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The Calderon problem for two-dimensional manifolds by the BC-method

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0 Introduction.

0.1 General statement.

Let (Ω, g) be a smooth [1] n-dimensional compact Riemann manifold with the border Γ consisting of a finite number of connected components; g the metric tensor; Δ_g the Beltrami–Laplace operator. Consider the Dirichlet problem

$$\Delta_g u = 0 \quad in \ \mathrm{int}\,\Omega \tag{0.1}$$

$$u = f \quad on \ \Gamma; \tag{0.2}$$

let $u = u^f(x)$ be the solution for a smooth$f$.

To the problem (0.1),(0.2) one associates the DN (Dirichlet-to-Neumann)–map $\Lambda_g : f \to \frac{\partial u^f}{\partial \nu}|_\Gamma$ (ν is the outward normal).

Denote g_Γ the metric on Γ induced by g. Assume that the border (Γ, g_Γ) is given and the operator Λ_g is known; *the Calderon problem* is to recover (Ω, g). Shortly speaking, one needs to find a manifold realizing the given DN-map.

Our paper tarnsfers a part of A.Nachman results ([7] ; see also [11]) on the case of manifolds. However, it should be noted that we use another technique because the exponentially growing solutions (Faddeev-Green functions) which are the main instrument of solving the Calderon problem in $\mathbf{R}^n$, are lacking on manifolds.

0.2 The case of $n = 2$. Main result.

The well known pecularity of the 2–dim case is the following. If g' and g'' are two metrics on Ω such that $g'' = \rho g'$ with a positive function ρ (so that g'' is the conformal deformation of g') then the DN-maps of the manifolds (Ω, g') and (Ω, g'') are connected through the relation $\Lambda_{g''} = \frac{1}{\sqrt{\rho}}\Lambda_{g'}$, whereas the additional condition $\rho|_\Gamma = 1$ implies $\Lambda_{g''} = \Lambda_{g'}$. This motivates the following definition.

Let (Ω', g') and (Ω'', g'') be two smooth orientable manifolds with the common border $\partial\Omega' = \partial\Omega'' = \Gamma$; we say them to be *conformally equivalent* if there exist a diffeomorphism $\beta : \Omega' \mapsto \Omega''$, $\beta(\Omega') = \Omega''$, $\beta|_\Gamma = \mathrm{id}$ and

[1] Everywhere in the paper 'smooth' means C^∞ -smooth.

a positive function $\rho \in C^\infty(\Omega')$, $\rho|_\Gamma = 1$ such that β is an isometry of $(\Omega', \rho g')$ onto (Ω'', g'').

The conformal equivalence of manifolds implies the coincidence of their DN-maps. The main result of the paper is the converse assertion.

Theorem 1 *Two 2-dim manifolds with common border are conformally equivalent iff their DN-maps coincide.*

In other words, in dimension 2 the class of conformally equivalent manifolds is determined by its DN-map.

0.3 A bit of philosophy.

Setting the goal to recover the manifold through its DN-map one comes up against the naive question: what the 'material' can it be constructed from?

A traditional situation in the boundary value inverse problems is the following: a domain $\Omega \subset \mathbf{R}^n$ *is given* whereas a function $c(x)$ on Ω (density, conductivity etc) has to be found through inverse data. Thus, we possess an initial reserve of points Ω, can mark an $x_0 \in \Omega$ and then discuss a possibility to determine $c(x_0)$.

It is not the case in the Calderon problem for manifolds: Ω is unknown in itself and has to be recovered, so that the question of 'material' turns out to be principal [2].

We propose to identify Ω to the spectrum (maximal ideal space or, the same, the set of multiplicative functionals) of some function algebra determined by inverse data. So, the 'material' proposed is the set of the Dirac measures $\{\delta_{x_0} \mid x_0 \in \Omega\}$.

0.4 The plan.

The following basic facts are in use:

(i) (Ω, g) carries the complex structure of the Riemann surface determining the class of metrics conformally equivalent to g;

[2] In the dynamic inverse problem this role is played by a subset of the space-time cylinder $\Gamma \times [0, T]$ (the so-called 'pattern'- see [1]). Some attempts to invoke the time for solving the Calderon problem were udertaken but, to the best of our knowledge, they were not successful.

(ii) the algebra $\mathcal{A}(\Omega)$ of functions continuous in Ω and holomorphic in $\operatorname{int}\Omega$ is nontrivial; functions $w \in \mathcal{A}(\Omega)$ (as local homeomorphisms $\Omega \mapsto C$) determine the complex structure;

(iii) algebra $\mathcal{A}(\Omega)$ is normal: its (topologized) spectrum is homeomorphic to the manifold: $\operatorname{sp}\mathcal{A}(\Omega) \asymp \Omega$ whereas the algebra itself is identical to its canonical realization (the Gelfand transform): $\mathcal{A}(\Omega) \equiv \hat{\mathcal{A}}(\Omega)$;

(iv) the algebra of the traces $\mathcal{A}(\Gamma) := \{w|_\Gamma \,|\, w \in \mathcal{A}(\Omega)\}$ is isometrically isomorphic to $\mathcal{A}(\Omega)$ (through the map $\operatorname{tr} : w \mapsto w|_\Gamma$); the isometry yields $\operatorname{sp}\mathcal{A}(\Gamma) \asymp \operatorname{sp}\mathcal{A}(\Omega), \quad \hat{\mathcal{A}}(\Gamma) \equiv \hat{\mathcal{A}}(\Omega)$ that leads to $\operatorname{sp}\mathcal{A}(\Gamma) \asymp \Omega$ and $\hat{\mathcal{A}}(\Gamma) \equiv \mathcal{A}(\Omega)$;

(v) algebra $\mathcal{A}(\Gamma)$ is determined by the DN-map Λ_g.

For solving the Calderon problem we use these facts in the inverse order:

(α) through the operator Λ_g one recovers the trace algebra $\mathcal{A}(\Gamma)$;

(β) finding its spectrum and canonical realization we get $\Omega \asymp \operatorname{sp}\mathcal{A}(\Gamma)$ and $\mathcal{A}(\Omega) \equiv \hat{\mathcal{A}}(\Gamma)$;

(γ) using functions of algebra $\mathcal{A}(\Omega)$ we endow Ω with the complex structure;

(δ) introducing a metric g on Ω conforming to the complex structure and having the preassigned g_Γ we get the manifold (Ω, g) whose DN-map coincides with Λ_g by construction.

The procedure $(\alpha) - (\delta)$ gives the canonical representator of the class of conformally equivalent manifolds which has the given DN-map. The assertion of Theorem 1 is a simple corollary of determinacy of this procedure.

0.5 A bit of philosophy more.

Let us clarify why we consider our approach as a variant of the BC–method. In dynamical problems the BC-method

creates a model (isometrical copy) of the initial system through inverse data;

proposes a mechanism (the amplitude formulas of Geometrical Optics) recovering the initial system through the model.

(see [1], [2]). In this paper the same is being done: DN-map determines trace algebra $\mathcal{A}(\Gamma)$ which is a model of algebra $\mathcal{A}(\Omega)$; the Gelfand transform recovers $\mathcal{A}(\Omega)$ (together with the manifold Ω). Moreover, there are some reasons to guess that the mechanism based upon the Geometrical Optics is in fact a kind of the Gelfand transform.

1 Harmonic functions and fields.

1.1 Harmonic functions.

So, we have a 2-dim smooth orientable compact (Ω, g) with the border Γ.

Let $\mathrm{Harm}_g\,\Omega := \{u \mid \Delta_g\, u = 0 \quad in \;\mathrm{int}\,\Omega\}$ be the set of functions harmonic in Ω; in local coordinates one has

$$\frac{\partial}{\partial x^i}\det^{\frac{1}{2}} g(x)\, g^{ij}(x)\frac{\partial u(x)}{\partial x^j} = 0. \tag{1.1}$$

The solutions of the problem (0.1),(0.2) are said to be *potentials*; the family of potentials $\mathcal{U} := \{u^f \mid f \in C^\infty(\Gamma)\}$ coincides with $\mathrm{Harm}_g\,\Omega \bigcap C^\infty(\Omega)$.

1.2 Rotation operator.

Introduce near Γ the semigeodesic coordinates γ, τ such that $ds^2 = Ed\gamma^2 + d\tau^2$, $\;E|_\Gamma = 1$ (so that γ is the natural parameter on Γ), and $\tau(x) = \mathrm{dist}(x, \Gamma)$. The field $\frac{\partial}{\partial\gamma}$ fixes an orientation of Γ and an orientation of Ω along with it; note that $\frac{\partial}{\partial\tau} = -\frac{\partial}{\partial\nu}$.

The orientation determines the family of rotations $\Phi(x) \in \mathrm{End}\, T_x\Omega$ such that $\langle\Phi(x)\, a, \Phi(x)\, b\rangle = \langle a, b\rangle$, $\langle\Phi(x)\, a, a\rangle = 0$ for $a, b \in T_x\Omega$ and $\Phi(\gamma)\frac{\partial}{\partial\nu} = \frac{\partial}{\partial\gamma}$ on Γ. Note that $\Phi^2(x) = -\mathrm{id}$, $x \in \Omega$.

Let $\vec{\mathcal{L}}$ be the (real) space of the square summable vectorial fields on (Ω, g) with the inner product $(a, b) = \int\limits_\Omega \mathrm{d}\,\Omega_x \langle a(x), b(x)\rangle$. The *rotation operator* $\Phi : \vec{\mathcal{L}} \mapsto \vec{\mathcal{L}}$, $(\Phi y)(x) := \Phi(x)y(x)$, $x \in \Omega$ is a unitary operator in $\vec{\mathcal{L}}$.

1.3 Harmonic fields.

Let $\vec{\mathcal{H}} := \{h \in \vec{\mathcal{L}} \,|\, \mathrm{div}h = \mathrm{div}\Phi h = 0 \quad in \;\mathrm{int}\,\Omega\}$ be the (sub) space of harmonic fields. The harmonic fields are smooth in $\mathrm{int}\,\Omega$; the set $\vec{\mathcal{H}}^\infty := \vec{\mathcal{H}} \bigcap \vec{C}^\infty(\Omega)$ is dense in $\vec{\mathcal{H}}$.

A harmonic field is determined by its trace: $h|_\Gamma = 0$ implies $h = 0$ in $\;\Omega$.

The space $\vec{\mathcal{H}}$ contains the subspace of potential fields $\vec{\mathcal{E}} := \{\nabla u \in \vec{\mathcal{H}} \,|\, u \in \mathrm{Harm}_g\,\Omega \bigcap H^1(\Omega)\}$. The set $\vec{\mathcal{E}}^\infty := \vec{\mathcal{E}} \bigcap \vec{C}^\infty(\Omega) = \nabla\mathcal{U}$ is dense in $\vec{\mathcal{E}}$. In order to formulate one of the properties of the potential fields let us adopt an agreement on notations:

in the sequel we denote by the common symbol 'tr' the reduction of fields and functions given in Ω on the border Γ;

for a set $\mathcal{B}$ of functions on Γ, $\dot{\mathcal{B}} := \{f \in \mathcal{B} \mid \int\limits_\Gamma d\gamma\, f(\gamma) = 0\}$ is its subset of functions with zero mean value;

Jf denotes the primitive function with zero mean value: $\frac{d}{d\gamma} Jf = f$, $\int\limits_\Gamma d\gamma (Jf)(\gamma) = 0$.

Lemma 1 *The representation*

$$\mathrm{tr}\, \vec{\mathcal{E}}^\infty = \{\alpha \frac{\partial}{\partial\gamma} + (\Lambda_g J\alpha) \frac{\partial}{\partial\nu} \ \mid \alpha \in \dot{C}^\infty(\Gamma)\}$$

is valid.

Proof.

(i) Let $k \in \mathrm{tr}\, \vec{\mathcal{E}}^\infty$; then

$$k = \mathrm{tr}\, \nabla u^f = \frac{\partial u^f}{\partial\gamma}\frac{\partial}{\partial\gamma} + \frac{\partial u^f}{\partial\tau}\frac{\partial}{\partial\tau} = \frac{\partial u^f}{\partial\gamma}\frac{\partial}{\partial\gamma} + \frac{\partial u^f}{\partial\nu}\frac{\partial}{\partial\nu} =$$

$$= \frac{df}{d\gamma}\frac{\partial}{\partial\gamma} + \Lambda_g f \frac{\partial}{\partial\nu} = \alpha\frac{\partial}{\partial\gamma} + (\Lambda_g J\alpha)\frac{\partial}{\partial\nu}$$

with $\alpha = \frac{df}{d\gamma} \in \dot{C}^\infty(\Gamma)$.

(ii) Let $k = \alpha\frac{\partial}{\partial\gamma} + (\Lambda_g J\alpha)\frac{\partial}{\partial\nu}$ with $\alpha \in \dot{C}^\infty(\Gamma)$. Denoting $f := J\alpha$ one has

$$\mathrm{tr}\, \vec{\mathcal{E}}^\infty \ni \mathrm{tr}\, \nabla u^f = \frac{\partial u^f}{\partial\gamma}\frac{\partial}{\partial\gamma} + \frac{\partial u^f}{\partial\tau}\frac{\partial}{\partial\tau} = \frac{df}{d\gamma}\frac{\partial}{\partial\gamma} + (\Lambda_g f)\frac{\partial}{\partial\nu} =$$

$$= \alpha\frac{\partial}{\partial\gamma} + (\Lambda_g J\alpha)\frac{\partial}{\partial\nu} = k.$$

The lemma is proved.

The subspace $\vec{\mathcal{D}} := \vec{\mathcal{H}} \ominus \vec{\mathcal{E}}$ has a finite dimension determined by topology of Ω; its elements are smooth fields: $\vec{\mathcal{D}} \subset \vec{C}^\infty(\Omega)$ [9] . The fields $b \in \vec{\mathcal{D}}$ are tangent on Γ; indeed, for $f \in C^\infty(\Gamma)$ we have

$$0 = \int\limits_\Omega d\Omega\, \langle \nabla u^f, b\rangle = \int\limits_\Gamma d\gamma\, f\, \langle b, \frac{\partial}{\partial\nu}\rangle$$

yielding $\langle b, \frac{\partial}{\partial\nu}\rangle = 0$.

1.4 Conjugate fields and potentials.

The rotation operator is invariant on $\vec{\mathcal{H}} : \Phi\vec{\mathcal{H}} = \vec{\mathcal{H}}$; its part acting in $\vec{\mathcal{H}}$ is denoted by the same symbol Φ.

To a field $h \in \vec{\mathcal{H}}$ we associate the *conjugate field* $h^* := \Phi h \in \vec{\mathcal{H}}$; note the equality $(h^*)^* = -h$.

A field $\nabla u \in \vec{\mathcal{E}}$ and its potential u are called *simple* if $(\nabla u)^* = \Phi\nabla u \in \vec{\mathcal{E}}$; in this case the potential u^* of the conjudate field $(\nabla u)^* = \nabla u^*$ is said to be *conjugate* to u. The conjugate potential is simple; it is determined by potential u up to constant, the equality $(u^*)^* = -u \ (\mathrm{mod\, const})$ being valid. Let $\mathcal{P} \subset \mathrm{Harm}_g\,\Omega$ be the set of simple potentials; denote $\mathcal{P}^\infty := \mathcal{P} \bigcap C^\infty(\Omega)$.

If Ω is homeomorphic to the disk, each potential is simple; it is not the case if the topology of Ω is not trivial [5].

1.5 Simple functions.

A function $f \in C^\infty(\Gamma)$ is said to be *simple* if the corresponding potential u^f is simple (has the conjugate $(u^f)^*$). Let $\mathcal{F}$ be the set of simple functions; note the equality $\mathcal{F} = \mathrm{tr}\,\mathcal{P}^\infty$.

Each simple function has the conjugate $f^* \in \mathcal{F} : f^* := \mathrm{tr}\,(u^f)^*$. The conjugate function is determined by the original one up to constant, and the relations $(f^*)^* = -f \ (\mathrm{mod\, const})$, $u^{f^*} = (u^f)^*$, $\frac{df^*}{d\gamma} = \Lambda_g J \frac{df}{d\gamma}$ hold, the last one following from the equalities

$$\frac{df^*}{d\gamma} = \langle \nabla u^{f*}, \frac{\partial}{\partial\gamma}\rangle = \langle \Phi\nabla u^f, \frac{\partial}{\partial\gamma}\rangle = \langle \nabla u^f, \Phi^* \frac{\partial}{\partial\gamma}\rangle =$$

$$\langle \nabla u^f, \frac{\partial}{\partial\nu}\rangle = \frac{\partial u^f}{\partial\nu} = \Lambda_g f = \Lambda_g J \frac{df}{d\gamma}.$$

The following fact plays the key role for the sequel: the set of simple functions is determined by the DN-map. Recall two of its properties: $\mathrm{Ker}\,\Lambda_g = \{\mathrm{const}\}$, $\mathrm{Ran}\,\Lambda_g \subset \dot{C}^\infty(\Gamma)$.

Lemma 2 *The relations $f \in \mathcal{F}$ and $\frac{df}{d\gamma} = -(\Lambda_g J)^2 \frac{df}{d\gamma}$ are equivalent.*

Proof. (i) If $f \in \mathcal{F}$ then $\Phi\nabla u^f \in \vec{\mathcal{E}}^\infty$, hence

$$\mathrm{tr}\,\Phi\nabla u^f = \Phi[\frac{\partial u^f}{\partial\gamma}\frac{\partial}{\partial\gamma} + \frac{\partial u^f}{\partial\tau}\frac{\partial}{\partial\tau}] = \Phi[\frac{df}{d\gamma}\frac{\partial}{\partial\gamma} + (\Lambda_g f)\frac{\partial}{\partial\nu}] =$$

$$= \frac{df}{d\gamma}\Phi\frac{\partial}{\partial\gamma} + (\Lambda_g f)\Phi\frac{\partial}{\partial\nu} = -\frac{df}{d\gamma}\frac{\partial}{\partial\nu} + (\Lambda_g f)\frac{\partial}{\partial\gamma} =$$

$$= (\Lambda_g J\frac{df}{d\gamma})\frac{\partial}{\partial\gamma} - \frac{df}{d\gamma}\frac{\partial}{\partial\nu} \in \mathrm{tr}\,\vec{\mathcal{E}}^{\infty}.$$

Comparing the last with the representation of Lemma 1 (with $\alpha = \Lambda_g J\frac{df}{d\gamma}$) we obtain

$$-\frac{df}{d\gamma} = \Lambda_g J(\Lambda_g J\frac{df}{d\gamma}) = (\Lambda_g J)^2\frac{df}{d\gamma}.$$

(ii) If a smooth $f \notin \mathcal{F}$ then $\Phi\nabla u^f \notin \vec{\mathcal{E}}^{\infty}$, and by virtue of the decomposition $\vec{\mathcal{H}} = \vec{\mathcal{E}} \oplus \vec{\mathcal{D}}$ one has

$$\Phi\nabla u^f = \nabla u^p + b \tag{1.2}$$

where $b \in \vec{\mathcal{D}}$, $\mathrm{tr}\, b = \kappa\frac{\partial}{\partial\gamma}$ with nonzero κ. Setting equal the traces of the fields and the coefficients of the base $\frac{\partial}{\partial\gamma}, \frac{\partial}{\partial\nu}$ one easily gets

$$\Lambda_g f = \frac{dp}{d\gamma} + \kappa; \;\; -\frac{df}{d\gamma} = \Lambda_g p,$$

that yields

$$\frac{df}{d\gamma} = -\Lambda_g J\frac{dp}{d\gamma} = -\Lambda_g J(\Lambda_g f - \kappa) =$$

$$= -(\Lambda_g J)^2\frac{df}{d\gamma} + \Lambda_g J\kappa.$$

The term $\Lambda_g J\kappa$ is a nonzero function, since $\Lambda_g J\kappa = 0$ would imply $J\kappa =$ const, $\kappa = 0$, and $\mathrm{tr}\, b = 0$; subsequently $b = 0$ and $\Phi\nabla u^f \in \vec{\mathcal{E}}$ in contadiction to the initial $f \notin \mathcal{F}$. Thus, $\frac{df}{d\gamma} \neq -(\Lambda_g J)^2\frac{df}{d\gamma}$. The lemma is proved.

On the harmonic fields the mapping tr is a linear isomorphism (on its range). This fact, together with representation (1.2), easily leads to the inequality

$$\mathrm{codim}_{\Phi\mathrm{tr}\,\vec{\mathcal{E}}^{\infty}}\ \mathrm{tr}\,\vec{\mathcal{E}}^{\infty} \leq \dim\vec{\mathcal{D}}.$$

Since the lineal set $\mathrm{tr}\,\vec{\mathcal{E}}^{\infty}$ is determined by the DN-map (Lemma 1), from this inequality some conclusions can be drawn regarding topology of Ω by Λ_g.

2 Algebras.

2.1 CBA-dictionary.

Begin with a minimal information about the Commutative Banach Algebras (CBA's); for details see [3], [6], [8]

A. The CBA is a (complex or real) Banach space $\mathcal{A}$ equipped with the multiplication operation ab satisfying $ab = ba$; $||ab|| \leq ||a||\,||b||$, $a, b \in \mathcal{A}$. Example: the algebra of functions $C(X)$ continuons on a (topological) compact X with the norm $||a|| = \max_\Omega |a(\cdot)|$; the subalgebras of $C(X)$ are called *functional algebras.*

CBA is said to be *uniform* if it has the unit $e : ea = a$ and the relation $||a^2|| = ||a||^2$ holds. All the function algebras are uniform.

B. A subspace $I \neq \mathcal{A}$ is called *ideal*, if $ja \in I$ for any $j \in I$, $a \in \mathcal{A}$. Ideal I is *maximal* if for any ideal $\tilde{I} \subset \mathcal{A}$ the relation $I \subset \tilde{I}$ implies $I = \tilde{I}$.

Let $\mathcal{I}$ be the set of maximal ideals of algebra $\mathcal{A}$. Every $I \in \mathcal{I}$ is closed; $\mathrm{codim} I = 1$; the quotient space $\mathcal{A}/I$ is isometrically isomorphic to the field of complex numbers $\mathbf{C}$.

C. Let $\mathcal{A}'$ be the space of continuons functionals on $\mathcal{A}$. A functional $\delta \in \mathcal{A}'$ is called *multiplicative* if $\delta(ab) = \delta(a)\delta(b)$. Example: the Dirac measure $\delta_{x_0} \in C'(X) : \delta_{x_0}(a) = a(x_0)$. The set of multiplicative functionals is denoted by $\mathcal{M}$.

D. There exists a canonical bijection between the sets $\mathcal{M}$ and $\mathcal{I}$: if $\delta \in \mathcal{M}$ then $I_\delta := \mathrm{Ker}\delta \in \mathcal{I}$; if $I \in \mathcal{I}$ then the projection $\delta_I : \mathcal{A} \mapsto \mathcal{A}/I = \mathbf{C}$ is an element of $\mathcal{M}$. In what follows we identify $\mathcal{M}$ to $\mathcal{I}$.

E. The *Gelfand transform* maps element $a \in \mathcal{A}$ into function $\hat{a}$ on $\mathcal{M}$ by the rule: $\hat{a}(\delta) := \delta(a)$, $\delta \in \mathcal{M}$. The Gelfand topology is the weakest topology on $\mathcal{M}$ in which all of $\hat{a}$ are continuous. The set $\mathcal{M}$ equipped with this topology is a compact; this compact is called *spectrum* of the algebra $\mathcal{A}$ (or the maximal ideal space) and denoted by $\mathrm{sp}\,\mathcal{A}$.

The Gelfand transform $\hat{\mathcal{A}} := \{\hat{a} \,|\, a \in \mathcal{A}\}$ of algebra $\mathcal{A}$ is a subalgebra of the algebra $C(\mathrm{sp}\,\mathcal{A})$. If $\mathcal{A}$ is a uniform algebra, the map $a \mapsto \hat{a}$ is an isometric isomorphism: $(\alpha a + \beta b + cd)^\wedge = \alpha\hat{a} + \beta\hat{b} + \hat{c}\hat{d}$, $||\hat{a}|| = ||a||$ for all $\alpha, \beta \in \mathbf{C}$ and $a, b, c, d, \in \mathcal{A}$; in this case $\hat{\mathcal{A}}$ is a canonical realization of $\mathcal{A}$ in the form of function algebra.

F. An isomorphism $t : A(X) \mapsto B(Y)$ of two function algebras is called *spatial* if there exists a bijection $\beta : X \mapsto Y$ such that $tw = w \circ \beta^{-1}$.

For function algebra $\mathcal{A} \subset C(X)$, to each point $x_0 \in X$ one associates the Dirac measure $\delta_{x_0} \in \operatorname{sp}\mathcal{A}$, due to that the natural embedding $\varepsilon : X \mapsto \operatorname{sp}\mathcal{A}$, $\varepsilon(x_0) = \delta_{x_0}$ occurs.

A function algebra $\mathcal{A} \subset C(X)$ is said to be *normal* if ε is a homeomorphism of X onto $\operatorname{sp}\mathcal{A}$. Normal algebra is spatially isomorphic to its canonical realization $\hat{\mathcal{A}}$: $\hat{w} = w \circ \varepsilon^{-1}$.

2.2 Algebra $\mathcal{A}(\Omega)$.

Recall that the set of the simple potentials $\mathcal{P}$ was introduced in sec.1.4. Assigne a simple potential u to the family $\mathcal{P}^0$ if $u, u^* \in C(\Omega)$.

Define the family of complex functions $\mathcal{A}(\Omega) := \{w = u + iu^* \,|\, u \in \mathcal{P}^0\}$ and mention some of its properties [5]:

(i) in proper (isothermal) local coordinates x^1, x^2 each $w \in \mathcal{A}(\Omega)$ is a holomorphic function of $z = x^1 + ix^2$; functions of $\mathcal{A}(\Omega)$, as local homeomorphisms of Ω into $\mathbf{C}$, determine on Ω a complex structure of the Riemann surface;

(ii) $\mathcal{A}(\Omega)$ is a closed subalgebra of the (complex) algebra $C(\Omega)$;

(iii) for each $w \in \mathcal{A}(\Omega)$ $|w|$ attains maximum on Γ.

Lemma 3 *The algebra $\mathcal{A}(\Omega)$ is normal.*

Proof. (1) The embedding $\varepsilon(\Omega) \subset \operatorname{sp}\mathcal{A}(\Omega)$ is a general fact (see F, sec.2.1) and one needs to prove the converse embedding only.

(2) Choose an $I \in \operatorname{sp}\mathcal{A}(\Omega)$; we are going to show that there exists a point $x_I \in \Omega$ such that $j(x_I) = 0$ for all $j \in I$. Assume the opposite: for any $x \in \Omega$ one can find $j_x \in I$ such that $j_x(x) \neq 0$. Due to continuity of j_x there exists a neighbourhood $U_x \ni x$ such that j_x doesn't vanish in U_x. Since $\bigcup\limits_x U_x$ covers Ω, we can select a finite subcover $U_{x_1}, ..., U_{x_p}$; in this case the corresponding $j_{x_1}, ..., j_{x_p}$ have no common zeros in Ω.

(3) Due to smoothnes of the border Γ the manifold (Ω, g) may be embedded into a wider manifold $(\Omega', g') : \Omega \subset \Omega', \operatorname{dist}_{g'}(\partial\Omega', \Omega) = r_0 > 0, g'|_\Omega = g$. One way to realize such the embedding is to use the semigeodesic coordinates, extending E smoothy on $\tau < 0$.

Denote $\Omega_r := \{x \in \Omega' \,|\, \operatorname{dist}_{g'}(x, \Omega) < r\}, 0 < r < r_0$. Assign a function $w \in \mathcal{A}(\Omega)$ to the family $\mathcal{O}(\Omega)$ if it has a holomorphic continuation in Ω_r

(with r depending on w). Using the smoothness of the border one can show that the family $\mathcal{O}(\Omega)$ is dense in $\mathcal{A}(\Omega)$ [3].

(4) Since $\operatorname{codim} I = 1$ the set $I \bigcap \mathcal{O}(\Omega)$ is dense in I. Due to this one can approximate $j_{x_1}, ..., j_{x_p}$ by functions $\tilde{j}_{x_1}, ..., \tilde{j}_{x_p} \in I \bigcap \mathcal{O}(\Omega)$ so that $\tilde{j}_{x_k} \neq 0$ in U_{x_k} and $\tilde{j}_{x_1}, ..., \tilde{j}_{x_p}$ have no common zeros in Ω.

Since the set $\{\tilde{j}_{x_k}\}$ is finite there exists $\Omega_r \supset \Omega$ such that $\tilde{j}_{x_1}, ..., \tilde{j}_{x_k}$ are defined and have no common zeros in Ω_r. Considering Ω_r as a non-compact Riemann surface one can find functions $g_1, ..., g_p$ holomorphic in Ω_r and satisfying the condition $g_1\tilde{j}_{x_1} + ... + g_p\tilde{j}_{x_p} = 1$ everywhere in Ω_r (see [5], p.205). Reducing functions on Ω we get $\tilde{g}_1\tilde{j}_{x_1} + ... + \tilde{g}_p\tilde{j}_{x_p} = 1$ in Ω for $\tilde{g}_k := g_k|_\Omega \in \mathcal{A}(\Omega)$ and $\tilde{j}_{x_k} \in I$ that yields the inclusion $1 \in I$. The last leads evidently to $I = \mathcal{A}(\Omega)$ that contradicts I to be a proper ideal.

Thus, all the functions belonging to I vanish in a point $x_I \in \Omega$ and the embedding $\varepsilon(\Omega) \supset \operatorname{sp} \mathcal{A}(\Omega)$ is established. So, $\varepsilon(\Omega) = \operatorname{sp} \mathcal{A}(\Omega)$ and it remains to show that ε is a homeomorphism.

(5) Identifying maximal ideals of $\mathcal{A}(\Omega)$ to multiplicative functionals we set $I_{x_0} \equiv \delta_{x_0}$ where δ_{x_0} is the Dirac measure. In terms of the bijection $\varepsilon : \Omega \mapsto \operatorname{sp} \mathcal{A}(\Omega)$, existence of which has been proved above, the Gelfand transform takes the form $\hat{w}(\delta) := \delta(w) = w(x_\delta) = (w \circ \varepsilon^{-1})(\delta)$ (is a spatial isomorphism).

Topologizing $\operatorname{sp} \mathcal{A}(\Omega)$ by Gelfand we endow it with the weakest topology in which all $\hat{w} \in \hat{\mathcal{A}}(\Omega)$ are continuous. Since the corresponding topology on Ω determined by functions $w \in \mathcal{A}(\Omega)$ coincides with the initial (metric) topology of (Ω, g), the bijection ε turns out to be a homeomorphism. The lemma is proved.

The density of $\mathcal{O}(\Omega)$ in $\mathcal{A}(\Omega)$ used in the proof gives one more important property of the algebra $\mathcal{A}(\Omega)$:

(iv) the smooth subalgebra $\mathcal{A}^\infty(\Omega) := \mathcal{A}(\Omega) \bigcap C^\infty(\Omega) = \{u + iu^* \,|\, u \in \mathcal{P}^\infty\}$ is dense in $\mathcal{A}(\Omega)$.

2.3 Simple topology.

Denote $D := \{z \in \mathbf{C} \,|\, |z| \leq 1\}$. The well know fact [8] is that the disc algebra $\mathcal{A}(D)$ has one generator (for instance, the function $w(z) = z$ gener-

[3] the proof of the density which we have belongs to G.M.Henkin (private communication).

ates $\mathcal{A}(D)$). This property is connected with topology of the manifold in the following way.

Lemma 4 *The algebra $\mathcal{A}(\Omega)$ has one generator iff Ω is homeomorphic to D.*

Proof. If Ω is homeomorphic to D it admits the *global* isothermic coordinates x^1, x^2 (such that $z = x^1 + ix^2 \in \mathcal{A}(\Omega)$) mapping Ω onto $z(\Omega) = D$ (see [12]), so that the map $w \mapsto w \circ z^{-1}$ sets an isometric isomorphism of $\mathcal{A}(\Omega)$ onto $\mathcal{A}(D)$. Since $\mathcal{A}(D)$ has one generator the same is true for $\mathcal{A}(\Omega)$.

If $w \in \mathcal{A}(\Omega)$ generates $\mathcal{A}(\Omega)$ then w maps Ω onto $w(\Omega) \subset \mathbf{C}$ injectively and, hence, homeomorphically; in this case $w(\Gamma) = \partial w(\Omega)$ is a Jardin curve in $\mathbf{C}$. By the Caratheodory theorem there exists a function $h \in C(w(\Omega))$ holomorphic in $\operatorname{int} w(\Omega)$ which maps $w(\Omega)$ onto D homeomorphically. The function $h \circ w$ sets a homeomorphism of Ω onto D. The lemma is proved.

2.4 Algebra $\mathcal{A}(\Gamma)$.

The set of traces $\mathcal{A}(\Gamma) := \operatorname{tr} \mathcal{A}(\Omega) = \{\operatorname{tr} w \mid w \in \mathcal{A}(\Omega)\}$ is a closed subalgebra of the (complex) algebra $C(\Gamma)$. The following is some of its properties.

(i) Since $\max|w|$ for $w \in \mathcal{A}(\Omega)$ is attained at Γ, the equality $||w||_{\mathcal{A}(\Omega)} = ||\operatorname{tr} w||_{\mathcal{A}(\Gamma)}$ holds. Therefore, the mapping tr is an isometric isomorphism of algebras.

(ii) Since $\mathcal{A}(\Omega)$ is normal, whereas $\mathcal{A}(\Gamma)$ is isometrically isomorphic to $\mathcal{A}(\Omega)$, the canonical realization $\hat{\mathcal{A}}(\Gamma) \subset C(\operatorname{sp} \mathcal{A}(\Gamma))$ turns out to be spatially isomorphic to $\mathcal{A}(\Omega)$. The corresponding spatial isomorphism of $\mathcal{A}(\Omega)$ onto $\hat{\mathcal{A}}(\Gamma)$ is $\wedge \operatorname{tr}$, whereas the corresponding space bijection is $\beta : \Omega \mapsto \operatorname{sp} \mathcal{A}(\Gamma)$, $\beta(x_0) = \operatorname{tr} I_{x_0}$.

(iii) In $\mathcal{A}(\Gamma)$ define the smooth subalgebra $\mathcal{A}^\infty(\Gamma) := \mathcal{A}(\Gamma) \bigcap C^\infty(\Gamma) = \operatorname{tr} \mathcal{A}^\infty(\Omega)$. Recalling the definition of the set $\mathcal{F}$ of the simple functions we have the representation $\mathcal{A}^\infty(\Gamma) = \{f + if^* \mid f \in \mathcal{F}\}$.

(iv) The density of $\mathcal{A}^\infty(\Omega)$ in $\mathcal{A}(\Omega)$ (see (iv), sec.2.2) leads to the density of $\mathcal{A}^\infty(\Gamma)$ in $\mathcal{A}(\Gamma)$ and gives the representation

$$\mathcal{A}(\Gamma) = \operatorname{clos}_{C(\Gamma)} \{f + if^* \mid f \in \mathcal{F}\}. \tag{2.1}$$

Since the set $\mathcal{F}$ is determined by the DN-map (Lemma 2), this representation determines $\mathcal{A}(\Gamma)$ through Λ_g.

2.5 Conforming metrics.

The algebra $\mathcal{A}(\Omega)$ as well as the complex structure on Ω are determined by the metric g, the structures corresponding to metrics g and ρg being identical since these metrics define one and the same reserve of harmonic functions [4].

The converse is true in the following sense. Say metric g to be *conforming* to the complex structure determined by the algebra $\mathcal{A}(\Omega)$ if $\mathrm{Re}\mathcal{A}(\Omega) \subset \mathrm{Harm}_g\,\Omega$. As may be shown, any two metrics conforming to a given structure are distinguished by a (function) multiplier.

To construct a metric conforming to a given structure one can use the following standart trick. Choose an atlas $\{U_k, \varphi_k\} : \bigcup\limits_k U_k \supset \Omega$; $\varphi_k : U_k \mapsto \mathbf{R}^2$, $\varphi_k(x) = \{\mathrm{Re}\, w_k(x), \mathrm{Im}\, w_k(x)\}, w_k \in \mathcal{A}(\Omega)$; let η_k be the subordinated partition of unity: $\eta_k \geq 0$, $\mathrm{supp}\,\eta_k \subset U_k$, $\sum\limits_k \eta_k = 1$. Define on Ω the tensors $g^{(k)}_{ij} := \eta_k \delta_{ij}$. The tensor $g := \sum\limits_k g^{(k)}$ defines a required metric.

Another way to find a conforming metric is to solve the equation (1.1) with respect to the entries $\det^{\frac{1}{2}} g\, g^{ij}$ for rich enough reserve of $u \in \mathrm{Re}\,\mathcal{A}(\Omega)$ that is to recover the metric g up to function multiplier.

3 Solving the problem.

3.1 Recovering the manifold.

So, let an operator Λ_g acting in (real) $C^\infty(\Gamma)$ be given and known to be the DN-map of a manifold; recall that g_Γ is also assumed given. The following procedure gives the canonical representator of the class of manifolds whose DN-maps coincide with Λ_g.

Step 1. Using the characterization of simple functions (Lemma 2) select the set $\mathcal{F}$ in $C^\infty(\Gamma)$. Through $\mathcal{F}$ recover the algebra $\mathcal{A}(\Gamma)$ by the representation (2.1).

Step 2. Find $\mathrm{sp}\,\mathcal{A}(\Gamma) =: \Omega$ (see (ii),sec.2.4). Identify the subset of ideals $\{I_{\gamma_0} \,|\, \gamma_0 \in \Gamma\} \subset \Omega$ to Γ by the rule $I_{\gamma_0} \equiv \gamma_0$.

[4]In dimension 2 the equalities $\Delta_g\, u = 0$ and $\Delta_{\rho g}\, u = 0$ are equivalent.

Step 3. Find the Gelfand transform $\hat{\mathcal{A}}(\Gamma) =: \mathcal{A}(\Omega)$ being a subalgebra of the (complex) $C(\Omega)$ [5].

Step 4. Using functions $w \in \mathcal{A}(\Omega)$ as local homeomorphisms $\Omega \mapsto \mathbf{C}$ recover on Ω the complex structure. Its realifying determines on Ω a structure of the smooth 2-dim manifold with the border Γ.

Step 5. Equip Ω with a metric $\tilde{g}$ conforming to the complex structure (see sec.2.5). Choose a positive function $\rho \in C^\infty(\Omega)$ such that the metric $g := \rho\tilde{g}$ satisfies $(\rho\tilde{g})_\Gamma = g_\Gamma$.

The manifold (Ω, g) solves the Calderon problem : its DN-map coincides with Λ_g by construction.

3.2 Proving Theorem 1.

The assertion of the theorem is just a corollary of the determinacy of the procedure sec.3.1 [6]. Indeed,

if (Ω', g') and (Ω'', g'') have the common DN-map, the corresponding sets of simple functions coincide: $\mathcal{F}' = \mathcal{F}''$. Whence, the trace algebras coincide: $\mathcal{A}'(\Gamma) = \mathcal{A}''(\Gamma)$;

by the property (ii),sec.2.4 the algebras $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$ are spatially isomorphic to the Gelfand transform of the common trace algebra:

$$[\mathrm{tr}'\mathcal{A}(\Omega')]^{\wedge'} = \hat{\mathcal{A}}'(\Gamma) = \hat{\mathcal{A}}''(\Gamma) = [\mathrm{tr}''\mathcal{A}(\Omega'')]^{\wedge''};$$

for the corresponding space bijections one has

$$\beta'(\Omega') = \mathrm{sp}\,\mathcal{A}'(\Gamma) = \mathrm{sp}\,\mathcal{A}''(\Gamma) = \beta''(\Omega''). \tag{3.1}$$

Therefore the algebras $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$ are spatially isomorphic (through the map $(\mathrm{tr}'')^{-1}(\wedge'')^{-1}\wedge'\,\mathrm{tr}')$ whereas the manifolds Ω' and Ω'' are connected through the bijection $\beta : \Omega' \mapsto \Omega''$, $\beta = \beta''^{-1}\beta'$;

since the differentiable structures on Ω' and Ω'' are determined by the algebras $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$, the bijection β is a diffeomorphism. Due to (3.1) one has $\beta'(\gamma_0) = I'_{\gamma_0} = I''_{\gamma_0} = \beta''(\gamma_0)$, $\gamma_0 \in \Gamma$ that follows to $\beta(\gamma_0) = \beta''^{-1}(\beta'(\gamma_0)) = \gamma_0$, i.e.$\beta|_\Gamma = \mathrm{id}$;

[5] Note that the 'border' $\{I_{\gamma_0} \,|\, \gamma_0 \in \Gamma\}$ may be identified as the Shilov boundary of $\mathcal{A}(\Omega)$.

[6] Such the situation is quite ordinary for the BC-method.

introduce on Ω' the (induced) metric $\tilde{g}''$ such that β to be isometry of $(\Omega', \tilde{g}'')$ onto (Ω'', g''). Since $\mathcal{A}(\Omega')$ and $\mathcal{A}(\Omega'')$ are isomorphic, the metric $\tilde{g}''$ (as well as g') turns out to be conforming to the structure determined by $\mathcal{A}(\Omega')$. Due to the last one has $\tilde{g}'' = \rho g'$ with a $\rho \in C^\infty(\Omega')$. By $\beta|_\Gamma = \mathrm{id}$ one has $\tilde{g}''_\Gamma = g''_\Gamma$; taking into account $g'_\Gamma = g''_\Gamma = g_\Gamma$ we obtain $\tilde{g}''_\Gamma = g'_\Gamma$ implying $\rho|_\Gamma = 1$.

So, β is an isometry of $(\Omega', \rho g')$ onto (Ω'', g'') i.e. (Ω', g') and (Ω'', g'') belong to one and the same conformal class. The theorem is proved.

3.3 Comments.

- In the case of the Riemann surface with the analytic border Γ homeomorphic to the circle the normality of $\mathcal{A}(\Omega)$ was established by J.Wermer [13]. If $\Omega \subset \mathbf{C}$ much stronger results (without assumptions on smoothness and connectedness of Γ) may be found in [3]. Probably, these results are valid for Riemann surfaces so that Lemmas 3,4 are well known for specialists in function algebras. Otherwise, we could not find exact references.

- As we hope, the area of applications of our approach may be extended due to the following notation taken from [4].

  If an operator L acting in $\Omega \subset \mathbf{C}$ admits a factorization $L = (\frac{\partial}{\partial z} + q)^*(\frac{\partial}{\partial z} + q)$ with $q = \frac{\partial p}{\partial z}$ then the complex solutions of the equation $Lw = 0$ satisfying $(\frac{\partial}{\partial z} + q)w = 0$ form an algebra with respect to the multiplication $v * w := e^p vw$ [7].

  One could attempt to use the last fact for recovering the operator by the scheme $\Lambda \Rightarrow \mathcal{A}(\Gamma) \Rightarrow \hat{\mathcal{A}}(\Gamma) \Rightarrow \mathcal{A}(\Omega)$ plus $\mathrm{sp}\,\hat{\mathcal{A}}(\Gamma) = \Omega \Rightarrow L$. In this case the question on normality of $\mathcal{A}(\Omega)$ would be put in the forefront: the presence of the corona (that is the hypothetic difference $\mathrm{sp}\,\mathcal{A}(\Omega) \setminus \varepsilon(\Omega)$ - see [6]) could complicate the recovering Ω.

- Perhaps, the operator $\Lambda_g J$ (see Lemmas 1,2) has an abstract analog in a class of the uniform algebras with nontrivial Shilov boundary.

[7] This rather promising observation is suggested by A.Ya.Kazakov.

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