}(1 \oplus N) \to Y$ be the projection and let $s : Y \to \mathbf{P}(1 \oplus N)$ be the section of the projection identifying Y with the subvariety $\mathbf{P}(1)$ in $\mathbf{P}(1 \oplus N)$. The following commutative diagram will be repeatedly used below in the text

$$\begin{tikzcd}
\mathbf{P}(1 \oplus N) \arrow[r, "j_0"] & X_t' & X \arrow[l, "j_1"'] \\
Y \arrow[u, "s"] \arrow[r, "k_0"] & Y \times \mathbf{A}^1 \arrow[u, "I_t"] & Y, \arrow[l, "k_1"'] \arrow[u, "i"']
\end{tikzcd}$$

where $I_t = e_t \circ i_t$ and j_0 is the inclusion of the exceptional divisor and $j_1 = e_t \circ i_1$ and k_0, k_1 are the closed imbedding given by $y \mapsto (y, 0)$ and $y \mapsto (y, 1)$ respectively.

2.2.12. Lemma (Useful lemma). *Under the notation from 2.2.7 let $j_t : V_t = X_t' - Y \times \mathbf{A}^1 \to X'$ be the inclusion. If the support extension operator $A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N)) \to A(\mathbf{P}(1 \oplus N))$ is injective then*

$$Ker(j_0^A) \cap Ker(j_t^A) = (0),$$

in the other words the operator

$$j_0^A \oplus j_t^A : A(X') \to A(\mathbf{P}(1 \oplus N)) \oplus A(V_t)$$

is monomorphism. In particular this holds if Y is a divisor on X.

Proof. Consider the commutative diagram

$$\begin{tikzcd} A(\mathbf{P}(1 \oplus N)) & A(X'_t) \arrow[l, "j_0^A"'] \\ A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus N)) \arrow[u, "\alpha"] & A_{Y \times \mathbf{A}^1}(X'_t). \arrow[l, "j_0^A"'] \arrow[u, "\alpha_t"'] \end{tikzcd}$$

where α and α_t are the support extension operators. The bottom operator j_0^A is an isomorphism by Corollary 2.2.9. The map α is injective by the very assumption (if Y is a divisor in X then α is injective by (8)). Since the composition $j_0^A \circ \alpha_t$ coincides with the one $\alpha \circ j_0^A$ it is injective as well.

The localization sequence for the pair (X'_t, V_t) shows that $Ker(j_t^A) = Im(\alpha_t)$. The Lemma follows. □

2.2.13. Let $i : \mathbf{P}(V) \hookrightarrow \mathbf{P}(W)$ and $j : \mathbf{P}(V) \hookrightarrow \mathbf{P}(W)$ be two linear imbeddings (imbeddings induced by linear imbeddings V into W). If the dimension of V is strictly less than the dimension of W, then $i^A = j^A : A(\mathbf{P}(W)) \to A(\mathbf{P}(V))$.

In fact, in this case there exists a linear automorphism ϕ of W which has the determinant 1 and such that $j = \phi \circ i$. Since ϕ is a composite of elementary matrices and each elementary matrix induces the identity automorphism $A(\mathbf{P}(W))$ (by the homotopy invariance of A) one gets the relation $\phi^A = id$. Therefore $j^A = i^A \circ \phi^A = i^A$.

2.3 Proof of Theorem 2.2.8.

Proof. We give the proof of the first assertion and will give a comment about the second one. We start with certain observations concerning elementary properties of the deformation to the normal cone construction. Namely, if U and V are Zarisky open subsets of X, then the following holds

(a) $U_t \cap V_t = (U \cap V)_t$;

(b) $U_t \cup V_t = (U \cup V_t)$;

(c) if an etale morphism $e : (\tilde{X}, \tilde{X} - \tilde{Y}) \to (X, X - Y)$ satisfies the hypotheses of the excision property 2.0.1, then the induced morphism $e_t : (\tilde{X}_t, \tilde{X} - \tilde{Y} \times \mathbf{A}^1) \to (X_t, X_t - Y \times \mathbf{A}^1)$ satisfies as well the hypotheses of the excision property.

To prove Theorem we need in Lemma and four Claims below.

2.3.1. Lemma. *If $X = Y \times \mathbf{A}^n$ and $Y = Y \times \{0\} \hookrightarrow Y \times \mathbf{A}^n$ then $i^A_{1,X} : A_{Y \times \mathbf{A}^1}(X_t) \to A_Y(X)$ is an isomorphism.*

2.3.2. Definition. *An open subset U in X is called good if there exists a diagram*

$$(U, U - Y_U) \xleftarrow{e} (T, T - S) \xrightarrow{f} (Y_U \times \mathbf{A}^n, Y_U \times \mathbf{A}^n - Y_U \times \{0\})$$

with $Y_U = Y \cap U$ and with morphisms e and f satisfying the hypotheses of the excision property 2.0.1.

2.3.3. Notation. *We write below in this proof $i_{1,U}$ for the imbedding $U \hookrightarrow U_t$ from the deformation to the normal cone construction for the pair (U, Y_U). We will write below in the proof of Theorem $i^A_{1,U}$ for the pull-back map $i^A_{1,U} : A_{Y_U \times \mathbf{A}^n}(U_t) \to A_{Y_U}(U)$.*

2.3.4. Claim. *If $U \subset X$ is good then the pull-back map $i^A_{1,U} : A_{Y_U \times \mathbf{A}^n}(U_t) \to A_{Y_U}(U)$ is an isomorphism.*

2.3.5. Claim. *If an open subset U in X is good then each open subset V in U is good as well.*

2.3.6. Claim. *If an open subset U in X is good and if $V \subset X$ is an open subset such that the pull-back map $i^A_{1,V}$ is an isomorphism, then the pull-back map $i^A_{1,U \cup V}$ is an isomorphism as well.*

2.3.7. Claim. *For each point $x \in X$ there exists a good Zarisky open neighbourhood U of the point x.*

Assuming for a moment Lemma and these four Claims one can complete the proof of Theorem as follows. By the fourth Claim there exists a finite Zariski open covering $X = \cup^n_{i=1} U_i$ with good open subsets U_i. The first Claim states that the pull-back map i^A_{1,U_1} is an isomorphism. Suppose for $V = \cup^k_{i=1} U_i$ the pull-back map $i^A_{1,V}$ is an isomorphism. Since the open subset U_{k+1} is good the third Claim shows that the pull-back map $i^A_{1,W}$ is an isomorphism for $W = \cup^{k+1}_{i=1} U_i$. The induction by k shows that the pull-back map i^A_1 is an isomorphism.

It remains to prove Lemma and four Claims.

□

Proof of Lemma. Let F/Y be a vector bundle and let F' be the blow-up of F at the zero section. The variety F' coincides with the total space of the line bundle $\mathcal{O}_F(-1)$ over $\mathbf{P}(F)$. Let $q_F : F' \to \mathbf{P}(F)$ be projection of the line bundle to its base $\mathbf{P}(F)$.

If $F = E \oplus 1$ for a vector bundle E over Y then one has the following commutative diagram

$$\begin{array}{ccccccc} E' & \longrightarrow & F' & \longleftarrow & F' - E' & \longleftarrow & \mathbf{P}(1) \times \mathbf{A}^1 \\ {\scriptstyle q_E}\big\downarrow & & {\scriptstyle q_F}\big\downarrow & & {\scriptstyle q}\big\downarrow & & {\scriptstyle pr}\big\downarrow \\ \mathbf{P}(E) & \longrightarrow & \mathbf{P}(F) & \longleftarrow & \mathbf{P}(F) - \mathbf{P}(E) & \longleftarrow & \mathbf{P}(1). \end{array}$$

in which all the vertical arrows are the projections of the line bundles to their bases. The homotopy invariance property 2.2.6 shows that the pull-back map $q^A : A_{\mathbf{P}(1)}(\mathbf{P}(F) - \mathbf{P}(E)) \to A_{\mathbf{P}(1)\times\mathbf{A}^1}(F' - E')$ is an isomorphism.

The projection q has two sections s_0 and s_1. The section s_0 is the zero section and the section s_1 is given by $x \mapsto (x, 1)$. Since $q \circ s_0 = id$ the pull-back map $s^A_0 : A_{\mathbf{P}(1)\times\mathbf{A}^1}(F' - E') \to A_{\mathbf{P}(1)}(\mathbf{P}(F) - \mathbf{P}(E))$ is an isomorphism. Since $q \circ s_1 = id$ the pull-back map $s^A_1 : A_{\mathbf{P}(1)\times\mathbf{A}^1}(F' - E') \to A_{\mathbf{P}(1)}(\mathbf{P}(F) - \mathbf{P}(E))$ is an isomorphism as well.

Now take $X = E$ and $Y =$ the zero section of E. Observe that the space X_t coincides with the variety $F' - E'$, the imbedding $i_1 : X \hookrightarrow X_t$ coincides with the section $s_1 : E \hookrightarrow F' - E'$. The normal bundle $N = N_{E/Y}$ to Y in E coincides with bundle E itself and the imbedding $i_0 : N \hookrightarrow X_t$ coincides with the section $s_0 : E \hookrightarrow X_t$. Finally the variety $Y \times \mathbf{A}^1$ coincides with $\mathbf{P}(1) \times \mathbf{A}^1$ and the imbedding $Y \times \mathbf{A}^1 \hookrightarrow X_t$ coincides with the imbedding $\mathbf{P}(1) \times \mathbf{A}^1 \hookrightarrow F' - E'$. Therefore both maps in the diagram

$$A_Y(E) = A_Y(N) \xleftarrow{i_0^A} A_{Y \times \mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(E)$$

are isomorphisms. In particular these two maps are isomorphisms for the case of the trivial bundle $E = \mathbb{A}^n \times Y$. Thus we proved Lemma.

Proof of the first Claim. The Claim follows immidiately from Lemma and the property (c) of the deformation to the normal cone construction.

Proof of the second Claim. To prove this Claim consider a diagram

$$(U, U - Y_U) \xleftarrow{e} (T, T - S) \xrightarrow{f} (Y_U \times \mathbf{A}^n, Y_U \times \mathbf{A}^n - Y_U \times \{0\})$$

with morphisms e and f satisfying the hypotheses of the excision property. Let $V \subset U$ be an open subset. Set $T = e^{-1}(V) \cap f^{-1}(Y_U \times \mathbf{A}^n)$ and $S = f^{-1}(Y_U)$. Then $S = f^{-1}(Y_U \times \{0\})$ and in the diagram

$$(V, V - Y_V) \xleftarrow{e_V} (T, T - S) \xrightarrow{f_V} (Y_V \times \mathbf{A}^n, Y_V \times \mathbf{A}^n - Y_V \times \{0\})$$

the morphisms e_V and f_V satisfy the hypotheses of the excision property as well. Thus the open subset V is good.

Proof of the third Claim. This Claim follows immidiately from the properties (a) and (b) and the first Claim compairing the Mayer-Vietoris sequence for $U \cup V$ with the one for $U_t \cup V_t$.*Proof of the fourth Claim.* This Claim is proved in [V3, Lemma ??].

Comment to the second assertion of Theorem.

Recall that a Nisnevich neighborhood of a closed subset Y in X is an etale morphism $\pi : X' \to X$ such that for $Y' = \pi^{-1}(Y)$ the restriction map $\pi : Y' \to Y$ is an isomorphism. Clearly if π is a Nisnevich neighborhood of Y then for each closed subset Z in Y the map π is a Nisnevich neighborhood of the subset Z as well.

Recall as well that for any vector bundle $p : E \to X$ and any its section s and any closed subset $Z \subset X$ the two pull-back maps $p^A : A_Z(X) \to A_{p^{-1}(Z)}(E)$ and $s^A : A_{p^{-1}(Z)}(E) \to A_Z(X)$ are isomorphisms inverse of each other.

These two observations shows that the proof of the first assertion of Theorem works as well for the second assertion of Theorem.

2.4 Ring cohomology theories

2.4.1. Definition. *Let $P = (X, U), Q = (Y, V) \in \mathcal{S}m\mathcal{O}p$. Set $P \times Q = (X \times Y, X \times V \cup U \times Y) \in \mathcal{S}m\mathcal{O}p$. This product is associative with the obvious associativity isomorphisms. The unit of this product is the variety pt.*

This product is commutative with the obvious isomorphisms $P \times Q \cong Q \times P$.

2.4.2. Definition. *One says that a cohomology theory A is a ring cohomology theory if for every $P, Q \in \mathcal{S}m\mathcal{O}p$ there is given a natural bilinear morphism called cross-product*

$$\times : A(P) \times A(Q) \to A(P \times Q)$$

which is functorial in both variables and satisfies the following properties

1. *associativity: $(a \times b) \times c = a \times (b \times c) \in A(P \times Q \times R)$ for $a \in A(P)$, $b \in A(Q)$, $c \in A(R)$;*

2. *there is given an element $1 \in A(pt)$ such that for any pair $P \in \mathcal{S}m\mathcal{O}p$ and any $a \in A(P)$ one has $1 \times a = a = a \times 1 \in A(P)$;*

3. *partial Leibnitz rule: $\partial_{P \times Y}(a \times b) = \partial_P(a) \times b \in A(X \times Y, U \times Y)$ for a pair $P = (X, U) \in \mathcal{S}m\mathcal{O}p$, smooth variety Y and elements $a \in A(U)$, $b \in A(Y)$.*

Given cross-products define cup-products $\cup : A_Z(X) \times A_{Z'}(X) \to A_{Z \cap Z'}(X)$ by

$$a \cup b = \Delta^A(a \times b), \tag{9}$$

where $\Delta : (X, U \cup V) \hookrightarrow (X \times X, X \times V \cup U \times X)$ is the diagonal. Clearly cup-products thus defined are bilinear and functorial in both variables. As well these cup-products are associative: $(a \cup b) \cup c = a \cup (b \cup c)$; the element $p^A(1) \in A(X)$, (here p is the projection $X \to pt$) is the unit for the cup-products $\cup : A_Z(X) \times A_(X) \to A_Z(X)$ and $\cup : A_(X) \times A_Z(X) \to A_Z(X)$; and a partial Leibnitz rule holds: $\partial(a \cup b) = \partial(a) \cup b$ for $a \in A(U)$, $b \in A(X)$.

Given cup-products one can construct cross-products by $a \times b = p_X^A(a) \cup p_Y^A(b)$ for $a \in A(X, U)$ and $b \in A(Y, V)$. Clearly these two constructions are inverse each to other. Thus having products of one kind we have products of the other kind and can use both products in the same time.

2.4.3. Definition. *A ring morphism $\varphi : (A, \partial_A, \times_A, 1_A) \to (B, \partial_B, \times_B, 1_B)$ of ring cohomology theories is a morphism $\varphi : (A, \partial_A) \to (B, \partial_B)$ of the underlying cohomology theories which takes the unit 1_A to the unit 1_B and commutes with the $\times$-products:*

$\varphi(1_A) = 1_B \in B(pt)$

and for every pairs $P, Q \in \mathcal{S}m\mathcal{O}p$ and every elements $a \in A(P)$, $b \in A(Q)$ one has

$\varphi_{P \times Q}(a \times b) = \varphi(a) \times \varphi(b) \in B(P \times Q)$.

2.4.4. Definition. *Let A be a ring cohomology theory and let X be a smooth variety. An element $a \in A(X)$ is called universally central if for any smooth variety $\tilde{X}$ and any morphism $f : \tilde{X} \to X$ the element $f^A(a)$ is central in $A(\tilde{X})$.*

Note, that if the theory A takes values in the category of graded abelian groups and is moreover a graded-commutative ring theory, i.e. for any $a \in A^p(X)$ and $b \in A^q(Y)$ one has the relation $a \cup b = (-1)^{pq} b \cup a$, then each *even degree* element is a universally central element.

One should remark as well that in the graded commutative case the second partial Leibnitz rule holds (if we assume as usual that for every pair (X,U) the operator $\partial_{X,U}$ is a graded operator of degree $+1$). Namely, if $a \in A^p(U)$ and $b \in A^q(Y)$ and U is open in a smooth X, then the relation $\partial_{Y\times X, Y\times U}(b \times a) = (-1)^q b \times \partial_{X,U}(a)$ in $A(Y \times X, Y \times U)$.

2.4.5. Proposition. *Let $f : (X,U) \to (X',U')$ be a morphism of pairs and let $\alpha \in A(X)$ and let $\alpha|_U = \alpha|_U \in A(U)$. The composition operator $(\cup\alpha \circ f^A : A(X') \to A(X))$ (respectively $(\cup\alpha_U \circ f^A : A(U') \to A(U))$ and $(\cup\alpha \circ f^A : A(X',U') \to A(X,U))$) denote for short α (respectively $\cup\alpha_U$, and $\cup\alpha$). Then these operators form a morphism of the localization sequences for the pairs (X',U') and (X,U), that is the diagram commutes*

$$\begin{tikzcd}
A(X) \arrow[r] & A(U) \arrow[r, "\partial_{X,U}"] & A(X,U) \arrow[r] & A(X) \arrow[r] & A(U) \\
A(X') \arrow[r] \arrow[u, "\cup\alpha"] & A(U') \arrow[r, "\partial_{X',U'}"] \arrow[u, "\cup\alpha_U"] & A(X',U') \arrow[r] \arrow[u, "\cup\alpha"] & A(X' \arrow[r] \arrow[u, "\cup\alpha"] & A(U')). \arrow[u, "\cup\alpha_U"]
\end{tikzcd}$$

One more statement of this type is useful as well. It follows straightforward from the construction of the Mayer-Vietoris sequence in 2.2.4.

2.4.6. Proposition. *Let $X = U_1 \cup U_2$ and let $X' = U_1' \cup U_2'$ be open coverings. Let $f : X \to X'$ be a morphism such that $f(U_i) \subset U_i'$ for $i = 1,2$.Let $\alpha \in A(X)$ be an element and let $\alpha_i = \alpha|_{U_i} \in A(U_i)$ and let $\alpha_{12} = \alpha|_{U_{12}} \in A(U_{12})$. The composition operators $(\cup\alpha \circ f^A : A(X') \to A(X))$ (respectively $(\cup\alpha_i \circ f^A : A(U_i') \to A(U_i))$ and $(\cup\alpha_{12} \circ f^A : A(U_{12}') \to A(U_{12}))$) denote for short $\cup\alpha$ (respectively $\cup\alpha_i$ and $\cup\alpha_{12}$). Then these operators form a morphism of the Mayer-Vietoris sequences corresponding to the coverings $X' = U_1' \cup U_2'$ and $X = U_1 \cup U_2$, that is the diagram commutes*

$$\begin{tikzcd}
A(U_1) \oplus A(U_2) \arrow[r] & A(U_{12}) \arrow[r, "\partial"] & A(X) \arrow[r] & A(U_1) \oplus A(U_2) \arrow[r] & A(U_1 2) \\
A(U_1') \oplus A(U_2') \arrow[r] \arrow[u, "(\cup\alpha_1, \cup\alpha_2)"] & A(U_{12}') \arrow[r, "\partial"] \arrow[u, "\cup\alpha_{12}"] & A(X') \arrow[r] \arrow[u, "\cup\alpha"] & A(U_1') \oplus A(U_2') \arrow[r] \arrow[u, "(\cup\alpha_1, \cup\alpha_2)"] & A(U_{12}')). \arrow[u, "\cup\alpha_{12}"]
\end{tikzcd}$$

The definition of a ring cohomology theory is equivalent to the following more technical but pretty useful one

2.4.7. Definition. *A ring cohomology theory is a weak morphism*

$$(A, \mu, e) : (\mathcal{S}m\mathcal{O}p, \times, pt) \to (\mathcal{A}b, \otimes, \mathbb{Z})$$

of the monoidal categories together with a functor transformation ∂ such that

the pair (A, ∂) is a cohomology theory 2.0.1 and

the boundary operator ∂ satisfies the partial Leibnitz rule saying that

$\partial_{P\times Y}(\mu_{U,Y}(a \otimes b)) = \mu_{P,Y}(\partial_P(a) \otimes b) \in A(X \times Y, U \times Y)$.

(Under this variant of the notation the cross-product $c \times d \in A(P \times Q)$ of elements $c \in A(P)$ and $d \in A(Q)$ is the element $\mu_{P,Q}(a \otimes b) \in A(P \times Q)$).

One could replace in this form of the definition the monoidal category $(\mathcal{A}b, \otimes, \mathbb{Z})$ by any other abelian monoidal category $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbf{1}_{\mathcal{C}})$ reformulating the partial Lebnitz rule as follows: for every pair $P \in \mathcal{S}m\mathcal{O}p$ and a smooth variety Y the relation holds

$$\partial_{P\times Y} \circ \mu_{U,Y} \circ (id_U \otimes_{\mathcal{C}} id_Y) = \mu_{P,Y} \circ (\partial_P \otimes_{\mathcal{C}} id_{A(Y)}).$$

Once again we left such a replacement to a reader to avoid technicalities as much as it is possible.

2.5 Examples

Consider following examples.

2.5.1. Etale cohomology. Let $A^*_Z(X) = \oplus_{q=-\infty}^{+\infty} H^*_Z(X, \mu_m^{\otimes q})$ be the etale cohomology theory, where m is an integer prime to char(k). The cup-products are described in [Mi, Ch.V, §1, 1.17].

2.5.2. K-theory. Let A be the algebraic K-theory from the example 2.1.8. So $A(X,U) = \oplus_{n=0}^{\infty} K_{-n}(X on Z)$, where $Z = X - U$. The idea of the definition of the products is given in [TT].

2.5.3. Motivic cohomology. Let $A_Z(X) = \oplus_{q=0}^{\infty} \oplus_{p=-\infty}^{\infty} H^p_{\mathcal{M},Z}(X, \mathbb{Z}_{tr}(q))$ be the motivic cohomology [SV]. Under the notation of 2.1.9 $A_Z(X)$ coincides with $\oplus_{q=0}^{\infty} A_Z^{\mathbb{Z}_{tr}(q)}(X)$. Take $\partial = \oplus_{q=0}^{\infty} \partial^{\mathbb{Z}_{tr}(q)}$. The products are defined in [SV, the text just below Lemma 3.3] and are induced by the canonical pairings $\mathbb{Z}_{tr}(r) \otimes_{tr} \mathbb{Z}_{tr}(s) \to \mathbb{Z}_{tr}(r+s)$. The products are associative and graded commutative, the unit 1 of this product is the element $1 \in H^0_{\mathcal{M}}(pt, \mathbb{Z}_{tr}(0)) = \mathbb{Z}$ [SV][??].

2.5.4. Semi-topological complex and real K-theories [FW]. If the ground field k is the field $\mathbb{R}$ of reals then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{semi}$ defined in[FW] is a ring cohomology theory as it is proved in [FW].

2.5.5. Algebraic cobordism theory. To introduce a ring structure on the algebraic cobordism theory (2.1.11) it would be convenient to enrich MGL with a symmetric ring structure [Ja, Sect.4]. For that we construct another T-spectra $\mathbb{MGL}$ which is a commutative symmetric ring spectra by the very construction and which is weekly equivalent to MGL as the T-spectra. Now we are going to describe $\mathbb{MGL}$. A ring structure on the algebraic cobordism theory was introduced as well in [Hu].

Let V be a countable dimension vector space and let $e \in V$ be a non-zero vector (for instance one can take $V = k^\infty$ and $e = (1, 0, \dots)$). Now set $V_n = V \oplus V \oplus \dots \oplus V$ (n summands) and let $e^n_i = (0, \dots, e, \dots, 0) \in V_n$ be a vector with the only non-zero term e on i-th position. For a finite dimensional vector space W and a positive m let $\mathbf{Gr}(m, W)$ be the Grassmannian of m-dimensional subspaces of W and Let $\mathcal{T}(m, W)$ be the tautological rank m vector bundle on $\mathbf{Gr}(m,W)$. For a vector bundle E over a smooth variety X we denote $Th(E)$ its the Thom space $E/(E-X)$ ([V1, p. 422]. In this subsection we identify $\mathcal{S}m$ with the full subcategory of spaces Spc [V1, p. 421].

Let $G(n) = \cup\mathbf{Gr}(n,W)$ and let $\mathcal{T} = \cup\mathcal{T}(n,W)$ and let $Th(\mathcal{T}(n)) = \cup Th(\mathcal{T}(n,W))$ where W runs over all finite-dimensional vector subspaces of V_n. Set

$$\mathbb{MGL}_n = Th(\mathcal{T}(n)).$$

The subspace of V_n generated by the vectors e_i^n $(i = 1,\dots,n)$ gives a distinguish point g_n of the space $G(n)$. The fiber of the vector bundle $\mathcal{T}(n)$ over the point g_n denote $\mathbb{A}^n$ (it is an n-dimensional vector bundle over the point g_n equipped with a free basis $e_1^n, e_2^n, \dots, e_n^n$).

We will identify the space $G(m)\times G(n)$ with a subspace of $G(m+n)$ identifying the point $(\{W_1\},\{W_2\}) \in G(m)\times G(n)$ with the point $\{W_1\oplus W_2\} \in G(m+n)$. Observe that the distinguish point (g_m, g_n) is identified with the distinguish point g_{m+n} under the mentioned identification. Observe furthermore that the restriction of the vector bundle $\mathcal{T}(m+n)$ to $G(m)\times G(n)$ coincides with the vector bundle $\mathcal{T}(m)\times\mathcal{T}(n)$. Thus one has the inclusion of spaces (see [V1, p. 422])

$$\mathbb{MGL}_m \wedge \mathbb{MGL}_n \subset \mathbb{MGL}_{m+n}. \tag{10}$$

The inclusion of the fiber $\mathbb{A}^m$ in $\mathcal{T}(m)$ induces an inclusion $Th(\mathbb{A}^m) \subset Th(\mathcal{T}(m)) = \mathbb{MGL}_m$. Composing it with the inclusion (10) one gets the following inclusion

$$Th(\mathbb{A}^m) \wedge \mathbb{MGL}_n \subset \mathbb{MGL}_{m+n}. \tag{11}$$

Consider the action of the permutation group Σ_k on the vector space V_k by the permutation of the summands. This action induces a Σ_k-action on the spaces $G(k)$, $\mathcal{T}(k)$ and $\mathbb{MGL}_k = Th(\mathcal{T}(k))$. As well this action permutes the vectors $e_1^k, e_2^k, \dots, e_k^k$ and thus acts on the fiber $\mathbb{A}^k$ and on the Thom space $Th(\mathbb{A}^k) = Th(\mathbb{A}^1)\wedge Th(\mathbb{A}^1)\wedge\cdots\wedge Th(\mathbb{A}^1)$ permuting the coordinates. If we consider the obvious inclusion of groups $\Sigma_m\times\Sigma_n \subset \Sigma_{m+n}$ then the inclusions (10) and (11) are $\Sigma_m\times\Sigma_n$-equivariant.

Now the family of spaces $\mathbb{MGL}_n$ together with the inclusions (11) forms a symmetric T-spectrum [Ja, Sect.4] which we denote $\mathbb{MGL}$. The family of inclusions (10) forms a structure of the commutative symmetric ring spectrum on the symmetric spectrum $\mathbb{MGL}$ (see [Ja, Sect.4.3]).

Considered as a T-spectra the spectra $\mathbb{MGL}$ is weakly equivalent to the spectra MGL from [V1, 6.3]. To prove this consider a subspace $V_n' =< e_1^n, e_2^n, \dots, e_{n-1}^n > \oplus V$ of the space V_n where the summand V is the very last copy of V in V_n. Let $G'(n) = \cup Gr(n,W)$ and $\mathcal{T}'(n) = \cup\mathcal{T}(n,W)$ where W runs over all finite-dimensional subspace of V_n'. Let $MGL_n = Th(\mathcal{T}'(n))$ be the Thom space of $\mathcal{T}'(n)$. It is exactly the space MGL_n from [V1, 6.3].

The imbedding $i_n : V_n \hookrightarrow V_{n+1}$ onto the last n copies of V in V_{n+1} induces an imbedding $i_n' : V_n' \hookrightarrow V_{n+1}'$ which takes the vector e_i^n to the vector e_{i+1}^{n+1} $(i = 1,2,\dots,n-1)$ and which is the identity on the summand V. These inclusions give rise to a commutative diagram of spaces

$$\begin{array}{ccc} Th(\mathbb{A}^1) \wedge Th(\mathfrak{T}'(n)) & \longrightarrow & Th(\mathfrak{T}'(n+1)) \\ \downarrow & & \downarrow \\ Th(\mathbb{A}^1) \wedge Th(\mathfrak{T}(n)) & \longrightarrow & Th(\mathfrak{T}(n+1)) \end{array}$$

Replacing just the notation one gets a commutative diagram of spaces

$$\begin{array}{ccc} Th(\mathbb{A}^1) \wedge MGL_n & \longrightarrow & MGL_{n+1} \\ \downarrow & & \downarrow \\ Th(\mathbb{A}^1) \wedge \mathbb{MGL}_n & \longrightarrow & \mathbb{MGL}_{n+1} \end{array}$$

Now observe that the top horizontal arrow in the last diagram is exactly the assembly map for the T-spectra MGL from [V1, 6.3]. Observe furthermore that the bottom horizontal arrow in this diagram is exactly the assembly map for the symmetric T-spectra $\mathbb{MGL}$ considered just as the T-spectra.Thus the family of inclusions $MGL_n \subset \mathbb{MGL}_n$ induced by the inclusions $V'_n \subset V_n$ is a T-spectra morphism $MGL \to \mathbb{MGL}$. The individual inclusions of spaces $MGL_n \subset \mathbb{MGL}_n$ is a weak equivalence. Thus the T-spectra morphism $MGL \to \mathbb{MGL}$ is a weak equivalence as well.

3 Orientations

In this section A is a ring cohomology theory. We introduce three following structures which A can be endowed with: an orientation, a Chern structure and a Thom structure. We show that there is a natural one-to-one correspondence between these structures (see Theorems 3.2.4, 3.7.4 and 3.7.5).

3.1 Orientations on a ring cohomology theory

Let us recall that for a vector bundle E over a variety X we identify X with $z(X)$ where $z : X \to E$ is the zero section.

3.1.1. Definition. *An orientation on the theory A is a rule assigning to each smooth variety X, to each its closed subset Z and to each vector bundle E/X an operator*

$$th^E_Z : A_Z(X) \to A_Z(E)$$

which is a two-sided $A(X)$-module isomorphism and satisfies the following properties

1. *invariance: for each vector bundle isomorphism $\varphi : E \to F$ the diagram commutes*

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{th^E_Z} & A_Z(F) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle \varphi^A} \\ A_Z(X) & \xrightarrow{th^E_Z} & A_Z(E) \end{array}$$

2. *base change: for each morphism $f : (X', X' - Z') \to (X, X - Z)$ with closed subsets $Z \hookrightarrow X$ and $Z' \hookrightarrow X'$ and for each vector bundle E/X and for its pull-back E' over X' and for the projection $g : E' = E \times_X X' \to E$ the diagram commutes*

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{th_Z^E} & A_Z(E) \\ {\scriptstyle f^A}\downarrow & & \downarrow{\scriptstyle g^A} \\ A_{Z'}(X') & \xrightarrow{th_{Z'}^{E'}} & A_{Z'}(E') \end{array}$$

3. *for each vector bundles $p : E \to X$ and $q : F \to X$ the following diagram commutes*

$$\begin{array}{ccc} A_Z(X) & \xrightarrow{th_Z^E} & A_Z(E) \\ {\scriptstyle th_Z^F}\downarrow & & \downarrow{\scriptstyle th_Z^{p^*(F)}} \\ A_Z(F) & \xrightarrow{th_Z^{q^*(E)}} & A_Z(E \oplus F) \end{array}$$

   and both compositions coincide with the operator $th_Z^{E\oplus F}$.

The operators th_Z^E are called Thom isomorphisms. The theory A is called orientable if there exists an orientation of A. The theory A is called oriented if an orientation is chosen and fixed.

Next we are going to describe a number of data which allow to orient A.

3.2 Chern and Thom structures on A

In this section A is a ring cohomology theory. If X is a smooth variety we write $\mathbf{1}_X$ for the trivial rank one bundle over X. Often we will just write $\mathbf{1}$ for $\mathbf{1}_X$ if it is clear from a context what the variety X is.

3.2.1. Definition. *A Chern structure on A is an assignment $L \mapsto c(L)$ which assosiate to each smooth X and each line bundle L/X a universally central element $c(L) \in A(X)$ satisfying the following properties*

1. *functoriality:*

   $c(L_1) = c(L_2)$ for isomorphic line bundles L_1 and L_2;

   $f^A(c(L)) = c(f^*(L))$ for each morphism $f : Y \to X$;

2. *nondegeneracy: the operator $(1, \xi) : A(X) \oplus A(X) \to A(X \times \mathbf{P}^1)$ is an isomorphism where $\xi = c(\mathcal{O}(-1))$ and $\mathcal{O}(-1)$ is the tautological line bundle on $\mathbf{P}^1$;*

3. *vanishing: $c(\mathbf{1}_X) = 0 \in A(X)$ for any smooth variety X.*

The element $c(L) \in A(X)$ is called Chern class of the line bundle L. (It will be proved below in 3.6.4 that the elements $c(L)$ are nilpotent).

Let E be a vector bundle over a smooth X and $m \in A_X(E)$ be an element. We will say below in the text that m is $A(X)$-*central*, if for any element $a \in A(X)$ one has the relations $m \cup a = a \cup m$ in $A_X(E)$ (this moment we consider elements of $A(X)$ as elements of $A(E)$ by means of the pull-back operator induced by the projection $E \to X$). We will say that m is *universally* $A(X)$-*central* if for any morphism $f : X' \to X$ and the vector bundle $E' = X' \times_X E$ and its projection $F : E' \to E$ the element $F^A(m) \in A_{X'}(E')$ is $A(X')$-central.

Observe that for a universally $A(X)$-central element $m \in A_X(E)$ the element $z^A(i^A(m))$ in $A(X)$ is universally central in the sense of Definition 2.4.4 (here $i^A : A_X(E) \to A(E)$ is the support extention operator and $z : X \to E$ is the zero section of E).

3.2.2. Definition. *One says that A is endowed with a Thom structure if for each smooth variety X and each line bundle L/X it is chosen and fixed a universally central element $th(L) \in A_X(L)$ satisfying the following properties*

1. *functoriality:*

   $\varphi^A(th(L_2)) = th(L_1)$ *for each isomorphism* $\varphi : L_1 \to L_2$ *of line bundles;*

   $f_L^A(th(L)) = th(L_Y)$ *for each morphism* $f : Y \to X$ *and each line bundle* L/X *where* $L_Y = L \times_X Y$ *is the line bundle over* Y *and* $f_L : L_Y \to L$ *is the projection to* L*;*

2. *nondegeneracy: the cup-product* $\cup th(1) : A(X) \to A_X(X \times \mathbf{A}^1)$ *is an isomorphism (here* X *is identified with* $X \times \{0\}$ *).*

The element $th(L) \in A_X(L)$ is called *Thom class* of the line bundle L. Now we are going to describe a one-to-one correspondence between Chern and Thom structures on A.

3.2.3. Lemma. *Assume A is endowed with a Chern structure $L \mapsto c(L)$. Let L be a line bundle over a smooth X and let $E = \mathbf{1}_X \oplus L$ and let $p : \mathbf{P}(E) \to X$ be the projection. Identify the group $A_{\mathbf{P(1)}}(\mathbf{P}(E))$ with a subgroup of $A(\mathbf{P}(E))$ via the support extension operator $A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A(\mathbf{P}(E))$ from the sequence (8). Then the element $c(\mathcal{O}_E(1) \otimes p^*L) \in A(\mathbf{P}(E))$ belongs to the subgroup $A_{\mathbf{P(1)}}(\mathbf{P}(E))$ of the group $A(\mathbf{P}(E))$. Below we will often write $\bar{th}(L)$ for $c(\mathcal{O}_E(1) \otimes p^*L)$.*

Proof. The projection to the base X identifies the closed subvariety $\mathbf{P}(L)$ with the variety X. The restriction of the line bundle $\mathcal{O}_E(1)$ to $\mathbf{P}(L)$ is coincides with $L^\vee$. Thus the restriction of $\mathcal{O}_E(1) \otimes p^*L$ to $\mathbf{P}(L)$ is the trivial bundle. Now if $i_L : \mathbf{P}(L) \hookrightarrow \mathbf{P}(E)$ is the inclusion from (8) then

$$i_L^A(c(\mathcal{O}_E(1) \otimes p^*L)) = c(i_L^*(\mathcal{O}_E(1) \otimes p^*L)) = 0.$$

The exactness of the sequence (8) completes the proof.

$\square$

Now we are ready to describe the mentioned one-to-one correspondence. Assuming that A is endowed with a Thom structure $L \mapsto th(L)$ endow A with a Chern structure as follows. For a line bundle L over a smooth X set

$$c(L) = [z^A \circ i^A](th(L)) \in A(X) \tag{12}$$

where $i^A : A_X(L) \to A(L)$ is the support extension operator (see 2.0.1) and $z^A : A(L) \to A(X)$ is the operator induced by the zero section $z : X \to L$.

Assuming that A is endowed with a Chern structure $L \mapsto c(L)$ endow A with a Thom structure as follows. For a line bundle L over a smooth X consider the vector bundle $E = \mathbf{1} \oplus L$, the projection $p : \mathbf{P}(E) \to X$, the natural inclusion $e : L \hookrightarrow \mathbf{P}(E)$ and the pull-back $e^A : A_{\mathbf{P}(1)}(\mathbf{P}(E)) \to A_X(L)$. The element $\bar{th}(L) = c(\mathcal{O}_E(1) \otimes p^*L) \in A(\mathbf{P}(E))$ belongs to the subgroup $A_{\mathbf{P}(1)}(\mathbf{P}(E))$ by Lemma above. Now set

$$th(L) = e^A(c(\mathcal{O}_E(1) \otimes p^*L)) \in A(L, L^0) = A_X(L) \tag{13}$$

3.2.4. Theorem. *For any ring cohomology theory A the following assertions hold.*

1. *If A is endowed with a Thom structure $L \mapsto th(L)$ then the assignment $L \mapsto c(L)$ given by (12) endows A with a Chern structure;*

2. *If A is endowed with a Chern structure $L \mapsto c(L)$ then the assignment $L \mapsto th(L)$ given by (13) endows A with a Thom structure;*

3. *The constructions described in the items 2 and in 1 are inverse of each other: namely if $L \mapsto c(L)$ is a Chern structure on A and $L \mapsto th(L)$ is a Thom structure on A, then the relation (12) valids for all line bundles if and only if the relation (13) valids for all line bundle.*

Let $L \mapsto c(L)$ be a Chern structure on A and let $L \mapsto th(L)$ be a Thom structure on A. In the case when (13) holds for all line bundles (or (12) holds for all line bundles which is the same) we say that the Chern structure and the Thom structure on A correspond to each other.

The item 1 describes the arrow γ from the Introduction. The item 2 describes a unique arrow inverse to the arrow γ.

Proof. We start with the following

3.2.5. Lemma. *Let $L \mapsto c(L)$ be a Chern structure on A. Let $\mathcal{O}(-1)$ be the tautological line bundle on the projective line $\mathbf{P}^1$ and let $\mathcal{O}(1)$ be the dual bundle and let $\xi = c(\mathcal{O}(-1))$, and $\zeta = \mathcal{O}(1)$. Then $\xi^2 = 0 = \zeta^2$ and $c(\mathcal{O}(1)) = -c(\mathcal{O}(-1))$.*

Proof of Lemma. Since $\mathcal{O}(-1)|_{\{0\}}$ is the trivial bundle one has $\xi|_{\{0\}} = 0$. Thus $\xi \in A_{\{0\}}(\mathbf{P}^1)$ and $\xi \in A_{\{\infty\}}(\mathbf{P}^1)$. Therefore the element $\xi^2 \in A(\mathbf{P}^1)$ is in the image of $A_{\{0\} \cap \{\infty\}}(\mathbf{P}^1) = A_{\emptyset}(\mathbf{P}^1)$. This last group vanishes by the property 2.2.1 and thus $\xi^2 = 0$. Similarly $\zeta^2 = 0$.

The $A(pt)$-module $A(\mathbf{P}^1\times\mathbf{P}^1)$ is a free module with the free bases 1, $\xi\otimes 1$, $1\otimes\xi$ and $\xi\otimes\xi$ by the property of the Chern classes. Consider an element $\alpha = c(p_1^*(\mathcal{O}(-1)\otimes p_2^*(\mathcal{O}(1)))$. Write it in the form $\alpha = a_{00}1\otimes 1 + a_{10}\xi\otimes 1 + a_{01}1\otimes\xi + a_{11}\xi\otimes\xi$, where a_{ij} are elements in $A(pt)$. Restricting the element α to $\{0\}\times\{0\}$, to $\mathbf{P}^1\times\{0\}$ and to the diagonal $\Delta(\mathbf{P}^1)$ one gets the following relations: $a_{00}=0$, $a_{00}+a_{10}\xi=\xi$ in $A(\mathbf{P}^1)$ and $a_{00}+(a_{10}+a_{01})\xi = 0$ in $A(\mathbf{P}^1)$. Thus $a_{10}=1$ and $a_{10}+a_{01}=0$. Therefore $a_{01}=-a_{10}=-1$. The chain of the relations $\zeta = \alpha_{\{0\}\times\mathbf{P}^1} = a_{00}+a_{01}\xi = -\xi$. completes the proof of Lemma.

Now we are ready to prove the assertion (2) of Theorem. The functoriality of the assignement $L\mapsto th(L)$ is obvious. Now to prove that the element $th(L)$ is universally central it suffices to prove that the element $th(L)$ is $A(L)$-central. Consider the element $\bar{th}(L) = c(\mathcal{O}_E(1)\otimes p^*L)) \in A_{\mathbf{P}(1)}(\mathbf{P}(E))$. This element is $A(\mathbf{P}(E))$-central because it is the Chern class. Since $th(L) = e^A(\bar{th}(L))$ and the pull-back map $A(\mathbf{P}(E))\to A(L)$ is surjective the element $th(L)$ is $A(L)$-central.

It remains to prove the nondegeneracy property of the element $th(1)$. For that consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_{X\times\{0\}}(X\times\mathbf{P}^1) & \longrightarrow & A(X\times\mathbf{P}^1) & \longrightarrow & A(X\times\mathbf{A}^1) & \longrightarrow & 0\\
 & & \uparrow{\scriptstyle \cup\bar{th}(1)} & & \uparrow{\scriptstyle (\cup\bar{th}(1),\cup 1)} & & \uparrow{\scriptstyle pr^A} & & \\
0 & \longrightarrow & A(X) & \longrightarrow & A(X)\oplus A(X) & \longrightarrow & A(X) & \longrightarrow & 0
\end{array}$$

The non-degeneracy property of the Chern class and the relation $\bar{th}(1) = -c(\mathcal{O}(-1)) = -\xi$ show that the middle vertical arrow is an isomorphism. The right vertical arrow is an isomorphism by the homotopy invariance property. Therefore the left vertical arrow is an isomorphism as well.

Now consider the commutative diagram

$$\begin{array}{ccc}
A(X) & \xrightarrow{\cup\bar{th}(1)} & A_{X\times\{0\}}(X\times\mathbf{P}^1)\\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle e^A}\\
A(X) & \xrightarrow{\cup th(1)} & A_{X\times\{0\}}(X\times\mathbf{A}^1)
\end{array}$$

The map e^A is an isomorphism by the excision property. Therefore the bottom arrow is an isomorphism as well. The non-degeneracy property of the class $th(1)$ is proved.

To prove the assertion (1) of Theorem we need in some preliminaries.

3.2.6. Notation. *Let M be a line bundle over a smooth variety X and let $e^A : A_{\mathbf{P}(1)}(\mathbf{P}(1\oplus M))\to A_X(M)$ be the excision isomorphism induced by the open inclusion $e: M\hookrightarrow \mathbf{P}(1\oplus M)$. For an element $\alpha\in A_X(M)$ set*

$$\bar{\alpha} = (e^A)^{-1}(\alpha)\in A_{\mathbf{P}(1)}(\mathbf{P}(1\oplus M))$$

Since the support extension map $A_{\mathbf{P}(1)}(\mathbf{P}(1\oplus M))\to A(\mathbf{P}(1\oplus M))$ is injective 8 we will often write $\bar{\alpha}$ for the image of this element in $A(\mathbf{P}(1\oplus M))$. If $\alpha = th(M)$ is the Thom class of M then we will often write $\bar{th}(M)$ for the element $\bar{\alpha}$.

The folowing two observations will be useful for the proof as well

- if $\varphi : X_1 \to X$ is a morphism of smooth varieties and $M_1 = \varphi^*(M)$ is the line bundle over X_1 and $\Phi : \mathbf{P}_{X_1}(1 \oplus M_1) \to \mathbf{P}_X(1 \oplus M) = \mathbf{P}(1 \oplus M)$ is the induced morphism of the projective bundles then for $\alpha_1 = \varphi^A(\alpha)$ one has the relation $\bar{\alpha}_1 = \Phi^A(\bar{\alpha})$.
- if $s : X \to \mathbf{P}(1 \oplus M)$ is the section identifying X with $\mathbf{P}(1)$ then one has $s^A(\bar{\alpha}) = z^A(i^A(\alpha))$, where z is the zero section of M and $i^A : A_X(M) \to A(M)$ is the support extension operator.

Now under Notation 3.2.6 one has the following

3.2.7. Lemma. *Let an assignement $L \mapsto th(L)$ be the Thom structure on A. Let $L \mapsto c(L)$ be the assignment given by the formular (12). Then for the line bundle $\mathcal{O}(1)$ on $\mathbf{P}^1$ one has the relation $\bar{th}(1) = c(\mathcal{O}(1))$ in $A(\mathbf{P}^1)$.*

Proof of Lemma. Let $\infty \in \mathbf{P}^2$ be a rational point and let $\sigma : \mathbf{P}' \to \mathbf{P}^2$ be the blow-up of the projective plane $\mathbf{P}^2$ at the point ∞. The linear projection $\mathbf{P}^2 - \infty \to \mathbf{P}^1$ extends canonically to a morphism $p : \mathbf{P}' \to \mathbf{P}^1$. Using this morphism the variety $\mathbf{P}'$ is natulally identified with the projective bundle $\mathbf{P}(1 \oplus L)$ over the projective line $\mathbf{P}^1$,where $L = \mathcal{O}(1)$. Under this identification the preimage $\sigma^{-1}(\infty)$ of the point ∞ coincides with the subvariety $\mathbf{P}(L)$ of the projective bundle $\mathbf{P}(1 \oplus L)$. The subvariety $\mathbf{P}(1) \subset \mathbf{P}(1 \oplus L)$ is the image of a section $s_1 : \mathbf{P}^1 \to \mathbf{P}(1 \oplus L)$ of the projection p. The image of $\mathbf{P}^1$ under the composite map $\sigma \circ s_1$ is a projective line l in $\mathbf{P}^2$ which avoids the point ∞. Let $x \in \mathbf{P}^1$ be a rational point and let $j : \mathbf{P}^1 = p^{-1}(x) \hookrightarrow \mathbf{P}(1 \oplus L)$ be the imbedding of the fibre into the total space. One can summurise these data in the following diagram

$$\begin{tikzcd}
 & \mathbf{P}^1 \arrow[r, "\sigma"] \arrow[d, "s_L"] & pt \arrow[d, "i_\infty"] \\
\mathbf{P}^1 \arrow[r, "j"] \arrow[d, "p_0"] & \mathbf{P}(1 \oplus L) \arrow[r, "\sigma"] \arrow[d, "p"] & \mathbf{P}^2 \\
pt \arrow[r, "i"] & \mathbf{P}^1 \arrow[ur, "\sigma \circ s_1"'] &
\end{tikzcd}$$

Now set $\alpha = th(L) \in A_X(L)$, then $\bar{\alpha} = \bar{th}(L) \in A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L))$. In the commutative diagram of the pull-backs

$$\begin{array}{ccc}
A_l(\mathbf{P}^2) & \xrightarrow{\sigma^A = u} & A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L)) \\
{\scriptstyle w}\downarrow & & \downarrow{\scriptstyle v} \\
A_l(\mathbf{P}^2 - \infty) & \xrightarrow{t} & A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L) - \mathbf{P}(L)),
\end{array}$$

the maps w, t and v are isomorphisms. In fact, w and v are isomorphisms by the excision property and t is isomorphism because the map σ identifies $\mathbf{P}(1 \oplus L) - \mathbf{P}(L)$ with $\mathbf{P}^2 - \infty$. Therefore the fourth arrow $u = \sigma^A$ is an isomorphism as well.

Therefore there exists an element $\beta \in A_l(\mathbf{P}^2)$ such that $\sigma^A(\beta) = \bar{\alpha}$. The mappings $\sigma \circ j, \sigma \circ s_1 : \mathbf{P}^1 \to \mathbf{P}^2$ are two linear imbeddings of the projective line into $\mathbf{P}^2$. Therefore by the property 2.2.13 one has the relation $(\sigma \circ j)^A(\beta) = (\sigma \circ s_1)^A(\beta)$ in $A(\mathbf{P}^1)$. Thus one gets the chain of relations in $A(\mathbf{P}^1)$

$$j^A(\bar{\alpha}) = (\sigma \circ j)^A(\beta) = (\sigma \circ s_1)^A(\beta) = s_1^A(\bar{\alpha}).$$

By the two observations mentioned just below 3.2.6 one gets the relations $j^A(\bar{\alpha}) = \bar{t}h(1)$ and $s_1^A(\bar{\alpha}) = c(L) = c(\mathcal{O}(1))$. Thus $\bar{t}h(1) = c(\mathcal{O}(1))$ and Lemma is proved.

Now we are ready to proof assertion (1) of Theorem. The functotiality of the class $L \mapsto c(L)$ given by the formula (12) is obvious. To prove that for the trivial line bundle 1 over a smooth variety X one has $c(1) = 0$ consider a section $s : X \to X \times \mathbf{A}^1$ of the trivial bundle 1 which takes a point $x \in X$ to the point $(x, 1)$. If z is the zero section of the same bundle then the pull-back mappings z^A and s^A coincides. In fact both are the inverse to the pull-back map $p^A : A(X) \to A(X \times \mathbf{A}^1)$ induced by the projection $p : X \times \mathbf{A}^1 \to X$. If $i^A : A_{X \times 0}(X \times \mathbf{A}^1) \to A(X \times \mathbf{A}^1)$ is the support extension map, then $c(1) = (z^A \circ i^A)(1) = (s^A \circ i^A)(1)$ and it remains to show that $s^A \circ i^A = 0$.

For that consider a commutative diagram

$$\begin{array}{ccc}
A_{X \times \{0\}}(X \times \mathbf{A}^1) & \xrightarrow{i^A} & A(X \times \mathbf{A}^1) \\
{\scriptstyle j^A}\downarrow & & \downarrow{\scriptstyle s^A} \\
A_{\emptyset}(X \times (\mathbf{A}^1 - \{0\})) & \longrightarrow & A(X),
\end{array}$$

where the pull-back map j^A is induced by the inclusion $j : X \times (\mathbf{A}^1 - \{0\}) \hookrightarrow X \times \mathbf{A}^1$ and the bottom horizontal arrow is the pull-back induced by the inclusion $s : (X, \emptyset) \to (X \times (\mathbf{A}^1 - \{0\}), X \times (\mathbf{A}^1 - \{0\}))$. The group $A_{\emptyset}(X \times (\mathbf{A}^1 - \{0\})$ vanishes by the vanishing property 2.2.1. Thus $s^A \circ i^A = 0$ which proves the relation $c(1) = 0$.

It remains to prove the non-degeneracy property of the class $L \mapsto c(L)$. For that consider the assignment $L \mapsto c'(L) = c(L^\vee)$. Clearly the class c' is functorial and satisfies the vanishing property. Moreover the map $(1, c'(\mathcal{O}(-1))) : A(X) \oplus A(X) \to A(X \times \mathbf{P}^1)$ is an isomorphism by the non-degeneracy property of the Thom class $L \mapsto th(L)$ and the very last Lemma.Thus the assignement $L \mapsto c'(L)$ is a Chern structure. Now the previous Lemma shows that $c'(\mathcal{O}(-1)) = -c'(\mathcal{O}(1))$. Thus $c(\mathcal{O}(1)) = -c(\mathcal{O}(-1))$ and therefore the map $(1, c(\mathcal{O}(-1))) : A(X) \oplus A(X) \to A(X \times \mathbf{P}^1)$ is an isomorphism as well. The non-degeneracy property of the class $L \mapsto c(L)$ is proved and hence the assertion (1) of Theorem is proved as well.

The third assertion of the Theorem is proved just after the section 3.3 because the proof of the third assertion presented in this text uses the projective bundle theorem 3.3.1.

□

3.3 Projective bundle theorem.

We are going to construct higher Chern classes for a ring cohomology theory A endowed with a Chern structure $L \mapsto c(L)$. Following the known Grothendieck's method one has to compute cohomology of a projective bundle.

3.3.1. Theorem (Projective Bundle Cohomology). *Let A be a ring cohomology theory endowed with a Chern structure $L \mapsto c(L)$ on A. Let X be a smooth variety and let E/X be a vector bundle with $rkE = n$. For $\xi_E = c(\mathcal{O}_E(-1)) \in A(\mathbf{P}(E))$ we have an isomorphism*

$$(1, \xi_E, \dots, \xi_E^{n-1}) : A(X) \oplus A(X) \cdots \oplus A(X) \to A(\mathbf{P}(E))$$

where (and elsewhere) we denote the operator of $\cup$-product with a universally central element by the symbol of the element.

Moreover, for trivial E we have $\xi_E^n = 0$. In addition, all the assertions hold if the element $\zeta_E = c(\mathcal{O}_E(1)) \in A(\mathbf{P}(E))$ is used instead of ξ_E.

Proof. This variant of the proof is based on an oral exposition of Suslin. Let $\{0\} = [1 : 0 : \cdots : 0] \in \mathbf{P}^n$ be a point and let $\mathbf{A}^n$ be an affine subspace in $\mathbf{P}^n$ defined by the inequality $x_0 \neq 0$. Let $\mathbf{P}^n_i$ be a hypersurface in $\mathbf{P}^n$ defined by $x_i = 0$ and let $\mathbf{A}^n_i = \mathbf{P}^n_i \cap \mathbf{A}^n$. Let $p_i : \mathbf{A}^n \to \mathbf{A}^1$ be the projection on the i-th axis and let $j_i : \mathbf{A}^1 \hookrightarrow \mathbf{A}^n$ be the i-th axis. Finally let $\bar{j}_i : \mathbf{P}^1 \hookrightarrow \mathbf{P}^n$ be the closed imbedding extending the imbedding j_i.

Let $res_i : A_{\mathbf{P}^n_i}(\mathbf{P}^n) \to A_{\mathbf{A}^n_i}(\mathbf{A}^n)$ be the pull-back map induced by the imbedding $\mathbf{A}^n \hookrightarrow \mathbf{P}^n$. Let $res : A_{\{0\}}(\mathbf{A}^n) \to A_{\{0\}}(\mathbf{P}^n)$ be the pull-back map induced by the same imbedding.

The element $\xi = c(\mathcal{O}(-1)) \in A(\mathbf{P}^1)$ vanishes being restcicted to $\mathbf{P}^1 - \{0\}$. Thus the element $\bar{t} = \xi$ belongs to the subgroup $A_{\{0\}}(\mathbf{P}^1)$ of the group $A(\mathbf{P}^1)$. Set

$$t = j^A(\bar{t}) \in A_{\{0\}}(\mathbf{A}^1)$$

where j^A is the pull-back map $A_{\{0\}}(\mathbf{P}^1) \to A_{\{0\}}(\mathbf{A}^1)$. Set

$$th_{(n)} = p_1^A(t) \cup p_2^A(t) \cup \cdots \cup p_1^A(t) \in A_{\{0\}}(\mathbf{A}^n).$$

Let $e_i : (\mathbf{P}^n, \emptyset) \hookrightarrow (\mathbf{P}^n, \mathbf{P}^n - \mathbf{P}^n_i)$ and let $e : (\mathbf{P}^n, \emptyset) \hookrightarrow (\mathbf{P}^n, \mathbf{P}^n - \emptyset)$ be the inclusions. The pull-back operators $e_i^A : A_{\mathbf{P}^n_i}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ and $e^A : A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ are just the support extension operators.

3.3.2. Lemma. *Let Y be a smooth variety and let $Z \subset Y$ be a closed subset. Let $pr : Y \times \mathbf{A}^1 \to \mathbf{A}^1$ and $p : Y \times \mathbf{A}^1 \to Y$ be the projections. Then the composition operator*

$$(\cup pr^A(t)) \circ p^A : A_Z(Y) \to A_{Z \times \{0\}}(Y \times \mathbf{A}^1)$$

is an isomorphism.

3.3.3. Lemma. *The map $\cup th_{(n)} : A(pt) \to A_{\{0\}}(\mathbf{A}^n)$ is an isomorphism.*

3.3.4. Lemma. *Let $\xi_n = c(\mathcal{O}(-1)) \in A(\mathbf{P}^n)$ and $\zeta_n = c(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. Then $\xi_n^{n+1} = 0$ and $\zeta_n^{n+1} = 0$.*

3.3.5. Lemma. *The support extension map $A_{\mathbf{P}^n_i}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is injective, the element $\xi_n = c(\mathcal{O}(-1))$ coincides with $e^A_i(\bar{t}_i)$ for an appropriative element $\bar{t}_i \in A_{\mathbf{P}^n_i}(\mathbf{P}^n)$ and the relation $res_i(\bar{t}_i) = p^A_i(t)$ holds in $A_{\mathbf{A}^n_i}(\mathbf{A}^n)$.*

3.3.6. Lemma. *The element $\xi^n \in A(\mathbf{P}^n)$ coincides with $e^A(\bar{t}h_{(n)})$ for an appropriative element $\bar{t}h_{(n)} \in A_{\{0\}}(\mathbf{P}^n)$.*

If the support extension operator $e^A : A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is injective, then the relation $th_{(n)} = res(\bar{t}h_{(n)})$ holds in $A_{\{0\}}(\mathbf{A}^n)$.

3.3.7. Remark. *The linear projection $\mathbf{P}^n - \{0\} \to \mathbf{P}^n_0 \to \mathbf{P}^n - \{0\}$ makes $\mathbf{P}^n - \{0\}$ a line bundle over $\mathbf{P}^n_0$ and the subvariety $\mathbf{P}^n_0$ is the zero section of this line bundle. By the homotopy invariance property the pull-back operator $A(\mathbf{P}^n - \{0\}) \to A(\mathbf{P}^n_0)$ is an isomorphism*

Given these five Lemmas complete the proof of Theorem as follows. The general case is reduced to the case of the trivial vector bundle E via the Mayer-Vietoris arguments using Proposition 2.4.6. If $E = 1^{k+1}$ then $\mathbf{P}(E) = X \times \mathbf{P}^k$. To use shorter notation we pove the theorem only for the case of the projective space $\mathbf{P}^k$ itself. We proceed the proof by the induction on the interger k. If $k = 1$, then the theorem holds by the very definition of the Chern structure on A. We will assume below that Theorem valids for all intergers $k < n$ and prove Theorem for $k = n$. Consider the localization sequence for the pair $(\mathbf{P}^n, \mathbf{P}^n - \{0\})$

$$\ldots \to A_{\{0\}}(\mathbf{P}^n) \xrightarrow{\alpha} A(\mathbf{P}^n) \xrightarrow{\beta} A(\mathbf{P}^n - \{0\}) \to \ldots$$

If $\xi_i \in A(\mathbf{P}^i)$ is the Chern class of the line bundle $\mathcal{O}(-1)$ on $\mathbf{P}^i$, then $\xi_n|_{\mathbf{P}^{n-1}} = \xi_{n-1}$ and $\xi^j_n|_{\mathbf{P}^{n-1}} = \xi^j_{n-1}$. By the inductive assumption the elements $1, \xi_{n-1}, \ldots, \xi^{n-1}_{n-1}$ form a free base of the $A(pt)$-module $A(\mathbf{P}^{n-1})$. Therefore the map $A(\mathbf{P}^n) \to A(\mathbf{P}^n_0)$ is a split surjection.

By Remark 3.3.7 the pull-back operator $A(\mathbf{P}^n - \{0\}) \to A(\mathbf{P}^n_0)$ is an isomorphism. Thus the pull-back operator $A(\mathbf{P}^n) \to A(\mathbf{P}^n - \{0\})$ is a split surjection as well. Now the localization sequence for the pair $(\mathbf{P}^n, \mathbf{P}^n - \{0\})$ shows that the support extention map $A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is an injection. Therefore one gets a short exact sequence

$$0 \to A_{\{0\}}(\mathbf{P}^n) \xrightarrow{\alpha} A(\mathbf{P}^n) \xrightarrow{\beta'} A(\mathbf{P}^n_0) \to 0$$

where β' is the pull-back map. One more consequence of the injectivity of the support extension operator $A_{\{0\}}(\mathbf{P}^n) \to A(\mathbf{P}^n)$ is the relation

$$p^A_1(t) \cup p^A_2(t) \cup \cdots \cup p^A_1(t) = res(\bar{t}h_n)$$

in $A_{\{0\}}(\mathbf{A}^n)$ which now holds by Lemma 3.3.6. An $A(pt)$-linear map $s : A(\mathbf{P}^n_0) \to A(\mathbf{P}^n)$ taking the element ξ^j_{n-1} to ξ^j_n $(j = 0, 1, \ldots, n-1)$ splits the surjection β'. The element $\xi^n_n \in A(\mathbf{P}^n)$ belongs to the subgroup $A_{\{0\}}(\mathbf{P}^n)$ because $\xi^n_{n-1} = 0$ in $A(\mathbf{P}^{n-1})$ by Lemma 3.3.4. It remains to show that the map $\cup \xi^n : A(pt) \to A_{\{0\}}(\mathbf{P}^n)$ is an isomorphism.

For that consider the diagram

$$\begin{array}{ccc} A(pt) & \xrightarrow{\cup \bar{th}_{(n)}} & A_{\{0\}}(\mathbf{P}^n) \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle res} \\ A(pt) & \xrightarrow{\cup th_{(n)}} & A_{\{0\}}(\mathbf{A}^n) \end{array}$$

It commutes by Lemma 3.3.6. The operator res is an isomorphism by the excision property. The operator $\cup th_{(n)}$ is an isomorphism by Lemma 3.3.3. Thus the operator $\cup \bar{th}_{(n)}$ is an isomorphism as well. To prove Theorem it remains to prove the five Lemmas 3.3.2 to 3.3.6.

Proof of Lemma 3.3.2. Let $\bar{pr} : Y \times \mathbf{P}^1 \to \mathbf{P}^1$ and $\bar{p} : Y \times \mathbf{P}^1 \to Y$ be the projections. We will write for short $\cup t$ for the operator $(\cup pr^A(t)) \circ p^A$ and will write in this proof $\cup\xi$ for the operator $\cup \bar{pr}^A(\xi) \circ \bar{p}^a : A_Z(Y) \to A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1)$ and write $\cup 1$ for the operator $\bar{p}^A : A_Z(Y) \to A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1)$. We begin with verifying that the operator

$$(\cup 1, \cup\xi) : A_Z(Y) \oplus A_Z(Y) \to A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1) \tag{14}$$

is an isomorphism. In fact, by Proposition 2.4.5 the diagram commutes (here U = Y - Z)

$$\begin{array}{ccccccc} A(Y \times \mathbf{P}^1) & \longrightarrow & A(U \times \mathbf{P}^1) & \xrightarrow{\partial} & A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1) & \longrightarrow & A(Y \times \mathbf{P}^1) \\ {\scriptstyle (\cup 1, \cup\xi)}\Big\uparrow & & {\scriptstyle (\cup 1, \cup\xi)}\Big\uparrow & & {\scriptstyle (\cup 1, \cup\xi)}\Big\uparrow & & {\scriptstyle (\cup 1, \cup\xi)}\Big\uparrow \\ A(Y) \oplus A(Y) & \longrightarrow & A(U) \oplus A(U) & \xrightarrow{\partial\oplus\partial} & A_Z(Y) \oplus A_Z(Y) & \longrightarrow & A(Y) \oplus A(Y). \end{array}$$

Since $\xi = c(\mathcal{O}(-1)) \in A(\mathbf{P}^1)$ the five-lemma proves that the operator 14 is an isomorphism.

The next step is to check that the operator

$$\cup\xi : A_Z(Y) \to A_{Z\times\{0\}}(Y \times \mathbf{P}^1) \tag{15}$$

is an isomorphism. For that consider the localization sequence

$$\ldots \to A_{Z\times\{0\}}(Y \times \mathbf{P}^1) \xrightarrow{\alpha} A_{Z\times\mathbf{P}^1}(Y \times \mathbf{P}^1) \xrightarrow{\beta} A_{Z\times\mathbf{A}^1}(Y \times \mathbf{P}^1 - Z \times \{0\}) \to \ldots$$

for the triple $(Y \times \mathbf{P}^1, Y \times \mathbf{P}^1 - Z \times \{0\}, Y \times \mathbf{P}^1 - Z \times \mathbf{P}^1)$. We claim that the operator β is always surjective (and thus the operator α is always injective and therefore the localization sequence splits in short exact sequences).

In fact, if $i : Y \times \mathbf{A}^1 \hookrightarrow Y \times \mathbf{P}^1 - Z \times \{0\}$ is the open inclusion and $q : Y \times \mathbf{P}^1 - Z \times \{0\}$ is the projection then $q \circ i = p : Y \times \mathbf{A}^1 \to Y$ and thus $i^A \circ q^A = p^A$. The pull-back operator $i^A : A_{Z\times\mathbf{A}^1}(Y \times \mathbf{P}^1 - Z \times \{0\}) \to A_{Z\times\mathbf{A}^1}(Y \times \mathbf{A}^1)$ is an isomorphism by the excision property and the pull-back operator $p^A : A_Z(Y) \to A_{Z\times\mathbf{A}^1}(Y \times \mathbf{A}^1)$ is an isomorphism by the strong homotopy invariance property. Thus $q^A : A_Z(Y) \to A_{Z\times\mathbf{A}^1}(Y \times \mathbf{P}^1 - Z \times \{0\})$ is an isomorphism. This proves the surjectivity of β and the injectivity of α.

We are ready to veryfy that the operator (15) is an isomorphism. For that consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & A_{Z\times\{0\}}(Y\times\mathbf{P}^1) \arrow[r, "\alpha"] & A_{Z\times\mathbf{P}^1}(Y\times\mathbf{P}^1) \arrow[r, "\beta"] & A_{Z\times\mathbf{A}^1}(Y\times\mathbf{P}^1 - Z\times\{0\}) \arrow[r] & 0 \\
0 \arrow[r] & A_Z(Y) \arrow[r, "in"] \arrow[u, "\cup\xi"] & A_Z(Y)\oplus A_Z(Y) \arrow[r, "pr"] \arrow[u, "(\cup\xi,\cup 1)"] & A_Z(Y) \arrow[r] \arrow[u, "q^A"] & 0
\end{tikzcd}$$

where the operator in is the inclusion to the fist summund and the operator pr is the projection on the second summund. The diagram commutes because the cup-product is functorial. The sequence on the top is short exact and the map q^A is an isomorphism as was just checked above. The operator $(\cup\xi,\cup 1)$ is an isomorphism as was checked as well above in this proof. Thus the operator (15) is an isomorphism as well.

The proof of the Lemma is completed as follows. Consider the diagram

$$\begin{tikzcd}
A_Z(Y) \arrow[r, "\cup\xi"] \arrow[d, "id"] & A_{Z\times\{0\}}(Y\times\mathbf{P}^1) \arrow[d, "\gamma"] \\
A_Z(Y) \arrow[r, "\cup t"] & A_{Z\times\{0\}}(Y\times\mathbf{A}^1)
\end{tikzcd}$$

where γ is the pull-back operator induced by the inclusion $Y\times\mathbf{A}^1 \hookrightarrow Y\times\mathbf{P}^1$. The operator γ is an isomorphism by the excision property. Since the operator is an isomorphism the operator $\cup t$ is an isomorphism as well. The Lemma 3.3.2 is proved.

Proof of Lemma 3.3.3. For every integer i let $p_{i,i}:\mathbf{A}^i\to\mathbf{A}^1$ be the projection of the affine space $\mathbf{A}^i$ to its last coordinate. Using the induction by n it straightforward to check that the cup-product operator $\cup th_{(n)}: A(pt)\to A_{\{0\}}(\mathbf{A}^n)$ coincides with the composition operator

$$A(pt)\xrightarrow{\cup t} A_{\{0\}}(\mathbf{A}^1)\xrightarrow{\cup p^A_{2,2}(t)}\dots\xrightarrow{\cup p^A_{n,n}(t)} A_{\{0\}}(\mathbf{A}^n).$$

Each arrow in this sequence of arrows is an isomorphism by Lemma 3.3.2. The Lemma follows.

Proof of Lemma 3.3.4. For every integer $i=0,1,\dots,n$ one has $\xi_n|_{\mathbf{P}^n-\mathbf{P}^n_i}=0$ because the Chern class of a trivial line bundle vanishes. Thus ξ_n belongs to the image of the support extension operator $e^A_i: A_{\mathbf{P}^n_i}(\mathbf{P}^n)\to A(\mathbf{P}^n)$, say $\xi_n=e^A_i(\bar t_i)$ for appropriative element $\bar t_i\in A_{\mathbf{P}^n_i}(\mathbf{P}^n)$. Now the element ξ_n^{n+1} coincides with the image of the cup-product $\bar t_0\cup\bar t_1\cup\dots\cup\bar t_n$ under the support extension map $A_{\mathbf{P}^n_0\cap\dots\cap\mathbf{P}^n_n}(\mathbf{P}^n)\to A(\mathbf{P}^n)$. The group $A_{\mathbf{P}^n_0\cap\dots\cap\mathbf{P}^n_n}(\mathbf{P}^n)$ vanishes because $\cap^n_0\mathbf{P}^n_i=\emptyset$. Thus $\xi_n^{n+1}=0$. Similarly one gets the relation $\zeta_n^{n+1}=0$. The Lemma is proved.

Proof of Lemma 3.3.5. The localisation sequence for the pair $(\mathbf{P}^n,\mathbf{P}^n-\mathbf{P}^n_i)$ cuts into short exact sequences

$$0\to A_{\mathbf{P}^n_n}(\mathbf{P}^n)\to A(\mathbf{P}^n)\to A(\mathbf{P}^n-\mathbf{P}^n_i)\to 0$$

because the composite map $A(pt)\xrightarrow{p^A} A(\mathbf{P}^n)\to A(\mathbf{P}^n-\mathbf{P}^n_i)$ is an isomorphism. This proves the first assertion of the Lemma.

The fact that $\xi_n = e_i^A(\bar{t}_i)$ for the element $\bar{t}_i \in A_{\mathbf{P}_i^n}(\mathbf{P}_i^n)$ is proved in the proof of the Lemma 3.3.4.

To prove the relation $res_i(\bar{t}_i) = p_i^A(t)$ consider the commutative diagram

$$\begin{tikzcd}
A(\mathbf{P}^n) \arrow[r, "\bar{j}_i^A"] & A(\mathbf{P}^1) \\
A_{\mathbf{P}_i^n}(\mathbf{P}^n) \arrow[u, "e_i^A"] \arrow[r, "\bar{j}_i^A"] \arrow[d, "res_i"'] & A_{\{0\}}(\mathbf{P}^1) \arrow[u, "i^A"'] \arrow[d, "j^A"] \\
A_{\mathbf{A}_i^n}(\mathbf{A}^n) \arrow[r, "j_i^A"] & A_{\{0\}}(\mathbf{A}^1)
\end{tikzcd}$$

where the maps e_i^A and i^A are the support extension operators. The relation $\bar{j}_i^A(\xi_n) = \xi$ in $A(\mathbf{P}^1)$ and the injectivity of the map i^A prove the relation $\bar{j}_i^A(\bar{t}_i) = \bar{t}$ in the group $A_{\{0\}}(\mathbf{P}^1)$. Since $t = j^A(\bar{t})$ in $A_{\{0\}}(\mathbf{A}^1)$ hence one gets the relation $j_i^A(res_i(\bar{t}_i)) = t$ in $A_{\{0\}}(\mathbf{A}^1)$. The pull-back homomorphisms $j_i^A : A_{\mathbf{A}_i^n}(\mathbf{A}^n) \to A_{\{0\}}(\mathbf{A}^1)$ and $p_i^A : A_{\{0\}}(\mathbf{A}^1) \to A_{\mathbf{A}_i^n}(\mathbf{A}^n)$ are inverse to each other isomorphisms by the homotopy invariance property. This proves the desired relation $res_i(\bar{t}_i) = p_i^A(t)$ in $A_{\mathbf{A}_i^n}(\mathbf{A}^n)$.

Proof of Lemma 3.3.6. Lemma 3.3.4(appplied to $\mathbf{P}^{n-1}$) shows that the element $\xi_n^n|_{\mathbf{P}_0^n}$ vanishes.By Remark 3.3.7 the pull-back operator $A(\mathbf{P}^n - \{0\}) \to A(\mathbf{P}_0^n)$ is an somorphism. Thus the element $\xi_n^n|_{\mathbf{P}^n-\{0\}}$ vanishes as well. Therefore $\xi_n^n = e^A(\bar{t}h_{(n)})$ for an appropriative element $\bar{t}h_{(n)} \in A_{\{0\}}(\mathbf{P}^n)$. Thi proves the first assertion of the Lemma.

To prove the last assertion of the Lemma consider the commutative diagram

$$\begin{tikzcd}
\prod_{i=1}^n A(\mathbf{P}^n) \arrow[r, "\cup"] & A(\mathbf{P}^n) \\
\prod_{i=1}^n A_{\mathbf{P}_i^n}(\mathbf{P}^n) \arrow[u, "\prod e_i^A"] \arrow[r, "\cup"] \arrow[d, "\prod res_i"'] & A_{\{0\}}(\mathbf{P}^n) \arrow[u, "e^A"'] \arrow[d, "res"] \\
\prod_{i=1}^n A_{\mathbf{A}_i^n}(\mathbf{A}^n) \arrow[r, "\cup"] & A_{\{0\}}(\mathbf{A}^n)
\end{tikzcd}$$

where the maps e_i^A and e^A are the support extension maps and the horizontal arrows are the cup-products. The commutativity of the upper square of this diagram and the injectivity of the map e^A show that the relation $\bar{t}_1 \cup \bar{t}_2 \cup \cdots \cup \bar{t}_n = \bar{t}h_{(n)}$ in the group $A_{\{0\}}(\mathbf{P}^n)$. Now the commutativity of the bottom square and the relations $res_i(\bar{t}_i) = p_i^A(t)$ prove the desired relation $p_1^A(t) \cup p_2^A(t) \cup \cdots \cup p_n^A(t) = res(\bar{t}h_{(n)})$ in the group $A_{\{0\}}(\mathbf{A}^n)$.

Now all the five Lemmas 3.3.2 to 3.3.6 are proved. The proof of Theorem 3.3.1 is completed.

□

3.3.8. Corollary (Projective Bundle Cohomology With Supports). *Under the hypotheses of Theorem 3.3.1 the map*

$$(1, \xi_E, \ldots, \xi_E^{n-1}) : A_Z(X) \oplus A_Z(X) \cdots \oplus A_Z(X) \to A_{\mathbf{P}(E_Z)}(\mathbf{P}(E))$$

is an isomorphism where $E_Z = E|_Z$ is the restriction of the vector bundle E to Z.

The short exact sequence (16) written-down below is useful as well. Namely, let X be a smooth variety and let M and N be two vector bundles over X. Let $\bar{i}_M : \mathbf{P}(M) \hookrightarrow \mathbf{P}(M \oplus N)$ and $\bar{i}_N : \mathbf{P}(N) \hookrightarrow \mathbf{P}(M \oplus N)$ be the closed imbeddings induced by the direct summunds M and N respectively. Let $p : \mathbf{P}(M \oplus N) \to X$ be the projection. Let $j_M^A : A_{\mathbf{P}(M)}(\mathbf{P}(M \oplus N)) \to A(\mathbf{P}(M \oplus N))$ be the support extension operator and let $\bar{i}_N^A : A(\mathbf{P}(M \oplus N)) \to A(\mathbf{P}(N))$ be the pull-back operator.

3.3.9. Corollary. *With these notation under the hypotheses of Theorem 3.3.1 the sequence*

$$0 \to A_{\mathbf{P}(M)}(\mathbf{P}(M \oplus N)) \xrightarrow{j_M^A} A(\mathbf{P}(M \oplus N)) \xrightarrow{\bar{i}_E^A} A(\mathbf{P}(N)) \to 0 \qquad (16)$$

is short exact.

To prove this consider $U = \mathbf{P}(M \oplus N) - \mathbf{P}(M)$ with the open inclusion $j : U \hookrightarrow \mathbf{P}(M \oplus N)$ and observe that U becomes a vector bundle over X by means of the linear projection $q : U \to \mathbf{P}(N)$. The obvious inclusion $i_N : \mathbf{P}(N) \hookrightarrow U$ is just the zero section of this vector bundle, $\bar{i}_N = j \circ i_N$ and the pull-back operator $i_N^A : A(U) \to A(\mathbf{P}(N))$ is an isomorphism (the inverse to the one q^A).

Now consider the pair $(\mathbf{P}(M \oplus N), U)$. By the localization property 2.0.1 the following sequence

$$\ldots \to A_{\mathbf{P}(M)}(\mathbf{P}(M \oplus N)) \xrightarrow{j_M^A} A(\mathbf{P}(M \oplus N)) \xrightarrow{j^A} A(U) \to \ldots$$

is exact.We claim that this sequence splits in short exact sequences with the surjective j^A and the injective j_M^A. To prove this claim observe that one has the relation

$$\xi_N = \bar{i}_N^A(\xi_{M \oplus N})$$

which holds because the restriction of the line bundle $\mathcal{O}_{M \oplus N}(-1)$ to $\mathbf{P}(N)$ is $\mathcal{O}_N(-1)$. Thus $\xi_N \in \bar{i}_N^A(A(\mathbf{P}(M \oplus N)))$ and by the projective bundle theorem (3.3.1) the operator $\bar{i}_N^A : A(\mathbf{P}(M \oplus N)) \to A(\mathbf{P}(N))$ surjects. The operator i_N^A is an isomorpism, $\bar{i}_N = j \circ i_N$ and thus $j^A : A(\mathbf{P}(M \oplus N)) \to A(U)$ surjects and the the support extension operator j_M^A injects. Now the sequence (16) is short exact because the operator i_N^A is an isomorpism and $\bar{i}_N = j \circ i_N$. The Corollary is proved.

The last Corollary and Lemma 2.2.12 prove the following

3.3.10. Corollary. *Under the hypotheses of Theorem 3.3.1 and the notation of Lemma 2.2.12 let $V'_t = X'_t - i_t(Y \times \mathbf{A}^1)$ and let $j_t : V'_t \hookrightarrow X'_t$ be the open inclusion. Then*

$$Ker(j_0^A) \cap Ker(j_t^A) = (0).$$

In the other words the operator $(j_0^A, j_t^A) : A(X'_t) \to A(\mathbf{P}(\mathbf{1} \oplus N)) \oplus A(V'_t)$ is a monomorphism.

3.4 End of the proof of Theorem 3.2.4.

The third assertion of Theorem 3.2.4 is proved in this subsection.

Assume we are given with a Chern structure $L \mapsto c(L)$ on A and let $L \mapsto th(L)$ be the Thom structure given by (13). We will check now that for each line bundle L over a smooth variety X one has $z^A(i^A(th(L))) = c(L)$. For that consider the commutative diagram

$$\begin{array}{ccc} A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L)) & \xrightarrow{\bar{i}^A} & A(\mathbf{P}(1 \oplus L)) \\ \downarrow{\scriptstyle e^A} & & \downarrow{\scriptstyle e^A} \\ A_X(L) & \xrightarrow{i^A} & A(L) \\ & & \downarrow{\scriptstyle z^A} \\ & & A(X) \end{array}$$

The chain of relations (here $\bar{z} = e \circ z$)

$$z^A(i^A(th(L))) = z^A(e^A(c(\mathcal{O}(1) \otimes p^*(L)))) = c(\bar{z}^*(\mathcal{O}(1) \otimes L)) = c(L)$$

proves the desired relation.

In the rest of the proof the notation 3.2.6 are used. Now suppose we are given with a Thom structure $L \mapsto th(L)$ on A and let $L \mapsto c(L)$ be the Chern structure on A given by the formula (12). For a line bundle L over a smooth X consider the vector bundle $E = 1 \oplus L$, the projection $p : \mathbf{P}(E) \to X$, the natural inclusion $e : L \hookrightarrow \mathbf{P}(E)$ and the pull-back $e^A : A_{\mathbf{P}(1)}(\mathbf{P}(E)) \to A_X(L)$. We have to check the relation (13). Since the operator e^A is an isomorphism it suffices to check the relation in $A(\mathbf{P}(1 \oplus L))$

$$\bar{t}h(L) = c(\mathcal{O}(1) \otimes q^*(L)) \tag{17}$$

To do this we need in some preliminary lemmas.

3.4.1. Lemma. *The elements $\bar{t}h(L)$ and $c(\mathcal{O}_E(1) \otimes p^*(L))$ are central in $A(\mathbf{P}(E))$. Moreover both elements belongs to the ideal $A_{\mathbf{P(1)}}(\mathbf{P}(E))$.*

Proof. Let $j : \mathbf{P}(E) - \mathbf{P}(\mathbf{1}) \hookrightarrow \mathbf{P}(\mathbf{E})$ be the inclusion. The restriction of the projection $p : \mathbf{P}(E) \to X$ to $\mathbf{P}(E) - \mathbf{P}(\mathbf{1})$ makes the last variety in a line bundle over X. The inclusion $s_L : X \hookrightarrow \mathbf{P}(E) - \mathbf{P}(\mathbf{1})$ identifying X with the subvariety $\mathbf{P}(L)$ in $\mathbf{P}(E)$ is the zero secton of the mentioned line bundle. By the strong homotopy property of the pretheory A the pull-back operator $s_L^A : A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1})) \to A(X)$ is an isomorphism. The line bundle $s_L^*(\mathcal{O}_E(1) \otimes p^*(L))$ coincides with the line bundle $L^\vee \otimes L$ and therefore it is the trivial line bundle. Thus the Chern class $c(s_L^*(\mathcal{O}_E(1) \otimes p^*(L))$ vanishes and the element $s_L^A(c(\mathcal{O}_E(1) \otimes p^*(L)))$ vanishes as well. Therefore $j^A(c(\mathcal{O}_E(1) \otimes p^*(L))) = 0$ in $A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1}))$, which proves the inclusion $c(\mathcal{O}_E(1) \otimes p^*(L)) \in A_{\mathbf{P(1)}}(\mathbf{P}(E))$.

The class $\bar{t}h(L)$ is in the subgroup $A_{\mathbf{P(1)}}(\mathbf{P}(E))$ by the very definition of the class $\bar{t}h(L)$.

The element $c(\mathcal{O}_E(1) \otimes p^*(L))$ is central in $A(\mathbf{P}(E))$ because it is a Chern class.

To prove that the element $\bar{t}h(L)$ is central recall that for every smooth variety X and every line bundle L over X the element $th(L) \in A_X(L)$ is $A(X)$-central. Now for every element $a \in A(\mathbf{P}(E))$ one has a chian of relations in $A_X(L)$

$$e^A(\bar{t}h(L) \cup a) = th(L) \cup e^A(a) = e^A(a) \cup th(L) = e^A(a \cup \bar{t}h(L)).$$

Since the support extension operator $e^A : A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A_X(L)$ is an isomorphism one gets the relation $\bar{t}h(L) \cup a = a \cup \bar{t}h(L)$ in $A_{\mathbf{P(1)}}(\mathbf{P}(E))$. Thus the element $\bar{t}h(L) \in A(\mathbf{P}(E))$ is central. The Lemma is proved.

3.4.2. Lemma. *The operator* $(\cup \bar{t}h(L)) : A(X) \to A_{\mathbf{P(1)}}(\mathbf{P}(E))$ *is an isomorphism.*

Proof. Consider the following commutative diagram

$$\begin{array}{ccc} A(X) & \xrightarrow{\cup \bar{t}h(L)} & A_{\mathbf{P(1)}}(\mathbf{P}(E)) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle e^A} \\ A(X) & \xrightarrow{\cup th(L)} & A_X(L) \end{array}$$

The operator e^A is an isomorphism by the excision property. The operator $\cup th(L)$ is an isomotphism because $th(L)$ is the Thom class. The Lemma follows.

3.4.3. Lemma. *The diagram commutes*

$$\begin{array}{ccc} A_{\mathbf{P(1)}}(\mathbf{P}(E)) & \xrightarrow{s^A} & A(X) \\ {\scriptstyle \cup \bar{t}h(L)}\uparrow & & \uparrow{\scriptstyle \cup c(L)} \\ A(X) & \xrightarrow{id} & A(X) \end{array}$$

and $s^A(\bar{t}h(L)) = s^A(c(\mathcal{O}_E(1) \otimes p^*(L)).$

Proof. Clearly $z^A \circ e^A = s^A : A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A(X)$, where $z : X \to L$ is the zero section of L. Thus $s^A(\bar{t}h(L)) = z^A(th(L)) = c(L)$ by the very definition of $c(L)$. The commutativity of the diagram is checked. It remains to check that $c(L) = s^A(c(\mathcal{O}_E(1) \otimes p^*(L))$. This is obvious because the line bundle $s^*(\mathcal{O}_E(1))$ is trivial and the line bundle $s^*(p^*(L))$ coincides with the line bundle L.

3.4.4. Claim. *For any variety X and any line bundle L over X there exists a finite-dimensional vector space V and a diagram of the form*

$$X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V) \tag{18}$$

in which X' is a torsor under a vector bundle over X and the morphism f is such that the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic.

Proof of the Claim. To construct the diagram (18) recall that by the Jouanalou trick [J] there is a torsor X'/X under a vector bundle over X such that X' is an affine variety. Now take the projection $p: X' \to X$ and consider the pull-back $p^*(L)$ of the line bundle L. Since the variety X' is affine the line bundle $p^*(L)$ can be induced from a projective space via a morphism $f: X \to \mathbf{P}(V)$. The Claim is proved.

Proof of the relation (17). . Take $X = \mathbf{P}^\infty$ and $L = \mathcal{O}_{\mathbf{P}^\infty}(1)$ and consider the commutative diagram from the last Lemma. By the Projective bundle theorem 3.3.1 the ring $A(pt)[[t]]$ of formal power series in one variable is identified with the ring $A(\mathbf{P}^\infty)$ identifying the variable t with the Chern class $c(L)$. Thus the operator $\cup c(L): A(X) \to A(X)$ is injective. The operator $\cup \bar{t}h(L)$ is an isomorphism by Lemma 3.4.2. Hence the operator $s^A: A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A(X)$ is injective. Now the relation $c(\mathcal{O}_E(1) \otimes p^*(L)) = \bar{t}h(L)$ in $A(X)$ holds by the last Lemma. The relation (17) is proved in the considered case. Clearly this implies the relation (17) in the case $X = \mathbf{P}(V)$ and $L = \mathcal{O}_V(1)$ for any finite-dimensional k-vector space V. The general case of the relation (17) will be reduced now to this particular case.

Let X be a variety and let L be a line bundle over X. By Claim 3.4.4 there exists a diagram of the form (18) such that the pull-back operator $p^A: A(X) \to A(X')$ is an isomorphism and the line bundles $L' = p^*(L)$ and $f^*(\mathcal{O}_V(1))$ are isomorphic.

Set $E' = \mathbf{1} \oplus L'$ and $E_V = \mathbf{1} \oplus \mathcal{O}_V(1)$ and let $p': \mathbf{P}(E') \to X'$ and $p_V: \mathbf{P}(E_V) \to \mathbf{P}(V)$ be the projections. A choice of a line bundle isomorphism $L' \to f^*(\mathcal{O}_V(1))$ gives rise to the following Cartesian diagram

$$\begin{array}{ccccc} \mathbf{P}(E) & \xleftarrow{P} & \mathbf{P}(E') & \xrightarrow{F} & \mathbf{P}(E_V) \\ {\scriptstyle p}\downarrow & & {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p_V} \\ X & \xleftarrow{p} & X' & \xrightarrow{f} & \mathbf{P}(V) \end{array}$$

Clearly one has relations

$$c(\mathcal{O}_{E'}(1) \otimes (p')^*(L')) = F^A(c(\mathcal{O}_{E_V}(1) \otimes (p_V)^*(L_V)))$$

and

$$c(\mathcal{O}_{E'}(1) \otimes (p')^*(L')) = P^A(c(\mathcal{O}_E(1) \otimes p^*(L))).$$

As we already know $c(\mathcal{O}_{E_V}(1) \otimes (p_V)^*(L_V)) = \bar{t}h(\mathcal{O}_V(1))$ in $A(\mathbf{P}(E_V))$. Now one has the chain of relations in $\mathbf{P}(E)$

$$P^A(c(\mathcal{O}_E(1) \otimes p^*(L))) = c(\mathcal{O}_{E'}(1) \otimes (p')^*(L')) = F^A(c(\mathcal{O}_{E_V}(1) \otimes (p_V)^*(L_V))) =$$

$$F^A(\bar{t}h(\mathcal{O}_V(1))) = \bar{t}h(f^*(\mathcal{O}_V(1))) = \bar{t}h(L') = P^A(\bar{t}h(L)).$$

Since the pull-back operator P^A is an isomorphism the desired relation follows. The Theorem is proved.

$\square$

3.5 Splitting Principle.

Let A be a ring cohomology theory endowed with a Chern structure $L \mapsto c(L)$ on A. Here a variant of splitting principle is given which will be used in the text below. It will be convinient to fix certain notation. Let $p : Y \to X$ and $f : X' \to X$ be morphisms. Then we will write Y' for the scheme $X' \times_X Y$ and write p' for the projection $X' \times_X Y \to X'$ and f' for the projection $X' \times_X Y \to Y$.

3.5.1. Lemma. *Let E be a rank n vector bundle over a smooth variety X. Then there exists a smooth morphism $r : T \to X$ such that the vector bundle $r^*(E)$ is a direct sum of line bundles and for each closed subset Z of X and for $S = r^{-1}(Z)$ the pull-back map $r^A : A_Z(X) \to A_S(T)$ is a split injection and moreover for a smooth variety X' and any morphism $f : X' \to X$ the pull-back map $(r')^A : A(X') \to A(T')$ is a split injection.*

Proof. Let E be a rank n vector bundle over a smooth variety X and let $p : \mathbf{P}(E) \to X$ be the associated projective bundle over X and let $\mathcal{O}_E(-1)$. Then there is the canonical short exact sequences of vector bundles on $\mathbf{P}(E)$ with the rank $n-1$ vector bundle E'

$$0 \to \mathcal{O}_E(-1) \to p^*(E) \to E' \to 0$$

and the pull-back map $p^A : A_Z(X) \to A_{p^{-1}(Z)}(\mathbf{P}(E))$ is a split injection by the projective bundle theorem.

Repeating this construction several times one gets a smooth variety Y , a morphism $q : Y \to X$ and a filtration $(0) \subset E_1 \subset E_2 \subset \cdots \subset E_n = q^*(E)$ of the vector bundle $q^*(E)$ such that all the quotients E_i/E_{i-1} are line bundles. Moreover the pull-back map $q^A : A_Z(X) \to A_{q^{-1}(Z)}(Y)$ is a split injection and for a smooth variety X' and any morphism $f : X' \to X$ the pull-back map $(q')^A : A(X') \to A(Y')$ is a split injection as well.

Claim. Let S be a smooth variety and let F_1, F_2 be two vector bundles over S and let $\alpha : F_1 \to F_2$ be a vector bundle epimorphism and let $K = ker(\alpha)$. Then there is an affine bundle $g : T \to S$ such that the epimorphism $g^*(\alpha)$ splits.

Assuming for a moment this Claim complete the proof of Lemma as follows. Take $S = Y$ and consider the filtration $(0) \subset E_1 \subset E_2 \subset \cdots \subset E_n = q^*(E)$ on the vector bundle $q^*(E)$. Applying several times Claim one gets an affine bundle $g : T \to S$ such that one has a direct sum decomposition $g^*(q^*(E)) = \oplus_{i=1}^n(g^*(E_i/E_{i-1}))$ of the vector bundle $(q \circ g)^*(E)$. Show that the morphism $r = q \circ g : T \to X$ have the desired property.

For each smooth variety S' an each morphism $S' \to S$ the pull-back map $(g')^A : A(S') \to A(T')$ is an isomorphism because T' is an affine bundle over S'. Now if X' is a smooth variety and $f : X' \to X$ is a morphism, then Y' is smooth over X' and therefore $S' = Y'$ is smooth as well. The pull-back map $(q')^A : A(X') \to A(Y')$ is a split injection by the projective bundle cohomology. Thus the composite map $(q' \circ g') : A(X') \to A(T')$ is a split injection as well.

Proof of Claim. Let $Hom(F_2, F_1)$ be the scheme representing the sheaf $\mathcal{H}om(F_2, F_1)$ and let $\phi : Hom(F_2, F_1) \to Hom(F_2, F_2)$ be the morphism corresponding to the morphism $\mathcal{H}om(F_2, F_1) \to \mathcal{H}om(F_2, F_2)$ induced by the vector bundle map $\alpha : F_1 \to F_2$. Let

$id : S \to Hom(F_2, F_2)$ be the section of the projection $Hom(F_2, F_2) \to S$ corresponding to the identity map $F_2 \to F_2$. Let $Sect(\alpha) = \phi^{-1}(id(S))$ be a closed subscheme of the scheme $Hom(F_2, F_1)$ and let $g : T = Sect(\alpha) \to S$ be the projection. The scheme T represents the sheaf of sections of the sheaf epimorphism α. Thus there exists a canonical section $s : g^*(F_2) \to g^*(F_1)$ of the epimorphism $g^*(\alpha) : g^*(F_1) \to g^*(F_2)$. This section gives rise by a standart way to a vector bundle isomorphism $g^*(F_1) \cong g^*(F_2) \oplus g^*(K)$.

To prove Claim it remains to observe that the variety $T = Sect(\alpha)$ is a torsor under the vector bundle $Hom(F_2, K)$. The Claim is proved.

□

3.6 Chern Classes.

Let A be a ring cohomology theory.

3.6.1. Definition. *A Chern classes theory on A is an assignment which associate to each smooth variety X and each vector bundle E on X certain elements $c_i(E) \in A(X)$ ($i = 0, 1, \dots$) which are universally central and satisfy the following properties*

1. $c_0(E) = 1$;

   the restriction of the assignment $L \mapsto c_1(L)$ to line bundles is a Chern structure on A;

2. *functoriality:*

   $c_i(E) = c_i(E')$ *for isomorphic vector bundles E and E';*

   $f^A(c_i(E)) = c_i(f^*(E))$ *for each morphism $f : Y \to X$;*

3. *Cartan formula:* $c_r(E) = c_0(E_1) \cup c_r(E_2) + \dots + c_r(E_1) \cup c_0(E_2)$ *for each short exact sequence* $0 \to E_1 \to E \to E_2 \to 0$ *of vector bundles;*

4. *Vanishing property:* $c_m(E) = 0$ *for* $m > rk(E)$.

3.6.2. Theorem. *Let A be endowed with a Chern structure $L \mapsto c(L)$. Then there exists a unique Chern classes theory on A such that for each line bundle L one has $c_1(L) = c(L)$. Moreover the Chern classes $c_i(E)$ are nilpotent for $i > 0$.*

Proof. First prove the uniqueness assertion. If there are two assignements $E/X \mapsto c_i'(E)$ and $E/X \mapsto c_i''(E)$ satisfying the required properties. Then they coincide on line bundles by the properties 1 and 4. Therefore they coincide on direct sums of line bundles by the Cartan formula 3. Thus they coincide on all vector bundles by the splitting principle 3.5.1.

It remains to construct a Chern classes theory. Let X be a smooth variety and E/X be a vector bundle with $\mathrm{rk} E = n$. Set $\xi = c(\mathcal{O}_E(-1))$. By (3.3.1) there are unique elements $c_i(E) \in A(X)$ such that

$$\xi^n - c_1(E)\xi^{n-1} + \dots + (-1)^n c_n(E) = 0. \qquad (19)$$

Set $c_0(E) = 1$ and $c_m(E) = 0$ if $m > n$.

3.6.3. Claim. *Classes $c_i(E)$ satisfy the theorem.*

The rest of the proof is devoted to the proof of this Claim. The property $c_0(E) = 1$ holds by the very definition. To prove the property $c_1(L) = c(L)$ for a line bundle L observe that $\mathbf{P}(L) = X$ and $\mathcal{O}_L(-1) = L$ over X. Thus $\xi = c(L)$ in $A(X)$ and the relation (19) shows that $c_1(L) = c(L)$.

3.6.4. Lemma. *For each line bundle L over a smooth X the class $c(L) \in A(X)$ is nilpotent.*

To prove this Lemma recall that by Claim 3.4.4 one can find a diagram of the form (18) with a torsor under a vector bundle $p : X' \to X$ and a morphism $f : X' \to \mathbf{P}(V)$ such that the line bundles $L' = p^*(L)$ and $f^*(\mathcal{O}_V(1))$ over X' are isomorphic.

The class $c(\mathcal{O}_V(1)) \in A(\mathbf{P}(V))$ is nilpotent by Lemma 3.3.4. Thus the class $c(L') = f^A(c(\mathcal{O}_V(1)))$ is nilpotent as well. The pull-back map $p^A : A(X) \to A(X')$ is an isomorphism by the strong homotopy invariance 2.2.6. Therefore the class $c(L) \in A(X)$ is nilpotent as well. The Lemma is proved.

Now prove the functoriality of the classes c_i. A vector bundle isomorphism $\phi : E \to E'$ induces an isomorphism $\Phi : \mathbf{P}(E) \to \mathbf{P}(E')$ of the projective bundles and a line bundle isomorphism $\Phi^*(\mathcal{O}_{E'}(-1)) \to \mathcal{O}_E(-1)$ over $\mathbf{P}(E)$. Therefore $\Phi^A(c(\mathcal{O}_{E'}(-1))) = c(\mathcal{O}_E(-1))$ in $A(\mathbf{P}(E))$. Now the relations $c_i(E) = c_i(E')$ follows immidiately from the projective bundle cohomology and the relation (19). The property $f^A(c_i(E)) = c_i(f^*(E))$ is proved similarly.

For the rest of the proof we need in the following

3.6.5. Claim. *For a rank r vector bundle F set $c_t(F) = 1 + c_1(F)t + \cdots + c_n(F)t^r$. Let T be a smooth variety and let $F = \oplus_{i=1}^r L_i$ for certain line bundles L_i over T. Then one has*

$$c_t(F) = \prod_{i=1}^{r} c_t(L_i).$$

In particular the elements $c_i(F)$ are universally central and nilpotent. (The nilpotence of the class $c_1(L)$ is proved just above).

Assuming for a moment Claim 3.6.5 complete the proof of Claim 3.6.3 as follows. By the splitting principle 3.5.1 there exists a smooth variety T and a morphism $r : T \to X$ such that each the vector bundle $r^*(E_i)$ is a sum of line bundles and the pull-back map $r^A : A(X) \to A(T)$ is injective.

The Claim 3.6.5 and the injectivity of the map r^A show the Cartan formula $c_t(E) = c_t(E_1)c_t(E_2)$. Furthermore the Claim 3.6.5 shows that the elements $r^A(c_i(E)) \in A(T)$ are universally central. In particular for a smooth variety X' and a morphism $f : X' \to X$ and for $T' = X' \times_X T$ the element $(f')^A(r^A(c_i(E)))$ is central in $A(T')$. By the same splitting principle the pull-back map $(r')^A : A(X') \to A(T')$ is injective. Now the relation $(f')^A(r^A(c_i(E))) = (r')^A(f^A(c_i(E)))$ and the injectivity of the map $(r')^A : A(X') \to A(T')$ show that the element $f^A(c_i(E))$ is central in $A(X')$. Thus the elements $c_i(E)$ are universally central.

Finally the Claim 3.6.5 shows that the elements $r^A(c_i(E))$ are nilpotent. The injectivity of the map r^A proves the nilpotence of the elements $c_i(E) \in A(X)$.

It remains to prove the Claim 3.6.5.

If $\xi = c(\mathcal{O}_F(-1))$ where $\mathcal{O}_F(-1)$ is the tautological line bundle on $\mathbf{P}(F)$ then it suffices to prove the relation $\prod(\xi - c_1(L_i)) = 0$ in $A(\mathbf{P}(F))$. To prove the very last relation set $F^i = L_1 \oplus \cdots \oplus \bar{L}_i \oplus \cdots \oplus L_n$, where the bar means that the corresponding summund has to be omitted. Since $\mathcal{O}_F(-1)|_{\mathbf{P}(L_i)} = L_i$ over X the element $\xi - c_1(L_i)$ vanishes being restricted to $\mathbf{P}(L_i)$. Therefore $\xi - c_1(L_i)$ belongs to the subgroup $A_{\mathbf{P}(F^i)}(\mathbf{P}(F))$ of the group $A(\mathbf{P}(F))$ (see (16)). Thus the cup-product $\prod_{i=1}^n(\xi - c_1(L_i))$ belongs to the subgroup $A_{\cap\mathbf{P}(F^i)}(\mathbf{P}(F))$ of the group $A(\mathbf{P}(F))$. Since the intersection $\cap_{i=1}^n \mathbf{P}(F^i)$ is empty the group $A_{\cap\mathbf{P}(F^i)}(\mathbf{P}(F))$ vanishes by the vanishing property. Hence indeed the relation $\prod_{i=1}^n(\xi - c_1(L_i)) = 0$ holds in $A(\mathbf{P}(F))$ and $\prod_{i=1}^n(1 + c_1(L_i)t) = c_t(F)$ in $A(X)$.

Finally since each of the elements $c_1(L_i)$ is universally central and nilpotent hence each of the elements $c_j(E)$ is universally central and nilpotent as well. The Claim 3.6.5 is proved.

□

3.6.6. Proposition. *Let X be a smooth variety and let E be a vector bundle over X of the constant rank n. Let $p : \mathbf{P}(E) \to X$ be the projection. Then the element $c_n(\mathcal{O}_E(1) \otimes p^*(E)) \in A(\mathbf{P}(E))$ vanishes.*

Proof. Define a rank $n - 1$ vector bundle Q over $\mathbf{P}(E)$ by the short exact sequence $0 \to \mathcal{O}_E(-1) \to p^*(E) \to Q \to 0$. Tensoring this short exact sequence with the line bundle $\mathcal{O}_E(1)$ one gets a short exact sequence $0 \to \mathcal{O} \to \mathcal{O}_E(1) \otimes p^*(E) \to \mathcal{O}_E(1) \otimes Q \to 0$. Now the Cartan formular for the Chern classes gives the relation $c_n(\mathcal{O}_E(1) \otimes p^*(E)) = c_1(\mathcal{O})c_{n-1}(\mathcal{O}_E(1) \otimes Q)$. Thus $c_n(\mathcal{O}_E(1) \otimes p^*(E)) = 0$. □

3.7 Orienting a theory

In this subsection A is a ring cohomology theory. Two theorems in this subsection shows how one can construct an orientation using a Chern structure (or a Thom structure) on A and how one can construct a Chern structure (or a Thom structure) using an orientation.

Before to state theorems it is convinient to fix a notion of Thom classes theory, which is equivalent to the notion of orientation but it is defined in terms of elements rather than in terms of homomorphisms.

The definition of $A(X)$-central elements in $A_X(E)$ (for a vector bundle E over a smooth variety X) is given just below the definition of a Chern structure.

3.7.1. Definition. *A Thom classes theory on A is an assignment which assosiate to each smooth variety X and to each vector bundle E over X an element $th(E) \in A_X(E)$ satisfying the following properties*

(1) $th(E)$ is $A(X)$-central;

(2) $\varphi^A(th(F))$=$th(E)$ for each vector bundle isomorphism $\varphi : E \to F$;

(3) $f^A(th(E)) = th(f^*(E))$ *for each morphism* $f : Y \to X$ *with a smooth variety* Y;

(4) *the operator* $A(X) \to A_X(E)$, $a \to th(E) \cup a$ *is an isomorphism;*

(5) *multiplicativity property: for the projections* $q_i : E_1 \oplus E_2 \to E_i$ $(i = 1, 2)$ *one has*

$$q_1^* th(E_1) \cup q_2^* th(E_2) = th(E_1 \oplus E_2) \in A_X(E_1 \oplus E_2) \tag{20}$$

The element $th(E)$ *is called Thom class of the vector bundle* E.

3.7.2. Lemma. *If* ω *is an orientation on the theory* A *then the assignment* $E \mapsto th^E_X(1) \in A_X(E)$ *is a Thom classes theory on* A. *We write* $th_X(E)$ *for the element* $th^E_X(1) \in A_X(E)$.

If an assignment $E/X \mapsto th(E) \in A_X(E)$ *is a Thom classes theory on* A, *then the family of homomorphisms* $\cup th(E) : A(X) \to A_X(E)$ *form an orientation on* A.

The two mentioned correspondences between orientations and Thom classes theories are inverse to each other.

Proof. It is obvious.

□

3.7.3. Lemma. *If an assignment* $E/X \mapsto A_X(E)$ *is a Thom classes theory on* A, *then its restriction to line bundles is a Thom structure on* A.

If two Thom classes theories coinside on each line bundle then they coincide.

Proof. The first assertion is obvious. To prove the second assertion consider two Thom classes theories $E \mapsto th(E) \in A_X(E)$ and $E \mapsto th'(E) \in A_X(E)$ which coincide on line bundles. To prove that for a vector bundle E one has the relation $th(E) = th'(E)$ one may assume by the splitting principle 3.5.1 that $E = \oplus L_i$ is a direct sum of line bundles. Let $q_i : E \to L_i$ be the projection to the i-th summund. Now the chain of relations

$$th(E) = \cup q_i^A(th(L_i)) = \cup q_i^A(th'(L_i)) = th'(E)$$

completes the proof of the assertion. □

3.7.4. Theorem. *Given a Chern structure* $L \mapsto c(L)$ *on* A *(or the corresponding by 3.2.4 Thom structure* $L \mapsto th(L)$ *on* A*) there exists an orientation* $(X, Z, E) \mapsto th^E_Z$ *on* A *such that the following properties hold*

1. *for each smooth variety* X *and each line bundle* L/X *one has* $th(L) = th^L_X(1)$;
2. *for each smooth* X *and each line bundle* L/X *one has* $z^A \circ i^A \circ th^L_X(a) = c(L) \cup a$ *where* $a \in A(X)$ *is any element,* $i^A : A_X(L) \to A(L)$ *is the support extension operator for the pair* $(L, L - X)$, $z : X \to L$ *is the zero section.*

Moreover the required orientation is uniquely determined both by the property (1) and by the property (2).

This Theorem describes the arrow δ and the composition $\delta \circ \gamma$ from the Introduction.

3.7.5. Theorem. *If $(X, Z, E) \mapsto th^E_Z$ is an orientation on A then the assignment $L \mapsto z^A \circ i^A \circ th^L_X(1)$ is a Chern structure on A, the assignment $L \mapsto th^L_X(1)$ is a Thom structure on A and so constructed Chern and Thom structures correspond to each other.*

Moreover the construction of an orientation by means of a Chern (or a Thom) structure given by Theorem 3.7.4 and the construction of a Chern and a Thom structure by means of an orientation are inverse of each other.

This Theorem describes the arrow ρ and the composition $\delta \circ \gamma$ from the Introduction. Moreover it states that the arrow ρ and the composition $\delta \circ \gamma$ are inverse to each other, and it states that the composition $\gamma \circ \rho$ and the arrow δ are inverse to each other.

Proof of Theorem 3.7.4. To construct an orientation on A it suffices (see 3.7.2) to construct a Thom classes theory $E \mapsto th(E) \in A_X(E)$.

Let E/X be a rank n vector bundle and let $F = E \oplus 1$. Let $p : \mathbf{P}(F) \to X$ be the projection. The support extension operator $A_{\mathbf{P(1)}}(\mathbf{P}(F)) \to A(\mathbf{P}(F))$ is injective because the sequence (16) is exact. The same exact sequence and Proposition 3.6.6 show that the element $c_n(\mathcal{O}_F(1) \otimes p^*E) \in A(\mathbf{P}(F))$ belongs to the subgroup $A_{\mathbf{P(1)}}(\mathbf{P}(F))$. Set

$$\bar{th}(E) = c_n(\mathcal{O}_F(1) \otimes p^*E) \in A_{\mathbf{P(1)}}(\mathbf{P}(F)). \tag{21}$$

and define the element $th(E) \in A_X(E)$ as follows

$$th(E) = e^A(\bar{th}(E)) = e^A(c_n(\mathcal{O}_F(1) \otimes p^*E)) \in A_X(E) \tag{22}$$

To show that the assignment $E \mapsto th(E) \in A_X(E)$ is a Thom classes theory it remains to check the properties (1) to (5) from 3.7.1.

The second and the third property follows immidiately from the functoriality of the Chern classes 3.6.2.

The element $\bar{th}(L) \in A(\mathbf{P}(F))$ is central because it is a Chern class. Since the pull-back map $e^A : A(\mathbf{P}(F)) \to A(E)$ is surjective the element $th(L) = e^A(\bar{th}(L))$ is $A(X)$-central. For a smooth variety Y and a morphism $f : Y \to X$ one has $th(f^*(E)) = f^A(th(E))$ in $A_Y(f^*(E))$. Thus the element $th(E)$ is universally $A(X)$-central. This proves the property (1).

To prove the fourth property consider the commutative diagram

$$\begin{tikzcd}
A_{\mathbf{P(1)}}(\mathbf{P}(F)) \arrow[r, "e^A"] & A_X(E) \\
A(X) \arrow[u, "\cup \bar{th}(E)"] \arrow[r, "id"] & A(X). \arrow[u, "\cup th(E)"']
\end{tikzcd}$$

The map e^A is an isomorphism by the excision property. Thus the right vertical arrow is an isomorphism if the cup-product with the class $\bar{th}(E)$ is an isomorphism. For that consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A_{\mathbf{P(1)}}(\mathbf{P}(F)) \arrow[r] & A(\mathbf{P}(F)) \arrow[r] & A(\mathbf{P}(E)) \arrow[r] & 0 \\
0 \arrow[r] & A(X) \arrow[u, "\cup \bar{th}(E)"] \arrow[r] & A(X) \oplus A(X)^n \arrow[u, "(\cup \bar{th}(E), \alpha)"] \arrow[r] & A(X)^n \arrow[u, "\beta"] \arrow[r] & 0
\end{tikzcd}$$

where $\beta = (\cup 1, \cup\zeta_E, \dots, \cup\zeta_E^{n-1})$, $\zeta_E = c(\mathcal{O}_E(1)) \in A(\mathbf{P}(E))$ and $\alpha = (\cup 1, \cup\zeta_F, \dots, \cup\zeta_F^{n-1})$, $\xi_F = c(\mathcal{O}_F(1)) \in \mathbf{P}(F)$.

The map β is an isomorphism by the projective bundle theorem. Thus to prove that the left vertical arrow is an isomorphism it suffices to check that the map $(\cup \bar{th}(E), \alpha)$ is an isomorphism. Using the Mayer-Vietoris property and 2.4.6 one may assume that the bundle E is the trivial rank n bundle. In this case one has $\bar{th}(E) = c_n(\mathcal{O}_F(1)^n) = \zeta_F^n$. Thus the map $(\bar{th}(E), \alpha)$ coincides in this case with the map $(\cup 1, \cup\zeta_F, \dots, \cup\zeta_F^n)$ and it is an isomorphism by the projective bundle theorem. The property (4) is proved.

Basically the property (5) follows from the Cartan formular for Chern classes. But to give a detailed prove one needs certain preliminaries.

Let $E = E_1 \oplus E_2$ be a vector bundle over a smooth variety X and let $F_r = E_r \oplus 1$ $(r = 1, 2)$ and let $F = E \oplus 1$ and let $p : \mathbf{P}(F) \to X$ be the projection.. Let $q_r : E \to E_i$ be the projection and let $i_r : E_r \hookrightarrow E$ be the imbedding. We will identify E with the open subset $\mathbf{P}(F) - \mathbf{P}(E)$ of $\mathbf{P}(F)$ and identify E_i with the open subset $\mathbf{P}(F_r) - \mathbf{P}(E_i)$ of $\mathbf{P}(F_r)$. Let $\mathbf{P}(F_i)$ be the subvariety in $\mathbf{P}(F)$ defined by the direct summund F_r of F. Let the closed imbedding $\bar{i}_r : \mathbf{P}(F_i) \hookrightarrow \mathbf{P}(F)$ be the one extending the imbedding i_r. We will write $p : \mathbf{P}(F) \to X$ for the projection to X.

Let $res_r : A_{\mathbf{P}(F_r)}(\mathbf{P}(F)) \to A_{E_r}(E)$ be the pull-back map induced by the imbedding $E \hookrightarrow \mathbf{P}(F)$. Let $res : A_{\mathbf{P}(1)}(\mathbf{P}(F)) \to A_X(E)$ be the pull-back map induced by the same imbedding. The support extension operators

$$A_{\mathbf{P}(F_r)}(\mathbf{P}(F)) \to A(\mathbf{P}(F))$$

are injective for $(r = 1, 2)$ because they are the operators from the short exact sequences of the form (16). The same exact sequences and Proposition 3.6.6 show that the elements

$$c_{n_1}(\mathcal{O}_F(1) \otimes p^*(E_1)) \in A(\mathbf{P}(F));\ c_{n_2}(\mathcal{O}_F(1) \otimes p^*(E_2)) \in A(\mathbf{P}(F))$$

belongs to the subgroups $A_{\mathbf{P}(F_2)}(\mathbf{P}(F))$ and $A_{\mathbf{P}(F_1)}(\mathbf{P}(F))$ respectively. Consider elements $x_1 = c_{n_2}(\mathcal{O}_F(1) \otimes p^*(E_2)) \in A_{\mathbf{P}(F_1)}(\mathbf{P}(F))$, $x_2 = c_{n_1}(\mathcal{O}_F(1) \otimes p^*(E_1)) \in A_{\mathbf{P}(F_2)}(\mathbf{P}(F))$. We claim that they satisfy the following relations

$$q_1^A(th(E_1)) = res_1(x_2);\ q_2^A(th(E_2)) = res_2(x_1). \tag{23}$$

In fact, the commutative diagram

$$\begin{tikzcd} A_{\mathbf{P}(1)}(\mathbf{P}(F_2)) \arrow[d, "e_2^A"'] & A_{\mathbf{P}(F_1)}(\mathbf{P}(F)) \arrow[l, "\bar{j}_2^A"'] \arrow[d, "res_2"] \\ A_X(E_2) & A_{E_1}(E). \arrow[l, "j_2^A"'] \end{tikzcd}$$

and the relation $\bar{j}_2^A(x_1) = \bar{th}(E_2)$ in $A_{\mathbf{P}(1)}(\mathbf{P}(F_2))$ show that $j_2^A(res_2(x_1)) = th(E_2)$. Now the relation $q_2^A \circ j_2^A = id$ proves the relation $res_2(x_1) = q_2^A(th(E_2))$. The second of the two relations (23) is proved. The first one is proved similarly. So the relations (23) are proved.

Consider one more commutative diagram

$$\begin{tikzcd}
A(\mathbf{P}(F)) \times A(\mathbf{P}(F)) \arrow[r, "\cup"] & A(\mathbf{P}(F)) \\
A_{\mathbf{P}(F_1)}(\mathbf{P}(F)) \times A_{\mathbf{P}(F_2)}(\mathbf{P}(F)) \arrow[r, "\cup"] \arrow[u, "I_1^A \times I_2^A"] \arrow[d, "res_1 \times res_2"'] & A_{\mathbf{P}(1)}(\mathbf{P}(F)) \arrow[u, "I^A"'] \arrow[d, "res"] \\
A_{E_1}(E) \times A_{E_2}(E) \arrow[r, "\cup"] & A_X(E)
\end{tikzcd}$$

The commutativity of the upper square of this diagram proves the relation $x_1 \cup x_2 = \bar{t}h(E)$ in $A_{\mathbf{P}(1)}(\mathbf{P}(E))$ because

$$c_{n_2}(\mathcal{O}_F(1) \otimes p^*(E_2)) \cup c_{n_1}(\mathcal{O}_F(1) \otimes p^*(E_1)) = c_n(\mathcal{O}_F(1) \otimes p^*(E))$$

in $A(\mathbf{P}(F))$. Now the chain of relations

$$th(E) = res(\bar{t}h(E)) = res(x_1 \cup x_2) = res_1(x_1) \cup res_2(x_2) = q_1^A(th(E_1)) \cup q_2^A(th(E_2)).$$

prove the property (5). Thus the assignment $E \mapsto th(E) \in A_X(E)$ is indeed a Thom theory on A.

We still have to check that the orientation corresponding to this Thom classes theory by Lemma 3.7.2 satisfies the requirements 1 and 2 of Theorem 3.7.4.

The property $th(L) = th_X^L(1)$ valids because the map th_X^L is defined as the cup-product with the class $th(L)$.

The requirement 2 is satisfied by the following reasons. The composite map $z^A \circ i^A \circ th_X^L : A(X) \to A(X)$ is a two-sided $A(X)$-module map. It takes the unit 1 to the class $c(L)$ by Theorem 3.2.4. The requirement is checked.

To complete the proof of Theorem it remains to prove the uniqueness of the orientation. To prove the uniqueness of the orientation satisfying the property 1 take two orientations ω and ω' on A satisfying the property 1. Then the assignments $E \mapsto th(E) = th_X^E(1)$ and $E \mapsto th'(E) = th_X'^E \in A_X(E)$ are two Thom classes theories on A by Lemma 3.7.2. To check that they coincide it suffices by Lemma 3.7.3 to check that their restrictions to line bundles coincide. This is the case by the requirement 1. Thus $\omega = \omega'$.

Now prove the uniqueness of the orientation satisfying the requirement 2. Let $L \mapsto c(L)$ be a Chern structure and let ω and ω' be two orientations satisfying the requirement 2. We will show that $\omega = \omega'$.

It suffices to check that the corresponding Thom classes theories $E \mapsto th_\omega(E)$ and $E \mapsto th_{\omega'}(E)$ coincide (see 3.7.2). By Lemma 3.7.3 the restriction of these Thom classes theories to line bundles are Thom structures on A. By the same Lemma the Thom classes theories coincide if the mentioned Thom structures on A coincide. To prove that the two Thom structures on A coincide it suffices by Theorem 3.2.4 to check that the two corresponding Chern structures $L \mapsto c_\omega(L)$ and $L \mapsto c_{\omega'}(L)$ coincide. This holds because $c_\omega(L) = c(L)$ and $c_{\omega'}(L) = c(L)$ by the requirement 2. The proof of the relation $\omega = \omega'$ is completed.

$\square$

Proof of Theorem 3.7.5. The assignment $E \mapsto th_X^E(1)$ is a Thom classes theory by Lemma 3.7.2. Its restriction to line bundles is a Thom structure on A by Lemma 3.7.3. The first assertion is proved.

The assignment $L \mapsto c(L) = z^A(i^A(th_X^L(1))) = z^A(i^A(th(L)))$ is a Chern structure by Theorem 3.2.4. The Chern structure $L \mapsto c(L)$ and the Thom structure $L \mapsto th(L)$ correspond to each other by the same Theorem 3.2.4. The first part of Theorem is proved.

Now verify that the correspondences between orientations and Thom structures in the two theorems are inverse to each other.

Now let $L \mapsto th(L)$ be a Thom structure and let ω be the corresponding by Theorem 3.7.4 orientation and let $L \mapsto th_X^L(1) = th'(L)$ be the Thom structure corresponding to ω by Theorem 3.7.5. We have to check that for each line bundle L one has $th'(L) = th(L)$.

If $L \mapsto c(L)$ is the Chern structure corresponding to the Thom structure $L \mapsto th(L)$, then $th'(L) = c_1(\mathcal{O}(1) \otimes p^*(L))$ by the very construction of the orientation ω.

From the other side the assignment $L \mapsto c_1(\mathcal{O}(1)) \otimes p^*(L)$ is exactly the Thom structure corresponding to the Chern structure $L \mapsto c(L)$. Thus $th'(L) = th(L)$ by Theorem 3.2.4.

Let ω be an orientation and let $L \to th_\omega(L)$ be the corresponding Thom structure and let ω' be the orientation corresponding to the Thom structure $L \to th_\omega(L)$. We have to check that $\omega' = \omega$.

It was proved just above that the Thom structure $L \mapsto th_{\omega'}(L)$ corresponding to ω' coincide with the Thom structure $L \mapsto th_\omega(L)$. Now by Lemma 3.7.3 the Thom classes theory $E \mapsto th_{\omega'}(E)$ corresponding to ω' coincides with the Thom classes theory $E \mapsto th_\omega(E)$ corresponding to ω. Thus $\omega' = \omega$ by Lemma 3.7.2.

It is verifed simultanously that the correspondences between orientations and Chern structures on A described in the two theorems are inverse to each other.

The theorem is proved.

$\square$

3.8 Examples

3.8.1. Let A be the algebraic K-theory (2.1.8). The rule $L \to [1] - [L^\vee]$ endows A with a Chern structure (the property (4) follows from [Qu2, §8, Th.2.1]) and thus orients A.

It's interesting to observe that the corresponding Chern class c_n of a rank n vector bundle E is exactly the known class $\lambda_{-1}(E^\vee) = [1] - [E^\vee] + [\wedge^2 E^\vee] + \cdots + (-1)^n[\wedge^n E^\vee]$.

3.8.2. Let A be the etale cohomology theory $A_Z^*(X) = \oplus_{q=-\infty}^{+\infty} H_Z^*(X, \mu_m^{\otimes q})$, where m is an integer prime to char(k). Consider the short exact sequence of the etale sheaves $0 \to \mu_m \to \mathbb{G} \xrightarrow{\times m} \mathbb{G} \to 0$ and denote by $\partial : H^1(X, \mathbb{G}_m) \to H^2(X, \mu_m)$ the boundary map. For a line bundle L over a smooth variety X let $[L] \in H^1(X, \mathbb{G}_m)$ be its isomorphism class. It is known [Mi] that the rule $L \mapsto \partial([L])$ endows A with a Chern structure. Thus A is oriented.

3.8.3. Let A be the motivic cohomology [SV]: $A^p_Z(X) = \oplus_{q=0}^{\infty} H^p_Z(X, \mathbb{Z}(q))$. Recall that $H^2_{\mathcal{M}}(X, \mathbb{Z}(1)) = CH^1(X)$ for a smooth X [SV]. For a line bundle L over a smooth variety X let $D(L) \in CH^1(X)$ be the associated class divisor. The rule $L \mapsto D(L)$ endows A with a Chern structure in the characteristic zero [SV, Cor. 4.12.1] (now it is known in any characteristic). Thus A is oriented.

3.8.4. Let A be the K-cohomology [Qu2, §7, 5.8]: $A^p_Z(X) = \oplus_{q=0}^{\infty} H^p_Z(X, \mathcal{K}_q))$, where $\mathcal{K}$ is the sheaf of K-groups. Recall that the sheaf $\mathcal{K}_1$ coincides with the sheaf $\mathcal{O}^*$ of invertible functions. For a line bundle L over a smooth variety X let $[L] \in H^1(X, \mathcal{K}_1) = H^1(X, \mathcal{O}^*)$ be the isomorphism class of L. The rule $L \mapsto [L]$ endows A with a Chern structure [Gi, Th.8.10] and thus orients A.

3.8.5. Let $k = \mathbb{R}$ and let A be the theory $A^p(X, U) = H^p(X(\mathbb{R}), U(\mathbb{R}); \mathbb{Z}/2\mathbb{Z})$. For a line bundle L consider the real line bundle $L(\mathbb{R})$ over the topological space $X(\mathbb{R})$ and set $c_1(L) = w_1(L(\mathbb{R}))$ (the first Stiefel-Whitney class). Since $\mathbf{P}^n(\mathbb{R}) = \mathbb{R}P^n$ is the real projective space the rule $L \mapsto c_1(L)$ endows A with a Chern structure and thus orients A.

3.8.6. Semi-topological complex and real K-theories [FW]. If the ground field k is the field $\mathbb{R}$ of reals then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{semi}$ defined in[FW] is an oriented theory as it is proved in [FW]. For a real variety X it interpolates between the algebraic K-theory of X and Atiyah's Real K-theory of the associated Real space of complex points, $X(\mathbb{C})$.

3.8.7. Orienting the algebraic cobordism theory. In this example the notation of 2.5.5 are used.

The identity morphism $\mathbb{MGL}_1$ to itself gives rise in the standard manner to an element $[id_1] \in \mathbb{MGL}^{2,1}(\mathbb{MGL}_1)$. By the very definition $\mathbb{MGL}_1 = Th(\mathcal{T}(1))$ and $\mathcal{T}(1)$ is the tautological line bundle $\mathcal{O}(-1)$ over the space $G(1) = \mathbf{P}(V) = \mathbf{P}^{\infty}$. Now set

$$th = [id_1] \in \mathbb{MGL}^{2,1}(\mathbb{MGL}_1) = \mathbb{MGL}^{2,1}_{\mathbf{P}^{\infty}}(\mathcal{O}(-1)).$$

Consider the fiber $\mathbb{A}^1$ of $\mathcal{T}(1)$ over the point $g_1 \in \mathbf{P}(V)$.

The restriction of the element th to the Thom space $Th(\mathbb{A}^1) = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$ coincides with the T-suspension $\sigma \in \mathbb{MGL}^{2,1}(Th(\mathbb{A}^1)) = \mathbb{MGL}^{2,1}_{\{0\}}(\mathbb{A}^1)$ of the unite $1 \in \mathbb{MGL}^{0,0}(pt)$. Thus the element th orients the algebraic cobordism theory $\mathbb{MGL}$ due to (**??**).

3.9 The formal group law F_ω

Let ω be an orientation of A. Thus A is endowed with the Chern structure which correspond to ω (see 3.7.5 and 3.7.4). Following [Qu1] we associate a formal group low F_ω with ω. This formal group low is defined over the ring $\bar{A}^{uc}$ and gives an expression of the first Chern class of $L_1 \otimes L_2$ in terms of the first Chern classes of line bundles L_1, L_2.

Using (3.3.1) identify the formal power series in one variable $\bar{A}[[u]]$ with the ring $A(\mathbf{P}^\infty)$ identifying u with $c_1(\mathcal{O}(1)) \in A(\mathbf{P}^\infty)$. The two "projections" $p_i : \mathbf{P}^\infty \times \mathbf{P}^\infty \to \mathbf{P}^\infty$ induce two pull-back maps $p_i^A : A(\mathbf{P}^\infty) \to A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$. Using (3.3.1) again identify $A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ with $\bar{A}[[u_1, u_2]]$ where $u_i = p_i^*(u) = c_1(p_i^*(\mathcal{O}(1))$. Set

$$F_\omega(u_1, u_2) = c_1(p_1^*(\mathcal{O}(1) \otimes p_2^*(\mathcal{O}(1))) \in \bar{A}[[u_1, u_2]] \tag{24}$$

Since the first Chern class is a universally central element $F_\omega \in \bar{A}^{uc}[[u_1, u_2]]$

3.9.1. Proposition. *For any $X \in \mathcal{S}m$ and line bundles L_1/X, L_2/X one has the following relation in $A(X)$*

$$c_1(L_1 \otimes L_2) = F_\omega(c_1(L_1), c_1(L_2))$$

Here the right hand side is well-defined since the first Chern classes are universally central and nilpotent (3.6.2).

Proof. Using the Jounalou device one may assume (compare with the proof of Lemma 3.6.4) that $L_i = f_i^*(\mathcal{O}(1)))$ for a maps $f_i : X \to \mathbf{P}^N$. Let $f = (f_1, f_2) : X \to \mathbf{P}^N \times \mathbf{P}^N$. The chain of relations

$$c_1(L_1 \otimes L_2) = f^A(c_1(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1))) = f^A(F_\omega(u_1, u_2)) = F_\omega(c_1(L_1), c_1(L_2)).$$

completes the proof. □

3.9.2. Proposition. *The formal power series $F_\omega \in \bar{A}^{uc}[[u_1, u_2]]$ is a commutative formal group law ([Ha]) with the "inverse element" $I_\omega(u) = c_1(\mathcal{O}(-1)) \in A^{uc}(\mathbf{P}^\infty) = \bar{A}^{uc}[[u]]$.*

Proof. One has to verify that the formal power series $F_\omega \in \bar{A}^{uc}[[u_1, u_2]]$ satisfies the following properties

- normalization: $F_\omega(u_1, u_2) \equiv u_1 + u_2$ modulo the degree 2 ;
- associativity: $F_\omega(F_\omega(u_1, u_2), F_\omega(u_3)) = F_\omega(u_1, F_\omega(u_2, u_3))$;
- commutativity: $F_\omega(u_1, u_2) = F_\omega(u_2, u_1)$;
- "inverse element": $F_\omega(u, I_\omega(u)) = 0$ for $I_\omega(u) = c_1(\mathcal{O}(-1)) \in A^{uc}(\mathbb{P}^\infty) = \bar{A}^{uc}[[u]]$.

Proposition 3.9.1 shows that the associativity and the commutativity follows immidiately from the corresponding properties of the tensor products of line bundles.

To prove the normalization property consider the element $\alpha = c_1(p_1^*(\mathcal{O}(1) \otimes p_2^*(\mathcal{O}(1))) \in A(\mathbf{P}^1 \times \mathbf{P}^1)$. The $A(pt)$-module $A(\mathbf{P}^1 \times \mathbf{P}^1)$ is a free module with the free bases 1, $\zeta \otimes 1$, $1 \otimes \zeta$ and $\zeta \otimes \zeta$ by the projective bundle theorem.

Write the element α in the form $\alpha = a_{00} 1 \otimes 1 + a_{10} \zeta \otimes 1 + a_{01} 1 \otimes \zeta + a_{11} \zeta \otimes \zeta$, where a_{ij} are elements in $A(pt)$. By the projective bundle theorem $A(\mathbf{P}^1 \times \mathbf{P}^1) = A[[u_1, u_2]]/(u_1^2, u_2^2)$ and thus it suffices to prove that $a_{00} = 0$ and $a_{10} = 1 = a_{01}$.

Restricting the element α to $\{0\} \times \{0\}$, to $\mathbf{P}^1 \times \{0\}$ and to $\{0\} \times \mathbf{P}^1$ and using the relation $\zeta|_{\{0\}} = 0$ one gets the relations: $a_{00} = 0$, $a_{00}+a_{10}\zeta = \zeta$ in $A(\mathbf{P}^1)$ and $a_{00}+a_{01}\zeta = \zeta$ in $A(\mathbf{P}^1)$. Thus $a_{10} = 1$ and $a_{01} = 1$ and the normalization property is proved.

The relation $F_\omega(u, I_\omega(u)) = 0$ follows from Proposition 3.9.1 applied to the line bundles $L_1 = \mathcal{O}(1)$ and $L_2 = \mathcal{O}(-1)$ on $\mathbf{P}^\infty$.

□

3.9.3. Definition. *The formal group low F_ω is called the formal group low associated with A endowed with the orientation ω. Its "inverse element" is I_ω.*

Let $E^-_\omega : \bar{A}[[u]] \to A(\mathbf{P}^\infty)$ be an isomorphism taking the variable u to the element $\xi_A = c_1^A(\mathcal{O}(-1)) \in A(\mathbf{P}^\infty)$. The formal power series

$$F^-_\omega(u_1, u_2) = (E^-_\omega \otimes E^-_\omega)^{-1}[c_1(p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\mathcal{O}(-1)))] \in A(pt)^{uc}[[u_1, u_2]] \qquad (25)$$

satisfies exactly the same property as the series $F_\omega(u_1, u_2)$ above. Namely, For any $X \in \mathcal{S}m$ and line bundles L_1/X, L_2/X one has the following relation in $A(X)$

$$c_1(L_1 \otimes L_2) = F^-_\omega(c_1(L_1), c_1(L_2)).$$

Taking $X = \mathbf{P}^\infty \times \mathbf{P}^\infty$ and line bundles $L_i = p_i^*(\mathcal{O}(-1))$ for $i = 1, 2$ one gets

$$F^-_\omega(u_1, u_2) = F_\omega(u_1, u_2).$$

So there is no difference which line bundle is used $\mathcal{O}(-1)$ or $\mathcal{O}(1)$. The formal group law $F_\omega(u_1, u_2)$ is the same in both cases. BELOW we usually for the purposes of definitions will use the tautological line bundle $\mathcal{O}(-1)$. However for certain computations it will be convinient to use the line bundle $\mathcal{O}(1)$.

3.9.4. Examples

- If $A = H^*_{\mathcal{M}}(-, \mathbb{Z}(*))$ with the first Chern class c_1^H then one has the relation $c_1^H(L_1 \otimes L_2) = c_1^H(L_1) + c_1^H(L_2)$;
- If $A = K$-theory with the first Chern class defined by $c_1^K(L) = [1] - [L^\vee]$ then one has the relation $c_1^K(L_1 \otimes L_2) = c_1^K(L_1) + c_1^K(L_2) - c_1^K(L_1).c_1^K(L_2)$.

4 Push-forwards

Let A be a ring cohomology theory.

4.1 Integrations on a ring cohomology theory

Here we define a notion of an integration on A and claim in 4.1.4 that an orientation of A gives rise to a unique integration on A subjecting certain normalization property. Visa versa an integration defines an orientation and these two constructions are inverse to each other. Let us recall a notion.

4.1.1. Definition. *Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties. Consider a Cartesian square (in the category of schemes)*

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} \\ {\scriptstyle \tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \varphi} \\ Y & \xrightarrow{i} & X \end{array}$$

consisting of smooth varieties. This square is called transversal if the canonical morphism $\tilde{\varphi}^*(N) \to \tilde{N}$ is an isomorphism (here N and $\tilde{N}$ are the normal bundle to Y in X and to $\tilde{Y}$ in $\tilde{X}$).

4.1.2. Definition. *Let A be the ring cohomology theory. An integration on A is a rule assigning to each projective morphism of smooth varieties $f : Y \to X$ a two-sided $A(X)$-module operator*

$$f_A : A(Y) \to A(X)$$

called the push-forward (for f) and satisfying the following properties

1. *$(f \circ g)_A = f_A \circ g_A$ for any projective morphisms $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ of smooth varieties;*
2. *for the transversal square from (4.1.1) the following diagram commutes*

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_A} & A(\tilde{X}) \\ {\scriptstyle \tilde{\varphi}^A}\uparrow & & \uparrow{\scriptstyle \varphi^A} \\ A(Y) & \xrightarrow{i_A} & A(X) \end{array}$$

3. *for any morphism of smooth varieties $f : Y \to X$ the following diagram commutes*

$$\begin{array}{ccc} A(\mathbf{P}^n \times Y) & \xleftarrow{(id \times f)^A} & A(\mathbf{P}^n \times X) \\ {\scriptstyle (p_Y)_A}\downarrow & & \downarrow{\scriptstyle (p_X)_A} \\ A(Y) & \xleftarrow{f^A} & A(X) \end{array}$$

   where $p_Y : \mathbf{P}^n \times Y \to Y$ and $p_X : \mathbf{P}^n \times X \to X$ are the natural projections;

4. *normalization: for any smooth variety X one has $(id_X)_A = id_{A(X)}$.*
5. *localization: for any closed imbedding of smooth varieties $i : Y \hookrightarrow X$ and the inclusion $j : X - Y \hookrightarrow X$ the sequence*

$$A(Y) \xrightarrow{i_A} A(X) \xrightarrow{j^A} A(X - Y)$$

   is exact. This sequence is often called Gysin sequence below in the text.

The line bundle $L(D)$ associated with an effective divisor D on a smooth variety X is defined in 1.1.

4.1.3. Definition. *Let ω be an orientation of A and let $L \mapsto c(L)$ be the corresponding by 3.7.5 Chern structure on A. One says that an integration $f \mapsto f_A$ on A is subjected to the orientation ω if for each smooth variety X and each smooth divisor $i : D \hookrightarrow X$ one has the following relation in $A(X)$*

$$i_A(1) = c(L(D)). \tag{26}$$

4.1.4. Theorem. *Let A be a ring cohomology theory.*

1. *Let ω be an orientation of A. Then there exists a unique integration $f \mapsto f_A$ on A subjecting to this orientation.*

2. *Let $f \mapsto f_A$ be an integration on A. Then there exists a unique orientation ω on A such that the integration is subjected to ω. Moreover the assignment $c(L) = z^A(z_A(1)) \in A(X)$ is a Chern structure on A and the required orientation ω is the one corresponding to this Chern structure by 3.7.4.*

3. *The correspondences described in the items* 1 *and* 2 *of this Theorem are inverse of each other.*

The first item of the Theorem describes the arrow α from the Introduction. The second item of the Theorem describes the composition $\delta \circ \gamma \circ \beta$ and the composition $\gamma \circ \beta$ from the Introduction. The arrow β from the Introduction is described rather easy as well. Namely for a line bundle L over a smooth variety X let $s : X \mapsto \mathbf{P}(1 \oplus L)$ be the section of the projection $p : \mathbf{P}(1 \oplus L) \to X$ identifying the variety X with the subvariety $\mathbf{P}(1)$ in the projective bundle $\mathbf{P}(1 \oplus L)$. Let $e : L \hookrightarrow \mathbf{P}(1 \oplus L)$ be the open inclusion and let $z : X \to L$ be the zero section of L (obviously $s = z \circ e$). Identify the group $A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L))$ with a subgroup of $A(\mathbf{P}(1 \oplus L))$ via the support extension operator from the exact sequence (8). Now set $\bar{th}(L) = s_A(1) \in A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L))$. Then the assignment $th(L) = e^A(\bar{th}(L)) \in A_X(L)$ is a Thom structure on A and moreover the corresponding by Theorem 3.2.4 Chern structure is the Chern structure from the item 2 of Theorem 4.1.4 (this is checked in the proof of the item 2 below).

Each theory from 3.8 is oriented and thus endowed with the integration given by (4.1.4). For any such theory A the integration coincides with the well-known one as follows from the known properties of push-forwards in these cases. In particular for the algebraic K-theory the push-forwards given by (4.1.4) and the orientation from 3.8 coincides with the push-forwards from [Qu2, §7, 2.7]. Note that the algebraic cobordism theory being oriented (see 3.8.7) is also endowed with an integration due to (4.1.4).

The proof of this Theorem is rather long. It ocupies all the Section 4 and is organized as follows. In this subsection the uniqueness assertion of the item 1 and the item 2 of the Theorem are proved. The proof of the uniqueness assertion of the item 1 is based on the first variant of Riemann-Roch type theorem proved in [Pa, Theorem 1.1.9]. The proof of the existence assertion is given below in subsection 4.8 and it requires a construction of an integration.

The construction of an integration subjecting to an orientation is rather long itself and it takes several subsections. Namely, in subsection 4.2 push-forwards are constructed for closed imbeddings of smooth varieties (they are called Gysin operators). In subsection 4.3 push-forwards are constructed for the case of projections $X \times \mathbf{P}^n \to X$ (they are called Quillen's operators). In the subsection 4.4 properties of Gysin operators are proved and in the subsection 4.5 properties of Quillen's operators are proved except of the consistence with a section of a trivial projective bundle, which is proved in the subsection 4.6. Push-forwards for any projective morphism are constructed in subsection 4.7. These push-forwards form an integration on A as it is stated in Theorem 4.7.1 from the same subsection.

Proof of the uniqueness assertion of the item 1. Let $f \mapsto f_A$ be an integration on A subjecting to the orientation ω. For each line bundle $p : L \to X$ and its zero section $z : X \to L$ the line bundle over L associated with the divisor X on L coincides with the line bundle $p^*(L)$. Thus $z^A(z_A(1)) = z^A(c(p^*(L))) = c(L)$ in $A(X)$. The assignment $L \mapsto z^A(z_A(1))$ is the Euler structure associated with the integration $f \mapsto f_A$ on the pretheory $A|_{\mathcal{S}m} : \mathcal{S}m \to Ab$ ([Pa, §1]). The projective bundle theorem 3.3.1 shows that the integration $f \mapsto f_A$ is a perfect integration on the pretheory $A|_{\mathcal{S}m}$ (in the sense of [Pa, Definition 1.1.6]).

If $f \mapsto f_A^{(1)}$ and $f \mapsto f_A^{(2)}$ are two integrations on A subjecting to the orientation ω, then the two Euler structures $L \mapsto z^A(z_A^{(1)}(1))$ and $L \mapsto z^A(z_A^{(2)}(1))$ coincide. By [Pa, Cor. 1.1.11] the two integrations coincide. The uniqueness assertion is proved. □

Proof of item 2 **of Theorem 4.1.4.** We begin with the uniqueness assertion. Assume the integration $f \mapsto f_A$ is subjected to two orientations ω and ω' on A. Prove that these two orientations coincide. Let $L \mapsto c(L)$ be the Chern structures on A corresponding to the orientation ω. For a line bundle L over a smooth variety X, consider its zero section $z : X \to L$ and the projection $p : L \to X$. The line bundle $L(z(X))$ associated with the divisor $z(X)$ on the variety L is isomorphic to the line bundle $p^*(L)$. Thus one has a chain of relations

$$z^A(z_A(1)) = z^A(c(L(z(X)))) = z^A(c(p^*(L))) = c(z^*(p^*(L))) = c(L).$$

Similarly one has the relation $z^A(z_A(1)) = c'(L)$, where $L \mapsto c'(L)$ is the Chern structure on A corresponding to the orientation ω'. Thus $c'(L) = c(L)$ and the relation $\omega = \omega'$ follows.

The nearest aim is to prove that the assignment $L \mapsto th(L)$ described just below Theorem 4.1.4 is a Thom structure. One has to check the functorial behavior of the class $th(L)$ and its non-degeneracy. To verify the functorial behavior of this class it suffices to check the functorial behavior of the class $\bar{t}h(L)$.

If $\phi : L_2 \to L_1$ is a line bundle isomorphism and $\Phi : \mathbf{P}(1 \oplus L_2) \to \mathbf{P}(1 \oplus L_1)$ is the induced map of the projective bundles then the diagram commutes

$$\begin{array}{ccc} A(\mathbf{P}(1 \oplus L_2)) & \xleftarrow{\Phi^A} & A(\mathbf{P}(1 \oplus L_1)) \\ {\scriptstyle (s_2)_A}\uparrow & & \uparrow{\scriptstyle (s_1)_A} \\ A(X) & \xleftarrow{id} & A(X). \end{array}$$

because the underlying diagram of varieties is transversal. The support extension operator $A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L_2)) \to A(\mathbf{P}(1 \oplus L_2))$ is injective by (8). Thus $\Phi^A(\bar{t}h(L_1)) = \bar{t}h(L_2)$ in $A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L_2))$.

Similarly for a morphism $f : X' \to X$ with smooth X' and for $L' = X' \times_X L'$ and for the induced map $\Phi : \mathbf{P}(1 \oplus L') \to \mathbf{P}(1 \oplus L)$ one gets the relation $\Phi^A(\bar{t}h(L)) = \bar{t}h(L')$. The functoriality of the class $th(L)$ is proved.

Now prove that the class $th(L)$ is $A(X)$-central (this notion is defined just below of the definition of the Chern structure). Clearly it suffices to check that the class $\bar{t}h(L)$ is $A(\mathbf{P}(1 \oplus L))$-central. The push-forward $s_A : A(X) \to A(\mathbf{P}(1 \oplus L))$ is a two-sided $A(\mathbf{P}(1 \oplus L))$-module operator and $\bar{t}h(L) = s_A(1)$. Therefore for each element $a \in A(\mathbf{P}(1 \oplus L))$ one has $a \cup s_A(1) = s_A(s^A(a)) = s_A(1) \cup a$. Thus the class $th(L)$ is $A(X)$-central.

To prove the no-degeneracy property recall that the Gysin sequence

$$0 \to A(X) \xrightarrow{s_A} A(X \times \mathbf{P}^1) \xrightarrow{j^A} A(X \times \mathbf{P}^1 - X \times \{0\}) \to 0$$

is exact in the middle term 4.1.2, where $s : X \to X \times \mathbf{P}^1$ is defined by $s(x) = (x, 0)$ and $j : X \times \mathbf{P}^1 - X \times \{0\} \hookrightarrow X \times \mathbf{P}^1$ is the open inclusion. The operator j^A is clearly surjective and the operator s_A is injective because $p_A \circ s_A = (id_X)_A = id_{A(X)}$, where $p : X \times \mathbf{P}^1 \to X$ is the projection. Hence the last five-terms sequence is short exact. Now the exactness of the sequence (8) shows that the operator s_A identifies $A(X)$ with the subgroup $A_{X \times \{0\}}(X \times \mathbf{P}^1)$ of the group $A(X \times \mathbf{P}^1)$.

We claim that the operator s_A coincides with the one $(\cup s_A(1)) \circ p^A$. In fact, for any element $a \in A(X)$ one has $s_A(1) \cup p^A(a) = s_A(s^A(p^A(a))) = s_A(a)$. Since $\bar{t}h(1) = s_A(1)$ the last observations shows that the operator $\cup \bar{t}h(1) : A(X) \to A(X \times \mathbf{P}^1)$ identifies $A(X)$ with the subgroup $A_{X \times \{0\}}(X \times \mathbf{P}^1)$ of the group $A(X \times \mathbf{P}^1)$.

Now we are ready to prove the non-degeneracy of property of the Thom class $th(L)$. For that consider the commutative diagram

$$\begin{array}{ccc} A_{X \times 0}(X \times \mathbf{A}^1) & \xleftarrow{e^A} & A_{X \times 0}(X \times \mathbf{P}^1) \\ {\scriptstyle \cup th(1)}\uparrow & & \uparrow{\scriptstyle \cup \bar{t}h(1)} \\ A(X) & \xleftarrow{id} & A(X). \end{array}$$

The pull-back map e^A is an isomorphism by the excision property. Thus the map $\cup th(1)$ is an isomorphism as well. The non-degeneracy property of the class $th(L)$ is proved.

Thus the assignment $L \mapsto th(L)$ with $th(L) = e^A(\bar{th}(L)) \in A_X(L)$ is a Thom structure on A.

Now show that the assignment $L \mapsto c(L)$ described in the item (2) is a Chern structure on A and MOREOVER it corresponds to the Thom structure $L \mapsto th(L)$ in the sence of 3.2.4. For that it suffices by Theorem 3.2.4 to check the relation $c(L) = [z^A \circ i^A](th(L))$, where $i^A : A_X(L) \to A(L)$ is the support extension operator.

To do this consider the transversal diagram

$$\begin{array}{ccc} L & \xrightarrow{e} & \mathbf{P}(1 \oplus L) \\ {\scriptstyle z_A}\uparrow & & \uparrow{\scriptstyle s_A} \\ X & \xrightarrow{id} & X \end{array}$$

and the corresponding commutative diagram

$$\begin{array}{ccc} A(L) & \xleftarrow{e^A} & A(\mathbf{P}(1 \oplus L)) \\ {\scriptstyle z}\uparrow & & \uparrow{\scriptstyle s} \\ A(X) & \xleftarrow{id} & A(X). \end{array}$$

Since $e \circ z = s$ one gets the relation

$$z^A(z_A(1)) = s^A(s_A(1)).$$

If $\bar{i}^A : A_{\mathbf{P}(1)}(\mathbf{P}(1 \oplus L)) \to A(\mathbf{P}(1 \oplus L))$ is the support extension operators then the chain of relations

$$c(L) = z^A(z_A(1)) = s^A(s_A(1)) = z^A[e^A(\bar{i}^A(\bar{th}(L)))] = z^A[i^A(e^A(\bar{th}(L)))] = [z^A \circ i^A](th(L))$$

proves the required relation $c(L) = [z^A \circ i^A](th(L))$. Thus the assignment $L \mapsto c(L)$ is a Chern structure and the Thom structure $L \mapsto th(L)$ corresponds to it in the sence of 3.2.4. The last chain of relations shows as well that one has the following one

$$c(L) = z^A(z_A(1)) = s^A(s_A(1)). \tag{27}$$

Let ω be the orientation of A corresponding to the Chern structure $L \mapsto c(L)$ by Theorem 3.7.4. We claim that this orientation is the required one. First of all the Chern structure $L \mapsto c(L)$ corresponds to the orientation ω by Theorem 3.7.5.

To check that the integration $f \mapsto f_A$ is subjected to the orientation ω recall that the assignment $L \mapsto z^A(z_A(1))$ is the Euler structure on A associated with the integration $f \mapsto f_A$ [Pa, 1.1.4]. The relations $c(L) = z^A(z_A(1))$ show that it coincides with the Chern structure $L \mapsto c(L)$. The projective bundle theorem 3.3.1 states that the integration $f \mapsto f_A$ is perfect in the sense of [Pa, Definition 1.1.6]. Now the relation (26) holds by Theorem [Pa, Theorem 1.1.8] and thus the integration $f \mapsto f_A$ is subjected to the orientation ω. The proof of the item (2) is completed.

$\square$

4.2 Push-forwards for closed imbeddings

Let A be a ring cohomology theory endowed with an orientation ω and with the corresponding by Theorem 3.7.5 Chern structure $L \mapsto c(L)$. Below we give a construction of push-forwards for closed imbeddings. The push-forward to be constructed for a closed imbedding i will be temporarily (until 4.7.1) denoted by i_{gys}. The deformation to the normal cone (see 2.2.7) plays an important role here (the deformation to the normal cone construction is a perfect substitute of the turbular neighbourhood in the differential topology).

For a closed imbedding $i : Y \hookrightarrow X$ of smooth varieties define an operator

$$i_{th} : A(Y) \to A_Y(X) \tag{28}$$

as the composition $i_{th} : A(Y) \xrightarrow{th_Y^N} A_Y(N) \xrightarrow{(i_0^A)^{-1}} A_{Y \times \mathbf{A}^1}(X_t) \xrightarrow{i_1^A} A_Y(X)$ where the notation for $N = N_{X/Y}$, i_0^A and i_1^A are taken from (2.2.7) and th_Y^N is the Thom operator corresponding to the orientation ω (3.1.1). The operator i_{th} is an isomorphism (see 3.7.2). Define the Gysin operator

$$i_{gys} : A(Y) \to A(X) \tag{29}$$

as the composition $i_{gys} : A(Y) \xrightarrow{i_{th}} A_Y(X) \xrightarrow{j^A} A(X)$ where j^A is the support extension operator for the pair $(X, X - Y)$ (see 2.0.1).

It will be checked below that the operator $i_{gys} : A(Y) \to A(X)$ is a *two-sided $A(X)$-module homomorphism.* In particular the composite operator $i_{gys} \circ i^A : A(X) \to A(X)$ coincides with the operator given by the cup-product with the element $i_{gys}(1)$:

$$i_{gys} \circ i^A = \cup i_{gys}(1) \tag{30}$$

The following properties of the Gysin operators will be proved below before the theorem 4.7.1 and are useful when proving this theorem.

4.2.1. Composition property: One has

$$i_{gys} \circ j_{gys} = (i \circ j)_{gys} \tag{31}$$

for closed imbeddings $Z \overset{j}{\hookrightarrow} Y \overset{i}{\hookrightarrow} X$ of smooth varieties.

4.2.2. Base change for Gysin operator: The Gysin operators commute with a transversal base change, i.e., for a transversal square from 4.1.1 the following diagram commutes

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_{gys}} & A(\tilde{X}) \\ {\scriptstyle \tilde{\varphi}^A}\uparrow & & {\scriptstyle \varphi^A}\uparrow \\ A(Y) & \xrightarrow{i_{gys}} & A(X) \end{array} \tag{32}$$

4.2.3. Additivity: Let j_1 and j_2 be the natural imbeddings of smooth varieties Y_1 and Y_2 to $Y = Y_1 \amalg Y_2$. For a closed imbedding $i : Y \hookrightarrow X$ one has

$$i_{gys} = (i_1)_{gys} \circ j_1^A + (i_2)_{gys} \circ j_2^A \tag{33}$$

where i_r is the composition $i \circ j_r$.

4.2.4. Identity to identity: for any smooth variety X one has

$$(id_X)_{gys} = id_{A(X)}; \tag{34}$$

4.2.5. Gysin sequence exact: for any closed imbedding of smooth varieties $i : Y \hookrightarrow X$ and the inclusion $k : X - Y \hookrightarrow X$ the sequence

$$A(Y) \overset{i_{gys}}{\to} A(X) \overset{k^A}{\to} A(X - Y) \tag{35}$$

is exact. This sequence is often called Gysin sequence below in the text.

4.2.6. Smooth divisor case: for a smooth divisor $i : D \hookrightarrow X$ one has

$$i_{gys}(1) = c(L(D)) \tag{36}$$

in $A(X)$ (see 1.1 for notation).

There are more properties of the Gysin operators which are useful themself and which are useful in proving some of the properties listed above. Recall that A is endowed with the Chern structure corresponding to the orientation ω (3.7.5) and with the corresponding higher Chern classes (3.6.2).

4.2.7. Let E be a rank n vector bundle over a smooth Y and let $s : Y \to \mathbf{P}(\mathbf{1} \oplus E)$ be the section of the projective bundle $p : \mathbf{P}(\mathbf{1} \oplus E) \to Y$ identifying Y with the closed subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(\mathbf{1} \oplus E)$. Then one has

$$s_{gys} = \cup \bar{t}h(E) \circ p^A, \tag{37}$$

where $\bar{t}h(E) = c_n(p^*(E) \otimes \mathcal{O}_E(1))$.

4.2.8. Let $z : Y \to E$ be the zero section of a vector bundle E/Y. Then the operator z_{th} coincides with the Thom operator th_Y^E.

4.2.9. Let F/Y be a rank m vector bundle and let $s : Y \to \mathbf{P}(F)$ be a section of the natural projection $p : \mathbf{P}(F) \to Y$. Consider the natural inclusion $\mathcal{O}_F(-1) \to p^*(F)$ and let Q be the factor-bundle $p^*(F)/\mathcal{O}_F(-1)$. In $A(\mathbf{P}(F))$ one has

$$s_{gys}(1) = c_{n-1}(\mathcal{O}_F(1) \otimes p^* s^* Q). \tag{38}$$

4.3 Push-forward for the projection $X \times \mathbf{P}^n \to X$

Let A be a ring cohomology theory endowed with an orientation ω and therefore endowed with the corresponding Chern structure ((3.7.5) and (3.7.4)) and with the corresponding higher Chern classes (3.6.2). Below we give a construction of push-forwards for projections. As above $\bar{A} = A(pt)$ and $\bar{A}^{uc}$ is the subring of all the universally central elements in $\bar{A}$.

We give two alternative ways to construct push-forwards for the projections. The first way is based on the fact that the cobordism ring MU is the coefficient ring of an universal formal group law. The second way uses no complicated facts and is based on residues theory. The push-forward to be constructed for the projection $p : X \times \mathbf{P}^n \to X$ will be temporarily (until 4.7.1) denoted by p_{quil}.

4.3.1. MU-approach. Consider the complex cobordism theory MU$(*)$ and the formal group low F_{MU} associated with this theory and its canonical Chern class for line bundles (Conner-Floyd class) [CF]. This low is defined over the ring $\mathrm{MU} = \mathrm{MU}(pt)$. According to a theorem of Quillen ([Qu1, Th.2]) F_{MU} is a universal commutative formal group low in one variable. This means that for any commutative ring R and any commutative formal group law in one variable F over R there exists a unique ring homomorphism $l_F : \mathrm{MU} \to R$ such that the coefficients of F coincide with the l_F-images of the corresponding coefficients of F_{MU}. For the theory A endowed with the orientation ω denote by

$$l_\omega : \mathrm{MU} \to \bar{A}^{uc} \tag{39}$$

the homomorphism l_F where $F = F_\omega$ is the formal group law associated with the orientation ω on A (3.9.3) and set

$$[\mathbf{P}^n]_\omega = l_\omega([\mathbb{C}P^n]),$$

where $[\mathbb{C}P^n]$ is the class of $\mathbb{C}P^n$ in MU.

For the projection $p : X \times \mathbf{P}^n \to X$ define the operator

$$p_{quil} : A(X \times \mathbf{P}^n) \to A(X) \tag{40}$$

as follows. Identify $A(X \times \mathbf{P}^n)$ with $A(X)[t]/(t^{n+1})$ taking t to the element $\zeta = c_1(\mathcal{O}(1))$ (see 3.3.1), consider the structural morphism $f : X \to pt$ and set p_{quil} to be the unique two-sided $A(X)$-module operator which takes the element t^i to the element $f^A([\mathbf{P}^{n-i}]_\omega) \in A(X)$ for $i = 0, 1, \ldots, n$.

By the very construction the operator $p_{quil} : A(X \times \mathbf{P}^n) \to A(X)$ is a *two-sided* $A(X)$-*module homomorphism.* In particular the composite operator $p_{quil} \circ p^A : A(X) \to A(X)$ coincides with the operator given by the cup-product with the element $p_{gys}(1)$:

$$p_{quil} \circ p^A = \cup p_{quil}(1) = \cup f^A([\mathbf{P}^{n-i}]_\omega) \tag{41}$$

The following properties of the operator p_{quil} can be proved before the theorem (4.7.1) and are useful when proving this theorem.

1. **Composition property:** the following diagram commutes

$$\begin{array}{ccc} A(X\times\mathbf{P}^n\times\mathbf{P}^m) & \xrightarrow{(\tilde{p}_n)_{quil}} & A(X\times\mathbf{P}^m) \\ {\scriptstyle (\tilde{p}_m)_{quil}}\downarrow & & \downarrow{\scriptstyle (p_m)_{quil}} \\ A(X\times\mathbf{P}^n) & \xrightarrow{(p_n)_{quil}} & A(X) \end{array}$$

where $p_n : X\times\mathbf{P}^n \to X$, $p_m : X\times\mathbf{P}^m \to X$, $\tilde{p}_n : X\times\mathbf{P}^n\times\mathbf{P}^m \to X\times\mathbf{P}^m$ and $\tilde{p}_m : X\times\mathbf{P}^n\times\mathbf{P}^m \to X\times\mathbf{P}^n$ are the natural projections.

2. **Base change property:** Let $\varphi : Y \to X$ be a morphism of smooth varieties and let $\tilde{\varphi}$ be the base change of φ by the natural projection $p_X : X\times\mathbf{P}^n \to X$. Then one has the relation $(p_Y)_{quil}\circ\tilde{\varphi}^A = \varphi^A\circ(p_X)_{quil}$ where $p_Y : Y\times\mathbf{P}^n \to Y$ is the natural projection.

3. **Consistence with linear imbeddings:** For a linear imbedding $i : \mathbf{P}^n \to \mathbf{P}^m$ the following diagram commutes

$$\begin{array}{ccc} A(X\times\mathbf{P}^n) & \xrightarrow{i_{gys}} & A(X\times\mathbf{P}^m) \\ {\scriptstyle (p_n)_{quil}}\downarrow & & \downarrow{\scriptstyle (p_m)_{quil}} \\ A(X) & \xrightarrow{id} & A(X) \end{array}$$

where $p_n : X\times\mathbf{P}^n \to X$ and $p_m : X\times\mathbf{P}^m \to X$ are the natural projections.

4. **Consistence with Gysin operators:** For a closed imbedding of smooth varieties $i : Y \hookrightarrow X$ the following diagram commutes

$$\begin{array}{ccc} A(Y\times\mathbf{P}^n) & \xrightarrow{(i\times id)_{gys}} & A(X\times\mathbf{P}^n) \\ {\scriptstyle (p_Y)_{quil}}\downarrow & & \downarrow{\scriptstyle (p_X)_{quil}} \\ A(Y) & \xrightarrow{i_{gys}} & A(X) \end{array}$$

where $p_X : X\times\mathbf{P}^n \to X$ and $p_Y : Y\times\mathbf{P}^n \to Y$ are the natural projections.

5. **Consistence with a section of trivial projective bundle:** For a section $s : X \to X\times\mathbf{P}^n$ of the projection $p : X\times\mathbf{P}^n \to X$ one has $p_{quil}\circ s_{gys} = id_{A(X)}$.

4.3.2. Approach by means of residues. Here we sketch another way to construct push-forwards for the projections. This way is based on an elementary residues theory and uses no complicated facts. Thus it could be applied in a more general situation. Now recall that for a commutative ring R and the ring of formal power series $R[[t]]$ in one variable t the ring of Laurent formal power series is defined as the localization of $R[[t]]$ at the element t: $R((t)) = R[[t]]_t$. Let $\Omega_{R[t]/R}$ be the module of Kehler differentials of $R[t]$ over R. Set $\Omega^{cont} = \Omega_{R[t]/R}\otimes_{R[t]} R[[t]]$ and $\Omega^{mer} = \Omega_{R[t]/R}\otimes_{R[t]} R((t))$. Since $\Omega_{R[t]/R}$

is a rank one free $R[t]$-module generated by the differential form dt we see that Ω^{cont} (respectively Ω^{mer}) is a rank one free $R[[t]]$-module (respectively a rank one free $R((t))$-module) generated by dt.Thus each element of Ω^{mer} can be uniquelly written in the form $\Sigma_{n=-N}^{\infty} a_n t^n dt$ for certain elements $a_n \in R$. The residue operator on Ω^{mer} is defined as an operator

$$\mathrm{Res}_{t=0} : \Omega^{mer} \to R \tag{42}$$

which takes a form $\Sigma_{n=-N}^{\infty} a_n t^n dt \in \Omega^{mer}$ to the element $a_{-1} \in R$. Let $\omega_{inv} \in \Omega^{cont}$ be the unique normalized F_A-invariant differential form (see ([Ha])). Define the element

$$[\mathbf{P}^n]_\omega \in \bar{A}^{uc} \tag{43}$$

as the coeficient at $t^{n-i}dt$ in $\omega_{inv}(t)$ of the form ω_{inv}. For the projection $p : \mathbf{P}^n \times X \to X$ define an operator

$$p_{quil} : A(\mathbf{P}^n \times X) = A(X)[\zeta]/(\zeta^{n+1}) \to A(X) \tag{44}$$

as a unique two-sided $A(X)$-module operator such that $p_{quil}(\zeta^r) = [\mathbf{P}^{n-r}]_\omega$ for each $r = 0, 1, \dots, n$. Using the residue operator (42) one can rewrite the operator p_{quil} for the projection $p : \mathbf{P}^n \times X \to X$ as follows

$$p_{quil}(a) = \mathrm{Res}_{t=0}(\frac{h}{t^{n+1}}\omega_{inv}) \tag{45}$$

where for an element $a \in A(\mathbf{P}^n \times X)$ we choose $h(t) \in A(X)[[t]]$ to be an arbitrary formal power series with $h(\zeta) = a \in A(\mathbf{P}^n \times X)$ and as above $\zeta = c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. Defined either by (44) or equivalently by (45) the operators p_{quil} are *two-sided* $A(X)$-module operators.

The operators p_{quil} defined by (40) and (45) coincide because by a theorem of Mischenko [**?**] the unique normalized F_{MU}-invariant differential form coincides with the form

$$\Sigma_{n=1}^{\infty} [\mathbf{P}^n]_{\mathrm{MU}} t^n dt$$

(it is just the differential form $dlog_{\mathrm{MU}}$).

The operators p_{quil} satisfy the properties 1 to 5 listed above. This is proved in Subsections 4.5 and 4.6 below in the text. Moreover the proofs of the properties 1 to 4 in the case of the residue approach are just the same as the proof of these properties in the case of the MU-approach (and it does not involve cobordism).

However if we take (45) as the definition of Quillen's operators then the proof of the property 5 is different of the proof of this property in the case of MU-Approach. And in fact it is purely algebraic and shorter (see the proof of Proposition 4.6.1).

4.3.3. Remark. *One concludes this subsection with the following observation: if $E^\vee = \oplus L_i$ is the direct sum of line bundles then one has (compare with [Qu1, Th.1])*

$$q_E = \prod_{i=1}^{d} (t -_F \lambda_i) \tag{46}$$

where $rkE = d$, $\lambda_i = c_1(L_i)$.

This relation can be proved but we skip this proof because this observation is never used in this text. (The definition of Quillen's operators via (45) allows to work easily in a relative situation (i.e. for varieties over a base)).

4.4 Proofs of Properties of the Gysin maps.

4.4.1. Identity to identity. The relation (34) is obvious because in this case $Y = X$ and $X_t = Y \times \mathbf{A}^1$ and $N = Y$ and $th^N_Y = id : A_Z(Y) \to A_Z(X)$.

4.4.2. Gysin sequence exact. This is clear, because the localization sequence

$$A_Y(X) \xrightarrow{j^A} A(X) \xrightarrow{k^A} A(X - Y)$$

for the pair $(X, X - Y)$ is exact and the operator $i_{th} : A(Y) \to A_Y(X)$ is an isomorphism and $i_{gys} = j^A \circ i_{th}$.

4.4.3. Two-sided projection formulae. In the deformation to the normal cone diagram 2.2.7 all morphisms are morphisms over X. Thus i_{gys} is a two-sided $A(X)$-module operator.

4.4.4. Base change property 4.2.2. The base change property for the Gysin maps i_{gys} follows from the one for the Thom maps i_{th}. We will prove it now.

Let $N = N_{X/Y}$ and $\tilde{N} = N_{\tilde{X}/\tilde{Y}}$. Since the square from 4.1.1 is transversal the canonical map $\tilde{Y} \times_Y N \to N$ is an isomorphism. We will identify $\tilde{Y} \times_Y N$ and N by means of this isomorphism and denote $\Phi : \tilde{Y} \times_Y N \to N$ the projection. The functoriality of the Thom class shows that $\Phi^A(th(N)) = th(\tilde{N})$. Therefore the diagram

$$\begin{tikzcd} A(\tilde{Y}) \arrow[r, "\cup th(\tilde{N})"] & A_{\tilde{Y}}(\tilde{N}) \\ A(Y) \arrow[u, "\tilde{\varphi}^A"] \arrow[r, "\cup th(N)"] & A_Y(N) \arrow[u, "\Phi^A"'] \end{tikzcd} \tag{47}$$

commutes. Since the square from 4.1.1 is transversal the map $\phi \times id : \tilde{X} \times \mathbf{A}^1 \to X \times \mathbf{A}^1$ gives rise to a commutative diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\tilde{z}"] \arrow[d, "\tilde{\varphi}"'] & \tilde{N} \arrow[r, "\tilde{i}_0"] \arrow[d, "\Phi"'] & \tilde{X}_t \arrow[d, "\varphi_t"'] & \tilde{X} \arrow[l, "\tilde{i}_1"'] \arrow[d, "\varphi"'] \\ Y \arrow[r, "z"] & N \arrow[r, "i_0"] & X_t & X \arrow[l, "i_1"'] \end{tikzcd}$$

in which rows coincide with the deformation to the normal cone diagrams for the pairs $(\tilde{X}, \tilde{Y})$ and (X, Y). The commutativity of the last diagram shows that the diagram consisting of pull-back operators commutes

$$\begin{tikzcd} A_{\tilde{Y}}(\tilde{N}) & A_{\tilde{Y} \times \mathbf{A}^1}(\tilde{X}_t) \arrow[l, "\tilde{i}_0^A"'] \arrow[r, "\tilde{i}_1^A"] & A_{\tilde{Y}}(\tilde{X}) \\ A_Y(N) \arrow[u, "\Phi^A"] & A_{Y \times \mathbf{A}^1}(X_t) \arrow[l, "i_0^A"'] \arrow[r, "i_1^A"] \arrow[u, "\varphi_t^A"] & A_Y(X) \arrow[u, "\varphi^A"] \end{tikzcd}$$

Gluwing this diagram with the commutative diagram (47) we get the commutativity of the diagram

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_{th}} & A_{\tilde{Y}}(\tilde{X}) \\ {\scriptstyle \tilde{\varphi}^A}\uparrow & & {\scriptstyle \varphi^A}\uparrow \\ A(Y) & \xrightarrow{i_{th}} & A_Y(X). \end{array}$$

The commutativity of the diagram (32) is proved. The property 4.2.2 follows.

4.4.5. Proof of the property 4.2.8. We consider here the zero section $z : Y \to E$ of a constant rank n vector bundle E and prove the relation $z_{th} = th^E_Y$. It suffices to prove that the composition

$$A_Y(E) \xrightarrow{(i_0^A)^{-1}} A_{Y\times\mathbf{A}^1}(E_t) \xrightarrow{i_1^A} A_Y(E)$$

is the identity map. For that we construct a morphism $q : E_t \to E$ such that

- the map q makes E_t into a line bundle over E and
- $q \circ i_1 = id$ and
- $q \circ i_0 = id$ and
- $q^{-1}(Y) = i_t(Y \times \mathbf{A}^1)$ in E_t,

where $i_t : Y \times \mathbf{A}^1 \hookrightarrow E_t$ is the imbedding from the deformation to the normal cone construction 2.2.7.

In this case one gets $q^{-1}(Y) = i_t(Y \times \mathbf{A}^1) \subset E_t$. Therefore the relations $i_0^A = (q^A)^{-1} = i_1^A$ holds by the strong homotopy invariance property 2.2.6 and we are done. It remains to construct the desired morphism $q : E_t \to E$.

Let F/Y be a vector bundle and let F' be the blow-up of F at the zero section. The variety F' coincides with the total space of the line bundle $\mathcal{O}_F(-1)$ over $\mathbf{P}(F)$. Let $q_F : F' \to \mathbf{P}(F)$ be projection of the line bundle to its base $\mathbf{P}(F)$.

If $F = E \oplus 1$ for a vector bundle E over Y then one has the following commutative diagram

$$\begin{array}{ccccccc} E' & \longrightarrow & F' & \longleftarrow & F' - E' = E_t & \longleftarrow & \mathbf{P}(1) \times \mathbf{A}^1 = Y \times \mathbf{A}^1 \\ {\scriptstyle q_E}\downarrow & & {\scriptstyle q_F}\downarrow & & {\scriptstyle q}\downarrow & & {\scriptstyle pr}\downarrow \\ \mathbf{P}(E) & \longrightarrow & \mathbf{P}(F) & \longleftarrow & \mathbf{P}(F) - \mathbf{P}(E) = E & \longleftarrow & \mathbf{P}(1) = Y. \end{array}$$

in which all the vertical arrows are the projections of the line bundles to their bases.

The projection q has two sections s_0 and s_1. The section s_0 is the zero section and the section s_1 is given by $x \mapsto (x, 1)$.

Observe that the variety $\mathbf{P}(F) - \mathbf{P}(E)$ coincides with E, the variety E_t coincides with the variety $F' - E'$, the imbedding $i_1 : E \hookrightarrow E_t$ coincides with the section $s_1 : E \hookrightarrow F' - E'$.

The normal bundle $N = N_{E/Y}$ to Y in E coincides with bundle E itself and the imbedding $i_0 : N \hookrightarrow E_t$ coincides with the section $s_0 : E \hookrightarrow F' - E'$. Finally the variety $Y \times \mathbf{A}^1$ coincides with $\mathbf{P}(1) \times \mathbf{A}^1$ and the imbedding $Y \times \mathbf{A}^1 \hookrightarrow E_t$ coincides with the imbedding $\mathbf{P}(1) \times \mathbf{A}^1 \hookrightarrow F' - E'$.

Thus the map $q : E_t \to E$ makes E_t into a line bundle over E and satisfies the relations $q \circ i_0 = id = q \circ i_1$, $q^{-1}(Y) = i_t(Y \times \mathbf{A}^1)$. Thus q is the desired map. The property is proved.

4.4.6. Proof of the relation (37). Since both sides of the relation (37) are $A(Y)$-linear it suffices to check the relation $s_{th}(1) = c_n(\mathcal{O}_F(1) \otimes p^*(E))$. Set $F = \mathbf{1} \oplus E$. Take the zero section z of the vector bundle E, the section s of the projective bundle $\mathbf{P}(F)$ which identifies Y with the closed subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(F)$ and consider the commutative diagram

$$\begin{array}{ccc} A(Y) & \xrightarrow{s_{th}} & A_{\mathbf{P}(1)}(\mathbf{P}(F)) \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle e^A} \\ A(Y) & \xrightarrow{z_{th}} & A_Y(E) \end{array}$$

where $e : E \to \mathbf{P}(\mathbf{1} \oplus E) = \mathbf{P}(F)$ is the open imbedding. As it was already proved $z_{th} = \cup th(E)$ and therefore $z_{th}(1) = th(E)$ in $A_Y(E)$. By the very definition of the Thom class one has $th(E) = e^A(c_n(\mathcal{O}_F(1) \otimes p^*(E)))$. Since the map e^A is an isomorphism one gets the desired relation $s_{th}(1) = c_n(\mathcal{O}_F(1) \otimes p^*(E))$. The property is proved.

4.4.7. Section of a projective bundle 38. Let F/Y be a rank m vector bundle and let $s : Y \to \mathbf{P}(F)$ be a section of the natural projection $p : \mathbf{P}(F) \to Y$. Consider the natural inclusion $\mathcal{O}_F(-1) \to p^*(F)$ and let Q be the factor-bundle $p^*(F)/\mathcal{O}_F(-1)$ and let $E = s^*(Q)$. We have to prove the relation $s_{gys}(1) = c_{m-1}(\mathcal{O}_F(1) \otimes p^*(E))$ in $A(\mathbf{P}(F))$.

Set $L = s^*(\mathcal{O}_F(-1))$. Then one has the obvious exact sequence of vector bundles on X

$$0 \to L \to F \to E \to 0.$$

If M is a line bundle over X then replacing F by $F \otimes M$ we does not change as the variety $\mathbf{P}(F)$ so the vector bundle $\mathcal{O}_F(1) \otimes p^*(E)$. Thus one may assume that L is the trivial line bundle.

By the slitting principle 3.5.1 one may assume further that $F = \mathbf{1} \oplus E$. In this case the relation $s_{gys}(1) = c_{m-1}(\mathcal{O}_F(1) \otimes p^*(E))$ is just the relation (37) which is proved just above ($n = m - 1$). The relation (38) is proved.

4.4.8. Proof of the relation (36). For a smooth divisor $i : D \hookrightarrow X$ one has to check the relation $i_{gys}(1) = c(L(D))$ in $A(X)$ (see 1.1 for notation). For that consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{P}(1 \oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle s}\uparrow & & {\scriptstyle I_t}\uparrow & & \uparrow{\scriptstyle i} \\ D & \xrightarrow{k_0} & D \times \mathbf{A}^1 & \xleftarrow{k_1} & D, \end{array}$$

from 2.2.11. Since both squares in this diagram are transversal the following diagram commutes

$$\begin{array}{ccccc} A(\mathbf{P}(1\oplus N)) & \xleftarrow{j_0^A} & A(X'_t) & \xrightarrow{j_1^A} & A(X) \\ {\scriptstyle s_{gys}}\uparrow & & {\scriptstyle (I_t)_{gys}}\uparrow & & \uparrow{\scriptstyle i_{gys}} \\ A(D) & \xleftarrow{k_0^A} & A(D\times \mathbf{A}^1) & \xrightarrow{k_1^A} & A(D), \end{array}$$

Now consider the line bundle $L_t = L(D \times \mathbf{A}^1)$ over X'_t (see the notation in 1.1). One can show that its restriction $j_0^*(L_t)$ to $\mathbf{P}(1 \oplus N)$ is isomorphic to the line bundle $\mathcal{O}_{1\oplus N}(1) \otimes p^*(N)$ and its restiction $j_1^*(L_t)$ to X is isomorphic to $L(D)$. Therefore one has the relation $j_0^A(c(L_t)) = \bar{t}h(N)$ in $A(\mathbf{P}(1 \oplus N))$ (see (21) for notation) and the relation $j_1^A(c(L_t)) = c(L(D))$ in $A(X)$. These two relations show that it remains to chech the relation

$$c(L_t) = I_{gys}(1) \tag{48}$$

in $A(X'_t)$.

To prove this relation observe that both elements vanish being restricted to the open subset $V_t = X' - D \times \mathbf{A}^1$. In fact this is clear for the element $c(L_t)$ and this holds for the element $(I_t)_{gys}(1)$ because the sequence (35) is a complex.

Now the relation (48) is proved as follows. The chain of relations in $A(\mathbf{P}(1 \oplus N))$

$$j_0^A(c(L_t)) = \bar{t}h(N) = s_{gys}(1) = s_{gys}(k_0^A(1)) = j_0^A((I_t)_{gys}(1)).$$

proves that $j_0^A(c(L_t)) = j_0^A((I_t)_{gys}(1))$. Since both elements $c(L_t)$ and $(I_t)_{gys}(1)$ vanish being restricted to the open subset V_t the relation (48) follows from Lemma 2.2.12.

The proof of the relation $i_{gys}(1) = c(L(D))$ in $A(X)$ is completed.

4.4.9. Proof of the additivity property (33) The proof of this property will be given just after three preliminary Lemmas 4.4.10, 4.4.11 and **??**. We beging with the following particular case.

4.4.10. Lemma. *Let $s = s_1 \amalg s_2 : S = S_1 \amalg S_2 \hookrightarrow T_1 \amalg T_2 = T$ be a closed imbedding of smooth varieties. Let $i_m : S_1 \hookrightarrow S$ and $j_m : T_m \hookrightarrow T$ be the open inclusions and let $r_m = j_m \circ s_m : S_m \hookrightarrow T$. Then*

$$s_A = (r_1)_A \circ i_1^A + (r_2)_A \circ i_2^A$$

To prove this Lemma consider the isomorphism $(j_1^A, j_2^A) : A(T) \to A(T_1) \oplus A(T_2)$. To prove the Lemma it suffices to check the following relations

$$j_1^A \circ s_A = j_1^A \circ (r_1)_A \circ i_1^A + j_1^A \circ (r_2)_A \circ i_2^A \tag{49}$$

$$j_2^A \circ s_A = j_2^A \circ (r_1)_A \circ i_1^A + j_2^A \circ (r_2)_A \circ i_2^A \tag{50}$$

The transversal square

$$\begin{tikzcd} T_1 \arrow[r, "j_1"] & T \\ S_1 \arrow[u, "s_1"] \arrow[r, "i_1"] & S \arrow[u, "s"'] \end{tikzcd}$$

prove relations $j_1^A \circ s_A = (s_1)_A \circ i_1^A$. The transversal square

$$\begin{tikzcd} T_1 \arrow[r, "j_1"] & T \\ S_1 \arrow[u, "s_1"] \arrow[r, "id"] & S_1 \arrow[u, "r_1"'] \end{tikzcd}$$

proves the relations $j_1^A \circ (r_1)_A = (s_1)_A$. The transversal square

$$\begin{tikzcd} T_1 \arrow[r, "j_1"] & T \\ \emptyset \arrow[u] \arrow[r] & S_2 \arrow[u, "r_2"'] \end{tikzcd}$$

proves the relation $j_1^A \circ (r_2)_A = 0$.

Now combining the last three relations one gets the chain of relations

$$j_1^A \circ (r_1)_A \circ i_1^A + j_1^A \circ (r_2)_A \circ i_2^A = (s_1)_A \circ i_1^A = j_1^A \circ s_A.$$

which proves the relation (49). The relation (50) is proved in a similar way. The Lemma is proved.

Now suppose we are given with varieties V and $U = U_1 \amalg U_2$ and a transversal square of the form

$$\begin{tikzcd} T_1 \amalg T_2 \arrow[r, "v"] & V \\ S_1 \amalg S_2 \arrow[u, "s=s_1 \amalg s_2"] \arrow[r, "u=u_1 \amalg u_2"] & U_1 \amalg U_2 \arrow[u, "t"'] \end{tikzcd}$$

4.4.11. Lemma. *Let $k_m : U_m \hookrightarrow U$ be the open inclusion and let $t_m = t \circ k_m : U_m \hookrightarrow V$. Then*

$$v^A \circ t_A = v^A \circ [(t_1)_A \circ k_1^A + (t_2)_A \circ k_2^A]$$

To prove this Lemma observe that the following squares are transversal $(m = 1, 2)$

$$\begin{tikzcd} T_1 \amalg T_2 \arrow[r, "v"] & V \\ S_m \arrow[u, "r_m"] \arrow[r, "u_m"] & U_m \arrow[u, "t_m"'] \end{tikzcd}$$

Thus $v^A \circ (t_m)_A = (r_m)_A \circ u^A_m$ for $m = 1, 2$ and one gets the following chain of relations using Lemma 4.4.10 in the last step

$$v^A \circ [(t_1)_A \circ k^A_1 + (t_2)_A \circ k^A_2 = (r_1)_A \circ u^A_1 \circ k^A_1 + (r_2)_A \circ u^A_2 \circ k^A_2 =$$

$$(r_1)_A \circ i^A_1 \circ u^A + (r_2)_A \circ i^A_2 \circ u^A = s_A \circ u^A.$$

Finally $s_A \circ u^A = v^A \circ t_A$ by the transversality of the diagram considered in the Lemma. The Lemma follows.

Now consider the deformation to the normal cone diagram from the subsection 2.2.7

$$\begin{tikzcd} \mathbf{P}(\mathbf{1} \oplus N) \arrow[r, "j_0"] & X'_t & X \arrow[l, hook', "j_1"'] \\ Y \arrow[u, "s"] \arrow[r, "k_0"] & Y \times \mathbf{A}^1 \arrow[u, "i_t"] & Y. \arrow[l, hook', "k_1"'] \arrow[u, "i"'] \end{tikzcd}$$

Since $Y = Y_1 \amalg Y_2$ one see that $Y \times \mathbf{A}^1 = Y_1 \times \mathbf{A}^1 \amalg Y_2 \times \mathbf{A}^1$ and $\mathbf{P}(\mathbf{1} \oplus N) = \mathbf{P}(\mathbf{1} \oplus N_1) \amalg \mathbf{P}(\mathbf{1} \oplus N_2)$ where N_i is the normal bundle of X to Y_i. Let $l_m : Y_m \times \mathbf{A}^1 \hookrightarrow Y \times \mathbf{A}^1$ be the inclusion for $m = 1, 2$ and let $i^m_t = i_t \circ l_m$. Since the left hand side square in the very last diagram is transversal Lemma 4.4.11 gives the following relation

$$j^A_0 \circ (i_t)_A = j^A_0 \circ [(i^1_t)_A \circ l^A_1 + (i^2_t)_A \circ l^A_2]. \tag{51}$$

To complete the proof of the Proposition we need in the following

4.4.12. Lemma. *Let $j_t : X'_t - Y \times \mathbf{A}^1 \hookrightarrow X'_t$ be the open inclusion. Then $j^A_t \circ (i_t)_A = 0$ and $j^A_t \circ (i^m_t)_A = 0$ for $m = 1, 2$.*

In fact, the first relation follows from the localization property 4.1.2. The remaning relations for $m = 1, 2$ follows from the first one because $(i^m_t)_A = (i_t)_A \circ (l_m)_A$.

By this Lemma $j^A_t \circ [(i_t)_A - (i^1_t)_A \circ l^A_1 - (i^2_t)_A \circ l^A_2] = 0$. By relation (51) one has the relation $j^A_0 \circ [(i_t)_A - (i^1_t)_A \circ l^A_1 - (i^2_t)_A \circ l^A_2] = 0$. Thus $[(i_t)_A - (i^1_t)_A \circ l^A_1 - (i^2_t)_A \circ l^A_2] = 0$ by Lemma 2.2.12 and

$$(i_t)_A = (i^1_t)_A \circ l^A_1 + (i^2_t)_A \circ l^A_2.$$

Since the right hand side square from the very last diagram is transversal one gets the relation

$$j^A_1 \circ (i_t)_A = i_A \circ k^A_1.$$

Since $k_1 = k_{11} \amalg k_{12} : Y_1 \amalg Y_2 \hookrightarrow Y_1 \times \mathbf{A}^1 \amalg Y_2 \times \mathbf{A}^1$ the squares

$$\begin{tikzcd} X'_t & X \arrow[l, hook', "j_1"'] \\ Y_m \times \mathbf{A}^1 \arrow[u, "i^{(m)}_t"] & Y_m. \arrow[l, hook', "k_{1m}"'] \arrow[u, "i_m"'] \end{tikzcd}$$

are transversal. Thus one has relations

$$j^A_1 \circ (i^m_t)_A = (i_m)_A \circ k^A_{1m}$$

for $m = 1, 2$.

Recall that $u_m : Y_m \hookrightarrow Y$ for $m = 1, 2$ are the inclusions and $i_m = i \circ u_m$. Combining last relations one gets a chain of relations

$$i_A \circ k_1^A = j_1^A \circ (i_t)_A = j_1^A \circ [(i_t^1)_A \circ l_1^A + (i_t^2)_A \circ l_2^A] = (i_1)_A \circ k_{11}^A \circ l_1^A + (i_2)_A \circ k_{12}^A \circ l_2^A =$$

$$(i_1)_A \circ u_1^A \circ k_1^A + (i_2)_A \circ u_2^A \circ k_1^A =$$

which proves the following one

$$i_A \circ k_1^A = [(i_1)_A \circ u_1^A + (i_2)_A \circ u_2^A] \circ k_1^A$$

Since the operator k_1^A is an isomorphism the additivity relation (33) follows.

4.4.13. Proof of the relation (31). We beging with proving the following generalisation of the smooth divisor case relation (36). Namely, Let $Y = \cap_{i=1}^n D_i$ be a transversal intersection of n smooth divisors D_r $(r = 1, 2, \dots, n)$ in X. Denoting $i : Y \hookrightarrow X$ the closed imbedding one has

$$i_A(1) = c_n(\oplus_{r=1}^n L_r), \tag{52}$$

where $L_r = L(D_r)$ for $(r = 1, 2, \dots, n)$. To prove this relation consider the commutative diagram

$$\begin{array}{ccccc} \mathbf{P}(1 \oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle s}\uparrow & & {\scriptstyle I_t}\uparrow & & \uparrow{\scriptstyle i} \\ Y & \xrightarrow{k_0} & Y \times \mathbf{A}^1 & \xleftarrow{k_1} & Y \end{array}$$

from 2.2.11 for the imbedding $i : Y \hookrightarrow X$. Observe that the normal bundle $N = N_{X/Y}$ to Y in X is the direct sum of the line bundles $L_r|Y$. Let $(D_r)'_t$ be the proper preimage of the divisor $D_r \times \mathbf{A}^1$ under the blow-up map $\sigma : X'_t \to X \times \mathbf{A}^1$. Clearly the subvariety $Y \times \mathbf{A}^1$ in X'_t is a complete intersection of the smooth divisors $(D_r)'_t$. Furthermore the divisor $(D_r)'_t$ intersect in X'_t the divisor $\mathbf{P}(\mathbf{1} \oplus N)$ transversally and their intersection coincides with the smooth divisor $P_r = \mathbf{P}(\mathbf{1} \oplus E_r)$ in $\mathbf{P}(\mathbf{1} \oplus N)$ where E_r is a direct summund of N which is the direct sum of all $L_s|Y$ except of the exactly $L_r|Y$. Clearly the line bundle $L(P_r)$ on $\mathbf{P}(1 \oplus N)$ is isomorphic to the line bundle $p^*(L_r|Y) \otimes \mathcal{O}_{\mathbf{1}\oplus N}(1)$. Thus $c_n(p^*(N) \otimes \mathcal{O}_{\mathbf{1}\oplus N}(1))$ coincides with the cup-product of classes $c_1(p^*(L_r|Y) \otimes \mathcal{O}_{\mathbf{1}\oplus N}(1))$.

Now consider the line bundle $(L_r)_t = L((D_r)'_t)$ over X'_t (see the notation in 1.1). The line bundle $j_0^*((L_r)_t)$ over $\mathbf{P}(\mathbf{1} \oplus N)$ is isomorphic to the line bundle $L(P_r)$ and thus it is isomorphic to the line bundle $\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(L_r|Y)$. The line bundle $j_1^*((L_r)_t)$ over X is isomorphic to L_r. Therefore one has the relation $j_0^A(c_1((L_r)_t)) = c_1(\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(L_r|Y))$ in $A(\mathbf{P}(1 \oplus N))$ (see 21 for notation) and the relation $j_1^A(c_1((L_r)_t)) = c_1(L_r)$ in $A(X)$. Recall that by (37) $s_A(1) = c_n(\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(N)$ and thus $s_A(1) = \cup_{r=1}^n c_1(\mathcal{O}_{\mathbf{1}\oplus N}(1) \otimes p^*(L_r|Y))$.

As an intermidiate step in the proof of the relation (52) we need to check the one

$$\cup_{r=1}^n c_1((L_r)_t) = I_{gys}(1) \tag{53}$$

in $A(X'_t)$.

To prove this relation observe that both elements vanish being restricted to the open subset $V_t = X'_t - Y \times \mathbf{A}^1$. In fact for every integer r the class $c_1((L_r)_t)$ vanishes on the complement to the divisor $(D_r)'_t$ and thus this class comes from the group $A_{(D_r)'_t}(X'_t)$ via the support extension operator.. Therefore the cup-product of the elements $c_1((L_r)_t)$ comes from the group $A_{Y\times\mathbf{A}^1}(X'_t)$ via the support extension operator. whence it this cup-product vanishes on V_t. The element $(I_t)_{gys}(1)$ comes from the group $A_{Y\times\mathbf{A}^1}(X'_t)$ via the support extension operator by the very definition of the Gysin operator. Now both elements $\cup_{r=1}^n c_1((L_r)_t)$ and $(I_t)_{gys}(1)$ vanishes on V_t by the localization property 2.0.1.

Now the relation (53) is proved as follows. The chain of relations in $A(\mathbf{P}(1\oplus N))$

$$j_0^A(\cup_{r=1}^n c_1((L_r)_t)) = \cup_{r=1}^n c_1(\mathcal{O}_{\mathbf{1}\oplus N}(1)\otimes p^*(L_r|Y)) = s_{gys}(1) = s_{gys}(k_0^A(1)) = j_0^A((I_t)_{gys}(1)).$$

proves that $j_0^A(\cup_{r=1}^n c_1((L_r)_t)) = j_0^A((I_t)_{gys}(1))$. Since both elements $\cup_{r=1}^n c_1((L_r)_t)$ and $(I_t)_{gys}(1)$ vanish being restricted to the open subset V_t the relation (53) follows from Lemma 3.3.10. The chain of relations in $A(X)$

$$\cup_{r=1}^n c_1(L_r) = j_1^A(\cup_{r=1}^n c_1((L_r)_t)) = j_1^A((I_t)_{gys}(1)) = i_{gys}(k_0^A(1)) = i_{gys}(1)$$

completed the proof of the relation (52).

The next step in the proof of the composition property is to prove the following relation. Let Y be a smooth variety and let $N \subset M$ be an imbedding of vector bundles on Y such that the quotient $F := M/N$ is a vector bundle as well. Suppose F is of a constant rank, say d. Let $l : \mathbf{P}(\mathbf{1}\oplus N) \hookrightarrow \mathbf{P}(\mathbf{1}\oplus M)$ be the obvious closed imbedding and let $p : \mathbf{P}(\mathbf{1}\oplus M) \to Y$ be the projection. Then one has the relation in $A(\mathbf{P}(\mathbf{1}\oplus M))$

$$l_A(1) = c_d(p^*(F)\otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)). \tag{54}$$

In fact, by Splitting principle 3.5.1 one may assume that the vector bundle F splits in a direct sum of line bundles $F = \oplus_{r=1}^m L_r$. Let F_r is a direct summund of F which is the direct sum of all line bundles L_s except of L_r. Let $M_r \subset M$ be the preimage of F_r under the canonical projection $M \to F$. Let $D_r = \mathbf{P}(\mathbf{1}\oplus M_r)$ be the divisor on $\mathbf{P}(\mathbf{1}\oplus M)$. Clearly the subvariety $\mathbf{P}(\mathbf{1}\oplus N)$ in $\mathbf{P}(\mathbf{1}\oplus M)$ is the transversal intersection of the smooth divisors D_r. And furthermore the line bundle $L(D_r)$ on $\mathbf{P}(\mathbf{1}\oplus M)$ is isomorphic to the line bundle $p^*(L_r)\otimes\mathcal{O}_{\mathbf{1}\oplus M}(1)$. The relation (52) shows that one has a chain of relations in $A(\mathbf{P}(\mathbf{1}\oplus M))$

$$l_A(1) = \cup_{r=1}^d c_1(L(D_r)) = \cup_{r=1}^d c_1(p^*(L_r)\otimes\mathcal{O}_{\mathbf{1}\oplus M}(1)) =$$

$$= c_d((\oplus_{r=1}^d p^*(L_r))\otimes\mathcal{O}_{\mathbf{1}\oplus M}(1)) = c_d(p^*(F)\otimes\mathcal{O}_{\mathbf{1}\oplus M}(1)).$$

The relation (54) is proved.

Now we are ready to prove the composition property in one particular case. Let $s_N : Y \to \mathbf{P}(\mathbf{1}\oplus N)$ be the section of the projective bundle $p_N : \mathbf{P}(\mathbf{1}\oplus E) \to Y$ identifying Y with the closed subvariety $\mathbf{P}(\mathbf{1})$. Let $s = l\circ s_E : Y \to \mathbf{P}(\mathbf{1}\oplus M)$. Clearly s

is a section of the projection $p : \mathbf{P}(\mathbf{1} \oplus M) \to Y$ identifying Y with the closed subvariety $\mathbf{P}(\mathbf{1})$. Assume that the variety Y is irreducible. We claim that

$$s_A = l_A \circ (s_N)_A. \tag{55}$$

In fact, both sides are two-sided $A(Y)$-module operators. Thus to prove this relation it suffices to prove that $s_A(1) = l_A(s_N)_A(1)$. Since Y is irreducible all vector bundles on Y are of constant rank. Let $n = rk(N)$. The relations (37) and (54) gives rise to a chain of relations in $A(\mathbf{P}(\mathbf{1} \oplus M)$

$$l_A[(s_N)_A(1)] = l_A[c_n(p_N^*(N) \otimes \mathcal{O}_{\mathbf{1}\oplus N}(1))] = l_A[l^A(c_n(p^*(N) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)))] =$$
$$= l_A(1) \cup c_n(p^*(N) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) = c_d(p^*(F) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) \cup c_n(p^*(N) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) =$$
$$= c_{d+n}(p^*(M) \otimes \mathcal{O}_{\mathbf{1}\oplus M}(1)) = s_A(1)$$

which proves the relation (55).

Now we are ready to prove the composition property. For that consider closed imbeddings of smooth varieties $j : Z \hookrightarrow Y$ and $i : Y \hookrightarrow X$. By the additivity property of the Gysin operators (33) one may assume that the variety Z is irreducible. Consider the diagram from 2.2.11 for the closed imbedding $j \circ i : Z \hookrightarrow X$.

$$\begin{tikzcd}
\mathbf{P}(\mathbf{1} \oplus N_{X/Z}) \arrow[r, "j_0"] & X'_t & X \arrow[l, "j_1"'] \\
Z \arrow[u, "s"] \arrow[r, "k_0"] & Z \times \mathbf{A}^1 \arrow[u, "(I\circ J)_t"] & Z. \arrow[l, "k_1"'] \arrow[u, "i\circ j"']
\end{tikzcd}$$

The proper preimage Y'_t of $Y \times \mathbf{A}^1$ under the blow-up map $\sigma : X'_t \to X \times \mathbf{A}^1$ and the projective bundle $\mathbf{P}(\mathbf{1} \oplus N_{Y/Z})$ fit in the this diagram making a big commutative diagram with transversal squares

$$\begin{tikzcd}
\mathbf{P}(\mathbf{1} \oplus N_{X/Z}) \arrow[r, "i_0"] & X'_t & X \arrow[l, "i_1"'] \\
\mathbf{P}(\mathbf{1} \oplus N_{Y/Z}) \arrow[u, "l"'] \arrow[r, "i_0^Y"] & Y'_t \arrow[u, "l_t"'] & Y \arrow[l, "i_1^Y"'] \arrow[u, "i"'] \\
Z \arrow[u, "s_Y"'] \arrow[r, "k_0"] & Z \times \mathbf{A}^1 \arrow[u, "J_t"'] & Z. \arrow[l, "k_1"'] \arrow[u, "j"']
\end{tikzcd}$$

in which the bottom and the middle rows form the deformation to the normal cone diagram for the closed imbedding $j : Z \hookrightarrow Y$ and $l \circ s_Y = s$, $l_t \circ J_t = (I \circ J)_t$. Using the transversality of the left hand side squares in the last diagram one gets a chain of relations

$$(l_A \circ (s_Y)_A) \circ k_0^A = l_A \circ (i_0^Y)^A \circ (J_t)_A = i^A \circ (l_t)_A \circ (J_t)_A$$

and

$$s_A \circ k_0^A = i_0^A \circ ((I \circ J)_t)_A = i_0^A \circ (l_t \circ J_t)_A$$

We already proved the relation

$$l_A \circ (s_Y)_A = s_A : A(Z) \to A(\mathbf{P}(\mathbf{1} \oplus N_{X/Z})).$$

Thus one has the relation

$$i_0^A \circ [(l_t \circ J_t)_A - (l_t)_A \circ (J_t)_A] = 0$$

Observe that for $V_t = X'_t - Z \times \mathbf{A}^1$ and the open inclusion $j_t : V_t \hookrightarrow X'_t$ the compositions $j_t^A \circ (l_t \circ J_t)_A$ and $j_t^A \circ (l_t)_A \circ (J_t)_A$ vanish. Now the useful Lemma 3.3.10 implies the relation

$$(l_t \circ J_t)_A - (l_t)_A \circ (J_t)_A = 0.$$

The trasversality of the two right hand side squares from the last diagram gives rise to the chain of relations

$$(i \circ j)_A \circ k_1^A = i_1^A \circ (l_t \circ J_t)_A = i_1^A \circ (l_t)_A \circ (J_t)_A =$$

$$= (i_A \circ (i_1^Y)^A) \circ (J_t)_A = i_A \circ (j_A \circ k_1^A) = (i_A \circ j_A) \circ k_1^A.$$

The operator k_1^A is an isomorphism by the homotopy invariance property. The required relation

$$(i \circ j)_A = i_A \circ j_A$$

follows.

4.5 Proofs of Properties of the Quillen's maps $(p_X)_{quil}$.

In this Subsection we prove properties of Quillen's operators stated in Subsection 4.3. The projection formular (41) holds by the very definition of the operator $(p_X)_{quil}$. The base change property 2 is obvious: in fact, take $M = A(X)$, $N = A(Y)$, $\phi = \varphi^A : M \to N$ and apply the final observation from the Remark **??**.

To prove the property 1 observe that all the maps in the diagram are two-sided $A(X)$-module maps. Thus it suffices to check the relations

$$((p_m)_{quil} \circ (\tilde{p}_n)_{quil})(\zeta_1^i \otimes \zeta_2^j) = ((p_n)_{quil} \circ (\tilde{p}_m)_{quil})(\zeta_1^i \otimes \zeta_2^j)$$

where $\zeta_1 \in A(X \times \mathbf{P}^n \times \mathbf{P}^m)$ is the pull-back of the element $c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n)$ under the projection $X \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^n$ and $\zeta_2 \in A(X \times \mathbf{P}^n \times \mathbf{P}^m)$ is the pull-back of the element $c_1(\mathcal{O}(1)) \in A(\mathbf{P}^m)$ under the projection $X \times \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^m$. Let $f : X \to pt$ be the structural morphism. Both sides of the relation coincide with the element $f^A([\mathbf{P}^{n-i}]_\omega \cup [\mathbf{P}^{m-j}]_\omega) \in A(X)$ as follows from the projection formula (41) and the base change property 2. The property 1 is proved.

Now prove the linear imbedding property 3. It suffices to check the case $n = m - 1$ because of the compatibility of the Gysin homomorphisms with the composition of closed imbeddings (31). Since all the homomorphisms in the considered diagram are two-sided $A(X)$-module maps one may assume $X = pt$. Let $\zeta_m = c_1(\mathcal{O}(1)) \in A(\mathbf{P}^m)$ and $\zeta_n =$

$c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. The projection formular for the Gysin maps (30) and the relation (36) prove the relations $i_{gys}(\zeta_n^r) = \zeta_m^{r+1})$ in $A(\mathbf{P}^m)$. Now the chain of relations

$$(p_m)_{quil}(i_{gys}(\zeta_n^r)) = (p_m)_{quil}(\zeta_m^{r+1}) = [\mathbf{P}^{m-r-1}]_\omega = [\mathbf{P}^{n-r}]_\omega = (p_n)_{quil}(\zeta_n^r)$$

completes the proof of the property 3.

Now prove the property 4. Consider the deformation to the normal cone diagrams (see 2.2.11) as for the closed imbedding $i : Y \hookrightarrow X$ so for the closed imbedding $i : \mathbf{P}^n \times Y \hookrightarrow \mathbf{P}^n \times X$. Put them together in the following diagram in which all the vertical arrows are of the form $p_{quil} : A(\mathbf{P}^n \times S) \to A(S)$ for various varieties S

$$\begin{tikzcd}
 & A(\mathbf{P}^n \times \mathbf{P}(\mathbf{1} \oplus N)) \arrow[d, "p_{quil}^{\mathbf{P}(\mathbf{1}\oplus N)}"] & & A(\mathbf{P}^n \times X'_t) \arrow[ll, "j_{0PX}^A"'] \arrow[rr, "j_{1PX}^A"] \arrow[d, "p_{quil}^{X'_t}"] & & A(\mathbf{P}^n \times X) \arrow[d, "p_{quil}^X"] \\
 & A(\mathbf{P}(\mathbf{1} \oplus N)) & & A(X'_t) \arrow[ll] \arrow[rr] & & A(X) \\
A(\mathbf{P}^n \times Y) \arrow[uur, "s_{gys}^P"] \arrow[d, "p_{quil}^Y"'] & & A(\mathbf{P}^n \times Y \times \mathbf{A}^1) \arrow[ll, "k_{0PY}^A"] \arrow[rr, "k_{1PY}^A"'] \arrow[uur, "I_{gys}^{Pt}"] \arrow[d, "p_{quil}^{Yt}"'] & & A(\mathbf{P}^n \times Y) \arrow[uur, "i_{gys}^P"] \arrow[d, "p_{quil}^Y"'] & \\
A(Y) \arrow[uur] & & A(Y \times \mathbf{A}^1) \arrow[ll, "k_{0Y}^A"'] \arrow[rr, "k_{1Y}^A"] \arrow[uur] & & A(Y) \arrow[uur, "i_{gys}"'] &
\end{tikzcd}$$

What do we need to prove is that the right hand side parallelogram commutes, i.e. we need to prove the relation $i_{gys} \circ p_{quil}^Y = p_{quil}^X \circ i_{gys}^P$. We first prove that the left hand side parallelogram commutes. Then using the Useful Lemma we check that the parallelogram in the middle commutes. And finally we conclude that the right hand side parallelogram commutes.

To check the commutativity of the left hand side parallelogram, i.e. to check the relation $s_{gys} \circ p_{quil}^Y = p_{quil}^{\mathbf{P}(\mathbf{1}\oplus N)} \circ s_{gys}^P$, recall that $s_{gys} = \cup \bar{t}h(N) \circ (p^Y)^A$ with N equal to the normal bundle $N_{X/Y}$ and $s_{gys}^P = \cup \bar{t}h(N_{\mathbf{P}^n \times X/\mathbf{P}^n \times Y}) \circ (id_{\mathbf{P}^n} \times p^Y)^A$. Clearly $N_{\mathbf{P}^n \times X/\mathbf{P}^n \times Y} = (p^Y)^*(N_{X/Y})$. Let $q : \mathbf{P}(\mathbf{1} \oplus N) \to Y$ be the projection. Then for any element $a \in A(\mathbf{P}^n \times Y)$ one has a chain of relations (here we write for short P for $p^{\mathbf{P}(\mathbf{1}\oplus N)}$)

$$P_{quil}[(id \times q)^A(a) \cup (P^A(\bar{t}h(N))] = P_{quil}[((id \times q)^A(a))] \cup \bar{t}h(N) = q^A(p_{quil}^Y(a)) \cup \bar{t}h(N)$$

which proves the relation $s_{gys} \circ p_{quil}^Y = P_{quil} \circ s_{gys}^P$.

Recall that $I_t : Y \times \mathbf{A}^1 \hookrightarrow X'_t$ and $I^{Pt} : \mathbf{P}^n \times Y \times \mathbf{A}^1 \hookrightarrow \mathbf{P}^n \times X'_t$ are the closed imbeddings for the deformation to the normal cone diagrams for the closed imbeddings $Y \hookrightarrow X$ and $\mathbf{P}^n \times Y \hookrightarrow \mathbf{P}^n \times X$. Now we prove the relation $(I_t)_{gys} \circ p_{quil}^{Yt} = p_{quil}^{X'_t} \circ I_{gys}^{Pt}$. To do this use first the base change property for Gysin and Quillens operators to get a chain of relations

$$i_0^A \circ (I_t)_{gys} \circ p_{quil}^{Yt} = s_{gys} \circ k_{0Y}^A \circ p_{quil}^{Yt} = s_{gys} \circ p_{quil}^Y \circ k_{0PY}^A = P_{quil} \circ s_{gys}^P \circ k_{0PY}^A =$$

$$= P_{quil} \circ j_{0PX}^A \circ I_{gys}^{Pt} = i_0^A \circ p_{quil}^{X'_t} \circ I_{gys}^{Pt}.$$

Let $V_t = X'_t - Y \times \mathbf{A}^1_t$ and $j_t : V_t \hookrightarrow X'_t$ be the open inclusion. Clearly $j_t^A \circ (I_t)_{gys} \circ p^{Yt}_{quil} = 0$ and $j_t^A \circ p^{X'_t}_{quil} \circ I^{Pt}_{gys} = 0$. Now the Useful Lemma 3.3.10 implies the relation $(I_t)_{gys} \circ p^{Yt}_{quil} = p^{X'_t}_{quil} \circ I^{Pt}_{gys}$.

We are ready to prove the required relation $i_{gys} \circ p^Y_{quil} = p^X_{quil} \circ i^P_{gys}$. To do this use first the base change property for Gysin and Quillens operators to get a chain of relations (here $j_{1X} : X \hookrightarrow X'_t$ and $j_{1PX} : \mathbf{P}^n \times X \hookrightarrow \mathbf{P}^n \times X'_t$ are the imbeddings from the deformation to the normal cone constructions)

$$i_{gys} \circ p^Y_{quil} \circ k^A_{1PY} = i_{gys} \circ k^A_{1Y} \circ p^{Yt}_{quil} = j^A_{1X} \circ (I_t)_{gys} \circ p^{Yt}_{quil} = j^A_{1X} \circ p^{X'_t}_{quil} \circ I^{Pt}_{gys} =$$

$$= p^X_{quil} \circ j^A_{1PX} \circ I^{Pt}_{gys} = p^X_{quil} \circ i^P_{gys} \circ k^A_{1PY}.$$

The operator k^A_{1PY} is an isomorphism by the homotopy invariance property. Thus we get the required relation $i_{gys} \circ p^Y_{quil} = p^X_{quil} \circ i^P_{gys}$.

The property 5 is proved in the next subsection.

4.6 Proof of the property 5 of Quillen's operator.

Let X be a smooth variety and let $p : X \times \mathbf{P}^n \to X$ be the projection and let $s : X \to X \times \mathbf{P}^n$ be a section of the projection p. One has to prove the relation

$$p_{quil} \circ s_{gys} = id_{A(X)} \tag{56}$$

Since both operators s_{gys} and p_{quil} are two-sided $A(X)$-module homomorphisms it suffices to check the relation in $A(X)$

$$p_{quil}(s_{gys}(1)) = 1. \tag{57}$$

Since s is a section of p it has a form (id_X, f) for a morphism $f : X \to \mathbf{P}^n$. Consider two diagrams

$$\begin{tikzcd} \mathbf{P}^n \times \mathbf{P}^n \arrow[d, "p_1"'] & X \times \mathbf{P}^n \arrow[l, "f \times id"'] \arrow[d, "p"] \\ \mathbf{P}^n & X \arrow[l, "f"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{P}^n \times \mathbf{P}^n & X \times \mathbf{P}^n \arrow[l, "f \times id"'] \\ \mathbf{P}^n \arrow[u, "\Delta"] & X, \arrow[l, "f"'] \arrow[u, "s"'] \end{tikzcd}$$

where Δ is the diagonal imbedding. Both these diagrams are Cartesian and transversal. The second diagram shows that $s_{gys} \circ f^A = (f \times id)^A \circ \Delta_{gys}$. In particular one has the relation

$$s_{gys}(1) = (f \times id)^A(\Delta_{gys}(1)). \tag{58}$$

Now this relation, the base change property for transversal squares and the Proposition stated just below give a chain of relations

$$p_{quil}(s_{gys}(1)) = p_{quil}((f \times id)^A(\Delta_{gys}(1))) = f^A((p_1)_{quil}(\Delta_{gys}(1))) = f^A(1) = 1$$

It remains to prove the following

4.6.1. Proposition. *One has the relation $(p_1)_{quil}(\Delta_{gys}(1)) = 1$ in the group $A(\mathbf{P}^n)$.*

Proof using the Residue approach. In this proof we assume that Quillen's operators p_{quil} are defined via (45). To compute $(p_1)_{quil}(\Delta_{gys}(1))$ in terms of (45) observe first that

$$\Delta_{gys}(1) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{59}$$

In fact this is just the relation (38) for the case $Y = \mathbf{P}^n$, $E = 1^{n+1}$, $s = \Delta$, $p = p_1$. To proceed further one needs to find a formal power series $h(t) \in A^{uc}(\mathbf{P}^n)[[t]]$ such that $h(\zeta) = \Delta_{gys}(1)$, where as usual $\zeta = c_1(\mathcal{O}(1)) \in A(\mathbf{P}^n)$.

For that consider the two projections $q_1 : \mathbf{P}^\infty \times \mathbf{P}^n \to \mathbf{P}^\infty$ and $q_2 : \mathbf{P}^\infty \times \mathbf{P}^n \to \mathbf{P}^n$, and the projections $p_r : \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^n$ for $r = 1, 2$. Set $R = A(\mathbf{P}^n)$ and identify the formal power series in one variable $R[[t]]$ with the ring $A(\mathbf{P}^\infty \times \mathbf{P}^n)$ taking the variable t to the element $c_1(q_1^*(\mathcal{O}(1)))$. This identification induces a ring isomorphism $R[[t]]/(t^{n+1})$ and $A(\mathbf{P}^n \times \mathbf{P}^n)$ which takes the variable t to the element $c_1(p_1^*(\mathcal{O}(1)))$. Now if $i : \mathbf{P}^n \hookrightarrow \mathbf{P}^\infty$ is the inclusion then under the mentioned identifications the ring morphism $(i \times id)^A : A(\mathbf{P}^\infty \times \mathbf{P}^n) \to A(\mathbf{P}^n \times \mathbf{P}^n)$ becomes the reduction modulo the ideal (t^{n+1}) morphism $R[[t]] \to R[[t]]/(t^{n+1})$. Let F_ω be the formal group law associated with the Chern structure on A and let $\lambda = c_1(\mathcal{O}(1)) \in R$. Write for short $t -_{F_\omega} \lambda$ for the element $F_\omega(t, I_\omega(\lambda)) \in R[[t]]$. If $R((t))$ is the localization of $R[[t]]$ with respect to the element t then $R[[t]] \subset R((t))$, the element $t -_{F_\omega} \lambda$ is a central unit in $R((t))$ and we claim that

$$c_n(q_1^*(\mathcal{O}(1) \otimes q_2^*(Q))) = t^{n+1}/(t -_{F_A} \lambda) \in R[[t]]. \tag{60}$$

This relation shows that the series $h(t) = t^{n+1}/(t -_{F_A} \lambda)$ belongs to $R[[t]]$. Furthermore since $(i \times id)^A(c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q))$ in $A(\mathbf{P}^\infty \times \mathbf{P}^n)$ the series $h(t)$ satisfies the relation

$$h(\zeta) = \Delta_{gys}(1) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{61}$$

This is the crucial point to finish the proof of the Proposition but first we check the relation (60).

For that define a rank n vector bundle Q over $\mathbf{P}^n$ via the short exact sequence of vector bundles $0 \to \mathcal{O}(-1) \to 1^{n+1} \to Q \to 0$. Then the short exact sequence of vector bundles

$$0 \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1)) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q) \to 0$$

on $\mathbf{P}^\infty \times \mathbf{P}^n$ and the Cartan formular for the total Chern class gives rise to the relation in $A(\mathbf{P}^\infty \times \mathbf{P}^n)$

$$c_1(q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1))) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = c_{n+1}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}))$$

With the notation fixed right above one can rewrite the last relation in terms of the formal group law F_ω and its "inverse element" I_ω (3.9.3) as follows

$$F_\omega(t, I_\omega(\lambda)) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = t^{n+1}$$

Since $F_\omega(t, I_\omega(\lambda)) = t -_{F_\omega} \lambda$ the relation (60) is proved.

Now with the relation (61) in hand and with p_{quil} defined via (45) one computes $(p_1)_{quil}(\Delta_{gys}(1)) \in R$ as follows

$$(p_1)_{quil}(\Delta_{gys}(1)) = \mathrm{Res}_{t=0}(\frac{h(t)}{t^{n+1}}\omega_{inv}) = \mathrm{Res}_{t=0}(\frac{\omega_{inv}}{t -_{F_A} \lambda}) \in R.$$

So to prove the Proposition it remains to prove the following

4.6.2. Claim. *Let S be a commutative ring and let $F = F(t_1, t_2) \in S[[t_1, t_2]]$ be a commutative formal group law and let $\omega \in \Omega_{S[t]/S} \otimes_{S[t]} S[[t]]$ be the unique normalized F-invariant differential form. Let $R = S[u]/(u^{n+1})$ and let $\lambda = \bar{u} \in R$ where $\bar{u}$ is the element u modulo the ideal (u^{n+1}). Then one has the relation in the ring R*

$$1 = Res_{t=0}(\frac{\omega_{inv}}{t -_F \lambda})$$

The proof of this Claim proceeds as follows. It straightforward to check it for the case of additive formal group law $F = t_1 + t_2$, because in this case the normalized invariant differential form is the form dt and $t -_F \lambda = t - \lambda$. Further if two formal group laws F_1 and F_2 over S are isomorphic and the required relation holds for one of them it holds as well for the other one. Thus the required relation holds for the formal group law $F_{un} \otimes \mathbb{Q}$ where F_{un} is the universal formal group law (over the Lazar ring L). Since L is a polinomial ring over the ring of integers $\mathbb{Z}$ the required relation holds for the universal formal group law F_{un} itself. If the required relation holds for a formal group law F' over a ring S' then for each ring homomorphism $f : S' \to S$ the required relation holds for the scalor extention F of the formal group law F' via the homomorphism f. Since we already know that the relation holds for the universal formal group law F_{un} one concludes that it does hold for the given formal group law F over the ring S from the Claim. The Claim is proved which completes the proof of Proposition 4.6.1. So we get a purely algebraic proof of the property 5 of Quillen's operators p_{quil}.

$\square$

Proof of 4.6.1 using the MU-approach. In this proof we assume that Quillen's operators p_{quil} are defined via (40). Let $\mathrm{MU}(*)$ be the compex cobordism theory (on the category of CW-complexes). Let as above $\mathrm{MU} = \mathrm{MU}(pt)$ be the coefficient ring of the theory $\mathrm{MU}(*)$. If E is a complex vector bundle of rank n over a space T, we let $c_i^{\mathrm{MU}}(E) \in \mathrm{MU}^{2i}(T)$, $(i = 1, \cdots, n)$ be the Chern classes of E in the sence of Conner-Floyd [CF]. For a projective morphism $f : S \to T$ of smooth algebraic varieties we let $f_{\mathrm{MU}} : \mathrm{MU}(S(\mathbb{C})) \to \mathrm{MU}(T(\mathbb{C}))$ the Gysin homomorphism [Qu2, 1.2 and 1.4] (if S and T are irreducible of dimensions m and n respectively, then this homomorphism changes the degree by $2(m - n)$). It respects the composition: if $g : Q \to S$ be a morphism of smooth complex algebraic varieties then $f_{\mathrm{MU}} \circ g_{\mathrm{MU}} = (f \circ g)_{\mathrm{MU}}$. For the identity map $id : S \to S$ the map id_{MU} is the identity. The compex cobordism theory satisfy the projective bundle theorem: $\mathrm{MU}[t]/(t^{n+1}) = \mathrm{MU}(\mathbb{C}P^n)$ (here t is identified with $c_1^{\mathrm{MU}}(\mathcal{O}(1))$).

Let $\bar{A} = A(pt)$ and let $l_\omega : \mathrm{MU} \to \bar{A}$ be the ring homomorphism (39) induced by the orientation ω on A (see 4.3.1). For a smooth variety X we will consider below in this proof the ring $A(X)$ as an MU-algebra through the homomorphism l_ω.

Let $l_\omega^n : \mathrm{MU}(\mathbb{C}P^n) \to A(\mathbf{P}^n)$ be the only MU-algebra homomorphism which takes the class $c_1^{\mathrm{MU}}(\mathcal{O}(1))$ to the class $c_1(\mathcal{O}(1))$. Then the diagram

$$\begin{tikzcd} \mathrm{MU}(\mathbb{C}P^n) \arrow[r, "l_\omega^n"] \arrow[d, "p_{\mathrm{MU}}"'] & A(\mathbf{P}^n) \arrow[d, "p_{quil}"] \\ \mathrm{MU} \arrow[r, "l_\omega"] & \bar{A}, \end{tikzcd}$$

commutes. In fact, one has a chain of relation (see (40) and [Qu1])

$$l_\omega(p_{\mathrm{MU}}(c_1^{\mathrm{MU}}(\mathcal{O}(1))^i)) = l_\omega([\mathbb{C}P^{n-i}]_{\mathrm{MU}}) = [\mathbf{P}^{n-i}]_\omega = p_{quil}(c_1(\mathcal{O}(1))^i) = p_{quil}(l_\omega^n(c_1^{\mathrm{MU}}(\mathcal{O}(1))^i)).$$

The projective bundles theorem and the projection formulas completes the proof of the commutativity. The projective bundle theorem for complex cobordism shows that the cup-product identifies the MU-algebras $\mathrm{MU}(\mathbb{C}P^n) \otimes_{\mathrm{MU}} \mathrm{MU}(\mathbb{C}P^n)$ and $\mathrm{MU}(\mathbb{C}P^n \times \mathbb{C}P^n)$. The projective bundle theorem for the theory A shows that the cup-product identifies the MU-algebras $A(\mathbf{P}^n) \otimes_{\bar{A}} A(\mathbf{P}^n)$ and $A(\mathbf{P}^n \times \mathbf{P}^n)$. Now if $p_1 : \mathbb{C}P^n \times \mathbb{C}P^n \to \mathbb{C}P^n$ is the projection to the first factor then the diagram commutes

$$\begin{tikzcd} \mathrm{MU}(\mathbb{C}P^n \times \mathbb{C}P^n) \arrow[r, "l_\omega^{n,n}"] \arrow[d, "(p_1)_{\mathrm{MU}}"'] & A(\mathbf{P}^n \times \mathbf{P}^n) \arrow[d, "(p_1)_{quil}"] \\ \mathrm{MU}(\mathbb{C}P^n) \arrow[r, "l_\omega^n"] & A(\mathbf{P}^n), \end{tikzcd}$$

where $l_\omega^{n,n} = l_\omega^n \otimes l_\omega^n$. This is proved exactly as the commutativity of the previos diagram using the projection formulas in both theories and the projective bundle theorem.

Let $\Delta : \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$ be the diagonal and let $\Delta : \mathbf{P}^n \to \mathbf{P}^n \times \mathbf{P}^n$ be the diagonal as well (the same symbal is used for both imbeddings). We claim that the following relation holds in $A(\mathbf{P}^n \times \mathbf{P}^n)$

$$l_\omega^{n,n}(\Delta_{\mathrm{MU}}(1)) = \Delta_{gys}(1). \tag{62}$$

Assuming for a moment this relation one gets a chain of relations

$$1 = l_\omega^n(1) = l_\omega^n((p_1)_{\mathrm{MU}}(\Delta_{\mathrm{MU}}(1))) = (p_1)_{quil}(l_\omega^{n,n}(\Delta_{\mathrm{MU}}(1))) = (p_1)_{quil}(\Delta_{gys}(1))$$

which proves Proposition 4.6.1. It remains to prove the relation (62).

For that we need in three other relation stated just below. Let Q be a rank n vector bundle over $\mathbf{P}^n$ defined by the short exact sequence

$$0 \to \mathcal{O}(-1) \to 1^{n+1} \to Q \to 0$$

and let $Q_{\mathbb{C}}$ be a rank n algebraic vector bundle over $\mathbb{C}P^n$ defined by the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^{n+1} \to Q_{\mathbb{C}} \to 0.$$

We claim the following three relations

$$\Delta_{gys}(1) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{63}$$

$$\Delta_{\mathrm{MU}}(1) = c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}})) \in \mathrm{MU}(\mathbb{C}P^n \times \mathbb{C}P^n). \tag{64}$$

$$l_\omega^{n,n}(c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{65}$$

Assuming for a minute the last three relations one gets a chain of relations

$$\Delta_{gys}(1) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) = l_\omega^{n,n}(c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = l_\omega^{n,n}(\Delta_{\mathrm{MU}}(1))$$

which prove the relation (62).

It remains to prove relations (63), (64), (65). The relation (63) is a particular case of the one (38) which is proved in 4.4.7. In fact take $Y = \mathbf{P}^n$, $E = 1^{n+1}$, $s = \Delta$, $p = p_1$.

Exactly the same arguments works for the complex cobordism $\mathrm{MU}(*)$ which proves the relation (64).

To prove the last relation consider the line bundle $\mathcal{O}(1)$ over $\mathbf{P}^\infty$ and consider on $\mathbf{P}^\infty \times \mathbf{P}^n$ the short exact sequence of vector bundles

$$0 \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1)) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q) \to 0.$$

where q_1 and q_2 are the projections of $\mathbf{P}^\infty \times \mathbf{P}^n$ to the first and the second factors respectively. This exact sequence and the Cartan formular for the total Chern class gives rise to the relation in $A(\mathbf{P}^\infty \times \mathbf{P}^n)$

$$c_1(q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1))) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = c_{n+1}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}))$$

Now setting $t_1 = c_1(q_1^*(\mathcal{O}(1)))$ and $t_2 = c_1(q_2^*(\mathcal{O}(1)))$ one can rewrite the last relation in terms of the formal group law F_ω and its "inverse element" I_ω (3.9.3) as follows

$$F_\omega(t_1, I_\omega(t_2)) \cup c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = t_1^{n+1} \tag{66}$$

Similarly one gets the following relation in $\mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n)$

$$F_{\mathrm{MU}}(t_1^{\mathrm{MU}}, I_{\mathrm{MU}}(t_2^{\mathrm{MU}})) \cup c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}})) = (t_1^{\mathrm{MU}})^{n+1}$$

Here q_1 and q_2 are the projections of $\mathbb{C}P^\infty \times \mathbb{C}P^n$ to the first and to the second factors respectively, and F_{MU} and I_{MU} are the formal group law and its "inverse element" for the complex cobordism 4.3.1. Applying the homomorphism

$$l_\omega^{\infty,n} = l_\omega^\infty \otimes l_\omega^n : \mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n) \to A(\mathbf{P}^\infty \times \mathbf{P}^n)$$

to the last relation in $\mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n)$ one gets in the group $A(\mathbf{P}^\infty \times \mathbf{P}^n)$ the relation

$$F_\omega(t_1, I_\omega(t_2)) \cup l_\omega^{\infty,n}(c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}}))) = t_1^{n+1} \tag{67}$$

Observe that the element $F_\omega(t_1, I_\omega(t_2)) = t_1 -_{F_\omega} t_2$ is not a zero-divisor in the ring $A(\mathbf{P}^\infty \times \mathbf{P}^n) = A[t_2][[t_1]]/(t_2^{n+1})$. Thus the relations (66) and (67) prove the one

$$l_\omega^{\infty,n}(c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}}))) = c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)).$$

Now consider a commutative diagram

$$\begin{array}{ccc}
\mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n) & \xrightarrow{l_\omega^{\infty,n}} & A(\mathbf{P}^\infty \times \mathbf{P}^n) \\
{\scriptstyle (i\times id)^{\mathrm{MU}}}\downarrow & & \downarrow{\scriptstyle (i\times id)^A} \\
\mathrm{MU}(\mathbb{C}P^n \times \mathbb{C}P^n) & \xrightarrow{l_\omega^{n,n}} & A(\mathbf{P}^n \times \mathbf{P}^n),
\end{array}$$

in which the vertical arrows are the pull-backs induced by the obvious imbeddings. Since

$$(i \times id)^{\mathrm{MU}}(c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}}))) = c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))$$

and

$$(i \times id)^A(c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \tag{68}$$

the commutativity of the last diagram proves the desired relation (65)

$$l_\omega^{n,n}(c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = c_n(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)).$$

The Proposition 4.6.1 is proved.

The proof of the property 5 of Quillen's operators p_{quil} is completed.

$\square$

4.7 Construction of an integration

Let A be a ring cohomology theory endowed with an orientation ω and with the corresponding by Theorem 3.7.5 Chern structure $L \mapsto c(L)$. The main aim of this subsection is to construct an integration required by Theorem 4.1.4. We will use the Gysin operators i_{gys} for closed imbeddings defined by (29) and the Quillen operators p_{quil} for projections defined by (40) or which is equivalent defined by (45).

Let $f : Y \to X$ be a projective morphism of smooth varieties. One can present f as a composition of a closed imbedding $i : Y \hookrightarrow X \times \mathbf{P}^n$ and the projection $p : X \times \mathbf{P}^n \to X$, i.e. $f = p \circ i$. Define now an operator $f_A : A(Y) \to A(X)$ by the formula

$$f_A = p_{quil} \circ i_{gys}.$$

4.7.1. Theorem. *The operator f_A does not depend on the particular choice of a decomposition of the morphism f in the form $f = p \circ i$, where $i : Y \hookrightarrow X \times \mathbf{P}^n$ is a closed imbedding and the projection $p : X \times \mathbf{P}^n \to X$.*

Moreover the assignment $f \mapsto f_A$ is an integration on A required by Theorem 4.1.4. Finally for a closed imbedding $i : S \hookrightarrow T$ of smooth varieties the operator i_A coincides with the Gysin operator i_{gys} and for the projection $p : T \times \mathbf{P}^n \to T$ the operator p_A coinsides with the Quillen operator p_{quil}.

Proof. For a projective morphism $f : Y \to X$ we first check that the map f_A does not depend on the particular choice of the decomposition of f.

Let $f = p' \circ i'$ be another decomposition of f, where $i' : Y \hookrightarrow X \times \mathbf{P}^m$ is a closed imbedding and $p' : X \times \mathbf{P}^m \to X$ is the projection. We have to check the relation

$$p_{quil} \circ i_{gys} = p'_{quil} \circ i'_{gys}. \tag{69}$$

For that consider the commutative diagram

$$\begin{tikzcd} X & X \times \mathbf{P}^m \arrow[l, "p'"'] \\ Y \arrow[u, "f"] \arrow[r, "I"] \arrow[d, "f"'] & X \times \mathbf{P}^n \times \mathbf{P}^m \arrow[u, "p_m"'] \arrow[d, "p_n"] \\ X & X \times \mathbf{P}^n \arrow[l, "p"] \end{tikzcd}$$

where I is a unique imbedding such that $p_m \circ I = i'$ and $p_n \circ I = i$.

Recall that $p_{quil} \circ p_{n,quil} = p'_{quil} \circ p_{m,quil}$ by the property 1 from 4.3.1. Now clearly it suffices to check two relations

- $p_{m,quil} \circ I_{gys} = i'_{gys}$
- $p_{n,quil} \circ I_{gys} = i_{gys}$.

Both these relations are particular cases of the one from the following Claim.

4.7.2. Claim. *Consider a commutative diagram*

$$\begin{CD} Y @>\tilde{j}>> T \times \mathbf{P}^k \\ @V{id}VV @VV{p}V \\ Y @>j>> T \end{CD}$$

with the closed imbeddings j and $\tilde{j}$. One has the following relation $p_{quil} \circ \tilde{j}_{gys} = j_{gys}$.

Proof of Claim. To prove this Claim consider the commutative diagram

$$\begin{CD} Y \times \mathbf{P}^k @>J>> T \times \mathbf{P}^k \\ @V{q}VV @VV{p}V \\ Y @>j>> T \end{CD}$$

with $J = j \times id$ and consider a section $s : Y \to Y \times \mathbf{P}^k$ of the projection q such that $J \circ s = \tilde{j}$.

The relations $\tilde{j}_{gys} = J_{gys} \circ s_{gys}$ (see 31) and $p_{quil} \circ J_{gys} = j_{gys} \circ q_{gys}$ (see 4) and $q_{quil} \circ s_{gys} = id_{A(Y)}$ give a chain of relations

$$p_{quil} \circ \tilde{j}_{gys} = p_{quil} \circ (J_{gys} \circ s_{gys}) = (p_{quil} \circ J_{gys}) \circ s_{gys} =$$

$$= (j_{gys} \circ q_{quil}) \circ s_{gys} = j_{gys} \circ (q_{quil} \circ s_{gys}) = j_{gys}.$$

The Claim is proved which completes the proof of the relation (69). Thus the operator f_A is well-defined.

The nearest goal is to prove the relation $f_A \circ g_A = (f \circ g)_A$. For that consider the commutative diagram

$$\begin{tikzcd} Z \arrow[r, "j"] \arrow[dr, "g"'] & Y \times \mathbf{P}^m \arrow[r, "I"] \arrow[d, "p_m"] & X \times \mathbf{P}^n \times \mathbf{P}^m \arrow[d, "q_n"] \\ & Y \arrow[r, "i"] \arrow[dr, "f"'] & X \times \mathbf{P}^n \arrow[d, "p_n"] \\ & & X \end{tikzcd}$$

with a smooth variety Z and a closed imbedding j and a closed imbedding i and with $I = i \times id$ and with the projection $q_n : X \times \mathbf{P}^n \times \mathbf{P}^m \to X \times \mathbf{P}^n$.

By the very definition one has relations $f_A = p_{n,quil} \circ i_{gys}$ and $g_A = p_{m,quil} \circ j_{gys}$. Thus one has a chain of relations

$$f_A \circ g_A = p_{n,quil} \circ (i_{gys} \circ p_{m,quil}) \circ j_{gys} = p_{n,quil} \circ (q_{n,quil} \circ I_{gys}) \circ j_{gys} =$$

$$= (p_{n,quil} \circ q_{n,quil}) \circ (I \circ j)_{gys}.$$

Now consider a closed imbedding $k : \mathbf{P}^n \times \mathbf{P}^m \hookrightarrow \mathbf{P}^N$ and the projection $p_N : X \times \mathbf{P}^N \to X$ and the commutative diagram

$$\begin{tikzcd}
Z \arrow[r, "I\circ j"] \arrow[dr, "f\circ g"'] & X \times \mathbf{P}^n \times \mathbf{P}^m \arrow[r, "id\times k"] \arrow[d, "p_n\circ q_n"] & X \times \mathbf{P}^N \arrow[dl, "p_N"] \\
& X &
\end{tikzcd}$$

Since

$$(f \circ g)_A = p_{N,quil} \circ ((id \times k) \circ I \circ j)_{gys} = p_{N,quil} \circ (id \times k)_{gys} \circ (I \circ j)_{gys}$$

it remains to prove

4.7.3. Claim. *The following relation holds*

$$p_{N,quil} \circ (id \times k)_{gys} = p_{n,quil} \circ q_{n,quil}. \tag{70}$$

Proof of Claim. First observe that each of the homomorphisms $p_{N,quil}$, $(id \times k)_{gys}$, $p_{n,quil}$ and $p_{m,n,quil}$ is $A(X)$-linear. Thus by the projective bundle theorem it suffices to prove Claim in the case $X = pt$.

First consider the projections $q_n : \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^n$ and $q_m : \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^m$ and consider the commutative diagram

$$\begin{tikzcd}
& \mathbf{P}^n \times \mathbf{P}^m \times \mathbf{P}^N \arrow[d, "q_{m,N}"] \\
\mathbf{P}^n \times \mathbf{P}^m \arrow[ur, "k_2"] \arrow[r, "k_1"] \arrow[dr, "k"'] & \mathbf{P}^m \times \mathbf{P}^N \arrow[d, "q_N"] \\
& \mathbf{P}^N
\end{tikzcd}$$

with the obvious projections $q_{m,N}$, q_N and with the closed imbeddings $k_1 = (q_m, k)$ and $k_2 = (id, k)$. The Claim 4.7.2 gives a chain of relations

$$k_{gys} = q_{N,quil} \circ k_{1,gys} = q_{N,quil} \circ q_{m,N,quil} \circ k_{2,gys}. \tag{71}$$

Now consider one more commutative diagram

$$\begin{tikzcd}
\mathbf{P}^n \times \mathbf{P}^m \arrow[d, "q_m"] & \mathbf{P}^n \times \mathbf{P}^m \times \mathbf{P}^N \arrow[l, "q_{n,m}"'] \arrow[d, "q_{m,N}"] \\
\mathbf{P}^m \arrow[d, "p_m"] & \mathbf{P}^m \times \mathbf{P}^N \arrow[l, "\bar{q}_m"'] \arrow[d, "q_N"] \\
pt & \mathbf{P}^N \arrow[l, "p_N"']
\end{tikzcd}$$

where $\bar{q}_m$ is the projection. It gives rise to the chain of relations

$$p_{N,quil} \circ q_{N,quil} \circ q_{m,N,quil} = p_{m,quil} \circ \bar{q}_{m,quil} \circ q_{m,N,quil} = p_{m,quil} \circ q_{m,quil} \circ q_{n,m,quil} \qquad (72)$$

Thus one gets the following relation

$$p_{N,quil} \circ k_{gys} = p_{m,quil} \circ q_{m,quil} \circ (q_{n,m,quil} \circ k_{2,gys}) \qquad (73)$$

To complete the proof of the Claim it remains to check the relation $id = q_{n,m,quil} \circ k_{2,gys}$ because $p_{m,quil} \circ q_{m,quil} = p_{n,quil} \circ q_{n,quil}$ by the property 1 of Quillen's operators p_{quil}.

To prove the relation $id = q_{n,m,quil} \circ k_{2,gys}$ observe that the map k_2 is a section of the projection $q_{n,m}$. Therefore this relation is just a particular case of the property 5 of the mappings p_{quil}.

The Claim is proved. Thus we checked the relation $(f_A \circ g_A) = f_A \circ g_A$. The relation $i_A = i_{gys}$ holds for a closed imbedding $i : Y \hookrightarrow X$ by the very construction of the map i_A. The relation $p_A = p_{quil}$ holds as well by the very construction of the map p_A for the projection $p : X \times \mathbf{P}^n \to X$.

To complete the proof of the Theorem it remains to check the properties 2 to 5 of the integration (see Definition 4.1.2) and to prove the relation $i_A(1) = c(L(D))$ for a smooth divisor $i : D \hookrightarrow X$. The property 2 coincides with the one 4.2.2. The property 3 coincides with the property 2 from 4.3.1. of the maps p_{quil}. The property 4 is obvious. The property 5 is obvious as well. In fact, the localization sequence $A_Y(X) \xrightarrow{k^A} A(X) \xrightarrow{j^A} A(X-Y)$ is exact (k^A is the support extension operator), $i_{gys} = k^A \circ i_{th}$ and the operator $i_{th} : A(Y) \to A_Y(X)$ defined by (28) is an isomorphism. The relation $i_A(1) = c(L(D))$ in $A(X)$ is exactly the relation (36) of the Gysin map i_{gys} which is already proved in Subsection 4.4.

$\square$

4.8 Proof of the assertions 1 and 3 of Theorem 4.1.4

Let A be a ring cohomology theory. Let ω be an orientation of the theory A. An integration subjecting to the orientation ω is constructed in Theorem 4.7.1. The uniqueness of an integration subjecting to the orientation ω is proved in subsection 4.1. The assertion 1 of Theorem 4.1.4 is proved.

Now prove the third assertion of the Theorem. Let ω be the orientation of A from the item 1 of the Theorem. Let $f \mapsto f_A$ be the integration on A from the same item 1. Let ω' be the orientation of A corresponding to the integration by the item 2 of the Theorem. The integration $f \mapsto f_A$ is subjected as to the orientation ω so to the orientation ω'. Thus $\omega = \omega'$ by the item (2) of the Theorem.

Now let $f \mapsto f_A$ be an integration on A. Let ω be the orientation of A corresponding to the integration by the item (2) of the Theorem. Let $f \mapsto f'_A$ be the integration subjecting to the orientation ω as it is described in the item (1) of the Theorem. Both integrations $f \mapsto f_A$ and $f \mapsto f'_A$ are subjected to the orientation ω. Hence they coincide by the item (1) of the Theorem.

This completes the proof of the Theorem 4.1.4.

4.9 All possible integrations on an orientable theory A

In this section we consider an orientable cohomology theory A and identify the set of all integrations on it with the set of universally central local parameters of the ring $A(\mathbf{P}^\infty)$ (Theorem 4.9.1). This Theorem one should consider as *the main result* of the preprint. To state this Theorem recall that an element $\pi \in A(\mathbf{P}^\infty)$ is called *universally central local parameter* of the ring $A(\mathbf{P}^\infty)$ if it is a universally central element (see 2.4.4) satisfying the two conditions:

(1) its restriction to a rational point vanishes and

(2) the ring homomorphism $A(pt)[[t]] \to A(\mathbf{P}^\infty)$ sending the variable t to the element π is an isomorphism.

4.9.1. Theorem (All Integrations). *The assignment associating to each integration on A the corresponding by the second item of Theorem 4.1.4 Chern class $c(\mathcal{O}(-1)) \in A(\mathbf{P}^\infty)$ is a bijection of the set of all integrations on A with the set of all universally central local parameters of the ring $A(\mathbf{P}^\infty)$.*

Proof. Consider the assignment from the item (2) of Theorem 4.1.4 which associate to an integration $f \mapsto f_A$ on A the Chern structure $L \mapsto c(L) = z^A(z_A(1))$. By Theorems 4.1.4 and 3.7.5 this assignment is bijection of the set of all integrations on A with the set of all Chern structures on A. To prove the Theorem it remains to check that the assignment

$$\xi : Chern\ Structures\ \to universally\ central\ local\ parameters\ of\ A(\mathbf{P}^\infty)$$

which takes a Chern structure $L \mapsto c(L)$ to the element $c(\mathcal{O}(-1)) \in A(\mathbf{P}^\infty)$ is bijective.

To prove the injectivity of ξ consider two Chern structures $L \mapsto c^{(1)}(L)$ and $L \mapsto c^{(2)}(L)$ on A and assume that $c^{(1)}(\mathcal{O}(-1)) = c^{(2)}(\mathcal{O}(-1))$. By Claim 3.4.4 for each variety X and each line bundle L over X there exists a finite-dimensional vector space V and a diagram of the form

$$X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V)$$

in which X' is a torsor under a vector bundle over X and the morphism f is such that the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic. The pull-back operator $p^* : A(X) \to A(X')$ is an isomorphism by the strong homotopy invariance of the theory A (see 2.2.6). Now $c^{(1)}(L) = c^{(2)}(L)$ by the functoriality of the classes $c^{(1)}$ and $c^{(2)}$.

To prove the surjectivity of ξ choose a Chern structure $L \mapsto c(L)$ on A. Then the element $c = c(\mathcal{O}(-1))$ is a universally central local parameter in $A(\mathbf{P}^\infty)$. Let $\pi \in A(\mathbf{P}^\infty)$ be one more universally central local parameter. Let $f(u) \in A^{uc}(pt)[[u]]$ be a formal power series such that $f(c) = \pi$ in $A(\mathbf{P}^\infty)$. Now consider an assignment $L \mapsto c^{new}(L) := f(c(L))$. Clearly it is a Chern structure on A and $c^{new}(\mathcal{O}(-1)) = f(c) = \pi$ in $A(\mathbf{P}^\infty)$. The Theorem is proved. □

References

[AH] *M.F. Atiyah, F. Hirzebruch.* Cohomologie-Operationen und charakteristische Klassen, Math.Zeitschr. 77 (1961), 149-187.

[Ad] *J. F. Adams.* Stable homotopy and generalised cohomology. University of Chicago Press, 1974.

[Ba] *P. Balmer.* Triangular Witt groups I: The 12-term localization sequence, K-theory, 19 (2000), 311-363.

[BFM] *P.Baum, W.Fulton, R.MacPherson.* Riemann-Roch and topological K-theory for singular varieties. Acta Math., 143 (1979), 155-192.

[Bl] *S. Bloch.* Algebraic cycles and higher K-theory, Adv. Math., 61 (1986), 267-304.

[Br] *P. Brosnan.* Steenrod operations in Chow theory, http://www.math.uiuc.edu/K-theory/0370/, 1999.

[BS] *A. Borel, J.P. Serre.* Le theoreme de Riemann-Roch. BSMF, 86 (1958), 97-136.

[CF] *P.E. Conner, E.E. Floyd.* The relation of cobordism to K-theory. Lect. Notes Math., 28 (1966), Springer-Verlag, Berlin.

[Dy] *E. Dyer.* Relations between cohomology theories. Col. Alg. Topology(1962), 1-10.

[FW] *E. Friedlander, M. Walker.* Semi-topological K-theory of real varieties, http://www.math.uiuc.edu/K-theory/0453/, 2000.

[Fu] *W. Fulton.* Intersection theory. Springer-Verlag, 1984.

[Ha] *M. Hazewinkel.* Formal groups and applications. Academic Press, NY, 1989.

[Har] *R. Hartshorne.* Algebraic geometry.Graduated texts in Mathematics 52,1977.

[H] *F. Hirzebruch.* Topological methods in algebraic geometry. Die Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, b. 131, 1966.

[Hu] *Po Hu.* S-modules in the category of schemes. http://www.math.uiuc.edu/K-theory/396 .

[Gi] *H. Gillet.* Riemann-Roch Theorems for Higher Algebraic K-Theory. Advances in Math. 40 (1981), 203-289.

[Gra] *D.Grayson.* Adams operations on higher K-theory. K-theory 6 (1992), no.2, 97-111.

[Gr] *A. Grothendieck.* Bull.Soc.Math.France, 86 (1958), 136-154.

[Ja] *J.F. Jardine.* Motivic symmetric spectra, http://www.math.uiuc.edu/K-theory/448 .

[J] *J. Jouanolou.* Une suite exacte de Mayer-Vietoris en K-the'orie algebrique, Lect. Notes Math., 341 (1973), 293-316.

[L] *M. Levine.* Motivic cohomology and algebraic cycles a categorical approach, http://www.math.uiuc.edu/K-theory/0107/, 1996.

[LM] *M.Levine, F. Morel.* Cobordisme algebrique I, C.R. Acad. Sci. Paris Se'r. I, Math. 332 (2001).

[M] *F. Morel.* Basic properties of the stable homotopy category of smooth schemes, taken from a Web site approximately in March, 2000.

[Me] *A. Merkurjev.* Algebraic oriented cohomology theories. http://www.math.ucla.edu/K-theory/0535.

[Mi] *J. S. Milne.* Etale Cohomology. Princeton University Press, 1980.

[MV] *F. Morel,V. Voevodsky.* Homotopy category of schemes over a base. 1997.

[Pa] *I. Panin* Riemann–Roch Theorem for Oriented Cohomology, http://www.math.uiuc.edu/K-theory/0552/ 2002.

[PS] *I. Panin, A. Smirnov* Push-forwards in Oriented Cohomology Theories of Algebraic Varieties http://www.math.uiuc.edu/K-theory/0459/ 2000.

[PY] *I. Panin, S. Yagunov.* Rigidity for orientable functors. MPI-preprint, 2000.

[Qu1] *D. Quillen.* On the formal group laws of unoriented and complex cobordism theory. Bull. AMS 75(1969), 1293-1298.

[Qu2] *D.Quillen.* Elementary proofs of some results of cobordism theory using Steenrod operations, Advances in Mathematics 7 (1971), 29-56.

[Qu3] *D.Quillen.* Higher algebraic K-theory:I. Lect.Notes Math. 341 (1973), 85-147.

[So] *Ch. Soule* Operations en K-theory algebrique. Canad. J. Math. 37 (1985), no.3, 488-550.

[SV] *A. Suslin, V. Voevodsky.* Bloch-Kato conjecture and motivic cohomology with finite coefficients. The arithmetics and geometry of algebraic cycles, 117-189, NATO Sci.Ser.C Math.Phys.Sci., 548, Kluver Acad.Publ., Dortrecht, 2000.

[Su] *A. Suslin.* On K-theory of algebraically closed fields. Inv.Math, 73(1983), 241-245.

[Sw] *R. M. Switzer.* Algebrai topology - homotopy and homology. Springer-Verlag, 1975.

[TT] *R. Thomason, T. Throbaugh.* Higher algebraic K-theory of schemes and of derived categories. In The Grothendieck festschift, 3, 247-436, Birkhauser, Boston, 1990.

[V1] *V. Voevodsky.* $\mathbf{A}^1$-Homotopy theory. Doc. Math., Extra Vol. ICM 1998(I), 417-442.

[V2] *V. Voevodsky.* The Milnor Conjecture, MPI-preprint, 1996.

[V3] *V. Voevodsky.* Homology of schemes, Selecta Mathematica, New Series, 2(1), 1996, 111-153.