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HARMONIC ANALYSIS ON TOTALLY DISCONNECTED GROUPS AND IRREGULARITIES OF POINT DISTRIBUTIONS

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ABSTRACT

The goal of this paper is to study point distributions in the multi-dimensional unit cube which possess the structure of finite abelian groups with respect to certain p-ary arithmetic operations. Such distributions can be thought of as finite subgroups in a compact totally disconnected group of the Cantor type. We apply the methods of L^q harmonic analysis to estimate very precisely the L^q-discrepancies for such distributions. Following this approach, we explicitly construct point distributions with the minimal order of the L^q-discrepancy for each q, $1 < q < \infty$.

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1. Introduction

A fundamental observation in the modern theory of irregularities of distributions is that most uniformly distributed point sets possess hidden regular structures. This general heuristic principle has been illustrated by a number of concrete examples. In fact, all known uniformly distributed point sets of low discrepancy have inner regularities. See Beck and Chen [BC], Drmota and Tichy [DT], and Matoušek [M].

In the present paper we shall study point distributions in the d-dimensional unit cube $U^d = [0,1)^d$ which possess the structure of finite abelian groups with respect to certain p-ary arithmetic operations. Such distributions can be thought of as subgroups in a compact totally disconnected abelian group U_p^d, the product of d copies of the Cantor type group

$$U_p = \prod_{i\geq 1} \mathbb{Z}_p = \{\mathbb{Z}_p\}^{\mathbb{N}}, \tag{1.1}$$

which is the full direct product of countably many copies of the additive cyclic group $\mathbb{Z}_p$ of order $p \geq 2$. In particular, U_2 is known as the diadic Cantor group. A detailed study of totally disconnected groups can be found in Hewitt and Ross [HR].

The group of characters for U_p^d is isomorphic to a discrete countable group $\hat{U}_p^d$, the product of d copies of the weak direct product

$$\hat{U}_p = \prod_{i\geq 1}^{*} \mathbb{Z}_p. \tag{1.2}$$

Furthermore, the characters of the group U_p can be identified with the Walsh functions in base p. See Golubov *et. al.* [GES] and Schipp *et al.* [SWSP].

By applying the methods of harmonic analysis on the group U_p^d, we shall estimate very precisely the L^q-discrepancies for point distributions possessing the group structure. Actually, we shall estimate an averaged rate of convergence of the Haar measures on finite subgroups $\mathcal{D} \subset U_p^d$ to the Haar measure on U_p^d.

Our main tool is the Littlewood-Paley theory for the multi-dimensional Fourier-Walsh series. See Edwards and Gaudry [EG]. Unfortunately, this theory can not be directly applied to the L^q-discrepancies. Nevertheless, the difficulty may be resolved if the Littlewood-Paley theory is used in conjunction with coding theory. Such an interaction between L^q harmonic analysis and coding theory is of crucial importance for our approach, and it is hoped that this idea may have wider applications.

Following this approach and using previous constructions from Skriganov [Sk1] and Chen and Skriganov [CS], we shall explicitly construct point distributions in the unit cube U^d with the minimal order of the L^q-discrepancy.

To be more specific in our discussion, let us briefly recall basic facts on irreqularities of point distributions. For details we refer to [BC], [DT], [M].

Let $\mathcal{D}_N$ be a distribution of $N > 1$ points, not necessary distinct, in the unit cube U^d. The *local discrepancy* $\mathcal{L}[\mathcal{D}_N; Y]$, $Y = (y_1, \ldots, y_d) \in U^d$, is defined by

$$\mathcal{L}[\mathcal{D}_N; Y] = \#\{\mathcal{D} \cap B_Y\} - N \mathrm{vol}\, B_Y, \tag{1.3}$$

where $B_Y = [0, y_1) \times \cdots \times [0, y_d) \subset V^d$ is a rectangular box of volume $\mathrm{vol}\, B_Y = y_1 \ldots y_d$. We write $\#\{S\}$ for the number of elements in a finite set S.

The L^q-*discrepancy* $\mathcal{L}_q[\mathcal{D}_N]$ is defined by

$$\mathcal{L}_q[\mathcal{D}_N] = \left(\int_{U^d} |\mathcal{L}[\mathcal{D}_N; Y]|^q dY \right)^{1/q}, \tag{1.4}$$

where $1 \leq q < \infty$.

For each q, $1 < q < \infty$, and for an arbitrary distribution D_N we have the following lower bound

$$\mathcal{L}_q[\mathcal{D}_N] > c_{d,q} (\log N)^{\frac{1}{2}(d-1)}, \tag{1.5}$$

where the positive constant $c_{d,q}$ is independent of N.

In 1954 Roth [Ro1] established (1.5) for $q = 2$, and therefore for every $q \geq 2$. In 1977 Schmidt [Sc] extended (1.5) to the interval $1 < q < 2$. In fact, this result of Roth and Schmidt was a starting point of the modern theory of irregularities of distributions in several dimensions.

The lower bound (1.5) is the best possible. More precisely, for each q, $1 \leq q < \infty$, and every $N > 1$ there exist distributions $\mathcal{D}_N$ of N points in U^d such that

$$\mathcal{L}_q[\mathcal{D}_N] < C_{d,q} (\log N)^{\frac{1}{2}(d-1)}, \tag{1.6}$$

where the constant $C_{d,q}$ is independent of N.

Constructions of distributions $\mathcal{D}_N$ satisfying (1.6) for $1 \leq q \leq 2$ were given in dimensions $d = 2$ and $d = 3$ by Davenport [D] and Roth [Ro2] respectively, and in arbitrary dimensions by Frolov [Fr] and Roth [Ro3]. Shortly thereafter, Chen [C1], [C2] extended (1.6) to the interval $2 < q < \infty$. At a later time, other constructions for arbitrary exponents q were also given by Skriganov [Sk1].

Notice that in the cited papers constructions of the distributions $\mathcal{D}_N$ satisfying (1.6) are based either on lattices in Euclidean spaces with very special diophantine properties (cf. [D], [F2], [Ro2], [Sk1]) or on specific sequences over finite fields and rings (cf. [C1], [C2], [Ro3]). In all instances such distributions have rich inner structures.

Hovewer, in dimensions $d \geq 3$ all these constructions also involve additional probabilistic arguments and therefore are not explicit. Known explicit constructions in two dimensions can be found in [BC], [DT], [M].

It must be emphasized that the problem of explicit constructions of distributions with the minimal order of the L^q-discrepancies is obviously of much importance not only for the theory of irregularities of distributions but also for applications to numerical integration and interpolation. Relationships with complexity in numerical analysis and effective derandomization in combinatorics should be also mentioned. See, for example, Nowak and Woźniakowski [NW], Ritter [R], Temlyakov [T], and references therein.

In the recent paper [CS] by Chen and Skriganov this problem was first solved in arbitrary dimensions for the L^2-discrepancy. We mention also the paper [LP] by Larcher and Pillichshammer where a similar problem was studied in three dimensions with an apper bound for the L^2-discrepancy of the order $\log N \ (\log \log N)^{1/2}$.

In the present paper the explicit construction problem will be solved completely. We shall prove the following

Proposition 1.1. *Suppose that a positive even integer $2t$ is given. Pick a prime $p > 2td^2$. Then for every $N > 1$ a distribution $\mathcal{D}_N$ of N points in the unit cube U^d can be constructed explicitly to satisfy the inequality*

$$\mathcal{L}_{2t}[\mathcal{D}_N] < 3p^{2d} t^{1/2} \gamma_p(d, 2t) (\log_p N + 2td + 1)^{\frac{1}{2}(d-1)}, \tag{1.7}$$

where $\log_p$ denotes the log in base p and $\gamma_p(d,2t)$ is the standard constant in the Paley inequality (4.11) for the d-dimensional Walsh functions in base p.

The explicit constructions of distributions $\mathcal{D}_N$ satisfying (1.7) will be given in Section 3 in terms of special finite subgroups of the group U_p^d. In the case $t=1$ Proposition 1.1 coinsides in essence with the main result of [CS].

It is interesting to note that for the L^2-discrepancy Theorem 1.1 can be improved in the following sense. It turns out that the L^2-discrepancies for finite subgroups $\mathcal{D} \subset U_p^d$ can be exactly evaluated in terms of certain weight enumerators for dual subgroups $\widehat{\mathcal{D}} \subset \hat{U}_p^d$. The corresponding explicit formulas will be given elsewhere.

In conclusion of the present discussion, we should point out that instead of the groups (1.1) we could study more general totally disconnected groups of the type

$$U_P = \prod_{i\geq 1} \mathbb{Z}_{p_i}, \tag{1.8}$$

where $P = (p_1, p_2, \dots)$ is an arbitrary sequence of integers $p_i \geq 2$.

The corresponding generalization of our approach to the groups (1.8) is strightforward if the integers p_i are bounded as $i \to \infty$. Otherwise, a considerably different state of affairs takes place. Rouhgly speaking, finite subgroups $\mathcal{D} \subset U_P^d$ no longer can meet the bound (1.6) if the integers p_i are unbounded as $i \to \infty$. In the present paper we are to leave aside these interesting questions.

The plan of the paper is the following. Section 2 contains the background material for the further discussion of distributions possessing the group sturcture. Our main result (Theorem 3.1) on the L^q-discrepancies for subgroups in U_p^d is stated in Section 3. In Section 3, we also describe explicit constructions of special subgroups in U_p^d and deduce Proposition 1.1 from Theorem 3.1.

The rest of the paper is concerned with the proof of Theorem 3.1. In Section 4, we recall necessary facts on L^q harmonic analysis on the group U_p^d. Our key results (Lemmas 5.1 and 5.2) on relationships between the Walsh functions and error-correcting codes are given in Section 5. These results will be used in Section 6 to complete the proof of Theorem 3.1.

2. Preliminary background

Throughout we shall use the following notation. We write $\mathbb{N}$ for the set of all positive integers, $\mathbb{N}_0$ for the set of all non-negative integers, $\mathbb{N}^d$ and $\mathbb{N}_0^d$ for the products of d copies of the corresponding sets. We fix an integer $p \geq 2$ and write $\mathbb{Z}_p = \{0, 1, \dots, p-1\}$ for the additive cyclic group of residues modulo p. If necessary, $\mathbb{Z}_p$ can be also thought of as a finite ring or, for a prime p, as the finite field of p elelents.

For $s \in \mathbb{N}_0$, we write

$$\mathbb{Q}(p^s) = \{x = mp^{-s} \in [0,1) : m = 0, 1, \dots, p^s - 1\},$$
$$\mathbb{Q}(p^s) = \{X = (x_1, \dots, x_d) \in U^d : x_j \in \mathbb{Q}(p^s), j = 1, \dots, d\},$$

and

$$\mathbb{Q}(p^\infty) = \bigcup_{s\geq 0} \mathbb{Q}(p^s), \quad \mathbb{Q}^d(p^\infty) = \bigcup_{s\geq 0} \mathbb{Q}^d(p^s).$$

Points in $\mathbb{Q}^d(p^\infty)$ are called the p-ary rational points.

Any $x \in [0,1)$ can be represented in the form

$$x = \sum_{i\geq 1} \eta_i(x) p^{-i}, \tag{2.1}$$

where the digits $\eta_i(x) \in \mathbb{Z}_p$, $i \in \mathbb{N}$. Furthermore, the p-ary expansion (2.1) is unique if we agree that the sum in (2.1) is finite for every p-ary rational points. With this agreetment, we have $\eta_i(x) = 0$ for $i > s$ if $x \in \mathbb{Q}(p^s)$.

For any two points x and y in $\mathbb{Q}(p^\infty)$ and any two scalars α and β in $\mathbb{Z}_p$, we define a linear combination $\alpha x \oplus \beta y$ by

$$\eta_i(\alpha x \oplus \beta u) = \alpha\eta_i(x) + \beta\eta_i(y) \mod p, \quad i \in \mathbb{N}.$$

Similarly, for any two vectors $X = (x_1, \dots, x_d)$ and $Y = (y_1, \dots, y_d)$ in $\mathbb{Q}^d(\rho^\infty)$, we define $\alpha X \oplus \beta Y$ by

$$\alpha X \oplus \beta Y = (\alpha x_1 \oplus \beta y_1, \dots, \alpha x_d \oplus \beta y_d).$$

With respect to the addition $\oplus$ the set $\mathbb{Q}^d(p^\infty)$ is an infinite abelian group while the sets $\mathbb{Q}^d(p^s)$ are subgroups of order p^{ds}. Furthermore, these sets can be thought of as modules over the ring $\mathbb{Z}_p$ or, for a prime p, as vector spaces over the finite field of p elements. The latter observation will be used only for explicit constructions in Section 2.

Finite subgroups $\mathcal{D}$ in $\mathbb{Q}^d(p^\infty)$ can be thought of as distributions in U^d. It turns out that the L^q-discrepancies for such distributions can be evaluated in terms of their dual subjects.

For $s \in \mathbb{N}_0$, we write

$$\mathbb{N}_0(p^s) = \{l \in \mathbb{N}_0 : 0 \le l < p^s\},$$
$$\mathbb{N}_0^d(p^s) = \{L = (l_1, \dots, l_d) \in \mathbb{N}_0^d : l_j \in \mathbb{N}_0(p^s), j = 1, \dots, d\}.$$

Any $l \in \mathbb{N}_0$ can be represented uniquely in the form

$$l = \sum_{i \ge 1} \lambda_i(l) p^{i-1}, \tag{2.2}$$

where the digits $\lambda_i(l) \in \mathbb{Z}_p$, $i \in \mathbb{N}$. Furthtrmore, $\lambda_i(l) = 0$ for $i > s$ if $l \in \mathbb{N}_0(p^s)$.

For any two integers l and k in $\mathbb{N}_0$ and any two scalars α and β in $\mathbb{Z}_p$, we define a linear combination $\alpha l \oplus \beta k$ by

$$\lambda_i(\alpha l \oplus \beta k) = \alpha\lambda_i(l) + \beta\lambda_i(k) \mod p, \quad i \in \mathbb{N}.$$

Similarly, for any two vectors $L = (l_1, \dots, l_d)$ and $K = (k_1, \dots, k_d)$ in $\mathbb{N}_0^d$, we define $\alpha L \oplus \beta K$ by

$$\alpha L \oplus \beta K = (\alpha l_1 \oplus \beta k_1, \dots, \alpha l_d \oplus \beta k_d).$$

With respect to the addition $\oplus$ the set $\mathbb{N}_0^d$ is an infinite abelian group, which will be denoted by $\mathbb{N}_0^d(p^\infty)$, while the sets $\mathbb{N}_0^d(p^s)$ are subgroups of order p^{ds} in $\mathbb{N}_0^d(p^\infty)$.

The group $\mathbb{N}_0(p^\infty)$ is isomorphic to the weak direct product (1.2) while the group $\mathbb{Q}(p^\infty)$ is isomorphic to a dense subgroup in the full direct product (1.1). We shall return to these questions in Section 3.

For every $l \in \mathbb{N}_0(p^s)$ and $x \in \mathbb{Q}(p^s)$, we define a pairing $\langle l, x \rangle$ by

$$\langle l, x \rangle = \sum_{i=1}^{s} \lambda_i(l)\eta_i(x) \mod p,$$

and for every $L=(l_1,\dots,l_d)\in \mathbb{N}_0^d(p^s)$ and $X=(x_1,\dots,x_d)\in \mathbb{Q}^d(p^s)$, we define a pairing $\langle L,X\rangle$ by

$$\langle L,X\rangle=\sum_{j=1}^{d}\langle l_j,x_j\rangle \mod p.$$

For each subgroup $\mathcal{D}\subset \mathbb{Q}^d(p^s)$ its *annihilator* $\widehat{\mathcal{D}}\subset \mathbb{N}_0^d(p^s)$ is defined by

$$\widehat{\mathcal{D}}=\{L\in \mathbb{N}_0^d(p^s):\langle L,X\rangle=0\quad \text{for every}\quad X\in\mathcal{D}\}.$$

It is easy to check that $\widehat{\mathcal{D}}$ is a subgroup in $\mathbb{N}_0^d(p^s)$ and the orders of $\mathcal{D}$ and $\widehat{\mathcal{D}}$ are related by $\#\{\mathcal{D}\}\#\{\widehat{\mathcal{D}}\}=p^{ds}$.

We introduce two metrics on the group $\mathbb{N}_0^d(p^\infty)$. For $l\in\mathbb{N}_0$ the *Hamming weight* $\varkappa(l)$ is defined as the number of non-zero digits in the expansion (2.2), and the *Rosenbloom–Tsfasman weight* $\rho(l)$ is defined by

$$\rho(l)=\begin{cases}0 & \text{if}\quad l=0\\ \max\{i\in\mathbb{N}:\lambda_i(l)\neq 0\} & \text{if}\quad l\in\mathbb{N}.\end{cases}\tag{2.3}$$

For $L=(l_1,\dots,l_d)\in\mathbb{N}_0^d$ we define the weights $\varkappa(L)$ and $\rho(L)$ by

$$\varkappa(L)=\sum_{j=1}^{d}\varkappa(l_j)\quad\text{and}\quad \rho(L)=\sum_{j=1}^{d}\rho(l_j).\tag{2.4}$$

It is easy to check that $\varkappa(L)=\rho(L)=0$ if and only if $L=0$ and that both weights (2.4) satisfy the triangle inequality. These give rise to metrics (or distances) on the group $\mathbb{N}_0^d(p^\infty)$. Furthermore, the definition (2.3) implies that

$$\rho(l\oplus k)\le\max\{\rho(l),\rho(k)\},\quad l,\,k\in\mathbb{N}_0.\tag{2.5}$$

Thus, the Rosenbloom–Tsfasman metric for $d=1$ is an ultrametric.

The Hamming metric is well known in coding theory and related fields (see, for example, MacWilliams and Sloane [MWS]) while the ρ-metric was introduced very recently by Rosenbloom and Tsfasman [RT]. It is remarkable that fundamental concepts related to the Hamming metric can be very naturally extended to the ρ-metric. See Dougherty and Skriganov [DS] and Martin and Stinson [MS]. We refer to [Sk2] for further comments on this matter.

For each subset $\mathcal{C}\subset\mathbb{N}_0^d(p^\infty)$ containing at least two points, we define the weights $\varkappa(\mathcal{C})$ and $\rho(\mathcal{C})$ by

$$\mathrm{wt}\,(\mathcal{C})=\min\{\mathrm{wt}\,(X\ominus Y):X,Y\in\mathcal{C},X\neq Y\},\tag{2.6}$$

where wt stands for any one of the weights ρ or $\varkappa$, and $\ominus$ denotes the subtraction related to the addition $\oplus$. If $\mathcal{C}$ is a subgroup in $\mathbb{N}_0^d(\rho^\infty)$, the relation (2.6) can be written in the form

$$\mathrm{wt}\,(\mathcal{C})=\min\{\mathrm{wt}\,(X):X\in\mathcal{C}\setminus\{0\}\}.\tag{2.7}$$

The appearance of the Rosenbloom–Tsfasman metric in the context of irregularities of distributions can be readily explained by the following (see [Sk2] for more details). Consider elementary rectangular boxes $\Delta_A^M\subset U^d$ of the form

$$\Delta_A^M=[m_1,p^{-a_1},(m_1+1)p^{-a_1})\times\cdots\times[m_jp^{-a_d},(m_j+1)p^{-a_d}),\tag{2.8}$$

where $A = (a_1, \ldots, a_d)$, $M = (m_1, \ldots, m_d) \in \mathbb{N}_0^d$ and $0 \le m_j < p^{a_j}$, $j = 1, \ldots, d$. Any such box has volume vol $\Delta_A^M = p^{-a_1-\ldots-a_d}$.

Recall the following definition (cf. Niederreiter [N1]): Let $0 \le \delta \le s$ be integers. A subset $\mathcal{D} \subset U^d$ consisting of p^s points in called a (δ, s, d)-*net* of deficiency δ (in base p) if every elementary box (2.8) of volume $p^{\delta-s}$ contains exactly p^δ points of $\mathcal{D}$.

It is well known that (δ, s, d)-nets fill out the unit cube very uniformly if $\delta = 0(1)$ as $s \to \infty$ (see, for example, [BC] or [N1]). First constructions of (δ, s, d)-nets in base 2 were given by Sobol [So] and in an arbitrary prime base p by Faure [Fa]. For the further results on this matter we refer to the survey [N2] by Niederreiter.

Lemma 2.1. *Let $\mathcal{D} \subset \mathbb{Q}^d(p^s)$ be a subgroup of order p^s and $\widehat{\mathcal{D}} \subset \mathbb{N}_0^d(p^s)$ be its annihilator. Then the following two statements are equivalent:*

(a) *$\mathcal{D}$ is a (δ, s, d)-net in base p;*

(b) $\rho(\widehat{\mathcal{D}}) \ge s + 1 - \delta$.

This is Theorem 4.2 of [Sk2] or Lemma 2C of [CS].

Lemma 2.1 implies that points of a finite subgroup $\mathcal{D}$ are well distributed in elementary boxes (2.8) if points of its annihilator $\widehat{\mathcal{D}}$ are well spaced in the Rosenbloom–Tsfasman metric.

Unfortunately the weight $\rho(\widehat{\mathcal{D}})$ alone is not sufficient to guarantee the minimal order of the L^q-discrepancies (1.6) (cf. Pillichshammer [P]). We shall see that the bound (1.6) can be established for subgroups $\mathcal{D} \subset \mathbb{Q}^d(p^s)$ if the Rosenbloom–Tsfasman weight $\rho(\widehat{\mathcal{D}})$ is considered in conjunction with the Hamming weight $\varkappa(\widehat{\mathcal{D}})$.

Now we wish to discuss some useful corollaries of the ultrametric inequality (2.5).

Consider the following boxes in $\mathbb{N}_0^d$. For each $A = (a_1, \ldots, a_d) \in \mathbb{N}_0^1$, we put

$$\mathbb{N}_0^d(p^A) = \mathbb{N}_0(p^{a_1}) \times \cdots \times \mathbb{N}_0(p^{a_d}) \tag{2.9}$$

and

$$\Pi_A = \Pi_{a_1} \times \cdots \times \Pi_{a_d}, \tag{2.10}$$

where Π_a, $a \in \mathbb{N}_0$, are the following intervals in $\mathbb{N}_0$

$$\Pi_0 = \{0\}, \quad \Pi_a = \{l \in \mathbb{N} : p^{a-1} \le l < p^a\}, \quad a \in \mathbb{N}.$$

Using (2.3), we can write

$$\mathbb{N}_0^d(p^A) = \{L = (l_1, \ldots, l_d) \in \mathbb{N}_0^d : \rho(l_j) \le a_j, j = 1, \ldots, d\} \tag{2.11}$$

and

$$\Pi_A = \{L = (l_1, \ldots l_d) \in \mathbb{N}_0^d : \rho(l_j) = a_j, j = 1, \ldots, d\}. \tag{2.12}$$

It is known that each ball in the ρ-metric $\mathcal{B}(r) = \{L \in \mathbb{N}_0^d : \rho(L) \le r\}$, $r \in \mathbb{N}_0$, is a union of the boxes $\mathbb{N}_0^d(p^A)$ with $a_1 + \ldots + a_d = r$ and each sphere in the ρ-metric $\mathcal{S}(r) = \{L \in \mathbb{N}_0^d : \rho(L) = r\}$ is a disjoint union of the boxes Π_A with $a_1 + \ldots + a_j = r$. See Lemmas 3.1 and 3.3 of [Sk2].

We observe that the ultrametric inequality (2.5) implies that each box $\mathbb{N}_0^d(p^A)$ is a subgroup of order $p^{a_1+\cdots+a_d}$ in the group $\mathbb{N}_0^d(p^\infty)$. Using this observation, we shall prove the following result, which will be needed in Section 6.

Lemma 2.2. *Let $\mathcal{C} \subset \mathbb{N}_0^d(p^s)$ be an arbitrary subgroup. Then for each $A = (a_1, \dots, a_d) \in \mathbb{N}_0^d$ with $0 \le a_j \le s$, $j = 1, \dots, d$, and $a_1 + \dots + a_d \ge \rho(\mathcal{C})$, we have*

$$\#\{\mathcal{C} \cap \Pi_A\} \le \#\{\mathcal{C} \cap \mathbb{N}_0^d(p^A)\} \le p^{a_1+\ldots+a_d-\rho(\mathcal{C})+1}$$

Proof. The left inequality is trivial, since $\Pi_A \subset \mathbb{N}_0^d(p^A)$ for each A. Let us consider the right inequality. Pick $B = (b_1, \dots b_d) \in \mathbb{N}_0^d$ to satisfy $0 \le b_j \le a_j$, $j = 1, \dots, d$, and $b_1 + \dots + b_d = \rho(\mathcal{C}) - 1$. The box $\mathbb{N}_0^d(p^B)$ is a subgroup in $\mathbb{N}_0^d(p^A)$ and the factor group $\mathbb{N}_0^d(p^A)/\mathbb{N}_0^d(p^B)$ has order $p^{a_1+\ldots+a_d-\rho(\mathcal{C})+1}$. Furthermore, translations $\mathbb{N}_0^d(p^B) \oplus K$ by vectors $K \in \mathbb{N}_0^d(p^A)/\mathbb{N}_0^d(p^B)$ form a partition of the box $\mathbb{N}_0^d(p^A)$. Therefore, it is sufficient to prove that each box $\mathbb{N}_0^d(p^B) \oplus K$ in this partition contains at most one point of the subgroup $\mathcal{C}$. Suppose on the contrary that a box $\mathbb{N}_0^d(p^B) \oplus K$ contains two distinct points L_1 and L_2 of $\mathcal{C}$. Then the box $\mathbb{N}_0^d(p^B)$ contains the point $L_0 = L_1 \ominus L_2 \in \mathcal{C} \setminus \{0\}$ and hence $\rho(\mathcal{C}) \le \rho(L_0) \le b_1 + \dots + b_d = \rho(\mathcal{C}) - 1$. This contradiction completes the proof. $\square$

3. The Main Theorem

Theorem 3.1. *Let $\mathcal{D}$ be a subgroup of order p^s in $\mathbb{Q}^d(p^s)$ and $\widehat{\mathcal{D}}$ be its annihilator in $\mathbb{N}_0^d(p^s)$. Suppose that*

$$\rho(\widehat{\mathcal{D}}) \ge s + 1 - \delta \quad \text{and} \quad \varkappa(\widehat{\mathcal{D}}) \ge 2td + 1,$$

where $t \ge 1$ and $0 \le \delta \le s$ are fixed integers.

Then, the L^{2t}-discrepancy of the distribution $\mathcal{D} \subset U^d$ dstisfies the bound

$$\mathcal{L}_{2t}[\mathcal{D}] < 3p^{d+\delta} t^{1/2} \gamma_p(d, 2t) (s+1)^{\frac{1}{2}(d-1)},$$

where $\gamma_p(d, 2t)$ is the standard constant in the Paley inequality (4.11) for the d-dimensional Walsh functions in base p.

The proof of Theorem 3.1 will be given in Section 6.

In order to deduce Proposition 1.1 from Theorem 3.1 we need to construct explicitly subgroups $\mathcal{D} \subset \mathbb{Q}^d(p^s)$ whose annihilators $\widehat{\mathcal{D}} \subset \mathbb{N}_0^d(p^s)$ have large weights (2.6) simultaneously in both metrics ρ and $\varkappa$. First constructions of such subgroups were given in the author's paper [Sk2]. Here we follow the paper [CS] where these constructions were slightly simplified.

Let p be a prime and let $\mathbb{Z}_p[z]$ denote the ring of polynomials

$$f(z) = \sum_{i \ge 0} f_i z^i$$

with coefficients $f_i \in \mathbb{Z}_p$. For a given polynomial $f(z)$ its k-th hyperderivative $\partial^k f(z) \in \mathbb{Z}_p[z]$ is defined by

$$\partial^k f(z) = \sum_{i \ge 0} \binom{i}{k} f_i z^{i-k},$$

where $\binom{i}{k}$ are the binomial coefficients modulo p, with the usual convention that $\binom{i}{k} = 0$ when $k > i$.

Suppose that the prime p satisfies the condition $p > gd$, where $g \in \mathbb{N}$ is fixed. Then there exist gd distict residues $\beta_{j,l} \in \mathbb{Z}_p$ with $1 \le j \le d$ and $1 \le l \le g$. For every $\sigma \in \mathbb{N}$ and $f \in \mathbb{Z}_p[z]$, let

$$X(f) = (x_1(f), \dots, x_d(f)),$$

where

$$x_j(f) = \sum_{l=1}^{g} p^{-(l-1)\sigma} \sum_{i=1}^{\sigma} \partial^{i-1} f(\beta_{j,l}) p^{-i} \tag{3.1}$$

for every $j = 1, \ldots, d$.

Consider the following set of $p^{g\sigma}$ points in U^d

$$\mathcal{D}(g,\sigma) = \{X(f) : f \in \mathbb{Z}_p[z] \quad \text{and} \quad \deg f < g\sigma\}. \tag{3.2}$$

In the special case $g = 1$ such distribution were studied earlier by Faure [Fa].

Lemma 3.1. *For every g, $\sigma \in \mathbb{N}$ and a prime $p > gd$ the set $\mathcal{D}(g,\sigma)$ is a subgroup of order $p^{g\sigma}$ in $\mathbb{Q}^d(p^{g\sigma})$. Furthermore, its annihilator $\widehat{\mathcal{D}}(g,\sigma)$ in $\mathbb{N}_0^d(p^{g\sigma})$ satisfies the inequalities*

$$\rho(\widehat{\mathcal{D}}(g,\sigma)) \geq g\sigma + 1 \quad \text{and} \quad \varkappa(\widehat{\mathcal{D}}(g,\sigma)) \geq g+1.$$

This is Lemma 2E of [CS] (see also Lemma 5.1 and Theorem 6.1 of [Sk2]).

Comparing Theorem 3.1 and Lemma 3.1, we obtain the following

Corollary 3.1. *For $g = 2td$ and each $\sigma \in \mathbb{N}$ the distribution $\mathcal{D}(g,\sigma)$ of $p^{g\sigma}$ points in v^d satisfies the inequality*

$$\mathcal{L}_{2t}[\mathcal{D}(g,\sigma)] < 3p^d t^{1/2} \gamma_p(d,2t)(g\sigma+1)^{\frac{1}{2}(d-1)}.$$

This proves Proposition 1.1 for all N of the form $p^{g\sigma}$, $\sigma \in \mathbb{N}$. For an arbitrary N we shall use the following known argument.

Suppose that an integer $N > 1$ is given. Take the unique $\sigma \in \mathbb{N}$ to satisfy

$$p^{g(\sigma-1)} < N \leq p^{g\sigma}.$$

By Lemmas 2.1 and 3.1 the set $\mathcal{D}(g,\sigma)$ is a $(d, g\sigma, 0)$-net. Therefore, the subset

$$\mathcal{D}_N^*(g) = \mathcal{D}(g,\sigma) \cap ([0, Np^{-g\sigma}) \times [0,1)^{d-1})$$

contains exactly N points.

Now we consider the following distribution of N points in U^d

$$\mathcal{D}_N(g) = \{(N^{-1}p^{g\sigma}x_1, x_2, \ldots, x_d) : (x_1, \ldots, x_d) \in \mathcal{D}_N^*(g)\}.$$

Corollary 3.2. *For $g = 2td$ and each $N > 1$ the distribution $\mathcal{D}_N(g)$ of N points in U^d satisfies the inequality*

$$\mathcal{L}_{2t}[\mathcal{D}_N(g)] < 3p^d t^{1/2} \gamma_p(d,2t)(\log_p N + 2td + 1)^{\frac{1}{2}(d-1)}.$$

Proof. We have

$$\begin{aligned} \int_{V^d} |\mathcal{L}[\mathcal{D}_N(g); Y]|^{2t} dY &= N^{-1}p^{g\sigma} \int_{[0,Np^{-g\sigma})\times[0,1)^{d-1}} \mathcal{L}[\mathcal{D}(g,\sigma); Y]|^{2t} dY \\ &\leq N^{-1}p^{g\sigma} \int_{U^d} |\mathcal{L}[\mathcal{D}(g,\sigma); Y]|^{2t} dY. \end{aligned}$$

Hence

$$\mathcal{L}_{2t}[\mathcal{D}_N(g)] < p^d \mathcal{L}_{2t}[\mathcal{D}(g,\sigma)].$$

Therefore, with Corollary 3.1 we have

$$\begin{aligned}\mathcal{L}_{2t}[\mathcal{D}_N(g)] &< 3p^{2d}t^{1/2}\gamma_p(d,2t)(g\sigma+1)^{\frac{1}{2}(d-1)}\\ &< 3p^{2d}t^{1/2}\gamma_p(d,2t)(\log_p N+2td+1)^{\frac{1}{2}(d-1)}. \quad \square\end{aligned}$$

Corollary 3.2 implies Proposition 1.1 for every $N>1$.

In conclusion of this section, it should be pointed out that the explicit constructions given by (3.1), and (3.2) can be essentially generalized. For example, instead of large primes $p>2td^2$ we could take an arbitrary prime p, say $p=2$, and consider the above constructions over an extension $\mathbb{F}$ of the finite field $\mathbb{Z}_p$. The degree of this extension has to satisfy the condition $p^{\deg\mathbb{F}}>2td^2$ (cf. [Sk2]). Furthermore, we can take an arbitrary integer $p\geq 2$, not necessary prime, and consider the corresponding construction over a Galois extension $\mathbb{K}$ of the ring $\mathbb{Z}_p$. The cardinality of this extension has to be sufficiently large to choose gd distict elements $\beta_{j,l}\in\mathbb{K}$, $1\leq j\leq d$, $1\leq l\leq g$, in such a way that all non-zero differences $\beta_{j,l}-\beta_{i,k}$ are invertible in the ring $\mathbb{K}$. For purposess of the present paper we have selected the simplest form of possible explicit constructions.

4. Harmonic Analysis on the Cantor-type Groups

In this section, we shall briefly discuss basic concepts and facts on harmonic analysis on the group U_p^d. For details we refer to [EG], [GES], [HR], and [SWSP].

The group U_p, defined by (1.1), consists of all one-way vectors $\xi=(\xi_1,\xi_2,\dots)$, $\xi_i\in\mathbb{Z}_p$, $i\in\mathbb{N}$, with coordinate-wise addition. The topology can be given by a p-adic metric $\varepsilon(\cdot)$, where $\varepsilon(0)=0$ and for $\xi\neq 0$ $\varepsilon(\xi)=p^{-v(\xi)}$ where $v(\xi)=\min\{i\in\mathbb{N}:\xi_i\neq 0\}$. The group U_p is compact and totally disconnected (the connected components are points). The group U_p^d consists of all d-tuples $\Xi=(\xi^{(1)},\dots,\xi^{(d)})$, $\xi^{(j)}\in U_p$, $j=1,\dots,d$.

We define a mapping $\boldsymbol{\eta}:U^d\to U_p^d$ by

$$\boldsymbol{\eta}:[0,1)\ni x\to\boldsymbol{\eta}(x)=(\eta_1(x),\eta_2(x),\dots)\in U_p$$

and

$$\boldsymbol{\eta}:U^d\ni X=(x_1,\dots,x_d)\to\boldsymbol{\eta}(X)=(\boldsymbol{\eta}(x_1),\dots,\boldsymbol{\eta}(x_d))\in U_p^d,$$

where $\eta_i(\cdot)\in\mathbb{Z}_p$, $i\in\mathbb{N}$, are digits in the p-ary expansion (2.1).

The mapping $\boldsymbol{\eta}$ is surjective. For $d=1$ the complement $U_p\setminus\boldsymbol{\eta}([0,1))$ is a countable set consisting of all vectors $\xi=(\xi_1,\xi_2,\dots)\in U_p$ where $\xi_i=p-1$ for every $i>i_0(\xi)$. For $d>1$ the complement $U_p^d\setminus\boldsymbol{\eta}(U^d)$ consists of all d-tuples $\Xi=(\xi^{(1)},\dots,\xi^{(d)})\in U_p^d$ where at least one of the coordinates $\xi^{(j)}$, $j=1,\dots,d$, belongs to $U_p\setminus\boldsymbol{\eta}([0,1))$.

Geometrically, the group U_p can be thought of as a modified interval $[0,1)^*$ where each p-ary rational point $x=mp^{-a}$, $m=1,\dots,p^a-1$, $a\in\mathbb{N}$, splits into a left part with a finite expansion (2.1) and a right part with an alterhative infinite expansion (2.1).

It is well known that the mapping $\boldsymbol{\eta}$ sends the Lebesgue measure on U^d to the unique Haar measure μ on the group U_p^d normalized by the condition $\mu(U_p^d)=1$. Therefore, for each q the mapping $\boldsymbol{\eta}$ generates an isometric isomorphism between the functional spaces $L^q(U^d)$ and $L^1(U_p^d)$.

Since the mapping $\boldsymbol{\eta}$ is not bijective, the group structure can not be transfered from U_p^d onto the unit cube U^d. For example, the points $x=p/p+1$ and $y=1/p+1$ have the p-ary expansions

$\boldsymbol{\eta}(x) = (p-1, 0, p-1, 0, \dots)$ and $\boldsymbol{\eta}(y) = (0, p-1, 0, p-1, \dots)$ but there not exists a point in $[0,1)$ with the p-ary expansion $\boldsymbol{\eta}(x) + \boldsymbol{\eta}(y) = (p-1, p-1, \dots)$.

Nevertheless, the group structures on U_p^d and $\mathbb{Q}^d(p^\infty)$ are in correct agreement. More precisely, we have

$$\boldsymbol{\eta}(X \oplus Y) = \boldsymbol{\eta}(X) + \boldsymbol{\eta}(Y)$$

for every X and Y in $\mathbb{Q}^d(p^\infty)$. Forthermore, it is easy to check that the image $\boldsymbol{\eta}(\mathbb{Q}^d(p^\infty))$ is a dense subgroup in U_p^d.

The dual group $\hat{U}_p^d$ can be described as follows. For each $L = (l_1, \dots, l_d) \in \mathbb{N}_0^d$ we define a character $\Omega_L(\cdot)$ of the group U_p^d by

$$\Omega_L(\Xi) = \prod_{j=1}^d \omega_{l_j}(\xi^{(j)}), \tag{4.1}$$

where

$$\omega_l(\xi) = \exp\left(\frac{2\pi\sqrt{-1}}{p} \sum_{i \geq 1} \lambda_i(l)\xi_i\right),$$

where $\xi(\xi_1, \xi_2, \dots) \in U_p$ and $\lambda_i(l)$, $i \in \mathbb{N}$, are digits in the p-ary expansion (2.2) for $l \in \mathbb{N}_0$.

Notice that the functions (4.1) are continuous on U_p^d and each character of the group U_p^d can be written in the form (4.1). This defines an isomorphism between the groups U_p^d and $\mathbb{N}_0^d(p^\infty)$.

Using the mapping $\boldsymbol{\eta}$, we define the following functions on the unit cube U^d

$$W_L(X) = \Omega_L(\boldsymbol{\eta}(X)), \quad L \in \mathbb{N}_0^d.$$

We have

$$W_L(X) = \prod_{j=1}^d w_{l_j}(x_j), \tag{4.2}$$

where

$$\omega_l(\xi) = \exp\left(\frac{2\pi\sqrt{-1}}{p} \sum_{i \geq 1} \lambda_i(l)\eta_i(x)\right), \quad l \in \mathbb{N}_0, \tag{4.3}$$

where $\eta_i(x)$, $i \in \mathbb{N}$, are digits in the p-ary expansion (2.1) for $x \in [0,1)$.

The function (4.3) and their multi-dimensional generalizations (4.2) are known as the Walsh functions if $p = 2$ and the Chrestenson or Chrestenson–Levy functions if $p > 2$. For simplisity we refer to them all as Walsh functions (in base p).

The Walsh functions W_L inherit the main properties of the characters Ω_L (exept those related to continuity). In particular, we have the following relations:

$$W_{L \oplus K}(X) = W_L(X) W_K(X) \tag{4.4a}$$

for every L, $K \in \mathbb{N}_0^d(p^\infty)$ and $X \in U^d$;

$$W_L(X \oplus Y) = W_L(X) W_L(Y) \tag{4.4b}$$

for every X, $Y \in \mathbb{Q}^d(p^\infty)$ and $L \in \mathbb{N}_0^d$;

$$\overline{W_L(X)} = W_{\overline{L}}(X) \tag{4.4c}$$

for every $L \in \mathbb{N}_0^d(p^\infty)$ and $X \in U^d$, here $\overline{W}_L(\cdot)$ is the complex conjugation and $\overline{L}$ denotes the unique vector in $\mathbb{N}_0^d(p^\infty)$ such that $L \oplus \overline{L} = 0$.

From (4.4b) we conclude that the Walsh functions $W_L(X)$ are characters of finite groups $\mathbb{Q}^d(p^s)$. It is easy to see that the set of Walsh functions $\{W_L(X) : L \in \mathbb{N}_0^d(p^s)\}$ forms the group of all characters for the group $\mathbb{Q}^d(p^s)$. Therefore, known results on characters of finite abelian groups imply the following

Lemma 4.1. *Let $\mathcal{D}$ be a subgroup in $\mathbb{Q}^d(p^s)$ and $\widehat{\mathcal{D}}$ denote its unnihilator in $\mathbb{N}_0^d(p^s)$. Then, for every $L \in \mathbb{N}_0^d(p^s)$ we have*

$$\sum_{X\in\mathcal{D}} \overline{W_L(X)} = \begin{cases} \#\{\mathcal{D}\} & \text{if} \quad L \in \widehat{\mathcal{D}}, \\ 0 & \text{if} \quad L \notin \widehat{\mathcal{D}}. \end{cases}$$

It is well know that the Walsh functions (4.2) form an orthonormal basis in the space $L^2(U^d)$. More precisely, we have

$$\int_{U^d} W_K(X)\overline{W_L(X)}dX = \int_{U^d} W_{K\oplus L}(X)dX = \delta(K \ominus L), \tag{4.5}$$

where Kroneker's delta $\delta(K) = 0$ if $K \in \mathbb{N}_0^d \setminus \{0\}$ and $\delta(0) = 1$.

For each function $f \in L^2(U^d)$, we have the Fourier–Walsh expansion

$$f(X) \simeq \sum_{L\in\mathbb{N}_0^d} \hat{f}_L \overline{W_L(X)}, \tag{4.6}$$

where the Fourier–Walsh coefficients

$$\hat{f}_L = \int_{U^d} W_L(X) f(X) dX, \tag{4.7}$$

and the symbol $\simeq$ denotes the L^2-convergence. We also have the identity

$$\|f\|_2^2 = \sum_{L\in\mathbb{N}_0^d} |\hat{f}_L|^2. \tag{4.8}$$

Hereinafter, we write $\|\cdot\|_q$ for the standard norm in the space $L^q(U^d)$,

$$\|f\|_q = \left(\int_{U^d} |f(X)|^q dX \right)^{1/q}, \quad 1 \le q < \infty.$$

Now we briefly discuss the Littlewood–Paley theory which gives the most adequate approach to study the Fourier–Walsh expansions in the L^q-spaces. For broad classes of totally disconnected abelian groups this theory was developed by Spector [Sp]. In our discussion we close follow Edwards and Gaudry [EG].

Suppose that a function $f \in L^2(U^d) \cap L^q(U^d)$ is given. Consider the *Littlewood-Paley decomposition*

$$f(X) \simeq \sum_{A\in\mathbb{N}_0^d} P_A f(X), \tag{4.9}$$

where $P_A f(X)$ are the following partial sums

$$P_A f(X) = \sum_{L\in\Pi_A} \hat{f}_L \overline{W_L(X)}$$

associated with boxes Π_A, $A \in \mathbb{N}_0^d$, see (2.10) and (2.12).

The *square function* $S[f(X)]$ is defined by

$$S[f(X)] = \Big(\sum_{A \in \mathbb{N}_0^d} |P_A f(X)|^2 \Big)^{1/2}. \tag{4.10}$$

It should be recognized that $S[f(X)]$ is correctly defined for each $X \in U^d$, since all terms of the sum in (4.11) are non-negative. Furthermore, the set of $X \in U^d$ where this sum is equal to infinity has zero measure, since $\|S[f]\|_2 = \|f\|_2 < \infty$.

With this background, we have the Paley inequality

$$\{\gamma_p(d,q)\}^{-1} \|S[f]\|_q \le \|f\|_q \le \gamma_p(d,q) \|S[f]\|_q, \tag{4.11}$$

where $1 < q < \infty$ and the constant $\gamma_p(d,q) \ge 1$ is independent of the function f.

We comment here that for $d = 1$ the inequality (4.11) is contained in Theorem 4.1.8 of [EG] applied to the group U_p, while for $d > 1$ this inequality follows from that for $d = 1$ and Theorem 1.3.4 of [EG] applied to the product U_p^d of d copies of the group U_p. It should be pointed out that in the special case $d = 1$ and $p = 2$ the inequality (4.11) is the classical result of Paley [Pa].

Notice also that the constants $\gamma_p(d,q)$ in the inequalities (4.11) can be evaluated efficiently (cf. [EG]) but it seems likely that in the case $d > 1$ their explicit expressions are absent in the literature.

In the following, we shall need only the right inequality in (4.11) for $q \ge 2$. Using Minkowski's inequality for the $L^{q/2}$-norm, we immediately obtain the following

Lemma 4.2. *For an arbitrary function $f \in L^q(U^d)$, $2 \le q < \infty$, we have*

$$\|f\|_q \le \gamma_p(d,q) \Big(\sum_{A \in \mathbb{N}_0^d} \|P_A f\|_q^2 \Big)^{1/2},$$

where the constant $\gamma_p(d,q) \ge 1$ is independent of the function f.

5. Walsh Functions and Error-Correcting Codes

We begin with the Fourier-Walsh expansion (4.6) for the characteristic function $\chi(B_Y, X)$ of the rectangular box $B_Y = [0, y_1) \times \cdots \times [0, y_d)$, $Y = (y_1, \dots, y_d) \in U^d$,

$$\chi(B_Y, X) = \begin{cases} 1 & \text{if} \quad X \in B_Y, \\ 0 & \text{if} \quad X \notin B_Y. \end{cases}$$

With (4.6) and (4.7) we have

$$\chi(B_Y, X) \simeq \sum_{L \in \mathbb{N}_0^d} \Big(\int_{B_Y} W_L(X) dX \Big) \overline{W_L(X)}.$$

Let us write this expansion in the form

$$\chi(B_Y, X) \simeq \sum_{L \in \mathbb{N}_0^d} p^{-\rho(L)} \Psi_L(Y) \overline{W_L(X)}, \tag{5.1}$$

where $\rho(L)$, $L = (l_1, \ldots, l_d) \in \mathbb{N}_0^d$, is the Rosenbloom–Tsfasman weight (2.4) and

$$\Psi_L(Y) = \prod_{j=1}^{d} \psi_{l_j}(y_j), \tag{5.2}$$

where

$$\psi_l(y) = p^{\rho(l)} \int_0^y w_l(x)dx. \tag{5.3}$$

With this normalization we have the bounds

$$c_1 < \|\Psi_L\|_q < c_2, \quad 1 \le q < \infty, \tag{5.4}$$

where the positive constants c_1 and c_2 are independent of $L \in \mathbb{N}_0^d$. We shall prove (5.4) somewhat below.

Our goal is to study the function system $\{\Psi_L : L \in \mathbb{N}_0^d\}$ in the spaces $L^q(U^d)$. In general, this is a very bad system. For example, the linear mapping $\Sigma c_L W_L \to \Sigma c_L \Psi_L$ is unbounded even in $L^2(U^d)$. However, it turns out that this function system contains very good subsystems $\{\Psi_L : L \in \mathcal{C}\}$, $\mathcal{C} \subset \mathbb{N}_0^d$, provided that vectors L in the subset (code) $\mathcal{C}$ are well spaced in the Hamming metric.

The functions (5.3) can be evaluated precisely. The corresponding explicit formulas were found by Fine [Fi] for $p = 2$ and by Price [P2] for $p > 2$ (see also Hellekalek [H]).

First of all, we have

$$\psi_0(y) = y = \frac{1}{2} + \sum_{i \ge 1} p^{-i} \sum_{j=1}^{p-1} (\zeta^{-j} - 1) w_{p^{i-1}}(y), \tag{5.5}$$

where $\zeta = \exp(2\pi\sqrt{-1}/p)$ is a primitive p-th root of unity. Notice that for $p = 2$ this formula is a well known expansion for the fractional part $\{y\}$, $y \in \mathbb{R}$, in terms of the Rademacher functions $r_i(y) = w_{2^i}(y)$, $i \in \mathbb{N}_0$.

Next, for every $l \in \mathbb{N}$ we write $l = \lambda(l)p^{\rho(l)-1} + \tau(l)$, where $\lambda(l) = \lambda_{\rho(l)}(l) \neq 0 \pmod p$ and $0 \le \tau(l) < p^{\rho(l)-1}$. With this notation we have the following Fine–Price formula

$$\psi_l(y) = (1 - \zeta^{\lambda(l)})^{-1} w_{\tau(l)}(y) + (\frac{1}{2} - \left(1 - \zeta^{\lambda(l)})^{-1}\right) w_l(y)$$
$$+ \sum_{i \ge 1} p^{-i} \sum_{j=1}^{p-1} \zeta^j (1 - \zeta^j)^{-1} w_{l + jp^{\rho(l)+i-1}}(y). \tag{5.6}$$

Notice that for $j \neq 0 \pmod p$ we have

$$|1 - \zeta^j| = 2|\sin \frac{\pi j}{p}| \ge 2 \sin \frac{\pi}{p} \ge \frac{4}{p}, \tag{5.7}$$

so that the series (5.5) and (5.6) converge absolutely and uniformly for $y \in [0, 1)$. In particular, we have the bounds $|\psi_l(y)| < p$ and

$$|\Psi_L(Y)| < p^d \tag{5.8}$$

for every $L \in \mathbb{N}_0^d$ and $Y \in U^d$. Using (5.7), we find also that $\|\Psi_L\|_1 \ge \left(\frac{1}{4}p\right)^d$. These imply the bounds (5.4).

Our key observation is contained in the following

Lemma 5.1. *For each $t \in \mathbb{N}$ we have the generalized orthogonality*

$$\int_{\mathfrak{v}^d} \overline{\Psi_{K_1}(Y)} \dots \overline{\Psi_{K_t}(Y)} \Psi_{L_1}(Y) \dots \Psi_{L_t}(Y) dY = 0, \tag{5.9}$$

provided that the Hamming distance $\varkappa(L \ominus K) \geq 2td + 1$, where $K = K_1 \oplus \dots \oplus K_t$ and $L = L_1 \oplus \dots \oplus L_t$.

In particular, for $t = 1$ we have the usual orthogonality

$$\int_{\mathfrak{v}^d} \overline{\Psi_K(Y)} \Psi_L(Y) dY = 0, \tag{5.10}$$

provided that the Hamming distance $\varkappa(L \ominus K) \geq 2d + 1$.

Proof. Let us consider the Walsh functions involved in the series (5.6). It is easy to see that

$$\tau(l) = l \ominus \lambda(l) p^{\rho(l)-1} = l \oplus \overline{\lambda(l)} p^{\rho(l)-1},$$

where $\lambda(l) \oplus \overline{\lambda(l)} = 0 \pmod p$, and

$$l + jp^{\rho(l)+i-1} = l \oplus jp^{\rho(l)+i-1}$$

for every $l \in \mathbb{N}$, $i \in \mathbb{N}$, $j \in \mathbb{Z}_p$. Using (4.4a) and (4.4c), we can write

$$w_{\tau(l)}(y) = w_l(y) w_{\overline{\lambda(l)} p^{\rho(l)-1}}(y)$$

and

$$w_{l+jp^{\rho(l)+i-1}}(y) = w_l(y) w_{jp^{\rho(l)+i-1}}(y).$$

Now, from (5.5) and (5.6), we conclude that each function $\psi_l(y)$, $l \in \mathbb{N}_0$, can be represented as the following absolutely and uniformly convergent series

$$\psi_l(y) = w_l(y) \sum_{\varkappa(a) \leq 1} c_{l,a} w_a(y) \tag{5.11}$$

with certain complex coefficients $c_{l,a}$ whose precise values are of no importance for the present discussion. The only crucial point here is that the summation in (5.11) is extended to integers $a \in \mathbb{N}_0$ with the Hamming weight $\varkappa(a) \leq 1$.

Substituting (5.11) to (5.2), we find that

$$\Psi_L(Y) = W_L(Y) \sum_{\varkappa(A) \leq d} c_{L,A} W_A(Y),$$

where the summation is extended to vectors $A \in \mathbb{N}_0^d$ with the Hamming weight $\varkappa(A) \leq d$.

Similarly, we find that the integrand in (5.9) has the form

$$\overline{\Psi_{K_1}(Y)} \dots \overline{\Psi_{K_t}(Y)} \Psi_{L_1}(Y) \dots \Psi_{L_t}(Y) = W_{K \ominus L}(Y) \sum_{\varkappa(A) \leq 2td} c_A W_A(Y), \tag{5.12}$$

where $c_A = c_A\ (K_1, \ldots, K_t, L_1, \ldots, L_t)$ are certain complex coefficients and the summation in (5.12) is extended to vectors $A \in \mathbb{N}_0^d$ with the Hamming weight $\varkappa(A) \leq 2td$.

Since the series (5.12) converges absolutely and uniformly, we conclude that the integral (5.9) is equal to

$$\sum_{\varkappa(A) \leq 2td} c_A \int_{U^d} W_{L \ominus K \oplus A}\ (Y) dY. \tag{5.13}$$

With the triangle inequality for the Hamming metric we have

$$\varkappa(L \ominus K \oplus A) \geq \varkappa(L \ominus K) - \varkappa(A) \geq 2td + 1 - 2td = 1.$$

Therefore, $L \ominus K \oplus A \neq 0$ always, and by (4.5) all integrals in (5.13) vanish. $\square$

Suppose that a finite subset $\mathcal{C} \subset \mathbb{N}_0^d$ is given. Consider functions of the form

$$f(Y) = \sum_{L \in \mathcal{C}} c_L \Psi_L(Y), \tag{5.14}$$

where c_L are arbitrary complex coefficients.

We define a quasi-linear mapping $f \to f^\natural$ by

$$f^\natural(Y) = \sum_{L \in \mathcal{C}} |c_L| W_L(Y). \tag{5.15}$$

Lemma 5.2. *Let $\mathcal{C} \subset \mathbb{N}_0^d(p^\infty)$ be a finite subgroup and $\varkappa(\mathcal{C})$ denote its Hamming weight (2.7). Then, for every $t \in \mathbb{N}$ with $2td < \varkappa(\mathcal{C})$ the L^{2t}-norms of the functions (5.14) and (5.15) are related by*

$$\|f\|_{2t} < p^d \|f^\natural\|_{2t}. \tag{5.16}$$

Proof. We have

$$|f(Y)|^{2t} = \sum \bar{c}_{K_1} \ldots \bar{c}_{K_t} c_{L_1} \ldots c_{L_t} \overline{\Psi_{K_1}(Y)} \ldots \overline{\Psi_{K_t}(Y)} \Psi_{L_1}(Y) \ldots \Psi_{L_t}(Y),$$

where the summation is extended to all $K_1, \ldots, K_t$, $L_1, \ldots L_t$ in $\mathcal{C}$. Therfore

$$\|f\|_{2t}^{2t} = \sum \bar{c}_{K_1} \ldots \bar{c}_{K_t} c_{L_1} \ldots c_{L_t} J(K_1, \ldots, K_t, L_1, \ldots L_t),$$

where $J(K_1, \ldots, K_t, L_1, \ldots, L_t)$ denotes the integral (5.9).

Since $\mathcal{C}$ is a subgroup in $\mathbb{N}_0^d(p^\infty)$, the vectors $K = K_1 \oplus \cdots \oplus K_t$ and $L = L_1 \oplus \cdots \oplus L_t$ belong to $\mathcal{C}$, and by Lemma 5.1 integrals $J(K_1, \ldots, K_t, L_1, \ldots, L_t)$ vanish whenever $K \neq L$. This gives

$$\|f\|_{2t}^{2t} = \sum \bar{c}_{K_1} \ldots \bar{c}_{K_t} c_{L_1} \ldots c_{L_t} J(K_1, \ldots, K_t, L_1, \ldots L_t) \delta(L \ominus K).$$

On the other hand, the bound (5.8) implies

$$|J(K_1, \ldots, K_t, K_1, \ldots, L_t)| < p^{2td}.$$

Therefore, we have

$$\|f\|_{2t}^{2t} < p^{2td} \sum |c_{K_1}| \ldots |c_{K_t}| |c_{L_1}| \ldots |c_{L_t}| \delta(L \ominus K). \tag{5.17}$$

Using (4.4a), (4.4c) and (4.5), we can write

$$\delta(L \ominus K) = \int_{U^d} \overline{W}_{K_1}(Y) \dots \overline{W}_{K_t}(Y) W_{L_1}(Y) \dots W_{L_t}(Y) dY.$$

Substituting this formula to (5.17), we find that

$$\|f\|_{2t}^{2t} < p^{2td} \int_{U^d} \left| \sum_{L \in \mathcal{C}} |c_L| W_L(Y) \right|^{2t} dY = p^{2td} \|f^{\natural}\|_{2t}^{2t}.$$

This completes the proof of Lemma 5.1. $\square$

It is interesting to note that in Lemma 5.2 the assumption that $\mathcal{C}$ is a subgroup in $\mathbb{N}_0^d(p^\infty)$ is essential only for the L^{2t}-norms with $t > 1$. Using the orthogonality relation (5.10), one can prove that for $t = 1$ the bound (5.16) holds with an arbitrary subset $\mathcal{C}$ in (5.14) if the Hamming weight $\varkappa(\mathcal{C})$ is defined by (2.6). Furthermore, in this case for each function (5.14) we have the identity

$$\|f\|_2^2 = \sum_{L \in \mathcal{C}} |c_L|^2 \|\Psi_L\|_2^2, \tag{5.18}$$

provided that $\varkappa(\mathcal{C}) \geq 2d+1$. The norms $\|\Psi_L\|_2$ in (5.18) can be easily evaluated by (5.5), (5.6), and (4.8).

It must be emphasized that the above Lemmas 5.1 and 5.2 are particular cases of much more general results which will be geven elsewhere.

6. Proof of the Main Theorem

Now we are ready to prove Theorem 3.1.

The local discrepancy (1.3) can be written in the form

$$\mathcal{L}[\mathcal{D}; Y] = \sum_{X \in \mathcal{D}} \Lambda(Y, X), \tag{6.1}$$

where $\Lambda(Y, X) = \chi(B_Y, X) - \operatorname{vol} B_Y$.

From (5.1) we conclude that $\Lambda(Y, X)$ as a function of $X \in U^d$ has the following Fourier–Walsh expansion

$$\Lambda(Y, X) \simeq \sum_{L \in \mathbb{N}_0^d \setminus \{0\}} p^{-\rho(L)} \Psi_L(Y) \overline{W_L(X)}.$$

To handle this series pointwise, we shall use the following argument. For any $s \in \mathbb{N}$ we write

$$\Lambda(Y, X) = \Lambda_s(Y, X) + \mathcal{R}_s(Y, X), \tag{6.2}$$

where

$$\Lambda_s(Y, X) = \sum_{L \in \mathbb{N}_0^d(p^s) \setminus \{0\}} p^{-\rho(L)} \Psi_L(Y) \overline{W_L(X)}$$

is a finite sum.

For $y \in [0, 1)$, let $\Delta_s(y) = [mp^{-s}, (m+1)p^{-s})$ denote the unique elementary interval of lenght p^{-s} containing y, and for $Y = (y_1, \dots, y_d) \in U^d$ and every $j = 1, \dots, d$, let

$$\Delta_{j,s}(Y) = [0, 1)^{j-1} \times \Delta_s(y_j) \times [0, 1)^{d-j}.$$

We write $\chi(\Delta_{j,s}(Y), X)$ for the characteristic function of the elementary box $\Delta_{j,s}(Y)$.

Lemma 6.1. *With the above notation we have*

$$\mathcal{R}_s(Y,X) = \sum_{j=1}^{d} r_{j,s}(Y,X),$$

where

$$0 \le r_{j,s}(Y,X) \le \chi(\Delta_{j,s}(Y),X)$$

for every $j = 1,\dots,d$. In particular, $r_{j,s}(Y,X) = 0$ whenever $X \notin \Delta_{j,s}(Y)$.

This is Lemma 5B of [CS].

Substituting (6.2) to (6.1), we find that

$$\mathcal{L}[\mathcal{D};Y] = \mathcal{L}_s[\mathcal{D};Y] + \mathcal{R}_s[\mathcal{D};Y], \tag{6.3}$$

where

$$\Lambda_s[\mathcal{D};Y] = \sum_{X\in\mathcal{D}} \Lambda_s(Y,X) = \sum_{L\in\mathbb{N}_0^d(p^s)\setminus\{0\}} p^{-\rho(L)}\Psi_L(Y) \sum_{X\in\mathcal{D}} \overline{W_L(X)} \tag{6.4}$$

and

$$\mathcal{R}_s[\mathcal{D};Y] = \sum_{X\in\mathcal{D}} \mathcal{R}_s(Y,X) = \sum_{j=1}^{d}\sum_{X\in\mathcal{D}} r_{j,s}(Y,X).$$

Furthermore, with Lemma 6.1 we have

$$0 \le \mathcal{R}_s[\mathcal{D};Y] \le \sum_{j=1}^{d} \#\{\mathcal{D}\cap\Delta_{j,s}(Y)\}.$$

From this point on, we suppose that the distribution $\mathcal{D}$ satisfies the hypothesis of Theorem 3.1. Thus, $\mathcal{D}$ is a subgroup of order p^s in $\mathbb{Q}^d(p^s)$ and its annihilator $\widehat{\mathcal{D}}$ in $\mathbb{N}_0^d(p^s)$ satisfies the conditions

$$\rho(\widehat{\mathcal{D}}) \ge s+1-\delta \tag{6.5a}$$

and

$$\varkappa(\widehat{\mathcal{D}}) \ge 2t+1, \tag{6.5b}$$

where $0 \le \delta \le s$ and $t \ge 1$ are fixed integers.

Applying Lemma 4.1 with $\mathcal{C} = \widehat{\mathcal{D}}$ to the inner sum in (6.4), we find that

$$\Lambda_s[\mathcal{D};Y] = \sum_{L\in\widehat{\mathcal{D}}\setminus\{0\}} p^{s-\rho(L)}\Psi_L(Y). \tag{6.6}$$

Using Lemma 2.1 together with the condition (6.5a), we observe that $\mathcal{D}$ is a (d,s,δ)-net. Each elementary box $\Delta_{j,s}(Y)$ has volume p^{-s} and is contained in some elementary box Δ_A^M of volume $p^{\delta-s}$. Therefore, $\#\{\mathcal{D}\cap\Delta_{j,s}(Y)\} \le p^\delta$ and

$$0 \le \mathcal{R}_s[\mathcal{D};Y] \le dp^\delta. \tag{6.7}$$

Combining (6.3) and (6.7) with the definition of the L^q-discrepancy (1.4), we obtain the bound

$$\mathcal{L}_{2t}[\mathcal{D}] \le \|\Lambda_s[\mathcal{D};\cdot]\|_{2t} + dp^\delta. \tag{6.8}$$

Hence, we need to estimate the L^{2t}-norm for the function (6.6).

Using Lemma 5.2 together with the condition (6.5b), we obtain the bound

$$\|\Lambda_s[\mathcal{D};\cdot]\|_{2t} < p^d \|\Lambda_s^\natural[\mathcal{D};\cdot]\|_{2t}, \tag{6.9}$$

where

$$\Lambda_s^\natural[\mathcal{D};Y] = \sum_{L\in\widehat{\mathcal{D}}\setminus\{0\}} p^{s-\rho(L)} W_L(Y). \tag{6.10}$$

Consider the Littlewood–Paley decomposition (4.9) for the function (6.10). We have

$$\Lambda_s^\natural[\mathcal{D};Y] = \sum_{A\in\mathcal{A}_s} P_A\Lambda_s^\natural[\mathcal{D};Y], \tag{6.11}$$

where for short we write

$$\mathcal{A}_s = \{A=(a_1,\dots,a_d)\in\mathbb{N}_0^d\setminus\{0\} : 0\le a_j\le s, j=1,\dots,d\},$$

and where

$$P_A\Lambda_s^\natural[\mathcal{D};Y] = \sum_{L\in\widehat{\mathcal{D}}\cap\Pi_A} p^{s-\rho(L)} W_L(Y).$$

Recall that all vectors $L\in\Pi_A$ have the same weight $\rho(L)=a_1+\dots+a_d$, see (2.12). Therefore we have

$$P_A\Lambda_s^\natural[\mathcal{D};Y] = p^{s-a_1-\dots-a_d}\Phi_A[\widehat{\mathcal{D}};Y],$$

where

$$\Phi_A[\widehat{\mathcal{D}};Y] = \sum_{L\in\widehat{\mathcal{D}}\cap\Pi_A} W_L(Y).$$

Now we apply Lemma 4.2 to the function (6.11). This gives

$$\begin{aligned}&\|\Lambda_s^\natural[\mathcal{D};\cdot]\|_{2t}\\ &\le \gamma_p(d,2t)\Bigg(\sum_{A\in\mathcal{A}_s} p^{2(s-a_1-\dots-a_d)}\,\|\Phi_A[\widehat{\mathcal{D}};\cdot]\|_{2t}^2\Bigg)^{1/2}.\end{aligned} \tag{6.12}$$

Lemma 6.2. *With the above notation we have*

$$\Phi_A[\widehat{\mathcal{D}};Y] = 0, \tag{6.13}$$

if $a_1+\dots+a_d < s+1-\delta$, *and*

$$\|\Phi_A[\widehat{\mathcal{D}};\cdot]\|_{2t} \le p^{\left(1-\frac{1}{2t}\right)(a_1+\dots+a_d-s+\delta)}, \tag{6.14}$$

if $a_1+\dots+a_d \ge s+1-\delta$.

Proof. Form the condition (6.5a), we conclude that boxes Π_A do not contain points of $\widehat{\mathcal{D}}$ at all if $A\in\mathcal{A}_s$ and $a_1+\dots a_d < s+1-\delta$. This implies (6.13).

To prove (6.14), we use the bound

$$\int_{U^d} |\Phi_A[\widehat{\mathcal{D}};Y]|^{2t} dY \leq \left(\sup_{Y\in U^d} |\Phi_A[\widehat{\mathcal{D}};Y]| \right)^{2t-2} \int_{U^d} |\Phi_A[\widehat{\mathcal{D}};Y]|^2 dY.$$

On the other hand, using Lemma 2.2 with $\mathcal{C} = \widehat{\mathcal{D}}$, we observe that

$$\sup_{Y\in V^d} |\Phi_A[\widehat{\mathcal{D}};Y]| \leq \#\{\widehat{\mathcal{D}} \cap \Pi_A\} \leq p^{a_1+\dots+a_d-s+\delta},$$

since $|W_L(Y)| = 1$, see (4.2) and (4.3), and also

$$\int_{U^d} |\Phi_A[\widehat{\mathcal{D}};Y]|^2 dY = \#\{\widehat{\mathcal{D}} \cap \Pi_A\} \leq p^{a_1+\dots+a_d-s+\delta},$$

since the Walsh functions are orthonormal, see (4.5) and (4.8).

Combining the above bounds, we find that

$$\int_{U^d} |\Phi_A[\widehat{\mathcal{D}};Y]|^{2t} dY \leq p^{(2t-1)(a_1+\dots+a_d-s+\delta)}.$$

This implies (6.14). $\square$

With Lemma 6.2 we can write the bound (6.12) in the form

$$\|\Lambda_s^{\#}[\mathcal{D};\cdot]\|_{2t} \leq \gamma_p(d,2t)p^{(1-1/2t)\delta} \left(\sum_{A\in\mathcal{B}_s} p^{-(a_1+\dots a_d-s)/t} \right)^{1/2}, \tag{6.15}$$

where for short we write

$$\mathcal{B}_s = \{A = (a_1,\dots,a_d) \in \mathbb{N}_0^d : 0 \leq a_j \leq s, j = 1,\dots,d, \quad \text{and} \quad a_1+\dots+a_d \geq s+1-\delta\}.$$

The set $\mathcal{B}_s$ can be represented as a disjoint union of the following subsets

$$\mathcal{B}_s = \bigcup_{k=s+1-\delta}^{ds} \mathcal{B}_{s,k},$$

where $\mathcal{B}_{s,k} = \{A \in \mathcal{B}_s : a_1+\dots+a_d = k\}$. It is easy to see that $\#\{\mathcal{B}_{s,k}\} \leq (s+1)^{d-1}$ for each k.

Therefore, the sum in (6.15) can be estimated as follows

$$\begin{aligned}
\sum_{A\in\mathcal{B}_s} p^{-(a_1+\dots+a_d-s)/t} &\leq (s+1)^{d-1} \sum_{k=s+1-\delta}^{ds} p^{-(k-s)/t} \\
&< (s+1)^{d-1} \sum_{k=s+1-\delta}^{\infty} p^{-(k-s)/t} \\
&= (s+1)^{d-1} p^{(\delta-1)/t}(1-p^{-1/t})^{-1} \\
&< 4tp^{\delta/t}(s+1)^{d-1}.
\end{aligned} \tag{6.16}$$

At the last step in (6.16) we used the following elementary inequality $(1-p^{-1/t})^{-1} < 4t$ for every $p \geq 2$ and $t \geq 1$.

Substituting (6.16) to (6.15), we get the bound

$$\|\Lambda_s^{\natural}[\mathcal{D};\cdot]\|_{2t} < 2t^{1/2}p^{\delta}\gamma_p(d,2t)(s+1)^{\frac{1}{2}(d-1)}. \tag{6.17}$$

Now we combine the bounds (6.17), (6.9) and (6.8). This gives

$$\begin{aligned}\mathcal{L}_{2t}[\mathcal{D}] &< 2t^{1/2}p^{d+\delta}\gamma_p(d,2t)(s+1)^{\frac{1}{2}(d-1)} + dp^{\delta} \\ &\leq 3t^{1/2}p^{d+\delta}\gamma_p(d,2t)(s+1)^{\frac{1}{2}(d-1)}.\end{aligned}$$

The proof of Theorem 3.1 is complete.

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