PDMI PREPRINT 06 / 2002

Riemann-Roch theorem for oriented cohomology

I. Panin

05.03.2002

Abstract

Notion of an oriented cohomology pretheory on algebraic varieties is introduced and a Riemann-Roch theorem for ring morphisms between oriented pretheories is proved. An explicit formula for the Todd genus related to a ring morphism is given. The theory is illustrated by classical and other examples.

CONTENTS

0.1 Introduction

This article is a report on the joint work with A.Smirnov. This article in particular contains proofs of Riemann-Roch type theorems stated in [PS]. All the key ideas and all the key results are either contained in [PS] or at least motivated by [PS]. Since the body of [PS] can be naturally subdivided in two parts (a construction of integrations and a use of integrations) one can present each part separately. Here we present the part devoted to Riemann-Roch type theorems stated in [PS]. In fact concerning this type theorems slightly more general results are stated and proved in this text. This is due to the fact that here we postulate strong homotopy invariance of a theory rather than deduce it from homotopy invariance and Mayer-Vietoris properties of the theory. Similarly we postulate existence of an integration on the theory rather than construct an integration. An integration on the theory gives rise in a standart way to an Euler class of line bundles and we further just postulate the projective bundle theorem (in terms of the Euler class) rather than prove it. An integration satisfying the projective bundle theorem is call *perfect* (recall that all the integrations constructed in [PS] satisfy the projective bundle theorem and thus they are perfect). This is the way how one can extract from [PS] certain minimum of data required for a formulation of Riemann-Roch type theorem. It turns out that this minimum of data gives already completely natural context as for the formulation so for the proof of such a theorem.

The text is pretty self contained. At least it does not depend on the theory of push-forwards developed in [PS]. Only considering some examples we refer to [PS] to get the required integrations. It's not surprising that the deformation to the normal cone construction is used repeatedly. However one should stress that we need not at all in any kind of local Chern classes. This is due to the systematical use of one simple lemma 1.4.2.

Let us describe now the main object of the text. Namely in this text we consider the category of smooth varieties X over a field k, *ring cohomology pretheories* on this category and especially *oriented ring cohomology pretheories*. By a ring cohomology pretheory we mean a contrvariant functor A from this category to the category of rings which satisfies additivity and a strong homotopy invariance. If a ring cohomology pretheory

A is equipped with a family of push-forward operators satisfying certain properties then the pretheory can be equipped with an *Euler structure* associating to each line bundle an invariant called its Euler class. If this Euler structure satisfies the *projective bundle axiom* then this family of push-forwards is called a *perfect integration* and the pretheory with this family of push-forwards is called *oriented*.

In this case the cohomology ring $A(\mathbf{P}^\infty)$ can be identified with the formal power series in one variable $A(pt)[[t]]$ over the cohomology ring of the point. This identification takes the variable t to the Euler class $\xi_A = e(\mathcal{O}(-1))$ of the tautological line bundle over the projective space. So the class ξ_A is a local parameter in the ring $A(\mathbf{P}^\infty)$. A perfect integration is uniquelly defined by the corresponding Euler structure 1.1.11 and even by the Euler class ξ_A 1.1.12. Namely two perfect integrations coincide if the corresponding Euler structures coincide or if just the two classes $\xi_A^{(1)}$ and $\xi_A^{(2)}$ coinside.

It turns out that an oriented cohomology pretheory can be equipped with many perfect integrations 2.3.2 and the set of all perfect integrations is in a bijection with the set of universally central local parameters of the ring $A(\mathbf{P}^\infty)$. This bijection is given by associating to a perfect integration the corresponding Euler class $\xi_A = e(\mathcal{O}(-1))$ of the mentioned line bundle 2.3.2.

A ring morphism $\varphi : A \to B$ of oriented cohomology pretheories A and B is by the very definition a morphism of the underlying functors. If this morphism respects to the Euler structures on A and B then it commutes with the integrations. This is stated by the first Riemann-Roch type theorem 1.1.9. A ring morphism of oriented cohomology theories commuting with the integrations is called *an oriented morphism*. In general the ring morphism φ is not oriented at all, so it does not commute with the perfect integrations.

However if the element $\varphi(\xi_A)$ is a universally central local parameter in $B(\mathbf{P}^\infty)$ then by Theorem 2.3.2 there exists a new perfect integration on B such that the corresponding Euler structure coincides with the one given by $L \mapsto \varphi(e^A(L))$. The first Riemann-Roch type theorem 1.1.9 states that the morphism φ commutes with this new perfect integration on B. Namely for any projective morphism of smooth varieties $f : Y \to X$ one has the relation

$$f_B^{new} \circ \varphi_Y = \varphi_X \circ f_A. \tag{1}$$

To express this result in an explicite way it is convinient to use a Todd genus of the morphism φ, which we are going to describe now.

Namely, with the ring morphism $\varphi : A \to B$ we associate an *inverse Todd series* $itd_\varphi(t)$ of φ. It is the image of the universally central unite

$$itd_\varphi = \varphi(\xi_A)/\xi_B \in B(\mathbf{P}^\infty)$$

under the isomorphism $B(\mathbf{P}^\infty)$ with the formal power series $B(pt)[[t]]$. In the other words $itd_\varphi(t)$ is the only series such that its evaluation on the class ξ_B coincides with the unite $\varphi(\xi_A)/\xi_B$. The rule $L/X \mapsto itd_\varphi(L) = itd_\varphi(e(L)) \in B^{uc}(X)$ defines a functorial assignment on line bundles. If E/X is a rank n vector bundle and $\mathbf{P}(E)$ is the corresponding projective bundle (of lines in E), then one has a universally central unit

$itd_\varphi(\mathcal{O}_E(-1)) = itd_\varphi(e(\mathcal{O}_E(-1))) \in B(\mathbf{P}(E))$. Since the integration on B is perfect the subring $B(\mathbf{P}(E))^{uc}$ of all universally central elements in $B(\mathbf{P}(E))$ is a free rank n $B(X)^{uc}$-module for the subring of all universally central elements in $B(X)$. The *inverse Todd genus* $itd_\varphi(E)$ of E is defined by

$$itd_\varphi(E) = det(itd_\varphi(\mathcal{O}_E(-1))) \in B(X)^{uc} \tag{2}$$

where $itd_\varphi(\mathcal{O}_E(-1)) : B(\mathbf{P}(E))^{uc} \to B(\mathbf{P}(E))^{uc}$ is the operator $a \mapsto a \cup itd_\varphi(\mathcal{O}_E(-1))$. If the series $itd_\varphi(t)$ is invertible (or equivalently if its free term is invertible in $B(pt)$) then the element $itd_\varphi(\mathcal{O}_E(-1))$ is invertible as well and we define *the Todd genus* of E by

$$td_\varphi(E) = det(td_\varphi(\mathcal{O}_E(-1))) \in B(X)^{uc} \tag{3}$$

where $td_\varphi(t) = (itd_\varphi(t))^{-1} \in B(pt)[[t]]$ is the multiplicative inverse of $itd_\varphi(t)$. Being a symmetric function of the "eigenvalues" of the operator $B(\mathbf{P}(E))^{uc} \xrightarrow{\cup e(\mathcal{O}_E(-1))} B(\mathbf{P}(E))^{uc}$ the Todd genus can be expressed as a function of the elementary symmetric polinomials of these "eigenvalues". These elementary symmetric polinomials of the "eigenvalues" are just *the coefficients of the characteristic polinomial* $\chi_E(t) \in B(X)^{uc}[t]$ of the operator "$\cup e(\mathcal{O}_E(-1))$". These coefficients are exactly *the Chern classes* $c_i^B(E)$ of E with values in B. This shows that the Todd genus $td_\varphi(E)$ of E is *a function of the Chern classes* of E. The same holds for the inverse Todd genus $itd_\varphi(E)$. Details one can find in Subsections 1.9.1 and 2.2 below.

Now the explicite expression for the new perfect integration on B looks as follows. For a projective morphism $f : Y \to X$ of smooth varieties (with tangent bundles T_Y and T_X) f_B^{new} is defined as the composite operator $(\cup itd(T_X)) \circ f_B \circ (\cup td(T_Y))$ (a conjugation of f_B by means of the inverse Todd genus of φ). Thus the relation (1) says that for any element $a \in A(Y)$ one has the relation in B(X)

$$f_B(\varphi(a) \cup td_\varphi(T_Y)) = \varphi(f_A(a)) \cup td_\varphi(T_X). \tag{4}$$

This is the statement of the Riemann-Roch type Theorem 2.5.4.

Clearly in general the class $\varphi(\xi_A)$ is not at all a universally central local parameter of the ring $B(\mathbf{P}^\infty)$. Thus in general there is no perfect integration on B with the corresponding Euler class $\varphi(\xi_A)$. However if the class $\varphi(\xi_A)$ is universally central a Riemann-Roch type theorem still holds for closed imbeddings. Namely for a closed imbedding of smooth varieties $i : Y \hookrightarrow X$ with the normal bundle N and for any element $a \in A(Y)$ one has the relation in $B(X)$

$$i_B(\varphi(a) \cup itd_\varphi(N)) = \varphi(i_A(a)). \tag{5}$$

This is the statement of Theorem 2.5.3.

Note that in the case when φ is the Chern character from K-theory to Chow ring the formulas (4) and (5) coincide with the classical Riemann-Roch-Hirzebruch theorem in the form of A.Grothendieck [BS, §7]. In particular, if the variety X is the point then the formulae (4) coincides with the original Riemann-Roch theorem of Hirzebruch [H, Ch.IV, 21.1.1]. In the case when φ is the Adams operation ψ^r from K-theory to itself formulae (4) coincides with the classical one as well (see [Ad1] for the topological context and for higher K-theory see [So]).

Another interesting example is the case of the operator $\psi^{-1}: K_0 \to K_0$ which takes the class of a vector bundle E to the class of its dual vector bundle $E^\vee$. The Todd genus of E is equal to the class $(-1)^{rk(E)} \cdot \wedge^{max}(E^\vee)$, where $\wedge^{max}(E^\vee)$ is the maximal exterior power of $E^\vee$. In particular, $td_{\psi^{-1}}(T_X) = (-1)^{dim(X)} \cdot \omega_X$, where ω_Y is the canonical line bundle on Y. If X is the point pt then the relation (4) is reduced to the following one

$$\chi(E^\vee \otimes \omega_Y) = (-1)^{dim(Y)} \cdot \chi(E). \tag{6}$$

where for a vector bundle F on Y the integer $\chi(F)$ is the Euler characteristic of the vector bundle F. The relation (6) is nothing else as the well-known "Serre's duality".

Now take $k = \mathbb{R}$ and consider the theory A sending a variety X to $H^*(X(\mathbb{R}); \mathbb{Z}/2)$ (the singular cohomology of the topological space $X(\mathbb{R})$). It turns out that the Euler class $e(L)$ of a line bundle L coincides in this case with the first Stiefel-Whitney class $w_1(L(\mathbb{R}))$ of the real line bundle $L(\mathbb{R})$. The total Steenrod square Sq is a ring operation from A to itself. The inverse Todd genus $itd(E)$ coincides in this case with the total Stiefel-Whitney class $w(E(\mathbb{R}))$ of the real vector bundle $E(\mathbb{R})$. Thus the Riemann-Roch theorem for a closed imbedding $i: Y \hookrightarrow X$ states that the following relation holds

$$i_*(w(N(\mathbb{R}))) = Sq([Y])$$

where N is the normal bundle to Y. This is the known Wu formula. The mentioned interpretation of the Wu formular in topology was originally presented in [AH]. These and other examples are discussed in details in Subsection 2.6 below.

For the concept of oriented cohomology theories in the topological context see [Ad2] and [Ad1]. The Riemann-Roch theorems (2.5.3) and (2.5.4) are inspired by a Riemann-Roch theorem from [Dy]. However the explicit definitions of the inverse Todd genus (2.5.1) and of the Todd genus (2.5.2) possibly are not mentioned even in the topological context.

The author would like to express his special gratitude to V.Voevodsky for the series of very inspiring conversations on motivic cohomology [V] and the homotopy theory of schemes [V1]; to S.Yagunov for his very stimulating interest in building up push-forwards for 1998-2000; to F.Morel as for his excellent and inspiring lectures (Muenster, May - June 1999) so for very open conversations on the homotopy theory of schemes [Mo]; to A.Suslin as for his great interest to the subject so for raising a question (Bonn, June 1999) about possible existence of well-defined push-forwards respecting the composition (namely Suslin's great interest and this question inspired the work [PS]); to E.Friedlander A.Merkurjev, U.Jannsen, M. Ojanguren and many, many others for their very stimulating interest to the subject.

To avoid certain confusions we would like to stress that the push-forwards constructed in [PY] do not form any integration because they do not respect in general the composition of morphisms. However even with the help of that push-forwards a far generalization of Suslin's rigidity theorem is proved in [PY].

The author thanks very much for the support the the TMR Network ERB FMRX CT-97-0107, the grant of the year 2001 of the " Support Fond of National Science" at the Russian Academy of Science, the grant INTAS-99-00817, and the RFFI-grant 00-01-00116.

0.2 Terminology and notation

Let k be a field. The term "variety" is used in this text for an arbitrary smooth quasi-projective variety over the field k. The term ring is used in this text for an arbitrary associative ring with a unit.

We fix the following notation:

- $\mathcal{R}ings$ - the category of rings;
- $\mathcal{S}m$ - the category of smooth varieties;
- $pt = \mathrm{Spec}(k)$;

  For a smooth X and an effective divisor $D \subset X$ we write $L(D)$ for a line bundle over X whose sheaf of sections is the sheaf $\mathcal{L}_X(D)$ (see [Har, Ch.II,§6, 6.13]).

  $\mathbf{P}(V) = Proj(S^*(V^\vee))$-the space of lines in a finite dimensional k-vector space V;

  $L_V = \mathcal{O}_V(-1)$-the tautological line bundle over $\mathbf{P}(V)$;
- $\mathbf{P}(E) = Proj(S^*(E^\vee))$ - the space of lines in a vector bundle E;

  $L_E = \mathcal{O}_E(-1)$ - the tautological line bundle on $\mathbf{P}(E)$;

  $\mathcal{O}_E(1)$ - the line bundle on $\mathbf{P}(E)$ dual to $\mathcal{O}_E(-1)$;

  E^0 - the complement to the zero section of E;

  $E^\vee$ - the vector bundle dual to a vector bundle E;

  $z : X \to E$ - the zero section of a vector bundle E;
- For a contravariant functor A on $\mathcal{S}m$ set

  $$A(\mathbf{P}^\infty) = \varprojlim A(\mathbf{P}(V)), \tag{7}$$

  where the projective system is induced by all the finite dimensional vector subspaces $V \hookrightarrow k^\infty$.

  For such V and $a \in A(\mathbf{P}^\infty)$ denote by a_V the canonical image of a in $A(\mathbf{P}(V))$.

  In particular, taking the Picard group Pic as A one gets the definition of $\mathrm{Pic}(\mathbf{P}^\infty)$.

1 Oriented cohomology pretheories

In this section the notion of an oriented cohomology pretheory is defined and first variant of Riemann-Roch type theorem 1.1.9 is stated and proved. One should point out here that in the following definition one could replace the category of rings by the category of monoids in an arbitrary tensor abelian category and one could extend (without any essential changes) all the results of this text to that more general context which could be quite intersting. However we prefer to keep certain simplicity of the exposition and to work with the category of rings which is just the category of monoids in the tensor category of abelian groups.

1.1 First Riemann-Roch type theorem

1.1.1. Definition. *A ring cohomology pretheory is a ring presheaf $A : \mathcal{S}m \to \mathcal{R}ings$ satisfying the following conditions*

(1) additivity: let $i_r : X_r \hookrightarrow X_1 \amalg X_2$ be the natural inclusion ($r = 1, 2$). Then the induced map $A(X_1 \amalg X_2) \to A(X_1) \oplus A(X_2)$ is an isomorphism;

(2) strong homotopy invariance: for any variety X and any vector bundle E over X and any E-torsor $\mathcal{E}$ with the projection $p : \mathcal{E} \to X$ the pull-back map $p^A : A(X) \to A(\mathcal{E})$ is an isomorphism;

(3) $A(\emptyset) = \{0\}$.

A ring morphism $\varphi : A \to B$ of ring pretheories is a ring presheaf morphism.

For a morphism of varieties $f : Y \to X$ we shall consider below in the text the ring $A(Y)$ as a two-sided $A(X)$-module via the ring homomorphism $f^A : A(X) \to A(Y)$.

Let us recall a notion.

1.1.2. Definition. *Let $i : Y \hookrightarrow X$ be a closed imbedding of varieties. Consider a Cartesian square (in the category of schemes)*

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ {\scriptstyle\tilde{\varphi}}\uparrow & & \uparrow{\scriptstyle\varphi} \\ \tilde{Y} & \xrightarrow{\tilde{i}} & \tilde{X} \end{array}$$

consisting of smooth varieties. This square is called transversal if the canonical morphism $\tilde{\varphi}^*(N) \to \tilde{N}$ is an isomorphism (here N and $\tilde{N}$ are the normal bundle to Y in X and to $\tilde{Y}$ in $\tilde{X}$).

1.1.3. Definition. *Let A be the ring cohomology pretheory. An integration on A is a rule assigning to each projective morphism of smooth varieties $f : Y \to X$ a two-sided $A(X)$-module operator*

$$f_A : A(Y) \to A(X)$$

called the push-forward (for f) and satisfying the following properties

1. *$(f \circ g)_A = f_A \circ g_A$ for any projective morphisms $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ of smooth varieties;*

2. *for the transversal square from (1.1.2) the following diagram commutes*

$$\begin{array}{ccc} A(Y) & \xrightarrow{i_A} & A(X) \\ {\scriptstyle \tilde{\varphi}^A}\downarrow & & \downarrow{\scriptstyle \varphi^A} \\ A(\tilde{Y}) & \xrightarrow{\tilde{i}_A} & A(\tilde{X}) \end{array}$$

3. *for any morphism of smooth varieties $f : Y \to X$ the following diagram commutes*

$$\begin{array}{ccc} A(\mathbf{P}^n \times Y) & \xleftarrow{(id \times f)^A} & A(\mathbf{P}^n \times X) \\ {\scriptstyle (p_Y)_A}\downarrow & & \downarrow{\scriptstyle (p_X)_A} \\ A(Y) & \xleftarrow{f^A} & A(X) \end{array}$$

   where $p_Y : \mathbf{P}^n \times Y \to Y$ and $p_X : \mathbf{P}^n \times X \to X$ are the natural projections;

4. *normalization: for any smooth variety X one has $(id_X)_A = id_{A(X)}$;*

5. *localization: for any closed imbedding $i : Y \hookrightarrow X$ and the inclusion $j : X - Y \hookrightarrow X$ the sequence $A(Y) \xrightarrow{i_A} A(X) \xrightarrow{j^A} A(X - Y)$ is exact.*

Observe that already the property (2) of push-forwards implies vanishing of the composite map $j^A \circ i_A : A(Y) \to A(X - Y)$, because $A(\emptyset) = 0$ and the square

$$\begin{array}{ccc} Y & \xrightarrow{i} & X \\ \uparrow & & \uparrow{\scriptstyle j} \\ \emptyset & \longrightarrow & X - Y \end{array}$$

is clearly transversal.

1.1.4. Euler class. Let A be a ring cohomology pretheory equipped with an integration $f \mapsto f_A$. For a variety X and a line bundle L over X set

$$e(L) = z^A(z_A(1)) \in A(X). \tag{8}$$

The assignment $L \mapsto e(L)$ is called the *Euler structure* on A corresponding to the integration $f \mapsto f_A$ and the element $e(L)$ is called the *Euler class* of L corresponding to the integration $f \mapsto f_A$ or just the Euler class of L if the integration is known from a context.

It's easy to see that the class $e(L)$ is a *central element* in $A(X)$. If fact, the element $z_A(1) \in A(L)$ is central because the operator $z_A : A(X) \to A(L)$ is a two-sided $A(L)$-module homomorphism. The pull-back operator $z^A : A(L) \to A(X)$ is an isomorphism by the homotopy invariance property of A. Thus the class $e(L)$ is central.

1.1.5. Notation. *Let A be a ring cohomology pretheory. For any variety X and any vector bundle F over X and any central element $\alpha \in A(\mathbf{P}(F))$ we often will write below in the text*

$$\alpha : A(X) \to A(\mathbf{P}(F))$$

for the operator given by $a \mapsto \alpha \cup p^A(a)$ where $p : \mathbf{P}(F) \to X$ is the projection.

1.1.6. Perfect integration. Let A be a ring cohomology pretheory. An integration $f \mapsto f_A$ on A is called *perfect* if the corresponding Euler structure $L \mapsto e(L)$ on A satisfies the following condition called below *projective bundle axiom*

(i) for any variety X and any rank n vector bundle E over X and for the Euler class $\xi_E = e(\mathcal{O}_E(-1)) \in A(\mathbf{P}(E))$ the operator

$$(1, \xi_E, \ldots, \xi_E^{n-1}) : A(X) \oplus A(X) \oplus \cdots \oplus A(X) \to A(\mathbf{P}(E))$$

is an isomorphism (see 1.1.5 for the notation).

It is proved below in 1.2.1 that for a perfect integration on A and for an n-dimensional k-vector space V one has the relation $\xi_V^n = 0$ in $A(\mathbf{P}(V))$. Thus the ring $A(\mathbf{P}(V))$ can be identified with the truncated polinomial ring $A(pt)[t]/(t^n)$. The identification takes the variable t to the class ξ_V.

1.1.7. Definition. *An oriented cohomology pretheory is a ring cohomology pretheory A equipped with a perfect integration $f \mapsto f_A$.*

A ring morphism of oriented cohomology pretheories is just a morphism of the underlying ring presheaves.

An oriented morphism of oriented cohomology pretheories is a ring morphism of the ring cohomology pretheories respecting the perfect integrations on the oriented cohomology pretheories.

The restriction $A|_{\mathcal{S}m}$ of an oriented cohomology theory A (in the sence of [PS]) to the category $\mathcal{S}m$ of smooth varieties is an oriented cohomology pretheory (with respect to the integration given by [PS][Corollary 4.1.4.1]). Moreover by [PS][Corollary 4.1.4.2] the corresponding Euler structure on the pretheory $A|_{\mathcal{S}m}$ is exactly the Chern structure on the theory A corresponding to the orientation of A.

1.1.8. Theorem (Normalization). *Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. Then for any variety X and any smooth divisor $i : D \hookrightarrow X$ on X the following relation holds in $A(X)$*

$$i_A(1) = e(L(D)), \tag{9}$$

where $L(D)$ is the line bundle over X defined in 0.2.

1.1.9. Theorem. *Let $\varphi : A \to B$ be a ring morphism of oriented cohomology pretheories. If for any variety X and any smooth divisor $i : D \hookrightarrow X$ on X the relation $\varphi(i_A(1)) = i_B(1)$ holds in $B(X)$, then for any projective morphism $f : Y \to X$ of varieties one has the relation*

$$f_B \circ \varphi_Y = \varphi_X \circ f_A. \tag{10}$$

The following Corollary is an obvious consequence of these two Theorems.

1.1.10. Corollary. *Let A and B be two oriented cohomology pretheories and let $L \mapsto e^A(L)$ and $L \mapsto e^B(L)$ be the corresponding Euler structures. Let $\varphi : A \to B$ be a ring morphism. If for any variety X and any line bundle L over X the relation $\varphi(e^A(L)) = e^B(L)$ holds in $B(X)$, then for any projective morphism $f : Y \to X$ of varieties one has the relation*

$$f_B \circ \varphi_Y = \varphi_X \circ f_A. \tag{11}$$

In the other words a ring morphism $\varphi : A \to B$ respecting the Euler structures is an oriented morphism of the oriented cohomology pretheories.

Taking $B = A$ and $\varphi = id_A$ in this Corollary one gets the following

1.1.11. Corollary. *Let A be a cohomology pretheory equipped with two perfect integrations $f \mapsto f_A^{(1)}$ and $f \mapsto f_A^{(2)}$. Let $L \mapsto e^{(1)}(L)$ and $L \mapsto e^{(2)}(L)$ be the corresponding Euler structures. If for any variety X and any line bundle L over X the relation $e^{(1)}(L) = e^{(2)}(L)$ holds in $A(X)$, then the two integrations coincide, i.e. for any projective morphism $f : Y \to X$ of varieties one has the relation*

$$f_A^{(1)} = f_A^{(2)}. \tag{12}$$

1.1.12. Remark. In the Corollary 1.1.11 it suffices to assume that for any vector space V finite-dimensional over k the relation $e^{(1)}(\mathcal{O}_V(-1)) = e^{(2)}(\mathcal{O}_V(-1))$ holds in $A(\mathbf{P}(V))$. Similarly in Corollary 1.1.10 it suffices to assume that the relation $\varphi(e^A(\mathcal{O}_V(-1))) = e^B(\mathcal{O}_V(-1))$ holds in $B(\mathbf{P}(V))$.

These replacements of the assumptions are available because the Euler class is functorial (see Lemma 1.2.1 below) and the line bundle $\mathcal{O}_{\mathbf{P}^\infty}(-1)$ is universal (see Claim 1.2.2 below): for any variety X and any line bundle L over X there exists a diagram of the form

$$X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V)$$

in which X' is a torsor under a vector bundle over X and the morphism f is such that the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic.

We conclude this subsection with the following observation

1.1.13. Lemma. *Let A be an oriented cohomology pretheory. Let E be a vector bundle over a variety X and let $p : \mathbf{P}(E) \to X$ be the associated projective bundle over X. For any morphism $f : \tilde{X} \to X$ of varieties set $\tilde{E} = \tilde{X} \times_X E$ and let $\tilde{p} : \mathbf{P}(\tilde{E}) \to \tilde{X}$ be the projective bundle over $\tilde{X}$ associated with $\tilde{E}$. Let $F : \mathbf{P}(\tilde{E}) \to \mathbf{P}(E)$ be a morphism induced by the projection $\tilde{E} \to E$. Then one has the relation*

$$f^A \circ p_A = \tilde{p}_A \circ F^A : A(\mathbf{P}(E)) \to A(\tilde{X}) \tag{13}$$

Proof. Consider a closed imbedding $i : \mathbf{P}(E) \hookrightarrow X \times \mathbf{P}^n$ of the X-varieties. Let $\tilde{i} : \mathbf{P}(\tilde{E}) \hookrightarrow \tilde{X} \times \mathbf{P}^n$ be its base change by means of the morphism $f : \tilde{X} \to X$. Now consider the diagram

$$\begin{array}{ccc} \tilde{X} \times \mathbf{P}^n & \xrightarrow{f \times id} & X \times \mathbf{P}^n \\ {\scriptstyle \tilde{i}}\uparrow & & \uparrow{\scriptstyle i} \\ \mathbf{P}(\tilde{E}) & \xrightarrow{F} & \mathbf{P}(E) \\ {\scriptstyle \tilde{p}}\downarrow & & \downarrow{\scriptstyle p} \\ \tilde{X} & \xrightarrow{f} & X \end{array}$$

By the property 2 from Definition 1.1.3 one has the relation $(f \times id)^A \circ i_A = \tilde{i}_A \circ F^A$. By the property 3 from the same Definition one has the relation $(p_{\tilde{X}})_A \circ (f \times id)^A = f^A \circ (p_X)_A$, where $p_X : X \times \mathbf{P}^n \to X$ and $\tilde{X} \times \mathbf{P}^n \to \tilde{X}$ are the projections. These two relations together with the property 1 completes the proof of the relation (13).

□

1.2 Euler structure

Let A be an oriented cohomology pretheory. Certain properties of the corresponding Euler structure $L \mapsto e(L)$ are proved in this subsection.

1.2.1. Lemma. *The Euler structure $L \mapsto e(L)$ on A satisfies the following properties*

(1) functoriality: $e(L_1) = e(L_2)$ for isomorphic line bundles L_1 and L_2 and furthermore $f^A(e(L)) = e(f^*(L))$ for any morphism $f : \tilde{X} \to X$ of varieties;

(2) vanishing: $e(\mathbf{1}) = \mathbf{0}$ for the trivial line bundle $\mathbf{1}$ over a variety X;

(3) the Euler class $e(L)$ is universally central ($f^A(e(L))$ is central for any morphism $f : Y \to X$ of varieties). The class $e(L)$ is nilpotent;

(4) nilportence: if V is a dimension n k-vector space then $\xi_V^n = 0$ in $A(\mathbf{P}(V))$.

Proof. To prove the first property consider a line bundle isomorphism $\alpha : L_1 \to L_2$ and the zero sections z_1 and z_2 of the line bundles L_1 and L_2 respectively. Then the diagram

$$\begin{array}{ccc} X & \xrightarrow{z_2} & L_2 \\ {\scriptstyle id}\uparrow & & \uparrow{\scriptstyle \alpha} \\ X & \xrightarrow{z_1} & L_1 \end{array}$$

is transversal. Thus one has the relation $\alpha^A \circ (z_2)_A = (z_1)_A$. The chain of relations completes the proof

$$e(L_2) = (z_2^A \circ (z_2)_A)(1) = (z_1^A \circ \alpha^A \circ (z_2)_A)(1) = (z_1^A \circ (z_1)_A)(1) = e(L_1).$$

To prove the functoriality set $\tilde{L} = \tilde{X} \times L$ and consider the following diagram

$$\begin{array}{ccc} X & \xrightarrow{z} & L \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f_L} \\ \tilde{X} & \xrightarrow{\tilde{z}} & \tilde{L} \end{array}$$

with the zero sections z and $\tilde{z}$ and with the projection f_L. Since the diagram is transversal one has the relation $f_L^A \circ z_A = \tilde{z}_A \circ f^A$. Thus

$$f^A(e(L)) = [f^A \circ z^A \circ z_A](1) = [\tilde{z}^A \circ f_L^A \circ z_A](1) = [\tilde{z}^A \circ \tilde{z}_A \circ f^A](1) = e(f^*(L)).$$

To prove the second property consider the zero section $z : X \hookrightarrow X \times \mathbf{A}^1$ of the trivial bundle, the open imbedding $j : X \times (\mathbf{A}^1 - \{0\}) \hookrightarrow X \times \mathbf{A}^1$ and the section $s : X \to X \times (\mathbf{A}^1 - \{0\})$ of the projection $X \times (\mathbf{A}^1 - \{0\}) \to X$ given by $s(x) = (x, 1)$. By the localization property from 1.1.3 one has $j^A \circ z_A = 0$. Thus $(j \circ s)^A \circ z_A = s^A \circ j^A \circ z_A = 0$. The operator $(j \circ s)^A : A(X \times \mathbf{A}^1) \to A(X)$ is an isomorphism by the homotopy invariance property. Hence $z_A = 0$ and $e(\mathbf{1}) = z^A(z_A(1)) = 0$.

The fourth property is proved induction by the dimension of V. If $dim(V) = 1$ then $\mathbf{P}(V) = pt$ and the line bundle $\mathcal{O}_V(-1)$ is trivial. Thus $\xi_V = 0$ by the vanishing property of the Euler class. Now assume that for each k-vector space W of dimension $m < n$ the element ξ_W^m vanishes in $A(\mathbf{P}(W))$ and consider a k-vector space V of dimension n. Let $W \subset V$ be a hyperplane and let $i : \mathbf{P}(W) \hookrightarrow \mathbf{P}(V)$ be the imbedding and let $j : \mathbf{P}(V) - \mathbf{P}(W) \hookrightarrow \mathbf{P}(V)$ be the inclusion. The line bundle $j^*(\mathcal{O}_V(-1))$ is trivial. Thus $j^A(\xi_V) = 0$ in $A(\mathbf{P}(V) - \mathbf{P}(W))$. By the localization property of push-forwards the kernel of the operator j^A coincides with the image of the operator i_A. Hence $\xi_V = i_A(a)$ for an element $a \in A(\mathbf{P}(W))$. Now the chain of relations

$$\xi_V^n = \xi_V \cdot \xi_V^{n-1} = i_A(a) \cup \xi_V^{n-1} = i_A(a \cup i^A(\xi_V^{n-1})) = i_A(a \cup \xi_W^{n-1}) = 0.$$

completes the proof of the property.

To prove the third property recall that the Euler class $e(L)$ is central as it is proved in 1.1.4. Now this class is universally central by the first property. To prove the nilpotence property recall that for any finite-dimensional k-vector space V the classes $e(\mathcal{O}_V(-1)) \in \mathbf{P}(V)$ is nilpotent by the projective bundle axiom 1.1.6. It remains to use the following

1.2.2. Claim. *For any variety X and any line bundle L over X there exists a finite-dimensional vector space V and a diagram of the form*

$$X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V) \tag{14}$$

in which X' is a torsor under a vector bundle over X and the morphism f is such that the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic.

With this diagram in hand one has a chain of relations

$$p^A(e(L)) = e(p^*(L)) = e(f^*(\mathcal{O}_V(-1))) = f^A(e(\mathcal{O}_V(-1))).$$

which proves the nilpotence of the class $p^A(e(L))$. Since p^A is an isomorphism the class $e(L)$ is nilpotent as well. To complete the proof of the third property it remains to prove the Claim.

Proof of the Claim. To construct the diagram (14) recall that by the Jouanolou trick [J] there is a torsor X'/X under a vector bundle over X such that X' is an affine variety. Now take the projection $p : X' \to X$ and consider the pull-back $p^*(L)$ of the line bundle L. Since the variety X' is affine the line bundle $p^*(L)$ can be induced from a projective space via a morphism $f : X \to \mathbf{P}(V)$. The Claim is proved.

□

1.2.3. Proposition. *Let E be a rank n vector bundle over a variety X and let $\mathbf{P}(E)$ be the corresponding projective bundle. The class $\zeta_E = e(\mathcal{O}_E(1)) \in A(\mathbf{P}(E))$ satisfies the following properties*

(i) the operator

$$(1, \zeta_E, \ldots, \zeta_E^{n-1}) : A(X) \oplus A(X) \cdots \oplus A(X) \to A(\mathbf{P}(E))$$

is an isomorphism (see 1.1.5 for the notation);

(ii) the relation $\zeta_E^n = 0$ holds if E is a trivial rank n bundle.

Proof. Let $E^\vee$ be the dual vector bundle and let $\mathbf{P}(E^\vee)$ be the corresponding projective bundle over X. Let $Q = \mathbf{P}(E) \times_X \mathbf{P}(E^\vee)$ and let $p_1 : Q \to \mathbf{P}(E)$ and $p_2 : Q \to \mathbf{P}(E^\vee)$ be the projections and let $q : Q \to X$ be the structural morphism. Consider a composite morphism of vector bundles

$$p_1^*(\mathcal{O}_E(-1)) \otimes p_2^*(\mathcal{O}_{E^\vee}(-1)) \to q^*(E) \otimes q^*(E^\vee) \to q^*(\mathbf{1}) \tag{15}$$

where the left hand side arrow is the tensor product of the two canonical imbeddings $p_1^*(\mathcal{O}_E(-1)) \subset q^*(E)$ and $p_2^*(\mathcal{O}_{E^\vee}(-1)) \subset q^*(E^\vee)$ and the right hand side arrow is the pull-back by means of q of the canonical pairing $E \otimes E^\vee \to \mathbf{1}$.

Now let $s : Q \to p_1^*(\mathcal{O}_E(1)) \otimes p_2^*(\mathcal{O}_{E^\vee}(1))$ be the section dual to the composite morphism (15). Let $H \subset Q$ be the vanishing locus of the section s. The projections $Q - H \to \mathbf{P}(E)$ and $Q - H \to \mathbf{P}(E^\vee)$ are still denoted by p_1 and p_2 respectively. The section s induces an isomorphism of line bundles $p_1^*(\mathcal{O}_E(1))$ and $p_2^*(\mathcal{O}_{E^\vee}(-1))$ over the variety $Q - H$. Thus $p_1^A(\zeta_E) = p_2^A(\xi_{E^\vee})$ in $A(Q - H)$.

Clearly $Q - H$ is an F-torsor over $\mathbf{P}(E)$ under a vector bundle F and $Q - H$ is an G-torsor over $\mathbf{P}(E^\vee)$ under a vector bundle G. Thus the pull-back operators $p_1^A : A(\mathbf{P}(E)) \to A(Q - H)$ and $p_2^A : A(\mathbf{P}(E^\vee)) \to A(Q - H)$ are isomorphisms by the strong homotopy invariance property of the pretheory A.

By the projective bundle axiom the family $1, \xi_E^\vee, \ldots, \xi_E^{\vee n-1}$ is a free basis of the two-sided $A(X)$-module $A(\mathbf{P}(E^\vee))$. As it was already observed $p_1^A(\zeta_E) = p_2^A(\xi_{E^\vee})$. Thus the family $1, \zeta_E, \ldots, \zeta_E^{n-1}$ is a free basis of the two-sided $A(X)$-module $A(\mathbf{P}(E))$.

If the vector bundle E is trivial, then $\xi_{E^\vee}^n = 0$ in $A(\mathbf{P}(E^\vee))$ and therefore $\zeta_E^n = 0$ in $A(\mathbf{P}(E))$. The proof is completed. □

1.3 Key relation

Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. Let X be a variety, let L be a line bundle over X and let $E = \mathbf{1} \oplus \mathbf{L}$. Let $\mathbf{P}(E)$ be the corresponding projective bundle over X and let $s : X \to \mathbf{P}(E)$ be a section of the projection $p : \mathbf{P}(E) \to X$ identifying X with the subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(E)$.

1.3.1. Theorem. *The following relation holds in* $A(\mathbf{P}(E))$

$$s_A(1) = e(\mathcal{O}_E(1) \otimes p^*(L)) \tag{16}$$

To prove this theorem it is convinient to fix certain notation.

1.3.2. Notation. *Set* $\bar{th}(L) = e(\mathcal{O}_E(1) \otimes p^*(L))$. *Let* $j : \mathbf{P}(E) - \mathbf{P}(\mathbf{1}) \hookrightarrow \mathbf{P}(E)$ *be the open inclusion. Denote* $Ker[j^A : A(\mathbf{P}(E)) \to A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1}))]$ *by* $A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$.

1.3.3. Lemma. *The elements* $\bar{th}(L)$ *and* $s_A(1)$ *are central in* $A(\mathbf{P}(E))$. *Moreover both elements belongs to the ideal* $A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$.

Proof. The restriction of the projection $p : \mathbf{P}(E) \to X$ to $\mathbf{P}(E) - \mathbf{P}(\mathbf{1})$ makes the last variety in a line bundle over X. The inclusion $s_L : X \hookrightarrow \mathbf{P}(E) - \mathbf{P}(\mathbf{1})$ identifying X with the subvariety $\mathbf{P}(L)$ in $\mathbf{P}(E)$ is the zero secton of the mentioned line bundle. By the strong homotopy property of the pretheory A the pull-back operator $s_L^A : A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1})) \to A(X)$ is an isomorphism. The line bundle $s_L^*(\mathcal{O}_E(1) \otimes p^*(L))$ coinsides with the line bundle $L^\vee \otimes L$ and therefore it is the trivial line bundle. Thus the Euler class $e(s_L^*(\mathcal{O}_E(1) \otimes p^*(L))$ vanishes and the element $s_L^A(\bar{th}(L))$ vanishes as well. Therefore $j^A(\bar{th}(L)) = 0$ in $A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1}))$, which proves the inclusion $j^A(\bar{th}(L)) \in A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$.

By the localization property of the pretheory A the composite map $j^A \circ s_A$ vanishes. Thus $s_A(1) \in A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$.

The element $\bar{th}(L)$ is central in $A(\mathbf{P}(E))$ because it is an Euler class. The element $s_A(1)$ is central because the operator s_A is a two-sided $A(\mathbf{P}(E))$-module homomorphism. The Lemma is proved.

1.3.4. Lemma. *The composite operator* $(\cup\bar{th}(L)) \circ p^A : A(X) \to A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E))$ *is an isomorphism.*

Proof. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A_{\mathbf{P}(\mathbf{1})}(\mathbf{P}(E)) \arrow[r] & A(\mathbf{P}(E)) \arrow[r, "j^A"] & A(\mathbf{P}(E) - \mathbf{P}(\mathbf{1})) \arrow[r] & 0 \\
0 \arrow[r] & A(X) \arrow[r, "i"] \arrow[u, "\cup\bar{th}(L)"] & A(X) \oplus A(X) \arrow[r, "q"] \arrow[u, "(\cup\bar{th}(L)\text{,}\cup 1)"] & A(X) \arrow[r] \arrow[u, "p^A"] & 0
\end{tikzcd}$$

where $i(a) = (a, 0)$ and $q(a_1, a_2) = a_2$. If $E' = E \otimes L^\vee$, then the variety $\mathbf{P}(E')$ coincides (tautologically) with the variety $\mathbf{P}(E)$ and the line bundle $\mathcal{O}_{E'}(1)$ over $\mathbf{P}(E')$ coincides with the line bundle $\mathcal{O}_E(1) \otimes p^*(L)$. Thus the element $\bar{th}(L)$ coincides with the Euler class $\zeta_{E'} = e(\mathcal{O}_{E'}(1))$ and middle vertical arrow is an isomorphism by Proposition 1.2.3. The pull-back operator p^A is an isomorphism by the strong homotopy invariance property. Thus the left vertical arrow is an isomorphism as well. The Lemma is proved.

1.3.5. Lemma. *The diagram commutes*

$$\begin{array}{ccc} A_{\mathbf{P(1)}}(\mathbf{P}(E)) & \xrightarrow{s^A} & A(X) \\ {\scriptstyle \cup \bar{t}h(L)}\uparrow & & \uparrow{\scriptstyle \cup e(L)} \\ A(X) & \xrightarrow{id} & A(X) \end{array}$$

and $s^A(\bar{t}h(L)) = s^A(s_A(1))$.

Proof. The diagram commutes because $s^A(\bar{t}h(L)) = e(s^*(\mathcal{O}_E(1) \otimes p^*(L))) = e(L)$. Now observe that the variety $\mathbf{P}(E) - \mathbf{P}(L)$ coincides with the line bundle L over X. Moreover if $i : L = \mathbf{P}(E) - \mathbf{P}(L) \hookrightarrow \mathbf{P}(E)$ is the open inclusion and $z : X \to L$ is the zero section, then $s = i \circ z$ and the diagram

$$\begin{array}{ccc} X & \xrightarrow{z} & L \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle i} \\ X & \xrightarrow{s} & \mathbf{P}(E) \end{array}$$

is obviously transversal. Thus $s^A = z^A \circ i^A$ and $z_A = i^A \circ s_A$. These two relations shows that $z^A \circ z_A = z^A \circ i^A \circ s_A = s^A \circ s_A$ and therefore $e(L) = s^A(s_A(1))$. On the other side $e(L) = s^A(\bar{t}h(L))$. Thus $s^A(\bar{t}h(L)) = s^A(s_A(1))$. The Lemma is proved.

Proof of the relation (16). . Take $X = \mathbf{P}^\infty$ and $L = \mathcal{O}_{\mathbf{P}^\infty}(1)$ and consider the commutative diagram from the last Lemma. By Proposition 1.2.3 the ring $A(pt)[[t]]$ of formal power series in one variable is identified with the ring $A(\mathbf{P}^\infty)$ identifying the variable t with the Euler class $e(L)$. Thus the operator $\cup e(L) : A(X) \to A(X)$ is injective. The operator $\cup \bar{t}h(L)$ is an isomorphism by Lemma 1.3.4. Hence the operator $s^A : A_{\mathbf{P(1)}}(\mathbf{P}(E)) \to A(X)$ is injective. Now the relation $s_A(1) = \bar{t}h(L)$ in $A(X)$ holds by the last Lemma. The relation (16) is proved in the considered case. Clearly this implies the relation (16) in the case $X = \mathbf{P}(V)$ and $L = \mathcal{O}_V(1)$ for any finite-dimensional k-vector space V. The general case of the relation (16) will be reduced now to this particular case.

Let X be a variety and let L be a line bundle over X. By Claim 1.2.2 there exists a diagram of the form (14) such that the pull-back operator $p^A : A(X) \to A(X')$ is an isomorphism and the line bundles $L' = p^*(L)$ and $\mathcal{O}_V(1)$ are isomorphic.

Set $E' = \mathbf{1} \oplus L'$ and $E_V = \mathbf{1} \oplus \mathcal{O}_V(1)$ and consider the section $s' : X' \to \mathbf{P}(E')$ identifying X' with the subvariety $\mathbf{P}(\mathbf{1})$ and the section $s_V : \mathbf{P}(V) \to \mathbf{P}(E_V)$ identifying $\mathbf{P}(V)$ with the subvariety $\mathbf{P}(\mathbf{1})$. A choice of a line bundle isomorphism $L' \to f^*(\mathcal{O}_V(1))$ gives rise to the following commutative diagram

$$\begin{array}{ccccc} \mathbf{P}(E) & \xleftarrow[P]{} & \mathbf{P}(E') & \xrightarrow{F} & \mathbf{P}(E_V) \\ {\scriptstyle s}\uparrow & & {\scriptstyle s'}\uparrow & & \uparrow{\scriptstyle s_V} \\ X & \xleftarrow[p]{} & X' & \xrightarrow{f} & \mathbf{P}(V) \end{array}$$

Both squares in this diagram are transversal. Thus one has relations $s'_A(1) = F^A((s_V)_A(1))$ and $s'_A(1) = P^A(s_A(1))$. As we already know $(s_V)_A(1) = \bar{t}h(\mathcal{O}_V(1))$ in $A(\mathbf{P}(V))$. Now the chain of relations in $\mathbf{P}(E)$

$$P^A(s_A(1)) = s'_A(1) = F^A((s_V)_A(1)) = F^A(\bar{t}h(\mathcal{O}_V(1))) = \bar{t}h(f^*(\mathcal{O}_V(1)))$$
$$= \bar{t}h(L') = P^A(\bar{t}h(L)).$$

Since the pull-back operator P^A is an isomorphism the desired relation follows. The Theorem is proved.

□

1.4 A useful lemma

In this subsection A is an oriented cohomology pretheory and $L \mapsto e(L)$ is the corresponding Euler structure on A.

1.4.1. Deformation to the normal cone. The deformation to the normal cone is a well-known construction (for example, see [Fu, Ch.5]). Since the construction and its property (see Lemma 1.4.2) play an important role in what follows we give here some details.

Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N. Let $\mathbf{P}(\mathbf{1} \oplus N)$ be the projective bundle over Y and let $s : Y \to \mathbf{P}(\mathbf{1} \oplus N)$ be a section of the projection $\mathbf{P}(\mathbf{1} \oplus N) \to Y$ identifying Y with the subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(\mathbf{1} \oplus N)$.

Now consider the blow-up X'_t of $X \times \mathbf{A}^1$ with the center $Y \times \{0\}$. Let $\sigma' : X'_t \to X \times \mathbf{A}^1$ be the blow-up map.

The preimage of $Y \times \{0\}$ under the blow-up map is the projective bundle $\mathbf{P}(\mathbf{1} \oplus N)$ and the base change of σ' by means of the inclusion $Y \times \{0\} \hookrightarrow X \times \mathbf{A}^1$ is the projection $p : \mathbf{P}(\mathbf{1} \oplus N) \to Y$. Denote by $j_0 : \mathbf{P}(\mathbf{1} \oplus N) \hookrightarrow X'_t$ the obvious closed inclusion.

The preimage of $Y \times \mathbf{A}^1$ under the blow-up morphism consists of two components: the proper preimage of $Y \times \mathbf{A}^1$ and the projective bundle $\mathbf{P}(\mathbf{1} \oplus N)$. The proper preimage of $Y \times \mathbf{A}^1$ under the blow-up map is mapped isomorphically to $Y \times \mathbf{A}^1$ under the blow-up map. Thus the inverse isomorphism gives a closed imbedding $i_t : Y \times \mathbf{A}^1 \hookrightarrow X'_t$.

Clearly $i_t(Y \times \mathbf{A}^1)$ crosses $\mathbf{P}(\mathbf{1} \oplus N)$ transversally along the subvariety $\mathbf{P}(\mathbf{1}) = s(Y)$, where $s : Y \to \mathbf{P}(\mathbf{1} \oplus N)$ is the section of p described above in this subsection.

The preimage of $X \times \{1\}$ under the blow-up map is mapped isomorphically to $X \times \{1\}$. The inverse isomorphism gives a closed imbedding $j_1 : X \hookrightarrow X'_t$. The base change of the closed imbedding i_t by means of j_1 coinsides with the closed imbedding $i : Y \hookrightarrow X$.

Thus we get a commutative diagram (deformation to the normal cone)

$$\begin{tikzcd}
\mathbf{P}(\mathbf{1} \oplus N) \arrow[r, "j_0"] & X'_t & X \arrow[l, "j_1"'] \\
Y \arrow[u, "s=i_0"] \arrow[r, "k_0"] & Y \times \mathbf{A}^1 \arrow[u, "i_t"] & Y, \arrow[l, "k_1"'] \arrow[u, "i=i_1"']
\end{tikzcd}$$

with in which the two squares are transversal. Therefore the following diagram commutes

$$\begin{array}{ccccc} A(\mathbf{P}(\mathbf{1}\oplus N)) & \xleftarrow{j_0^A} & A(X'_t) & \xrightarrow{j_1^A} & A(X) \\ {\scriptstyle s_A}\uparrow & & {\scriptstyle (i_t)_A}\uparrow & & \uparrow{\scriptstyle i_A} \\ A(Y) & \xleftarrow{k_0^A} & A(Y\times\mathbf{A}^1) & \xrightarrow{k_1} & A(Y), \end{array}$$

The following lemma will be repeatedly used below in the text.

1.4.2. Lemma. *Let $j_t : X'_t - i_t(Y\times\mathbf{A}^1) \hookrightarrow X'_t$ be the open inclusion. Then*

$$Ker(j_0^A)\cap Ker(j_t^A) = (0).$$

In the other words the operator $(j_0^A, j_t^A) : A(X'_t) \to A(\mathbf{P}(\mathbf{1}\oplus N))\oplus A(X'_t - Y\times\mathbf{A}^1)$ is a monomorphism.

Proof. Let $a \in Ker(j_t^A)$. By the localization property of the pretheory A there is an element $b \in A(Y\times\mathbf{A}^1)$ such that $a = (i_t)_A(b)$. If furthermore $a \in Ker(j_0^A)$, then $(j_0^A\circ(i_t)_A)(b) = 0$ and thus $(s_A\circ k_0^A)(b) = 0$. The pull-back operator k_0^A is an isomorphism by the homotopy invariance property of A. The operator s_A is injective because $p_A\circ s_A = id_{A(Y)}$. Thus $b = 0$ and $a = 0$ as well. The Lemma is proved.

□

1.5 Normalization property $i_A(1) = e(L(D))$

In this subsection Theorem 1.1.8 is proved. Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. Let X be a variety and let $i : D \hookrightarrow X$ be a smooth divisor on X.

Proof of Theorem 1.1.8. Consider the deformation to the normal cone diagram from 1.4.1 for the closed imbedding $i : D \hookrightarrow X$.

$$\begin{array}{ccccc} \mathbf{P}(\mathbf{1}\oplus N) & \xrightarrow{j_0} & X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle s=i_0}\uparrow & & {\scriptstyle i_t}\uparrow & & \uparrow{\scriptstyle i=i_1} \\ D & \xrightarrow{k_0} & D\times\mathbf{A}^1 & \xleftarrow{k_1} & D, \end{array}$$

And consider the corresponding commutative diagram of groups

$$\begin{array}{ccccc} A(\mathbf{P}(\mathbf{1}\oplus N)) & \xleftarrow{j_0^A} & A(X'_t) & \xrightarrow{j_1^A} & A(X) \\ {\scriptstyle s_A}\uparrow & & {\scriptstyle (i_t)_A}\uparrow & & \uparrow{\scriptstyle i_A} \\ A(D) & \xleftarrow{k_0^A} & A(D\times\mathbf{A}^1) & \xrightarrow{k_1} & A(D). \end{array}$$

Let $L(D\times\mathbf{A}^1)$ be line bundle over X'_t corresponding to the divisor $i_t(D\times\mathbf{A}^1)$ on X'_t. Since the left hand side square in the geometric diagram is transversal the pull-back

$j^*(L(D \times \mathbf{A}^1))$ of the line bundle $L(D \times \mathbf{A}^1)$ is isomorphic to the line bundle $L(s(Y))$ on $\mathbf{P}(\mathbf{1} \oplus N)$. Therefore the following relation holds for the Euler classes

$$j_0^A(e(L(D \times \mathbf{A}^1))) = e(L(s(D))).$$

Let $E = \mathbf{1} \oplus N$. The line bundle $L(s(D))$ is isomorphic to the line bundle $\mathcal{O}_E(1) \otimes p^*(L)$ over $\mathbf{P}(E)$. In fact, both line bundles are of the form " $\mathcal{O}(1)$ " and both are trivial being restrected to the section $\mathbf{P}(N)$ of the projection p. By the relation (16) one has $s_A(1) = e(\mathcal{O}_E(1) \otimes p^*(L))$ in $A(\mathbf{P}(E))$. Thus $s_A(1) = e(L(s(D)))$. Since $s_A \circ k_0^A = j_0^A \circ (i_t)_A$ one has the relation $j_0^A((i_t)_A(1)) = e(L(s(D)))$ in $A(\mathbf{P}(E))$. Thus

$$j_0^A((i_t)_A(1) - e(L(D \times \mathbf{A}^1))) = 0.$$

If $j_t : X'_t - i_t(D \times \mathbf{A}^1) \hookrightarrow X'_t$ is the open imbedding, then obviously $j_t^A(e(L(D \times \mathbf{A}^1))) = 0$. Furthermore by the localization property of the pretheory A one has $j_t^A((i_t)_A(1)) = 0$. Thus by Lemma 1.4.2 one has

$$(i_t)_A(1) - e(L(D \times \mathbf{A}^1)) = 0$$

and $(i_t)_A(1) = e(L(D \times \mathbf{A}^1))$.

Since the right hand side square in the geometric diagram is transversal the line bundles $j_1^*(L(D \times \mathbf{A}^1))$ and $L(D)$ over X are isomorphic. Therefore $j_1^A(e(L(D \times \mathbf{A}^1))) = e(L(D))$ in $A(X)$. Now the chain of relations in $A(X)$

$$i_A(1) = (i_A \circ k_1^A)(1) = (j_1^A \circ (i_t)_A)(1) = j_1^A(e(L(D \times \mathbf{A}^1))) = e(L(D))$$

completes the proof of the Theorem.

$\square$

1.6 Splitting principle

Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. Here a variant of splitting principle is given which will be used in the text below. It will be convinient to fix certain notation. Let $p : Y \to X$ and $f : X' \to X$ be morphisms. Then we will write Y' for the scheme $X' \times_X Y$ and write p' for the projection $X' \times_X Y \to X'$ and f' for the projection $X' \times_X Y \to Y$.

1.6.1. Lemma. *Let E be a rank n vector bundle over a smooth variety X. Then there exists a smooth morphism $r : T \to X$ such that the vector bundle $r^*(E)$ is a direct sum of line bundles and the pull-back map $r^A : A(X) \to A(T)$ is a split injection and moreover for a smooth variety X' and any morphism $f : X' \to X$ the pull-back map $(r')^A : A(X') \to A(T')$ is a split injection.*

Proof. Let E be a rank n vector bundle over a smooth variety X and let $p : \mathbf{P}(E) \to X$ be the associated projective bundle over X and let $\mathcal{O}_E(-1)$. Then there is the canonical short exact sequences of vector bundles on $\mathbf{P}(E)$ with the rank $n-1$ vector bundle E'

$$0 \to \mathcal{O}_E(-1) \to p^*(E) \to E' \to 0$$

and the pull-back map $p^A : A(X) \to A(\mathbf{P}(E))$ is a split injection by the projective bundle axiom.

Repeating this construction several times one gets a smooth variety Y , a morphism $q : Y \to X$ and a filtration $(0) \subset E_1 \subset E_2 \subset \cdots \subset E_n = q^*(E)$ of the vector bundle $q^*(E)$ such that all the quotients E_i/E_{i-1} are line bundles. Moreover the pull-back map $q^A : A(X) \to A(Y)$ is a split injection and for a smooth variety X' and any morphism $f : X' \to X$ the pull-back map $(q')^A : A(X') \to A(Y')$ is a split injection as well.

Claim. Let S be a smooth variety and let F_1, F_2 be two vector bundles over S and let $\alpha : F_1 \to F_2$ be a vector bundle epimorphism and let $K = ker(\alpha)$. Then there is an affine bundle $g : T \to S$ such that the epimorphism $g^*(\alpha)$ splits.

Assuming for a moment this Claim complete the proof of Lemma as follows. Take $S = Y$ and consider the filtration $(0) \subset E_1 \subset E_2 \subset \cdots \subset E_n = q^*(E)$ on the vector bundle $q^*(E)$. Applying several times Claim one gets an affine bundle $g : T \to S$ such that one has a direct sum decomposition $g^*(q^*(E)) = \oplus_{i=1}^n (g^*(E_i/E_{i-1}))$ of the vector bundle $(q \circ g)^*(E)$. Show that the morphism $r = q \circ g : T \to X$ have the desired property.

For each smooth variety S' an each morphism $S' \to S$ the pull-back map $(g')^A : A(S') \to A(T')$ is an isomorphism because T' is an affine bundle over S'. Now if X' is a smooth variety and $f : X' \to X$ is a morphism, then Y' is smooth over X' and therefore $S' = Y'$ is smooth as well. The pull-back map $(q')^A : A(X') \to A(Y')$ is a split injection by the projective bundle cohomology. Thus the composite map $(q' \circ g') : A(X') \to A(T')$ is a split injection as well.

Proof of Claim. Let $Hom(F_2, F_1)$ be the scheme representing the sheaf $\mathcal{H}om(F_2, F_1)$ and let $\phi : Hom(F_2, F_1) \to Hom(F_2, F_2)$ be the morphism corresponding to the morphism $\mathcal{H}om(F_2, F_1) \to \mathcal{H}om(F_2, F_2)$ induced by the vector bundle map $\alpha : F_1 \to F_2$. Let $id : S \to Hom(F_2, F_2)$ be the section of the projection $Hom(F_2, F_2) \to S$ corresponding to the identity map $F_2 \to F_2$. Let $Sect(\alpha) = \phi^{-1}(id(S))$ be a closed subscheme of the scheme $Hom(F_2, F_1)$ and let $g : T = Sect(\alpha) \to S$ be the projection. The scheme T represents the sheaf of sections of the sheaf epimorphism α. Thus there exists a canonical section $s : g^*(F_2) \to g^*(F_1)$ of the epimorphism $g^*(\alpha) : g^*(F_1) \to g^*(F_2)$. This section gives rise by a standart way to a vector bundle isomorphism $g^*(F_1) \cong g^*(F_2) \oplus g^*(K)$.

To prove Claim it remains to observe that the variety $T = Sect(\alpha)$ is a torsor under the vector bundle $Hom(F_2, K)$. The Claim is proved.

□

1.7 Additivity of push-forwards

Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. One could require the additivity of push-forwards just including this property in the very definition 1.1.3. However we prefer to prove this property. Namely, the aim of this subsection is to prove the following

1.7.1. Proposition. *Let u_1 and u_2 be the natural imbeddings of varieties Y_1 and Y_2 to $Y = Y_1 \amalg Y_2$. For a closed imbedding $i : Y \hookrightarrow X$ set $i_r = i \circ u_r : Y_r \hookrightarrow X$. Then*

$$i_A = (i_1)_A \circ u_1^A + (i_2)_A \circ u_2^A : A(Y) \to A(X)$$

Proof. We begin with a particular case of the Proposition.

1.7.2. Lemma. *Let $s = s_1 \amalg s_2 : S = S_1 \amalg S_2 \hookrightarrow T_1 \amalg T_2 = T$ be a closed imbedding of smooth varieties. Let $i_m : S_1 \hookrightarrow S$ and $j_m : T_m \hookrightarrow T$ be the open inclusions and let $r_m = j_m \circ s_m : S_m \hookrightarrow T$. Then*

$$s_A = (r_1)_A \circ i_1^A + (r_2)_A \circ i_2^A$$

To prove this Lemma consider the isomorphism $(j_1^A, j_2^A) : A(T) \to A(T_1) \oplus A(T_2)$. To prove the Lemma it suffices to check the following relations

$$j_1^A \circ s_A = j_1^A \circ (r_1)_A \circ i_1^A + j_1^A \circ (r_2)_A \circ i_2^A \tag{17}$$

$$j_2^A \circ s_A = j_2^A \circ (r_1)_A \circ i_1^A + j_2^A \circ (r_2)_A \circ i_2^A \tag{18}$$

The transversal square

$$\begin{tikzcd} T_1 \arrow[r, "j_1"] & T \\ S_1 \arrow[u, "s_1"] \arrow[r, "i_1"] & S \arrow[u, "s"'] \end{tikzcd}$$

prove relations $j_1^A \circ (s_1)_A = (s_1)_A \circ i_1^A$. The transversal square

$$\begin{tikzcd} T_1 \arrow[r, "j_1"] & T \\ S_1 \arrow[u, "s_1"] \arrow[r, "i_1"] & S_1 \arrow[u, "r_1"'] \end{tikzcd}$$

proves the relations $j_1^A \circ (r_1)_A = (s_1)_A$. The transversal square

$$\begin{tikzcd} T_1 \arrow[r, "j_1"] & T \\ \emptyset \arrow[u] \arrow[r] & S_2 \arrow[u, "r_2"'] \end{tikzcd}$$

proves the relation $j_1^A \circ (r_2)_A = 0$.

Now combining the last three relations one gets the chain of relations

$$j_1^A \circ (r_1)_A \circ i_1^A + j_1^A \circ (r_2)_A \circ i_2^A = j_1^A \circ (s_1)_A = (s_1)_A \circ i_1^A.$$

which proves the relation (17). The relation (18) is proved in a similar way. The Lemma is proved.

Now suppose we are given with varieties V and $U = U_1 \amalg U_2$ and a transversal square of the form

$$\begin{tikzcd} T_1 \amalg T_2 \arrow[r, "v"] & V \\ S_1 \amalg S_2 \arrow[u, "s = s_1 \amalg s_2"] \arrow[r, "u = u_1 \amalg u_2"] & U_1 \amalg U_2 \arrow[u, "t"'] \end{tikzcd}$$

1.7.3. Lemma. *Let $k_m : U_m \hookrightarrow U$ be the open inclusion and let $t_m = t \circ k_m : U_m \hookrightarrow V$. Then*

$$v^A \circ t_A = v^A \circ [(t_1)_A \circ k_1^A + (t_1)_A \circ k_1^A]$$

To prove this Lemma observe that the following squares are transversal ($m = 1, 2$)

$$\begin{tikzcd} T_1 \amalg T_2 \arrow[r, "v"] & V \\ S_m \arrow[u, "r_m"] \arrow[r, "u_m"] & U_m \arrow[u, "t_m"'] \end{tikzcd}$$

Thus for $v^A \circ (t_m)_A = (s_m)_A \circ u_m^A$ for $m = 1, 2$ and one gets the following chain of relations using Lemma 1.7.2 in the last step.

$$v^A \circ [(t_1)_A \circ k_1^A + (t_1)_A \circ k_1^A = (r_1)_A \circ u_1^A \circ k_1^A + (r_2)_A \circ u_2^A \circ k_2^A =$$

$$(r_1)_A \circ i_1^A \circ u^A + (r_2)_A \circ i_2^A \circ u^A = s_A \circ u^A.$$

Finally $s_A \circ u^A = v^A \circ t_A$ by the transversality of the diagram considered in the Lemma. The Lemma follows.

Now consider the deformation to the normal cone diagram from the subsection 1.4.1

$$\begin{tikzcd} \mathbf{P}(\mathbf{1} \oplus N) \arrow[r, "j_0"] & X'_t & X \arrow[l, "j_1"'] \\ Y \arrow[u, "s"] \arrow[r, "k_0"] & Y \times \mathbf{A}^1 \arrow[u, "i_t"] & Y. \arrow[l, "k_1"'] \arrow[u, "i"'] \end{tikzcd}$$

Since $Y = Y_1 \amalg Y_2$ one see that $Y \times \mathbf{A}^1 = Y_1 \times \mathbf{A}^1 \amalg Y_2 \times \mathbf{A}^1$ and $\mathbf{P}(\mathbf{1} \oplus N) = \mathbf{P}(\mathbf{1} \oplus N_1) \amalg \mathbf{P}(\mathbf{1} \oplus N_2)$ where N_i is the normal bundle of X to Y_i. Let $l_m : Y_m \times \mathbf{A}^1 \hookrightarrow Y \times \mathbf{A}^1$ be the inclusion for $m = 1, 2$ and let $i_t^m = i_t \circ l_m$. Since the left hand side square in the very last diagram is transversal Lemma 1.7.3 gives the following relation

$$j_0^A \circ (i_t)_A = j_0^A \circ [(i_t^1)_A \circ l_1^A + (i_t^2)_A \circ l_2^A]. \tag{19}$$

To complete the proof of the Proposition we need in the following

1.7.4. Lemma. *Let $j_t : X'_t - Y \times \mathbf{A}^1 \hookrightarrow X'_t$ be the open inclusion. Then $j_t^A \circ (i_t)_A = 0$ and $j_t^A \circ (i_t^m)_A = 0$ for $m = 1, 2$.*

In fact, the first relation follows from the localization property 1.1.3. The remaning relations for $m = 1, 2$ follows from the first one because $(i_t^m)_A = (i_t)_A \circ (l_m)_A$.

By this Lemma $j_t^A \circ [(i_t)_A - (i_t^1)_A \circ l_1^A - (i_t^2)_A \circ l_2^A] = 0$. By relation (19) one has the relation $j_0^A \circ [(i_t)_A - (i_t^1)_A \circ l_1^A - (i_t^2)_A \circ l_2^A] = 0$. Thus $[(i_t)_A - (i_t^1)_A \circ l_1^A - (i_t^2)_A \circ l_2^A] = 0$ by Lemma 1.4.2 and

$$(i_t)_A = (i_t^1)_A \circ l_1^A + (i_t^2)_A \circ l_2^A.$$

Since the right hand side square from the very last diagram is transversal one gets the relation

$$j_1^A \circ (i_t)_A = i_A \circ k_1^A.$$

Since $k_1 = k_{11} \amalg k_{12} : Y_1 \amalg Y_2 \hookrightarrow Y_1 \times \mathbf{A}^1 \amalg Y_2 \times \mathbf{A}^1$ the squares

$$\begin{array}{ccc} X'_t & \xleftarrow{j_1} & X \\ {\scriptstyle i_t^{(m)}}\uparrow & & \uparrow{\scriptstyle i_m} \\ Y_m \times \mathbf{A}^1 & \xleftarrow{k_{1m}} & Y_m . \end{array}$$

are transversal. Thus one has relations

$$j_1^A \circ (i_t^m)_A = (i_m)_A \circ k_{1m}^A$$

for $m = 1, 2$.

Recall that $u_m : Y_m \hookrightarrow Y$ for $m = 1, 2$ are the inclusions and $i_m = i \circ u_m$. Combining last relations one gets a chain of relations

$$i_A \circ k_1^A = j_1^A \circ (i_t)_A = j_1^A \circ [(i_t^1)_A \circ l_1^A + (i_t^2)_A \circ l_2^A] = (i_1)_A \circ k_{11}^A \circ l_1^A + (i_2)_A \circ k_{12}^A \circ l_2^A =$$

$$(i_1)_A \circ u_1^A \circ k_1^A + (i_2)_A \circ u_2^A \circ k_1^A =$$

which proves the following one

$$i_A \circ k_1^A = [(i_1)_A \circ u_1^A + (i_2)_A \circ u_2^A] \circ k_1^A$$

Since the operator k_1^A is an isomorphism the Proposition follows.

□

1.8 Gysin Structures

Let A be a ring cohomology pretheory. In this subsection a general theorem is proved which is very useful for our proof of Riemann-Roch type theorems.

1.8.1. Definition. *A Gysin structure on A is an assignment which associate to each closed imbedding $i : Y \hookrightarrow X$ of varieties a two-sided $A(X)$-module operator $i_A : A(Y) \to A(X)$ satisfying the following properties*

(1) $(f \circ g)_A = f_A \circ g_A$ for any projective morphisms $Z \xrightarrow{g} Y$ and $Y \xrightarrow{f} X$ of smooth varieties;

(2) for the transversal square from (1.1.2) the following diagram commutes

$$\begin{array}{ccc} A(\tilde{Y}) & \xrightarrow{\tilde{i}_A} & A(\tilde{X}) \\ {\scriptstyle \tilde{\varphi}^A}\uparrow & & \uparrow{\scriptstyle \varphi^A} \\ A(Y) & \xrightarrow{i_A} & A(X) \end{array}$$

(3) normalization: for any smooth variety X one has $(id_X)_A = id_{A(X)}$.

The restriction of an integration on A to the family of closed imbeddings is a Gysin structure on A. A Gysin structure on A looks as a partially defined integration on A. Therefore notation $i_{\mathcal{A}}$ is used in contrast with the notation i_A which is used for an integration. The operator $i_{\mathcal{A}}$ is called sometime *Gysin operator*.

The following lemma will be useful below in this subsection.

1.8.2. Lemma. *Let A be equipped with a Gysin structure $i \mapsto i_{\mathcal{A}}$. Let $i : S \hookrightarrow T$ be a closed imbedding of smooth varieties and let $j : T - S \hookrightarrow T$ be the open inclusion. Then the composite mapping*

$$A(S) \xrightarrow{i_{\mathcal{A}}} A(T) \xrightarrow{j^A} A(T - S)$$

vanishes.

In fact, the property (2) of the Gysin operators $i_{\mathcal{A}}$ implies vanishing of the composite map $j^A \circ i_{\mathcal{A}} : A(Y) \to A(X - Y)$, because $A(\emptyset) = 0$ and the square

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X - Y \arrow[d, "j"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

is clearly transversal.

1.8.3. Theorem. *Let A and B be two oriented cohomology pretheories and let $\varphi : A \to B$ be a ring morphism. Let an assignment $i \mapsto i_{\mathcal{A}}$ be a Gysin structure on A and let an assignment $i \mapsto i_{\mathcal{B}}$ be a Gysin structure on B. Suppose for each smooth divisor $j : D \hookrightarrow X$ on a smooth variety X the relation $j_{\mathcal{B}}(1) = \varphi(j_{\mathcal{A}}(1))$ valids in $B(X)$. Then for each closed imbedding of smooth varieties $i : Y \hookrightarrow X$ one has the relation*

$$i_{\mathcal{B}} \circ \varphi_Y = \varphi_X \circ i_{\mathcal{A}}.$$

Proof of Theorem 1.8.3. The proof consists of several many steps. It's convinient to begin with the following general lemma.

1.8.4. Lemma. *Let $i : Y \hookrightarrow X$ be a closed imbedding of varieties with the normal bundle $N = N_{X/Y}$. Let $s : Y \to \mathbf{P}(\mathbf{1} \oplus N)$ be a section of the projection $p : \mathbf{P}(\mathbf{1} \oplus N)$ identifying Y with the subvariety $\mathbf{P}(\mathbf{1})$ in $\mathbf{P}(\mathbf{1} \oplus N)$. If the Theorem holds for the closed imbedding s then it holds for the closed imbedding $i : Y \hookrightarrow X$ as well.*

Consider the deformation to the normal cone diagram from 1.4.1

$$\begin{tikzcd} \mathbf{P}(1 \oplus N) \arrow[r, "j_0"] & X_t' & X \arrow[l, "j_1"'] \\ Y \arrow[u, "s"] \arrow[r, "k_0"] & Y \times \mathbf{A}^1 \arrow[u, "i_t"] & Y. \arrow[l, "k_1"'] \arrow[u, "i"'] \end{tikzcd}$$

Since both squares in this diagram are transversal the following diagram commutes

$$\begin{array}{ccccc} B(\mathbf{P}(1\oplus N)) & \xleftarrow{j_0^B} & B(X_t') & \xrightarrow{j_1^B} & B(X) \\ {\scriptstyle\gamma_0}\uparrow & & {\scriptstyle\gamma_t}\uparrow & & \uparrow{\scriptstyle\gamma_1} \\ A(Y) & \xleftarrow{k_0^A} & A(Y\times\mathbf{A}^1) & \xrightarrow{k_1^A} & A(Y). \end{array}$$

where $\gamma_0 = s_{\mathcal{B}}\circ\varphi_Y - \varphi_{\mathbf{P}(F)}\circ s_{\mathcal{A}}$, $\gamma_t = (i_t)_{\mathcal{B}}\circ\varphi_{Y\times\mathbf{A}^1} - \varphi_{X_t'}\circ(i_t)_{\mathcal{A}}$ and $\gamma_1 = i_{\mathcal{B}}\circ\varphi_Y - \varphi_X\circ i_{\mathcal{A}}$.

Let $V_t = X_t' - Y\times\mathbf{A}^1$ and let $j_t : V_t \hookrightarrow X_t'$ be the open inclusion. We claim that

$$j_t^B \circ \gamma_t = 0.$$

In fact, the operators $j_t^B\circ(i_t)_{\mathcal{B}}$ and $j_t^A\circ(i_t)_{\mathcal{A}}$ vanish by Lemma 1.8.2. Thus the operator $j_t^B\circ(i_t)_{\mathcal{B}}\circ\varphi_{Y\times\mathbf{A}^1}$ vanishes. The operator $j_t^B\circ\varphi_{X_t'}\circ(i_t)_{\mathcal{A}}$ vanishes because it coincides with the one $\varphi_{V_t}\circ j_t^A\circ(i_t)_{\mathcal{A}}$.

The operator γ_0 vanishes by the very assumption of the Lemma. Thus $j_0^B\circ\gamma_t$ vanishes as well. The composite operator $j_t^B\circ\gamma_t$ vanishes as was explained just above. Thus Lemma 1.4.2 implies the relation $\gamma_t = 0$.

Since $\gamma_1\circ k_1^A = j_1^B\circ\gamma_t$ the operator $\gamma_1\circ k_1^A$ vanishes. Since k_1^A is an isomorphism the operator γ_1 vanishes as well. The required relation

$$i_{\mathcal{B}}\circ\varphi_Y = \varphi_X\circ i_{\mathcal{A}}.$$

follows. The proof of the Lemma is completed.

1.8.5. Lemma. *Let $i : D\hookrightarrow X$ be a smooth divisor on a smooth X and let $\alpha = i^A(a)$ for an element $a\in A(X)$. Then $i_{\mathcal{B}}(\varphi(\alpha)) = \varphi(i_{\mathcal{A}}(\alpha))$.*

This Lemma follows immidiately from the following two chain of relations

$$i_{\mathcal{B}}(1)\cup\varphi(a) = i_{\mathcal{B}}(i^B(\varphi(a))) = i_{\mathcal{B}}(\varphi(i^A(a))) = i_{\mathcal{B}}(\varphi(\alpha))$$

$$\varphi(i_{\mathcal{A}}(1))\cup\varphi(a) = \varphi(i_{\mathcal{A}}(1)\cup a) = \varphi(i_{\mathcal{A}}i^A(a)) = \varphi(i_{\mathcal{A}}(\alpha))$$

because $i_{\mathcal{B}}(1) = \varphi(i_{\mathcal{A}}(1))$ by the very assumption of the Theorem.

1.8.6. Lemma. *Let L be a line bundle over a smooth Y and let $p : \mathbf{P}(1\oplus L)\to Y$ be the projective bundle over Y and let $s : Y\to\mathbf{P}(1\oplus L)$ be the section of p identifying Y with the subvariety $\mathbf{P}(1)$ in $\mathbf{P}(1\oplus L)$. Then the Theorem holds for the closed imbedding s.*

Since $s^A\circ p^A : A(Y)\to A(Y)$ is the identity the pull-back map s^A is surjective. Thus this Lemma follows from the previous one.

1.8.7. Lemma. *Let $i : D\hookrightarrow X$ be a smooth divisor on a smooth X. Then the Theorem valids for the imbedding i.*

This lemma follows from the previous one and the general Lemma 1.8.4. In fact, the normal bundle $N = N_{X/D}$ is a line bundle over Y and the Theorem holds for the closed imbedding $s : Y\to\mathbf{P}(\mathbf{1}\oplus N)$. Thus the Theorem holds for the imbedding $i : D\hookrightarrow X$.

1.8.8. Lemma. *Let Y be a smooth variety and let $E = L_1 \oplus L_2 \oplus \cdots \oplus L_n$ be the direct sum of line bundles and let $p : \mathbf{P}(1 \oplus E) \to Y$ be the projective bundle over Y. Let $s : Y \to \mathbf{P}(1 \oplus E)$ be the section of p identifying Y with the subvariety $\mathbf{P}(1)$ in $\mathbf{P}(1 \oplus E)$. Then the Theorem valids for the closed imbedding s.*

Let $F_i = 1 \oplus L_1 \oplus \cdots \oplus L_i$ and let $s_i : \mathbf{P}(F_{i-1}) \hookrightarrow \mathbf{P}(F_i)$ be the obvious closed imbedding. Then $s = s_n \circ \cdots \circ s_2 \circ s_1 : Y \to \mathbf{P}(F_n) = \mathbf{P}(1 \oplus E)$ and each s_i is a closed imbedding of a smooth divisor. Thus for each integer $i = 1, \ldots, n$ one has the relation

$$(s_i)_{\mathcal{B}} \circ \varphi_Y = \varphi_X \circ (s_i)_{\mathcal{A}}.$$

The Lemma follows.

1.8.9. Lemma. *Let Y be a smooth variety and let E be a rank n vector bundle over Y and let $p : \mathbf{P}(1 \oplus E) \to Y$ be the projective bundle over Y. Let $s : Y \to \mathbf{P}(1 \oplus E)$ be the section of p identifying Y with the subvariety $\mathbf{P}(1)$ in $\mathbf{P}(1 \oplus E)$. Then the Theorem valids for the closed imbedding s.*

This Lemma is reduced to the previous one using the splitting principle 1.6.1. Namely, by the slitting principle there exists a smooth morphism $\phi : X' \to X$ such that

- the vector bundle $E' = \phi^*(E)$ is a direct sum of line bundles and
- for any morhism $f : Y \to X$ with smooth Y and for the projection $\phi' : Y' = Y \times_X X' \to Y$ the pull-back map $(\phi')^A : A(Y) \to A(Y')$ is injective.

Now take $F = 1 \oplus E$, $Y = \mathbf{P}(F)$ and $f = p : \mathbf{P}(F) \to X$ then $Y' = \mathbf{P}(F')$ for $F' = 1 \oplus E'$ and $\phi' : \mathbf{P}(F') \to \mathbf{P}(F)$ and is the base change of ϕ by means of the projection p. The projection $p' : \mathbf{P}(F') \to X'$ is the base change of p by means of ϕ. Let $s' : X' \to \mathbf{P}(F')$ be the section of the projection p' identifying X' with the subvariety $\mathbf{P}(1)$ of the variety $\mathbf{P}(F')$. Then the following square is transversal (see 1.1.2)

$$\begin{array}{ccc}
\mathbf{P}(F) & \xleftarrow{\phi'} & \mathbf{P}(F') \\
{\scriptstyle s}\uparrow & & \uparrow{\scriptstyle s'} \\
X & \xleftarrow{\phi} & X'.
\end{array}$$

Therefore it gives rise to the commutative square

$$\begin{array}{ccc}
B(\mathbf{P}(F)) & \xrightarrow{(\phi')^A} & B(\mathbf{P}(F')) \\
{\scriptstyle \gamma}\uparrow & & \uparrow{\scriptstyle \gamma'} \\
A(X) & \xrightarrow{\phi^A} & A(X')
\end{array}$$

where $\gamma = s_{\mathcal{B}} \circ \varphi_Y - \varphi_{\mathbf{P}(F)} \circ s_{\mathcal{A}}$, $\gamma' = s'_{\mathcal{B}} \circ \varphi_Y - \varphi_{\mathbf{P}(F)} \circ s'_{\mathcal{A}}$. By the previous Lemma $\gamma' = 0$. Since $(\phi')^A$ is injective one gets $\gamma = 0$. The Lemma follows.

Now we are ready to complete the proof of the Theorem. Let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle $N = N_{X/Y}$. If Y consists of several many components, then each component is smooth. By the Proposition 1.7.1 it suffices to check the Theorem for each component separately. Thus one may assume that Y is connected. In this case the normal bundle has a constant rank and by the previuos Lemma the Theorem holds for the closed imbedding $s : Y \to \mathbf{P}(\mathbf{1} \oplus N)$. Thus the Theorem holds for the imbedding $i : Y \hookrightarrow X$ by the general Lemma 1.8.4. The proof of the Theorem is completed.

$\square$

1.9 Several Lemmas

Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. Recall that for any variety X and any smooth divisor $i : D \hookrightarrow X$ on X one has the relation $i_A(1) = e(L(D))$ by Theorem 1.1.8.

Set $\{\mathbf{P}^r\} = (p_r)_A(1) \in A(pt)$, where $p_r : \mathbf{P}^r \to pt$ is the projection. Observe that $\{\mathbf{P}^0\} = 1$ because $p_0 = id$ and $(id)_A = id$ by the normalization property 4. Set $\zeta_m = e(\mathcal{O}(1)) \in A(\mathbf{P}^m)$.

(Till the end of the proof of Theorem 2.8 we will write $\{\mathbf{P}^r\}$ for the class $(p_r)_A(1)$ in contrast with the notation $[\mathbf{P}^r]_A$ used in [PS] in the context of the oriented cohomology theory. Theorem 2.8 shows that for every non-negative integer r one has the relation $[\mathbf{P}^r]_A = \{\mathbf{P}^r\}$ if A is an oriented cohomology pretheory).

1.9.1. Lemma. *Let $f : Y \to X$ be a projective morphism of varieties. Then for any central element $a \in A(Y)$ the element $f_A(a) \in A(X)$ is central as well. In particular, all elements $\{\mathbf{P}^r\} \in A(pt)$ are central.*

In fact, if $a \in A(Y)$ and $b \in A(X)$ then one has a chain of relations

$$f_A(a) \cup b = f_A(a \cup f^A(b)) = f_A(f^A(b) \cup a) = b \cup f_A(a)$$

because f_A is a two-sided $A(X)$-module operator. The Claim follows.

1.9.2. Lemma. *Let $i_{r,n} : \mathbf{P}^r \hookrightarrow \mathbf{P}^n$ be the inclusion. Then $(i_{r,n})_A(\zeta_r^j) = \zeta_n^{j+n-r}$ in $A(\mathbf{P}^n)$. In particular, $(i_{0,n})_A(1) = \zeta_n^n$.*

1.9.3. Lemma. *One has $(p_n)_A(\zeta_n^r) = \{\mathbf{P}^{n-r}\}$. In particular, $(p_n)_A(\zeta_n^r) = 1$.*

1.9.4. Lemma. *Let $\Delta : \mathbf{P}^n \hookrightarrow \mathbf{P}^n \times \mathbf{P}^n$ be the diagonal imbedding. Identify $A(\mathbf{P}^n) \otimes_{A(pt)} A(\mathbf{P}^n)$ with $A(\mathbf{P}^n \times \mathbf{P}^n)$ by means of the cup-product. Then one has the relation in $A(\mathbf{P}^n \times \mathbf{P}^n)$*

$$\Delta_A = \zeta_n^n \otimes 1 + 1 \otimes \zeta_n^n + \Sigma_{i>0,j>0} a_{i,j} \zeta_n^i \otimes \zeta_n^j$$

for certain elements $a_{i,j}$ in $A(pt)$.

1.9.5. Lemma. *Let S and $T = \mathbf{P}^n$ be smooth projective varieties and let $f : S \to pt$, $g : T \to pt$ be the structural morphisms. Let $p_S : S \times T \to S$ and $p_T : S \times T \to T$ be the projections. Then for any elements $a \in A(S)$ and $b \in A(T)$ one has the relation $(p_S)_A((p_S)^A)(a) \cup (p_T)^A)(b)) = a \cup f_S^A(g_A(b))$. We will write $a \cup g_A(b)$ for the element $a \cup f_S^A(g_A(b))$.*

Proof of Lemma 1.9.2. Let $i_m : \mathbf{P}^{m-1} \hookrightarrow \mathbf{P}^m$ be the inclusion. Then one has a chain of relations in $A(\mathbf{P}^m)$

$$(i_m)_A(\zeta^j_{m-1}) = (i_m)_A((i^A_m)_A(\zeta^j_m)) = (i_m)_A(1) \cup \zeta^j_m = e(\mathcal{O}(1)) \cup \zeta^j_m = \zeta^{j+1}_m.$$

Thus

$$(i_{r,n})_A(\zeta^j_r) = (i_n \circ \cdots \circ i_{r+1})_A(\zeta^j_r) = \zeta^{j+n-r}_n.$$

The Lemma is proved.

Proof of Lemma 1.9.3. Using Lemma 1.9.2 one gets in $A(pt)$ a chain of relations

$$(p_n)_A(\zeta^r_n) = (p_n)_A((i_{n-r,n})_A(1)) = (p_n \circ i_{n-r,n})_A(1) = (p_{n-r})_A(1) = \{\mathbf{P}^{n-r}\}.$$

The Lemma is proved.

Proof of Lemma 1.9.4. Let $j : pt \hookrightarrow \mathbf{P}^n$ be an inclusion. Then the square

$$\begin{array}{ccc}
\mathbf{P}^n & \xrightarrow{\Delta} & \mathbf{P}^n \times \mathbf{P}^n \\
{\scriptstyle j}\uparrow & & \uparrow{\scriptstyle (j\times id)} \\
pt & \xrightarrow{(id,j)} & pt \times \mathbf{P}^n
\end{array}$$

is transversal (see 1.1.2 for the definition). Therefore $(j \times id)^A(\Delta_A(1)) = (id, j)_A(j^A(1)) = (id, j)_A(1) = \zeta^n$ in $A(\mathbf{P}^n)$.The first of these relation holds by the base change property 2 of push-forwards and the last relation holds by Lemma 1.9.2. Considering a similar transversal square one gets in $A(\mathbf{P}^n)$ the relation $(id \times j)^A(\Delta_A(1)) = \zeta^n$.

Now consider the commutative diagram

$$\begin{array}{ccc}
A(\mathbf{P}^n \times \mathbf{P}^n) & \xrightarrow{\alpha} & A(\mathbf{P}^n) \oplus A(\mathbf{P}^n) \\
{\scriptstyle \cup}\uparrow & & \uparrow{\scriptstyle id} \\
A(\mathbf{P}^n) \otimes_{A(pt)} A(\mathbf{P}^n) & \xrightarrow{\beta} & A(\mathbf{P}^n) \oplus A(\mathbf{P}^n)
\end{array}$$

where $\alpha = ((j \times id)^A, (id \times j)^A)$ and $\beta(a \otimes b) = j^A(a)b + aj^A(b)$. The kernel of j^A is generated as the $A(\mathbf{P}^n)$-module by ζ_n. Thus the kernel of β is generated as the $A(pt)$-module by the elements $\zeta^i_n \otimes \zeta^j_n$ with $i > 0$ and $j > 0$. Since $\alpha(\Delta_A(1)) = \alpha(\zeta^n_n \otimes 1 + 1 \otimes \zeta^n_n)$ The Lemma is proved.

Proof of Lemma 1.9.5. The push-forward $(p_S)_A$ is a two-sided $A(S)$-module operator. Thus $(p_S)_A((p_S)^A)(a) \cup (p_T)^A)(b) = a \cup (p_S)_A((p_T)^A)(b))$. Since $T = \mathbf{P}^n$ the base change property 3 shows that $(p_S)_A((p_T)^A)(b)) = f^A_S(g_A(b))$. The Lemma is proved.

1.10 Proof of Theorem 1.1.9

The aim of this subsection is to prove Theorem 1.1.9. Let $\varphi : A \to B$ be the ring morphism of oriented cohomology pretheories satisfying the hypotheses of Theorem 1.1.9. We begin with two lemmas.

1.10.1. Lemma. *For any variety X and any line bundle L over X one has the relation*

$$\varphi(e^A(L)) = e^B(L) \tag{20}$$

In fact, if $z : X \to L$ is the zero section of L then the chain of relation

$$\varphi(e^A(L)) = \varphi(z^A(z_A(1))) = z^B(\varphi(z_A(1))) = z^B(z_B(1)) = e^B(L)$$

proves the relation (20).

1.10.2. Lemma. *Let $p_n : \mathbf{P}^n \to pt$ be the structural morphism. Set $\{\mathbf{P}^n\}_A = (p_n)_A(1)$ and $\{\mathbf{P}^n\}_B = (p_n)_B(1)$. Then one has the relation in $B(pt)$*

$$\varphi(\{\mathbf{P}^n\}_A) = \{\mathbf{P}^n\}_B. \tag{21}$$

Proof of the Lemma. We proceed induction by m. Since $\{\mathbf{P}^0\}_A = id(1) = 1$ where $id : pt \to pt$ is the identity map and similarly $\{\mathbf{P}^0\}_B = id(1) = 1$ the case $m = 0$ is obvious. Now assume that the relation $\varphi(\{\mathbf{P}^m\}_A) = \{\mathbf{P}^m\}_B$ holds for all $m < n$ and prove the relation $\varphi(\{\mathbf{P}^n\}_A) = \{\mathbf{P}^n\}_B$.

Let $\zeta_A = e^A(\mathcal{O}(1)) \in A(\mathbf{P}^n)$ and let $\zeta_B = e^B(\mathcal{O}(1)) \in B(\mathbf{P}^n)$. By Lemma 1.9.4 one has the relation in $A(\mathbf{P}^n \times \mathbf{P}^n)$

$$\Delta_A(1) = \zeta_A^n \otimes 1 + 1 \otimes \zeta_A^n + \Sigma_{i>0,j>0} a_{i,j}^{(n)} \zeta_A^i \otimes \zeta_A^j \tag{22}$$

and one has the relation in $B(\mathbf{P}^n \times \mathbf{P}^n)$

$$\Delta_B(1) = \zeta_B^n \otimes 1 + 1 \otimes \zeta_B^n + \Sigma_{i>0,j>0} b_{i,j}^{(n)} \zeta_B^i \otimes \zeta_B^j \tag{23}$$

for certain elements $a_{i,j}^{(n)}$ in $A(pt)$ and $b_{i,j}^{(n)}$ in $B(pt)$.

The restriction of an integration to the family of closed imbeddings of varieties is a Gysin structure. Thus for any closed imbedding $i : Y \hookrightarrow X$ of varieties one has the relation $i_B \circ \varphi_Y = \varphi_X \circ i_A$. Taking the diagonal imbedding $\Delta : \mathbf{P}^n \hookrightarrow \mathbf{P}^n \times \mathbf{P}^n$ one gets the relation $\Delta_B \circ \varphi_Y = \varphi_X \circ \Delta_A$. By the relation (20) one has $\varphi(\zeta_A) = \zeta_B$. Now the Proposition 1.2.3 proves that

$$\varphi(a_{i,j}^{(n)}) = b_{i,j}^{(n)} \tag{24}$$

Let $pr_1 : \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P}^n$ be the projection to the 1-st factor. Observe that $pr_1 \circ \Delta = id_{\mathbf{P}^n}$. Thus applying the push-forward $(pr_1)_A$ to the relations (22) and using the composition property of push-forwars and Lemmas 1.9.3 and 1.9.5 one gets in $A(\mathbf{P}^n)$ the relation

$$1 = 1 + \{\mathbf{P}^n\}_A \zeta_n^n + \Sigma_{i>0,j>0} a_{i,j}^{(1)} \{\mathbf{P}^{n-i}\}_A \zeta_n^j$$

The family $\{1, \zeta_A, \ldots, \zeta_A^{n-1}\}$ is a free basis of the two-sided $A(pt)$-module $A(\mathbf{P}^n)$. Thus equating coefficients at ζ_A^n in the last relation one gets the following one in $A(pt)$

$$\{\mathbf{P}^n\}_A = -a_{1,n}^{(n)}\{\mathbf{P}^{n-1}\}_A - a_{2,n}^{(n)}\{\mathbf{P}^{n-2}\}_A - \cdots - a_{n,n}^{(1)}\{\mathbf{P}^0\}_A \tag{25}$$

Simultanously one gets the following relation in $B(pt)$

$$\{\mathbf{P}^n\}_B = -b_{1,n}^{(n)}\{\mathbf{P}^{n-1}\}_B - b_{2,n}^{(n)}\{\mathbf{P}^{n-2}\}_B - \cdots - b_{n,n}^{(1)}\{\mathbf{P}^0\}_B \tag{26}$$

By the inductive hypothesis $\varphi(\{\mathbf{P}^m\}_A) = \{\mathbf{P}^m\}_B$ for $m < n$ and it was already mentioned that $\varphi(a_{i,j}^{(n)}) = b_{i,j}^{(n)}$ in $B(pt)$. Thus applying the homomorphism $\varphi(pt) : A(pt) \to B(pt)$ to the relation (25) one gets the required relation $\varphi(\{\mathbf{P}^n\}_A) = \{\mathbf{P}^n\}_B$ in $B(pt)$.

□

Now we are ready to prove Theorem 1.1.9.

Proof of Theorem 1.1.9. Since push-forwards respect the composition of projective morphisms it suffices to prove the following two Claims

1.10.3. Claim. *For a closed imbedding of smooth varieties $i : Y \hookrightarrow X$ one has*

$$i_B \circ \varphi_Y = \varphi_X \circ i_A.$$

1.10.4. Claim. *For a smooth variety X and the projection $p : X \times \mathbf{P}^n \to X$ one has*

$$p_B \circ \varphi_{X\times\mathbf{P}^n} = \varphi_X \circ p_A.$$

To prove Claim 1.10.3 recall that the restriction of an integration on A (resp. on B) to the family of closed imbeddings is a Gysin structure on A (resp. on B). Thus the Claim follows from Theorem 1.8.3.

To prove Claim 1.10.4 recall that the push-forwards p_A is a two-sided $A(X)$-module operators and by Proposition 1.2.3 $A(X\times\mathbf{P}^n) = A(X)[\zeta_A]/(\zeta_A^{n+1})$, where $\zeta_A = e^A(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. Let $\zeta_B = e^B(\mathcal{O}(1)) \in B(\mathbf{P}^n)$. By the Lemma 1.9.5 it suffices to show that for the projection $p_n : \mathbf{P}^n \to pt$ and any non-negative integer i the elements $\varphi((p_n)_A(\zeta_A^i))$ and $(p_n)_B(\varphi(\zeta_A^i))$ coincide in $B(pt)$.

If $\zeta_B = e^B(\mathcal{O}(1)) \in B(\mathbf{P}^n)$ then the relation (20) states that $\varphi(\zeta_A) = \zeta_B$ in $B(\mathbf{P}^n)$ and therefore $\varphi(\zeta_A^i) = \zeta_B^i$ for any non-negative integer i. By Lemma 1.9.3 one has $(p_n)_A(\zeta_A^i) = \{\mathbf{P}^{n-i}\}_A$ and $(p_n)_B(\zeta_B^i) = \{\mathbf{P}^{n-i}\}_B$ Thus it remains to check the relation

$$\varphi(\{\mathbf{P}^m\}_A) = \{\mathbf{P}^m\}_B \tag{27}$$

for all non-negative integer m. This is just the relation (21) proved above in this subsection. The proof of Theorem 1.1.9 is completed.

□

2 Riemann-Roch type theorems

In this section we introduce Chern classes of vector bundles with values in an oriented cohomology pretheory. For a ring morphism of oriented cohomology pretheories an inverse Todd genus is defined and a Riemann-Roch theorem is proved. The results are illustrated by a series of examples including the classical Riemann-Roch-Hirzebruch theorem in the form of Grothendieck.

2.1 Ring of universally central elements

Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. The main aim of this subsection is to prove projective bundle theorem for central elements and for universally central elements. This will be used as in the proof of the Cartan formulae for Chern classes so in the proof of Riemann-Roch theorem stated below in the text.

If for any variety X the ring $A(X)$ *is commutative* then *the only result* of this subsection which is useful below in the text is Proposition 2.1.3.

2.1.1. Definition. *Let X be a variety. An element $a \in A(X)$ is called universally central if for any morphism $f : Y \to X$ of varieties the element $f^A(a)$ is central in the ring $A(Y)$.*

The subset af all universally central elements forms a subring of the ring $A(X)$, which will be denoted below by $A(X)^{uc}$. The subring of all central elements in $A(X)$ will be denoted by $A(X)^c$.

If L is a line bundle over a variety X then the Euler class $e(L)$ is a universally central element in A(X) by Lemma 1.2.1. Now one can state the first main result of this subsection.

2.1.2. Theorem. *Let E be a rank n vector bundle over a variety X and let $\mathbf{P}(E)$ be the corresponding projective bundle and let $\xi_E = \mathcal{O}_E(-1) \in A(\mathbf{P}(E))$ Then*

(i) the operators

$$(1, \xi_E, \dots, \xi_E^{n-1}) : A(X)^c \oplus A(X)^c \cdots \oplus A(X)^c \to A(\mathbf{P}(E))^c$$

$$(1, \xi_E, \dots, \xi_E^{n-1}) : A(X)^{uc} \oplus A(X)^{uc} \cdots \oplus A(X)^{uc} \to A(\mathbf{P}(E))^{uc}$$

are isomorphisms (see 1.1.5 for notation);

(ii) the canonical operators

$$A(X)^c \otimes_{A(X)^{uc}} A(\mathbf{P}(E))^{uc} \to A(\mathbf{P}(E))^c$$

$$A(X) \otimes_{A(X)^{uc}} A(\mathbf{P}(E))^{uc} \to A(\mathbf{P}(E))$$

are isomorphisms as well (see 1.1.5 for notation).

2.1.3. Proposition. *Let X be a variety and let E_1, E_2 be two vector bundles over X. Let $E = E_1 \oplus E_2$ and let $i_1 : \mathbf{P}(E_1) \hookrightarrow \mathbf{P}(E)$ and $i_2 : \mathbf{P}(E_2) \hookrightarrow \mathbf{P}(E)$ be the imbeddings of the corresponding projective bundles. Then the following sequences are well-defined and exact*

$$0 \to A(\mathbf{P}(E_1)) \xrightarrow{(i_1)_A} A(\mathbf{P}(E)) \xrightarrow{i_2^A} A(\mathbf{P}(E)) \to 0. \tag{28}$$

$$0 \to A(\mathbf{P}(E_1))^c \xrightarrow{(i_1)_A} A(\mathbf{P}(E))^c \xrightarrow{i_2^A} A(\mathbf{P}(E))^c \to 0. \tag{29}$$

To prove Theorem 2.1.2 we need in some preliminaries. We begin with the following

2.1.4. Claim. *Let $f : Y \to X$ be a projective morphism of varieties. Then one has the inclusion*

$$f_A(A(Y)^c) \subset A(X)^c \tag{30}$$

In fact, if $a \in A(Y)$ and $b \in A(X)$ then one has a chain of relations

$$f_A(a) \cup b = f_A(a \cup f^A(b)) = f_A(f^A(b) \cup a) = b \cup f_A(a)$$

because f_A is a two-sided $A(X)$-module operator. The Claim follows.

2.1.5. Claim. *Let $p_i : \mathbf{P}(E) \times_X \mathbf{P}(E) \to \mathbf{P}(E)$ be the projection on the i-th factor. Then the operator*

$$A(\mathbf{P}(E)) \otimes_{A(X)} A(\mathbf{P}(E)) \to A(\mathbf{P}(E) \times_X \mathbf{P}(E)) \tag{31}$$

given by $a \otimes b \mapsto p_1^A(a) \otimes p_2^A(b)$ is an isomorphism of the two-sided $A(X)$-modules. We will identify these two two-sided $A(X)$-modules by means of the mentioned isomorphism.

In fact, by the projective bundle axiom the family $(1, \xi_E, \dots, \xi_E^{n-1})$ is a free basis of the two-sided $A(X)$-module $A(\mathbf{P}(E))$. If $p : \mathbf{P}(E) \to X$ is the projection then by the same projective bundle axiom the family $(1, \xi_{p_1^*(E)}, \dots, \xi_{p_1^*(E)}^{n-1})$ is a free basis of the two-sided $A(\mathbf{P}(E))$-module $A(\mathbf{P}(E) \times_X \mathbf{P}(E))$, where the $A(\mathbf{P}(E))$-module structure on $A(\mathbf{P}(E) \times_X \mathbf{P}(E))$ is defined by means of the pull-back map p_2^A. Since $\xi_{p_1^*(E)} = p^A(\xi_E)$ the Claim follows.

Now let the family $\{x_1, x_2, \dots, x_n\}$ in $A(\mathbf{P}(E))$ be a free $A(X)$-module basis consisting of central elements. Then the family $\{x_1 \otimes 1, x_2 \otimes 1, \dots, x_n \otimes 1\}$ is a free basis of $A(\mathbf{P}(E) \times_X \mathbf{P}(E))$ considered as the two-sided $A(\mathbf{P}(E))$-module by means of the pull-back map p_2^A. Thus there exists a unique family $\{y_1, y_2, \dots, y_n\}$ of elements in $A(\mathbf{P}(E))$ such that

$$\Delta(1) = \sum_{i=1}^{n} x_i \otimes y_i, \tag{32}$$

where $\Delta : \mathbf{P}(E) \hookrightarrow \mathbf{P}(E) \times_X \mathbf{P}(E)$ is the diagonal.

2.1.6. Claim. *The elements $y_i \in A(\mathbf{P}(E))$ are all central.*

To prove this consider an element $a \in A(\mathbf{P}(E))$. Then one has a chain of relations

$$\Delta_A(a) = \Delta_A(\Delta^A(p_2^A(a))) = \Delta_A(1) \cup p_2^A(a) = \sum_{i=1}^{n} x_i \otimes (y_i \cup a). \tag{33}$$

From the other side the element $\Delta(1)$ is central in $A(\mathbf{P}(E) \times_X \mathbf{P}(E))$ and the elements x_i are central in $A(\mathbf{P}(E))$. Thus one has another chain of relations

$$\Delta_A(a) = \Delta_A(\Delta^A(p_1^A(a))) = p_1^A(a) \cup \Delta(1) = \sum_{i=1}^{n} (a \cup x_i) \otimes y_i =$$

$$\sum_{i=1}^{n} (x_i \cup a) \otimes y_i = \sum_{i=1}^{n} x_i \otimes (a \cup y_i).$$

Comparing these relations one gets the new one $\sum_{i=1}^{n} x_i \otimes (y_i \cup a) = \sum_{i=1}^{n} x_i \otimes (a \cup y_i)$. Recall that the family $\{x_1 \otimes 1, x_2 \otimes 1, \dots, x_n \otimes 1\}$ is a free basis of $A(\mathbf{P}(E)) \otimes_{A(X)} A(\mathbf{P}(E))$ considered as the $A(\mathbf{P}(E))$-module by means of the second factor. Thus for each $i = 1, 2, \dots, n$ one has $y_i \cup a = a \cup y_i$. The Claim follows.

2.1.7. Claim. *Let an element $a \in A(\mathbf{P}(E))$ be presented in the form $a = \sum_{i=1}^{n} x_i \cup a_i$ with certain $a_i \in A(X)$. Then*

$$a_i = p_A(y_i \cup a) \tag{34}$$

where the elements y_i are from Claim 2.1.6 and $p : \mathbf{P}(E) \to X$ is the projection.

In fact, one has a chain of relations

$$a = (p_2)_A(\Delta_A(a)) = (p_2)_A(\sum_{i=1}^{n} x_i \otimes (y_i \cup a)) = \sum_{i=1}^{n} x_i \cup p_A(y_i \cup a).$$

The family $\{x_1, x_2, \dots, x_n\} \in A(\mathbf{P}(E))^c$ is a free $A(X)$-basis of $A(\mathbf{P}(E))$. The Claim follows.

2.1.8. Claim. *Let an element $a \in A(\mathbf{P}(E))^c$ be presented in the form $a = \sum_{i=1}^{n} x_i \cup a_i$ with $a_i \in A(X)$. Then $a_i \in A(X)^c$ for each $i = 1, 2, \dots, n$*

In fact, by the previous Claim $a_i = p_A(y_i \cup a)$. Since both elements y_i and a are central in $A(\mathbf{P}(E))$ the element $p_A(y_i \cup a)$ is central as well by Claim 2.1.4.

2.1.9. Claim. *Let the elements y_i be from Claim 2.1.6 and let $p : \mathbf{P}(E) \to X$ be the projection. Then*

$$p_A(y_i \cup x_j) = \delta_{ij},$$

where δ_{ij} is the Kroneker symbol.

In fact, for any element $a \in A(X)$ one has a chain of relations

$$a = (p_2)_A(\Delta_A(a)) = (p_2)_A(\sum_{i=1}^{n} x_i \otimes (y_i \cup a)) = \sum_{i=1}^{n} x_i \cup p_A(y_i \cup a),$$

because of the relation 33. Taking $a = x_j$ one gets the relation $x_j = \sum_{i=1}^{n} x_i \cup p_A(y_i \cup x_j$. Since the family $\{x_1, x_2, \ldots, x_n\} \in A(\mathbf{P}(E))^c$ is a free $A(X)$-basis of $A(\mathbf{P}(E))$ the Claim follows.

Now consider an $A(X)^c$-module operator

$$\varphi : \oplus_{i=1}^{n} A(X)^c \to A(\mathbf{P}(E))^c$$

given by $\varphi((a_1, a_2, \ldots, a_n)) = \sum_{i=1}^{n} x_i \cup a_i$. Consider as well an $A(X)^{uc}$-module operator

$$\psi = \oplus_{i=1}^{n} \psi_i : A(\mathbf{P}(E))^c \to \oplus_{i=1}^{n} A(X)^c$$

given by $\psi_i(a) = p_A(y_i \cup a)$. The element $p_A(y_i \cup a)$ is central by Claims 2.1.6 and 2.1.4.

2.1.10. Claim. *The operators φ and ψ are isomorphism inverse to each other.*

Clearly this Claim proves that the first of the two operators from the item (i) of the Theorem is an isomorphism.

To prove this Claim observe first that for each $i, j \in \{i = 1, 2, \ldots, n\}$ the element $\alpha_{ij} = p_A(x_i \cup y_j) \in A(X)^c$ is central by Claims 2.1.4 and 2.1.6. The composite map $\psi \circ \varphi$ is given exactly by the matrix (α_{ij}). Hence the composite operator $\psi \circ \varphi$ is the identity by Claim 2.1.9. Thus the operator φ is injective. The operator φ is surjective by Claims 2.1.7 and 2.1.8. Thus φ is an isomorphism. The relation $\psi \circ \varphi$ shows that the operator ψ is the inverse to the operator φ.

Proof of Theorem 2.1.2. Suppose the free basis $\{x_1, x_2, \ldots, x_n\}$ of the two-sided $A(X)$-module $A(\mathbf{P}(E))$ consists of universally central elements. Let $\{y_1, y_2, \ldots, y_n\}$ be the elements from Claim 32. These elements are central by Claim 2.1.6.

2.1.11. Claim. *Let $a \in A(\mathbf{P}(E))^{uc}$. Then $p_A(y_i \cup a) \in A(X)^{uc}$.*

Assuming for a moment this Claim complete the proof as follows. Consider two $A(X)^{uc}$-module operators

$\varphi^{uc} : \oplus_{i=1}^{n} A(X)^c \to A(\mathbf{P}(E))^c$

$\psi^{uc} = \oplus_{i=1}^{n} \psi_i^{uc} : A(\mathbf{P}(E))^c \to \oplus_{i=1}^{n} A(X)^c$ given by

$\varphi((a_1, a_2, \ldots, a_n)) = \sum_{i=1}^{n} x_i \cup a_i$ and

$\psi_i^{uc}(a) = p_A(y_i \cup a)$. The operator ψ^{uc} is well-defined by Claims 2.1.6 and 2.1.4. We claim that the operators φ^{uc} and ψ^{uc} are isomorphisms inverse to each other.

In fact, as just above the composite operator $\psi^{uc} \circ \varphi^{uc}$ is given by the matrix (α_{ij}) with $\alpha_{ij} = p_A(y_i \cup x_j)$ and moreover the elements α_{ij} are the Kroneker symbols by Claim 2.1.9. Thus $\psi^{uc} \circ \varphi^{uc}$ is the identity operator and the operator φ^{uc} is injective. It is surjective by Claims 2.1.7 and 2.1.11. Hence the operator φ^{uc} is an isomorphism and ψ^{uc} is an isomorphism inverse to the operator φ^{uc}. The first assertion of Theorem 2.1.2 is proved. The second assertion of the Theorem follows immidiately from the first one and the projective bundle axiom 1.1.6.

To complete the proof of Theorem 2.1.2 it remains to prove Claim 2.1.11. To do this consider a morphism $f : X' \to X$ of varieties. Let $E' = X' \times_X \mathbf{P}(E)$ and let $p' : \mathbf{P}(E') \to X'$ be the projective bundle over X' and let $F : \mathbf{P}(E') \to \mathbf{P}(E)$ be the obvious projection. Let $\Delta' : \mathbf{P}(E') \to \mathbf{P}(E') \times_{X'} \mathbf{P}(E')$ be the diagonal. Then the diagram of varieties

$$\begin{array}{ccc}
\mathbf{P}(E') \times_{X'} \mathbf{P}(E') & \xrightarrow{F \times F} & \mathbf{P}(E) \times_X \mathbf{P}(E) \\
{\scriptstyle \Delta'}\uparrow & & \uparrow{\scriptstyle \Delta} \\
\mathbf{P}(E') & \xrightarrow{F} & \mathbf{P}(E)
\end{array}$$

is clearly transversal. Thus one has the relation

$$\Delta'_A \circ F^A = (F \times F)^A \circ \Delta_A.$$

In particular, one has $\Delta'_A(1) = (F \times F)^A(\Delta_A(1))$. Hence $\Delta'_A(1) = \sum_{i=1}^n F^A(x_i) \otimes F^A(y_i)$.

By the projective bundle axiom the canonical map $A(X') \otimes_{A(X)} A(\mathbf{P}(E)) \to A(\mathbf{P}(E'))$ is an isomorphism of the left $A(X')$-modules. Thus the family $\{F^A(x_1), F^A(x_2), \ldots, F^A(x_n)\}$ is a free basis of the left $A(X')$-module $A(\mathbf{P}(E'))$. Since the elements $F^A(x_i) \in A(X')$ are central the mentioned family is a free basis of the two-sided $A(X')$-module $A(\mathbf{P}(E'))$.

Now Claim 2.1.6 shows that the elements $F^A(y_i) \in A(\mathbf{P}(E'))$ are central. If $a \in A(\mathbf{P}(E))$ then

$$f^A[p_A(y_i \cup a)] = p'_A[F^A(y_i \cup a)] = p'_A[F^A(y_i) \cup F^A(a)]$$

by Lemma 13. If $a \in A(\mathbf{P}(E))^{uc}$ then $F^A(a) \in A(\mathbf{P}(E'))^{uc}$ and $p'_A[F^A(y_i) \cup F^A(a)] \in A(\mathbf{P}(E'))$ is a central element by Claim 2.1.4. Thus the elements $f^A[p_A(y_i \cup a)]$ are central as well which proves that the elements $p_A(y_i \cup a)$ are universally central. Claim 2.1.11 is proved which completes the proof of the Theorem.

□

Proof of Proposition 2.1.3. We need in the following

2.1.12. Claim. *The sequence* $0 \to A(\mathbf{P}(E_1)) \xrightarrow{(i_1)_A} A(\mathbf{P}(E)) \xrightarrow{i_2^A} A(\mathbf{P}(E_2)) \to 0$ *is exact.*

Assuming this Claim for a moment complete the proof of the Proposition as follows. Let $a \in A(\mathbf{P}(E)^{uc})$ be killed by the operator i_2^A. By the Claim there exists an element $a_1 \in A(\mathbf{P}(E_1))$ such that $(i_1)_A(a_1) = a$. For any element $b \in A(\mathbf{P}(E))$ one has a chain of relations

$$(i_1)_A[a_1 \cup i_1^A(b)] = (i_1)_A(a_1) \cup b = a \cup b = b \cup a = b \cup (i_1)_A(a_1) = (i_1)_A[i^A(b) \cup a_1]$$

The operator i_1 is injective by the Claim. Thus $a_1 \cup i_1^A(b) = i_1^A(b) \cup a_1$. By the projective bundle axiom the pull-back operator i_1^A is surjective. Thus the element a_1 is central.

It remains to prove the Claim. Let $j : \mathbf{P}(E) - \mathbf{P}(E_1) \hookrightarrow \mathbf{P}(E)$ be the open inclusion and let $s : \mathbf{P}(E_2) \to \mathbf{P}(E) - \mathbf{P}(E_1)$ be the obvious inclusion such that $j \circ s = i_2$. Observe that the variety $\mathbf{P}(E) - \mathbf{P}(E_1)$ is a vector bundle over the variety $\mathbf{P}(E_2)$ and moreover the inclusion s coincides with the zero section of this bundle. Consider the commutative diagram

$$\begin{tikzcd}
A(\mathbf{P}(E_1)) \arrow[r, "(i_1)_A"] & A(\mathbf{P}(E)) \arrow[r, "i_2^A"] & A(\mathbf{P}(E_2)) \\
A(\mathbf{P}(E_1)) \arrow[r, "(i_1)_A"] \arrow[u] & A(\mathbf{P}(E)) \arrow[r, "j_2^A"] \arrow[u, "id"'] & A(\mathbf{P}(E) - \mathbf{P}(E_1)) \arrow[u, "s^A"']
\end{tikzcd}$$

The bottom row is exact by the localization axiom 1.1.3. The pull-back operator s^A is an isomorphism by the strong homotopy invariance property of the pretheory A 1.1.1. Thus the top row is exact as well. By the projective bundle axiom the pull-back operator i_2^A is surjective. Thus it remains to prove that the operator $(i_1)_A$ is injective.

We proceed induction by the rank of the bundle E_1. By the splitting principle 1.6.1 one may assume that the vector bundle E_1 is a direct sum of line bundles: $E_1 = \oplus_{i=1}^{n_1} L_i$.

If E_1 is of rank one, then $\mathbf{P}(E_1) = X$ and $p \circ i_1 = id_X$ for the projection $p : \mathbf{P}(E) \to X$. The relation $p_A \circ (i_1)_A = id_{A(X)}$ proves the injectivity of $(i_1)_A$ in this case.

If $n > 1$, then set $F_1 = \oplus_{i=2}^{n_1} L_i$, $F = F_1 \oplus E_2$, and let $j : \mathbf{P}(F) \hookrightarrow \mathbf{P}(E)$, $j_1 : \mathbf{P}(F_1) \hookrightarrow \mathbf{P}(E_1)$ let $l : \mathbf{P}(L_1) \hookrightarrow \mathbf{P}(E)$, $l_1 : \mathbf{P}(L_1) \hookrightarrow \mathbf{P}(E_1)$, $k_1 : \mathbf{P}(F_1) \hookrightarrow \mathbf{P}(E_1)$ be the imbeddings induced by the inclusions of the vector bundles.

Now consider the commutative diagram

$$\begin{tikzcd}
A(\mathbf{P}(L_1)) \arrow[r, "l_A"] & A(\mathbf{P}(E)) \arrow[r, "j^A"] & A(\mathbf{P}(F)) \\
A(\mathbf{P}(L_1)) \arrow[r, "(l_1)_A"] \arrow[u, "id"] & A(\mathbf{P}(E_1)) \arrow[r, "j_1^A"] \arrow[u, "(i_1)_A"'] & A(\mathbf{P}(F_1)) \arrow[u, "(k_1)_A"']
\end{tikzcd}$$

The rows are exact and the operator l_A is injective because L_1 is a line bundle. The operator $(k_1)_A$ is injective by the inductive assumption. Thus the operator $(i_1)_A$ is injective. and the Claim follows.

$\square$

2.2 Chern classes

Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. In this section Chern classes with values in A are defined and certain properties of this classes are proved. In particular the Cartan formular is proved. We follow here the known construction of A.Grothendieck [Gr,].

2.2.1. Definition. *Let E be a rank n vector bundle over X and let $\mathbf{P}(E)$ be the corresponding projective bundle and let $\xi_E = \mathcal{O}_E(-1) \in A(\mathbf{P}(E))$. Consider the multiplication by the central element ξ_E on the rank n free $A(X)^{uc}$-module $A(\mathbf{P}(E))^{uc}$. Let $\chi^{uc}_E(t) \in A(X)^{uc}[t]$ be the characteristic polinomial of this operator and write down it in the form*

$$\chi^{uc}_E(t) = t^n - c_1(E)t^{n-1} + \cdots + (-1)^n c_n(E). \tag{35}$$

The elements $c_i(E)$ are in $A(X)^{uc}$ by the very definition. They are called Chern classes of E. Set $c_0(E) = 1$ and $c_m(E) = 0$ for $m < 0$ and for $m > n$. If the variety $X = \amalg X_i$ then $E = \amalg E_i$, where E_i is a vector bundle over X_i. In this case set $c_r(E)$ to be the element $(c_r(E_i))$ in $A(X) = \oplus A(X_i)$.

One can take the multiplication by the element ξ_E on the free rank n $A(X)^c$-module $A(\mathbf{P}(E))^c$ and consider the characteristic polinomial $\chi^c_E(t)$ of this operator. Then $\chi^c_E(t) = \chi^{uc}_E(t)$ by the item (ii) of Theorem 2.1.2.

2.2.2. Theorem. *Chern classes satisfy the following properties*

1. *$c_0(E) = 1$;*

   $c_1(L) = e(L)$ for any line bundle;

2. *functoriality:*

   $c_i(E) = c_i(E')$ for isomorphic vector bundles E and E';

   $f^A(c_i(E)) = c_i(f^*(E))$ for each morphism $f : Y \to X$;

3. *the Chern classes $c_i(E)$ are universally central and nilpotent;*

4. *Cartan formula: $c_r(E) = c_0(E_1) \cup c_r(E_2) + \cdots + c_r(E_1) \cup c_0(E_2)$ for each short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of vector bundles;*

5. *Vanishing property: $c_m(E) = 0$ for $m > rk(E)$.*

Proof. The relation $c_0(\mathbf{1}) = 0$ holds by the very definition. If L is a line bundle, then $\mathbf{P}(L) = X$ and $\mathcal{O}_L(-1) = L$ and $\chi_L(t) = t - e(L)$. Thus $c_1(L) = e(L)$. The functoriality of Chern classes is obvious.

To prove the Cartan formulae one may assume by the splitting principle 1.6.1 that $E = E_1 \oplus E_2$. In this case consider the closed imbeddings $i_s : \mathbf{P}(E_s) \hookrightarrow \mathbf{P}(E)$ for $s = 1, 2$ and recall that $A(\mathbf{P}(E_s))$ is a two-sided $A(\mathbf{P}(E))$-module by means of the pull-back operator i^A_s.

Now consider an endomorphism of the short exact sequence (29) given by the multiplication by ξ_E

$$\begin{tikzcd}
0 \arrow[r] & A(\mathbf{P}(E_1))^c \arrow[r, "(i_1)_A"] & A(\mathbf{P}(E))^c \arrow[r, "i^A_2"] & A(\mathbf{P}(E_2))^c \arrow[r] & 0 \\
0 \arrow[r] & A(\mathbf{P}(E_1))^c \arrow[r, "(i_1)_A"] \arrow[u, "\cup i^A_1(\xi_E)"] & A(\mathbf{P}(E))^c \arrow[r, "i^A_2"] \arrow[u, "\cup \xi_E"] & A(\mathbf{P}(E_2))^c \arrow[r] \arrow[u, "\cup i^A_2(\xi_E)"] & 0
\end{tikzcd}$$

Obviously $i_s^A(\xi_E) = \xi_{E_s}$ in $A(\mathbf{P}(E_s))^c$ for $s = 1, 2$. Thus $\chi^c_E(t) = \chi^c_{E_1}(t)\chi^c_{E_2}(t)$. As it is mentioned just after Definition 2.2.1 one has the relation $\chi^c_F(t) = \chi^{uc}_F(t)$ for any vector bundle F. Hence $\chi^{uc}_E(t) = \chi^{uc}_{E_1}(t)\chi^{uc}_{E_2}(t)$. This implies that the total Chern class $c_t(F) = 1 + c_1(F)t + c_2(F)t^2 + \dots$ satisfies the relation $c_t(E) = c_t(E_1)c_t(E_2)$. The Cartan formular follows.

The Chern classes are universally central by the very definition. To prove that classes $c_i(E)$ are nilpotent one may assume by the splitting principle 1.6.1 that E is a direct sum of line bundles. In this case the nilpotence follows from the Cartan formulae.

The vanishing over the rank property holds by the very definition.

□

2.2.3. Proposition. *Let $r(t) \in A(pt)^{uc}[[t]]$. There exists a unique assignment associating to each vector bundle E over a variety X an element $r(E)$ in the ring $A(X)^{uc}$ and satisfying the following properties*

(i) functoriality: $f^A(r(E)) = r(f^*(E))$ for any morphism of varieties $f : Y \to X$;

(ii) multiplicativity: $r(E) = r(E_1)r(E_2)$ for any short exact sequence of vector bundles $0 \to E_1 \to E \to E_2 \to 0$;

(iii) normalization: $r(L) = r(e(L))$- the evaluation of the series $r(t)$ on the Euler class $e(L)$.

This assignment satisfies as well the following property

(iv) if the series $r(t)$ is invertible and $r^{-1}(t)$ is its multiplicative inverse in $A(pt)^{uc}[[t]]$, then for any vector bundle E one has the relation $r(E)r^{-1}(E) = 1$ in $A(X)$.

Proof. The uniqueness of the assignment follows from the splitting principle. 1.6.1.

We give here two constructions of the required assignment $E \mapsto r(E)$. The first one is not traditional and it does not use Chern classes. The second one is traditional and it uses Chern classes. The uniqueness of the required assignment $E \mapsto r(E)$ shows that these two constructiones gives the same assignment $E \mapsto r(E)$.

First construction. Let $A(pt)^{uc}[[t]] \cong A(\mathbf{P}^\infty)^{uc}$ be the isomorphism sending the variable t to the Euler class $e(\mathcal{O}(-1))$. Let $r^- = r(e(\mathcal{O}(-1))) \in A(\mathbf{P}^\infty)^{uc}$ be the image of $r(t)$ under the isomorphism $A(pt)^{uc}[[t]] \cong A(\mathbf{P}^\infty)^{uc}$. For a variety X and a line bundle L over X there is a finite dimensional k-vector space V and a diagram $X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V)$ (see Claim 1.2.2) such that $p : X' \to X$ is a torsor over a vector bundle and the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic. Set $r(L) = (p^A)^{-1}(f^A(r^-|_{\mathbf{P}(V)})) \in A(X)^{uc}$. One can check that the element $r(L) \in A(X)^{uc}$ is well-defined, i.e. it does not depend on the particular choice of the diagram and moreover the element $r(L)$ depends only of the isomorphism class of the line bundle L (we skip here a careful proof of the well-definess of the element $r(L)$ because the second construction below is described in all the details). Clearly $r(\mathcal{O}(-1)) = r^- = r(e(\mathcal{O}(-1))) \in A(\mathbf{P}^\infty)^{uc}$. Thus for every variety X and every line bundle L over X one has $r(L) = r(e(L)) \in A(X)^{uc}$.

If E is a rank n vector bundle over X then set $r_E = r(\mathcal{O}_E(-1)) \in A(\mathbf{P}(E))^{uc}$. Consider an $A(X)^c$-module operator $A(\mathbf{P}(E))^c \xrightarrow{r_E} A(\mathbf{P}(E))^c$ given by $a \mapsto a \cup r_E$. By Theorem

2.1.2 the $A(X)^c$-module $A(\mathbf{P}(E))^c$ is free of rank n. Set

$$r(E) = det(r_E) \in A(X)^c \tag{36}$$

We claim that the assignment $E \mapsto r(E)$ is the required one. In fact, the functoriality of the class $r(E)$ is clear because of the functoriality of the class $r(L)$ (on line bundles). The functoriality of the class $r(E)$ shows that the elements $r(E) \in A(X)^c$ are in fact in $A(X)^{uc}$. The normalization holds because for a line bundle L over X one has $\mathbf{P}(L) = X$ and $\mathcal{O}_E(-1) = L$ over X and $r_L = r(L) = r(e(L)) \in A(X)^{uc}$. To prove the multiplicativity of the assignment $r(E)$ one may assume by Splitting principle 1.6.1 that $E = E_1 \oplus E_2$. In this case consider the closed imbeddings $i_s : \mathbf{P}(E_s) \hookrightarrow \mathbf{P}(E)$ for $s = 1, 2$ and recall that $A(\mathbf{P}(E_s))$ is a two-sided $A(\mathbf{P}(E))$-module by means of the pull-back operator i_s^A. Consider an endomorphism of the short exact sequence (29) given by the multiplication by ξ_E

$$\begin{tikzcd}
0 \arrow[r] & A(\mathbf{P}(E_1))^c \arrow[r, "(i_1)_A"] & A(\mathbf{P}(E))^c \arrow[r, "i_2^A"] & A(\mathbf{P}(E_2))^c \arrow[r] & 0 \\
0 \arrow[r] & A(\mathbf{P}(E_1))^c \arrow[r, "(i_1)_A"] \arrow[u, "\cup i_1^A(r_E)"] & A(\mathbf{P}(E))^c \arrow[r, "i_2^A"] \arrow[u, "\cup r_E"] & A(\mathbf{P}(E_2))^c \arrow[r] \arrow[u, "\cup i_2^A(r_E)"] & 0
\end{tikzcd}$$

Obviously $i_s^A(r_E) = r_{E_s}$ in $A(\mathbf{P}(E_s))^c$ for $s = 1, 2$. Thus $r(E) = r(E_1)r(E_2)$. The multiplicativity follows.

Second construction. Let $r(t) \in A(pt)^{uc}[[t]]$ be the formal power series in one variable and let $t_1, t_2, \ldots, t_n$ be independent variables and let $\sigma_1, \sigma_2, \ldots \sigma_n$ be the symmetric polynomial in these variables: $\sigma_1 = t_1 + t_2 + \cdots + t_n$, $\sigma_2 = t_1 t_2 + \cdots + t_{n-1} t_n$, ... , $\sigma_n = t_1 t_2 \ldots t_n$. Set

$$r(\sigma_1, \sigma_2, \ldots, \sigma_n) = \prod_{i=1}^{n} r(t_i) \in A(pt)^{uc}[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$$

Define an invariant $r(E)$ of a vector bundle E with $\mathrm{rk}E = n$ over a smooth X as the evaluation of the series $r(\sigma_1, \sigma_2, \ldots, \sigma_n)$ on the Chern classes of E

$$r(E) = r(c_1^A(E), c_2^A(E), \ldots, c_n^A(E)) \in A^{uc}(X).$$

Here the right hand side is well-defined because the Chern classes are universally central and nilpotent (see 2.2.2).

Now check that the assignment $E \mapsto r(E)$ just defined is the required one. The normalization property holds by the very construction. The functoriality is obvious because Chern classes are functorial.

If a vector bundle F is a sum of line bundles $F = \oplus_{i=1}^n L_i$. then

$$r(F) = \prod_{i=1}^{n} r(L_i) \tag{37}$$

by the Cartan formular for Chern classes. To prove the multiplicativity one may assume by the splitting principle 1.6.1 that the vector bundle E is a direct sum of vector bundles

E_1 and E_2 and moreover E_s is a direct sum of line bundles for $s = 1, 2$. In this case the multiplicativity follows from the relation (37) which is already proved. Thus the assignment $E \mapsto r(E)$ is the required one.

Now verify that the property (iv) holds for the assignment $E \mapsto r(E)$. This property follows from the relation (37) in the case when the vector bundle E is a direct sum of line bundles. The general case is reduced to the mentioned one by the splitting principle 1.6.1.

□

2.2.4. Corollary. *If the series $r(t) \in A(pt)^{uc}[[t]]$ from Proposition 2.2.3 is invertible then the element $r \in A(\mathbf{P}^\infty)$ from Proposition 2.2.3 is invertible as well. Moreover for any closed imbedding $i : Y \hookrightarrow X$ with tangent bundles T_Y and T_X and the normal bundle $N = N_{X/Y}$ one has the relation*

$$(\cup r^{-1}(T_X)) \circ i_A \circ (\cup r(T_Y)) = i_A \circ r(N) : A(Y) \to A(X) \tag{38}$$

Proof. The short exact sequence $0 \to T_Y \to i^*(T_X) \to N \to 0$ of vector bundles on Y gives by Proposition 2.2.3 the relation $i^*(r^{-1}(T_X))r(T_Y) = r(N)$ in $A(Y)$. Let $a \in A(Y)$. Then one has a chain of relations in $A(X)$

$$i_A[a \cup r(T_Y)] \cup r^{-1}(T_X) = i_A[a \cup r(T_Y) \cup i^*(r^{-1}(T_X)))] = i_A[a \cup r(N)]$$

The Corollary follows. □

2.3 How to form new orientations and new Gysin structures

Let A be an oriented cohomology pretheory and let $L \mapsto e(L)$ be the corresponding Euler structure on A. Let $r(t) \in A(pt)^{uc}[[t]]$ be a formal power series in one variable. For a vector bundle E over a variety X let $r(E) \in A(X)$ be the invariant from Proposition 2.2.3. An element $\pi \in A(\mathbf{P}^\infty)$ is called *universally central local parameter* of the ring $A(\mathbf{P}^\infty)$ if it is a universally central element 2.1.1 satisfying the two conditions:

(1) its restriction to a rational point vanishes and

(2) the ring homomorphism $A(pt)[[t]] \to A(\mathbf{P}^\infty)$ sending the variable t to the element π is an isomorphism.

(Clearly the set of all universally central local parameters of the ring $A(\mathbf{P}^\infty)$ is a principal homogeneous set over the group of units of the commutative ring $A(\mathbf{P}^\infty)^{uc}$, however this observation will not be used below in the text).

2.3.1. Theorem. *Let $i \mapsto i_{\mathcal{A}}$ be a Gysin structure on A. For a closed imbedding $i : Y \hookrightarrow X$ of varieties let $N = N_{X/Y}$ be the normal bundle and let*

$$i^{new}_{\mathcal{A}} = i_{\mathcal{A}} \circ (\cup r(N)) : A(Y) \to A(X)$$

Then the assignment $i \mapsto i^{new}_{\mathcal{A}}$ is a new Gysin structure on A.

Suppose the series $r(t)$ is invertible and let $r^{-1}(t)$ be its multiplicative inverse. Then by Proposition 2.2.3 for any vector bundle E one has the relation $r(E)r^{-1}(E) = 1$ which shows that the class $r(E)$ is invertible in this case.

2.3.2. Theorem.

(i) For a projective morphism of varieties $f : Y \to X$ let T_Y and T_X be the tangent bundles to Y and X respectively. Let

$$f_A^{new} = (\cup r^{-1}(T_X)) \circ f_A \circ (\cup r(T_Y)) : A(Y) \to A(X). \tag{39}$$

Then the assignment $f \mapsto f_A^{new}$ is a new perfect integration on A. The corresponding Euler structure is given by $e^{new}(L) = e(L) \cup r(L)$.

(ii) the assignment associating to each perfect integration on A the corresponding Euler class $e(\mathcal{O}(-1)) \in A(\mathbf{P}^\infty)$ is a bijection of the set of all perfect integrations with the set of all universally central local parameters in the ring $A(\mathbf{P}^\infty)$.

Proof of Theorem 2.3.1. We have to verify that the assignment $i \mapsto i_A^{new}$ satisfies properties (1) to (3) from Definition 1.8.1.

Proof of Property (1). Let $j : Z \hookrightarrow Y$ and $i : Y \hookrightarrow X$ be closed imbeddings of smooth varieties. Consider the diagram

$$\begin{tikzcd}
A(Z) \arrow[r, "\cup r(N_{Y/Z})"] \arrow[dr, "j_A^{new}"] & A(Z) \arrow[r, "\cup j^B[r(N_{X/Y})]"] \arrow[d, "j_A"] & A(Z) \arrow[d, "j_A"] \\
& A(Y) \arrow[r, "\cup r(N_{X/Y})"] \arrow[dr, "i_A^{new}"] & A(Y) \arrow[d, "i_A"] \\
& & A(X)
\end{tikzcd}$$

The only square in this diagram commutes because the push-forward j_A is a two-sided $A(Y)$-module operator. The invariant $E \mapsto r(E)$ is multiplicative by Proposition 2.2.3. Thus the composite of the two horizontal arrows coincides with the cup-product by the class $r(N_{X/Z})$. Now the property $(i \circ j)_A = i_A \circ j_A$ proves the required relation $(i \circ j)_A^{new} = i_A^{new} \circ j_A^{new}$.

Proof of Property (2). To do this ëonsider a transversal square (see 1.1.2)

$$\begin{tikzcd}
\tilde{Y} \arrow[r, "\tilde{i}"] \arrow[d, "\tilde{\phi}"'] & \tilde{X} \arrow[d, "\phi"] \\
Y \arrow[r, "i"] & X
\end{tikzcd}$$

with closed imbeddings i and $\tilde{i}$ and the normal bundles $N = N_{X/Y}$ and $\tilde{N} = N_{\tilde{X}/\tilde{Y}}$. One has to check the relation

$$\phi^A[i_A(a \cup r(N))] = \tilde{i}_A[\tilde{\phi}^A(a) \cup r(\tilde{N})].$$

Since the diagram is transversal the canonical map $\tilde{\phi}^*(N) \to \tilde{N}$ is an isomorphism. Thus one has a chain of relations

$$\phi^A[i_A(a \cup r(N))] = \tilde{i}_A[\tilde{\phi}^A(a) \cup \tilde{\phi}^A(r(N))] = \tilde{i}_A[\tilde{\phi}^A(a) \cup r(\phi^*(N))] = \tilde{i}_A[\tilde{\phi}^A(a) \cup r(\tilde{N})]$$

The property 2 is proved.

Proof of Property (3). Given a smooth variety X one has to verify the relation $id^{new} = id_{A(X)}$. This property is obvious, because $id_A = id_{A(X)}$ and the normal bundle $N_{X/X}$ is the zero vector bundle and its the inverse Todd genus is the unit $1 \in A(X)$.

The properties (1) to (3) from 1.8.1 are proved. The Theorem is proved.

$\square$

Proof of Theorem 2.3.2. First we deduce the assertion (ii) from the assertion (i). The assignment taking a perfect integration on A to the Euler class $e(\mathcal{O}(-1))$ is injective by Theorem 1.1.11 and Remark 1.1.12. It remains to prove the surjectivity of the assignment.

For that consider a universally central local parameter $e \in A(\mathbf{P}^\infty)$. The ratio $u = e/e(\mathcal{O}(-1))$ is a universally central unite in $A(\mathbf{P}^\infty)$. Thus there exists a series $r(t) \in \bar{A}^{uc}[[t]]$ such that $r(e(\mathcal{O}(-1))) = u$. Consider an assignment $f \mapsto f_A^{new}$, where $f_A^{new} = (\cup r^{-1}(T_X)) \circ f_A \circ (\cup r(T_Y)) : A(Y) \to A(X)$. It is a perfect integration on A by the assertion (i) of the Theorem and moreover the Euler class $e^{new}(\mathcal{O}(-1))$ coincides with the element $e(\mathcal{O}(-1)) \cup r(\mathcal{O}(-1)) = e(\mathcal{O}(-1)) \cup u = e$. Thus the assignment under the consideration is bijective. To complete the proof of the Theorem it remains to prove the assertion (i).

To prove the first assertion of the Theorem we need to check that the assignment $f \mapsto f^{new}$ satisfies properties (1) to (5) from 1.1.3 and to check the projective bundle axiom 1.1.6.

Proof of Property (1). Let $g : Z \to Y$ and $f : Y \to X$ be projective morphisms of smooth varieties. Consider the two diagrams

$$\begin{tikzcd} A(Z) \arrow[r, "g_A"] & A(Y) \arrow[d, "\cup r(T_Y)"] \\ A(Z) \arrow[u, "\cup r^{-1}(T_Z)"] \arrow[r, "g_A^{new}"] & A(Y) \end{tikzcd} \qquad \begin{tikzcd} A(Y) \arrow[r, "f_A"] & A(X) \arrow[d, "\cup r(T_X)"] \\ A(Y) \arrow[u, "\cup r^{-1}(T_Y)"] \arrow[r, "f_A^{new}"] & A(X) \end{tikzcd}$$

Attaching the right hand side square to the left hand side one we get the required relation $(f \circ g)_A^{new} = f_A^{new} \circ g_A^{new}$.

Proof of Property (2). To do this recall that for a closed imbedding $j : S \hookrightarrow T$ of smooth varieties with the normal bundle N the operator j_A^{new} coincides with the one $j_A \circ (\cup r(N))$. Thus the property (2) from 1.1.3 coincides with the property (2) from 1.8.1 which was checked above in the proof of Proposition 2.3.1.

Proof of Property (3). To do this consider a smooth variety T the projections $p : T \times \mathbf{P}^n \to T$ and $q : T \times \mathbf{P}^n \to \mathbf{P}^n$ and set $S = T \times \mathbf{P}^n$. Then for an element $a \in A(S)$ one has a chain of relations

$$p_A[a \cup r^{-1}(T_S)] \cup r(T_T) = p_A[a \cup r^{-1}(T_S) \cup p^A(r(T_T))] = p_A[a \cup r^{-1}(T_S) \cup r(p^*(T_T))]$$

$$= p_A[a \cup r^{-1}(q^*(T_{\mathbf{P}^n}))].$$

Now consider a morphism of smooth varieties $f : Y \to X$, the Cartesian diagram

$$\begin{array}{ccc} Y \times \mathbf{P}^n & \xrightarrow{F} & X \times \mathbf{P}^n \\ {\scriptstyle p_Y}\downarrow & & \downarrow{\scriptstyle p_X} \\ Y & \xrightarrow{f} & X \end{array}$$

with projections p_Y and p_X and with $F = f \times id$. Let $q : X \times \mathbf{P}^n \to X$ and $s : Y \times \mathbf{P}^n \to Y$ be the projections. To prove the property (3) one has to check the relation

$$f^A[(p_X)_A(a \cup r^{-1}(q^*(T_{\mathbf{P}^n})))] = (p_Y)_A[F^A(a) \cup r^{-1}(s^*(T_{\mathbf{P}^n}))].$$

Clearly $F^*(q^*(T_{\mathbf{P}^n})) = s^*(T_{\mathbf{P}^n})$. Thus one has a chain of relations

$$f^A[(p_X)_A(a \cup r^{-1}(q^*(T_{\mathbf{P}^n})))] = (p_Y)_A[F^A(a) \cup F^A(r^{-1}(q^*(T_{\mathbf{P}^n})))] =$$
$$(p_Y)_A[F^A(a) \cup (r^{-1}((F^* \circ q^*)(T_{\mathbf{P}^n})))] = (p_Y)_A[F^A(a) \cup r^{-1}(s^*(T_{\mathbf{P}^n}))]$$

The property (3) is proved.

Proof of Property (4). Given a smooth variety X one has to verify the relation $id_A = id_{A(X)}$. This property is obvious, because $id_A = id_{A(X)}$ and $r(T_X)^{-1} = r^{-1}(T_X)$ by 2.2.3.

The properties (1) to (4) are proved. It remains to prove the property (5) and to check the projective bundle axiom 1.1.6.

To check the property (5) consider a closed imbedding $i : Y \hookrightarrow X$ of varieties. If T_Y, T_X are the tangent bundles to Y and X and $N = N_{X/Y}$ is the normal bundle, then $i_A^{new} = i_A \circ (\cup r(N))$ by (38). The element $r(N) \in A(X)$ is invertible because the series $r(t)$ is invertible. Now the commutative diagram

$$\begin{array}{ccccc} A(Y) & \xrightarrow{i_A} & A(X) & \xrightarrow{j^A} & A(X-Y) \\ {\scriptstyle \cup r(N)}\uparrow & & \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle id} \\ A(Y) & \xrightarrow{i_A^{new}} & A(X) & \xrightarrow{j^A} & A(X-Y). \end{array}$$

with the exact top row shows that the bottom row is exact as well. The property (5) is proved.

To prove the the projective bundle axiom 1.1.6 one needs to compute the new Euler class. We claim that for a line bundle L over a variety X $e^{new}(L) = e(L) \cup r(L)$. In fact, if $z : X \to L$ is the zero section and $p : L \to X$ is the projection then the normal bundle $N_{L/X}$ coincides with the line bundle L and one has a chain of relation computing the class $e^{new}(L)$

$$e^{new}(L) = z^A(z_A^{new}(1)) = z^A(z_A(1 \cup r(L))) =$$
$$z^A[z_A(1 \cup z^A(p^A(r((L)))))] = z^A[z_A(1) \cup p^A(r(L))] = e(L) \cup r(L).$$

Now the nilpotence property $e^{new}(\mathcal{O}_V(-1))^n = 0$ is obvious, because the class $r(?)$ is central.

To check the property (i) from 1.1.6 for the class $\xi^{new} = e^{new}(\mathcal{O}_E(-1))$ we need in some preliminaries. Namely we need in two algebraic lemmas.

2.3.3. Lemma. *Let R be a commutative ring and let $R[[t]]$ be the formal power series in one variable. Let $r(t) \in R[[t]]$ be a unite then for any integer $n > 0$ the element $t \cdot r(t)$ generates the R-algebra $R[[t]]/(t^n)$.*

2.3.4. Lemma. *Let R be a commutative ring and let $R \subset A$ be a commutative R-algebra which is free of rank n as the R-module. Let $x \in A$ be which generates A as the R-algebra. Then the family $\{1, x, \dots, x^{n-1}\}$ is a free basis of the R-module A.*

The first of this lemmas is obvious. To prove the second one consider the characteristic polinomial $\chi(t) \in R[t]$ of the endomorphism $x : A \to A$ given by the rule $a \mapsto ax$. The polinomial $\chi(t)$ is a unitary polinomial of degree n and the evaluation of $\chi(t)$ at the element x vanishes. Thus the family $\{1, x, \dots, x^{n-1}\}$ generates A as the R-module. Now consider an R-module homomorphism

$$\varphi : R^n \to A$$

given by $\varphi(e_i) = x^{i-1}$. Since it is an epymorphism of free R-modules of the same rank it is an isomorphism. The Lemma follows.

With these two lemmas in hand we are ready to check the property (i) from the projective bundle axiom 1.1.6.

Let V be a dimension n k-vector space. Then the new Euler class $\xi_V^{new} = e^{new}(\mathcal{O}_V(-1))$ of the line bundle $\mathcal{O}_V(-1)$ coincides with the element $\xi_V \cup r(\mathcal{O}_V(-1))$ in $A(\mathbf{P}(V))$. The elements ξ_V and $r(\mathcal{O}_V(-1))$ are universally central and $r(\mathcal{O}_V(-1))$ is invertible.

Let X be a variety. Consider the projection $p : X \times \mathbf{P}(V) \to \mathbf{P}(V)$. Identifying the variable t with the element $p^A(\xi_V)$ in $A(\mathbf{P}(V))^{uc}$ one gets by Theorem 2.1.2 an isomorphism of the $A(X)^{uc}$-algebras $A(X \times \mathbf{P}(V))^{uc} = A(X)^{uc}[t]/(t^n)$ Now Lemma 2.3.3 shows that the element $p^A(\xi_V^{new}) = p^A(\xi_V^{new}) \cup p^A(r(\mathcal{O}_V(-1)))$ generates the $A(X)^{uc}$-algebra $A(X \times \mathbf{P}(V))^{uc}$.

Now let X be an affine variety and let E be a rank n vector bundle over X. There is a vector bundle F over X such that $E \oplus F = \mathbf{1}^m$ is the trivial vector bundle. Let $i : \mathbf{P}(E) \to \mathbf{P}(E \oplus F)$ be the obvious closed imbedding. The pull-back operator $A(\mathbf{P}(E \oplus F))^{uc} \to A(\mathbf{P}(E))^{uc}$ is surjective by the projective bundle axiom because $i^A(\xi_{E\oplus F}^{new}) = \xi_E^{new}$ in $A(\mathbf{P}(E))$. Thus the class ξ_E^{new} generates the $A(X)^{uc}$-algebra $A(\mathbf{P}(E))^{uc}$. By Theorem 2.1.2 the $A(X)^{uc}$-module $A(\mathbf{P}(E))^{uc}$ is free of rank n. Now by Lemma 2.3.4 the family $\{1, \xi_E^{new}, \dots, (\xi_E^{new})^{n-1}\}$ is a free basis of the $A(X)^{uc}$-module $A(\mathbf{P}(E))^{uc}$. Applying Theorem 2.1.2 once again one concludes that the family $\{1, \xi_E^{new}, \dots, (\xi_E^{new})^{n-1}\}$ is a free basis of the two-sided $A(X)$-module $A(\mathbf{P}(E))$.

If X is any variety then by Jouanolou trick [J] there exists a vector bundle F over X and a torsor $q : X' \to X$ under this vector bundle such that the variety X' is affine. Let $E' = X' \times_X E$ be the pull-back of E by means of q and let $Q : \mathbf{P}(E') \to \mathbf{P}(E)$ be the obvious projection. Clearly $\mathbf{P}(E')$ is a torsor under the vector bundle $p^*(F)$, where $p : \mathbf{P}(E) \to X$ is the projection. Thus the pull-back operators $Q^A : A(\mathbf{P}(E)) \to \mathbf{P}(E')$ and $q^A : A(X) \to A(X')$ are isomorphisms. By the functoriality of the new Euler class one has the relation $Q^A(\xi_E^{new}) = \xi_{E'}^{new}$. Since the variety X' is affine the family $\{1, \xi_{E'}^{new}, \dots, (\xi_{E'}^{new})^{n-1}\}$ is a free basis of the two-sided $A(X')$-module $A(\mathbf{P}(E'))$. Therefore the family $\{1, \xi_E^{new}, \ldots, (\xi_E^{new})^{n-1}\}$ is a free basis of the two-sided $A(X)$-module $A(\mathbf{P}(E))$. The property (i) of the projective bundle axiom 1.1.6 is checked. Theorem 2.3.2 is proved. $\square$

2.4 Riemann-Roch theorem

Let A and B be oriented ring cohomology pretheories and let $L \mapsto e^A(L)$ and $L \mapsto e^B(L)$ be the corresponding Euler structures. For a vector bundle E let $c_i^A(E)$ and $c_i^B(E)$ be the corresponding Chern classes 2.2.1.

Let $\bar{A} = A(pt)$ and $\bar{B} = B(pt)$. Let $\bar{A}^{uc}$ be the ring of universally central elements in $A(pt)$ (2.1.1) and let $\bar{B}^{uc}$ be the ring of universally central elements in $B(pt)$. Let

$$E_B^- : \bar{B}[[t]] \to B(\mathbf{P}^\infty)$$

be the isomorphism defined by $E_B^-(t) = \xi_B \in B(\mathbf{P}^\infty)$ with $\xi_B = e^B(\mathcal{O}(-1)) \in B(\mathbf{P}^\infty)$. Let

$$E_B^+ : \bar{B}[[t]] \to B(\mathbf{P}^\infty)$$

be the isomorphism defined by $E_B^+(t) = \zeta_B \in B(\mathbf{P}^\infty)$ with $\zeta_B = e^B(\mathcal{O}(1)) \in B(\mathbf{P}^\infty)$.

2.5 Todd genus

Let $\varphi : A \to B$ be a ring morphism of the oriented cohomology pretheories (see 1.1.7) which for any finite-dimensional k-vector space V takes the Euler class $\xi_A \in A(\mathbf{P}(V))$ to a universally central element of $B(\mathbf{P}(V))$ (see 2.1.1). Consider the operation φ on $\mathbf{P}^\infty$:

$$\varphi_\infty : A(\mathbf{P}^\infty) \to B(\mathbf{P}^\infty)$$

and define two series $\Phi^-(t) \in \bar{B}[[t]]$ and $\Phi^+(t) \in \bar{B}[[t]]$ as follows:

$E_B^-(\Phi^-(t)) = \varphi_\infty(\xi_A)$ and $E_B^+(\Phi^+(t)) = \varphi_\infty(\zeta_A)$. It's straightforward to check that for every variety X and every line bundle L over X one has the relations $\Phi^-(e^B(L)) = \varphi(e^A(L)) = \Phi^+(e^B(L))$ (compair with the arguments from 2.2.3). Thus $\Phi^-(t) = \Phi^+(t)$ and we will use below the shorter notation

$$\Phi(t) \in \bar{B}[[t]]$$

for both of them.

Note that all the coefficients of the formal power series $\Phi(t)$ are in $\bar{B}^{uc}$. The series $\Phi(t)$ is divisible by t, because the classes $\xi_A, \zeta_A \in A(\mathbf{P}^\infty)$ vanish being restricted to a point.

2.5.1. Definition.

1) Define the inverse Todd genus of the morphism φ as the formal power series

$$itd_\varphi(t) = \Phi(t)/t \in \bar{B}^{uc}[[t]]$$

(clearly $itd_\varphi(\xi_B) = \varphi(\xi_A)/\xi_B$ and $itd_\varphi(\zeta_B) = \varphi(\zeta_A)/\zeta_B$ in $B(\mathbf{P}^\infty)^{uc}$ and for every variety X and every line bundle L over X one has $e^B(L) \cup itd_\varphi(L) = \varphi(e^A(L)))$;

2) following 2.2.3 define the inverse Todd genus of a vector bundle E as the invariant $itd_\varphi(E) \in B(X)^{uc}$.

With respect to the second construction of the assignment $E \mapsto r(E)$ given in the proof of Proposition 2.2.3 the inverse Todd genus of a rank n vector bundle E over a variety X can be computed as follows. Set

$$itd_\varphi(\sigma_1, \sigma_2, \ldots, \sigma_n) = \prod_{i=1}^{n} itd_\varphi(t_i) \in \bar{B}^{uc}[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$$

Then $itd_\varphi(E) \in B(X)^{uc}$ is the evaluation of the series $itd_\varphi(\sigma_1, \sigma_2, \ldots, \sigma_n)$ on the Chern classes of E

$$itd_\varphi(E) = itd_\varphi(c_1^B(E), c_2^B(E), \ldots, c_n^B(E)) \in B^{uc}(X).$$

2.5.2. Definition. *Assume that the series $itd_\varphi(t)$ is invertible in $\bar{B}^{uc}[[t]]$ (this is equivalent to the assumption that its free term is a unit in the ring $\bar{B}^{uc}$ or, in the other words, that $\Phi(t)$ is a local parameter in $\bar{B}^{uc}[[t]]$).*

1) Define the Todd genus of φ as the multiplicative inverse of $itd_\varphi(t)$

$$td_\varphi(t) = t/\Phi(t) \in \bar{B}^{uc}[[t]]$$

(clearly $td_\varphi(\xi_B) = \xi_B/\varphi(\xi_A)$ and $td_\varphi(\zeta_B) = \zeta_B/\varphi(\zeta_A)$ in $B(\mathbf{P}^\infty)^{uc}$) and for every variety X and every line bundle L over X one has $e^B(L) = \varphi(e^A(L)) \cup td_\varphi(L))$;

2) following 2.2.3 define the Todd genus of a vector bundle E as the invariant $td_\varphi(E) \in B(X)^{uc}$.

With respect to the second construction of the assignment $E \mapsto r(E)$ given in the proof of Proposition 2.2.3 the Todd genus of a rank n vector bundle E over a variety X can be computed as follows. Set

$$td_\varphi(\sigma_1, \sigma_2, \ldots, \sigma_n) = \prod_{i=1}^{n} td_\varphi(t_i) \in \bar{B}^{uc}[[\sigma_1, \sigma_2, \ldots, \sigma_n]]$$

Then $td_\varphi(E) \in B(X)^{uc}$ is the evaluation of the series $td_\varphi(\sigma_1, \sigma_2, \ldots, \sigma_n)$ on the Chern classes of E

$$td_\varphi(E) = td_\varphi(c_1^B(E), c_2^B(E), \ldots, c_n^B(E)) \in B^{uc}(X).$$

By the property (iv) from Proposition 2.2.3 one has the relation $td_\varphi(E)^{-1} = itd_\varphi(E) \in B(X)^{uc}$ (this holds if the series $itd_\varphi(t)$ is invertible in $\bar{B}^{uc}[[t]]$).

2.5.3. Theorem. *Let $\varphi : A \to B$ be the ring morphism of the oriented cohomology pretheories and let $i : Y \hookrightarrow X$ be a closed imbedding of smooth varieties with the normal bundle N. Then for each element $\alpha \in A(Y)$ one has the relation in $B(X)$*

$$i_B(\varphi(\alpha) \cup itd_\varphi(N)) = \varphi(i_A(\alpha)).$$

2.5.4. Theorem. *Let $\varphi : A \to B$ be the ring morphism of the oriented cohomology pretheories. Suppose that the series $itd_\varphi(t)$ is invertible in $\bar{B}^{uc}[[t]]$. Then for each projective morphism of smooth projective varieties $f : Y \to X$ and each element $\alpha \in A(Y)$ one has the relation in $B(X)$*

$$f_B(\varphi(\alpha) \cup td_\varphi(T_Y)) = \varphi(f_A(\alpha)) \cup td_\varphi(T_X)$$

where T_Y (resp. T_X) is the tangent bundle to Y (resp. to X).

2.5.5. Corollary. *Under the hypothesis of Theorem 2.5.4 for each smooth projective variety X and each $a \in A(X)$ the following relation holds in $B(pt)$*

$$f_B(\varphi(a) \cup td_\varphi(T_X)) = \varphi(f_A(a))$$

where T_X is the tangent bundle to X and f is the structural morphism $X \to pt$.

2.5.6. Corollary. *Let A be a ring cohomology pretheory endowed with two perfect integrations $f \mapsto f'_A$ and $f \mapsto f''_A$. Let $L \mapsto e'(L)$, $e \mapsto e''(L)$ be the corresponding Euler structures on A. Let $\varphi : A \to A$ be the identity morphism. Then the inverse Todd genus $itd_\varphi = e'(\mathcal{O}(-1))/e''(\mathcal{O}(-1))$ is invertible in $\bar{A}[[t]]$ and thus the Todd genus $td_\varphi(t)$ is well-defined and for each $\alpha \in A(Y)$ one has the relation in $A(X)$*

$$f''_A[\alpha \cup td(T_Y)] \cup itd(T_X) = f'_A(\alpha).$$

2.5.7. Corollary. *Let φ be the ring morphism. Suppose the inverse Todd genus $itd_\varphi(t)$ is invertible in $\bar{B}[[t]]$. Then for each finite etale morphism $f : Y \to X$ one has the relation*

$$f_B \circ \varphi_Y = \varphi_X \circ f_A.$$

Proof of Theorem 2.5.3. For a closed imbedding $j : S \hookrightarrow T$ with the normal bundle $N_{T/S}$ set

$$j_B^{new} = j_B \circ (\cup itd_\varphi(N_{T/S}) : B(S) \to B(T).$$

Then by Theorem 2.3.1 the assignment $j \mapsto j_B^{new}$ is a Gysin structure on B. Moreover for any smooth divisor $j : D \hookrightarrow T$ one has the relation

$$\varphi(j_A(1)) = j_B^{new}(1). \tag{40}$$

In fact, by (9) $j_A(1) = e^A(L(D))$ and $j_B(1) = e^B(L(D))$. Thus one has the chain of relations

$$j_B^{new}(1) = j_B(itd_\varphi(N)) = j_B[itd_\varphi(j^*(L(D)))] = j_B[j^B(itd_\varphi(L(D)))] = j_B(1) \cup itd_\varphi(L(D)) =$$
$$= e^B(L(D)) \cup itd_\varphi(e^B(L(D))) = \Phi(e^B(L(D))) = \varphi(e^A(L(D))) = \varphi(j_A(1)).$$

Since the relation (40) holds for any smooth divisor the Theorem follows from Theorem 2.3.1.

□

Proof of Theorem 2.5.4. For a projective morphism $g : S \to T$ of varieties with the tangent bundles T_S and T_T set

$$g_B^{new} = (\cup itd_\varphi(T_T)) \circ g_B \circ (\cup td_\varphi(T_S)) : B(S) \to B(T).$$

Then by Theorem 2.3.2 the assignment $g \mapsto g_B^{new}$ is a perfect integration on B. Moreover by the same Theorem the corresponding Euler structure on B is given by $(e^B)^{new}(L) = e^B(L) \cup itd_\varphi(L(D))$. We claim that for any line bundle L one has

$$\varphi(e^A(L)) = (e^B)^{new}(L). \tag{41}$$

In fact, one has a chain of relations

$(e^B)^{new}(L) = e^B(L) \cup itd_\varphi(L(D)) = e^B(L) \cup itd_\varphi(e^B(L)) =$

$= (t \cdot itd_\varphi(t))(e^B(L)) = \Phi(e^B(L)) = \varphi(e^A(L))$

which proves the relation. Now the Theorem follows from the relation (41) by Corollary 1.1.10.

□

Proof of Corollary 2.5.6. The Corollary follows directly from Theorem 2.5.4.

□

Proof of Corollary 2.5.7. Let $a \in A(Y)$. Then

$$\varphi_X(f_A(a)) \cup td_\varphi(T_X) = f_B(\varphi_Y(a) \cup td_\varphi(T_Y))$$

by Theorem 2.5.4. The chain of relations

$$f_B(\varphi_Y(a) \cup td_\varphi(T_Y)) = f_B(\varphi_Y(a) \cup f^B(td_\varphi(T_X))) = f_B(\varphi_Y(a)) \cup td_\varphi(T_X)$$

shows that $\varphi_X(f_A(a)) \cup td_\varphi(T_X) = f_B(\varphi_Y(a)) \cup td_\varphi(T_X)$. Since the Todd genus $td_\varphi(T_X)$ is invertible one gets the required relation $\varphi_X(f_A(a)) = f_B(\varphi_Y(a))$.

□

2.6 Examples and applications

Here we mainly consider well-known ring cohomology pretheories equipped with the known perfect integrations. In other cases we often refer to [PS] where perfect integrations are constructed begining with a very few data (like a functorial first Chern class of line bundles, called in [PS] a Chern structure).

2.6.1. Suppose $k = \mathbb{C}$. Let $A(X) = K_0(X)$-theory and let $B(X) = H^{ev}(X, \mathbb{Q})$ be the usual singular cohomology (even degree part) with rational coefficients and let

$$ch : K_0 \to H^{ev}_{\mathbb{Q}}$$

be the Chern character [H, Ch.III]. For the pretheory K_0 push-forwards are defined in [BS][§5.d], their properties (1) to (5) from Definition 1.1.3 are well-known and can be extracted from [BS]. The corresponding Euler structure is given by $e^A(L) = [1] - [L^{-1}]$. Projective bundle axiom is proved (for instance) in [Qu2][§8, Th.2.1]. So the pretheory $A = K_0$ is an oriented cohomology pretheory.

For the pretheory $B = H^{ev}_{\mathbb{Q}}$ push-forwards are known as the Thom-Gysin maps. Their properties (1) to (5) from Definition 1.1.3 are well-known. The corresponding Euler structure is given by $e^B(L(D)) = [D] \in H^2(-;\mathbb{Q})$ where $[D]$ is the class of the divisor D. The projective bundle theorem is well-known. Thus the integration on B is perfect and with his definition B is an oriented cohomology pretheory.

If we replace H^{ev} by the Chow ring pretheory CH^* then the book [Fu] is a good reference for the definition of the ring structures on $CH^*(X)$, for the definition of push-forwards and for properties (1) to (5) from Definition 1.1.3. The Euler structure is given in this case by $e^B(L(D)) = [D] \in CH^1(-)_{\mathbb{Q}}$ where $[D]$ is the class of the divisor D. The projective bundle theorem one can find again in [Fu]. With this definition the functor $B = CH^*_{\mathbb{Q}}$ is an oriented cohomology pretheory.

The Chern character ch is a ring morphism of the ring pretheories (see [BS] or see again [Fu] or 2.7.2 below). To compute the Todd genus of the Chern character ch observe that $e^B(L^{-1}) = -e^B(L)$. Therefore if $t = e^B(\mathcal{O}(-1))$ as in the beginning of this Section then one has a chain of relations

$$ch(e^A(\mathcal{O}(-1)) = ch([1] - [\mathcal{O}(1)]) = \exp(e^B(1)) - \exp(e^B(\mathcal{O}(1)) = 1 - \exp(-t)$$

Thus $td_{ch}(t) = t/(1 - e^{-t})$ and the Todd genus (2.5.2) coincides in the case with the classical Todd genus [H, Ch.III].

Now consider Quillen's higher K-groups $K_*(-)$ (see [Qu2]) with the ring structure defined in [Gi], the K-cohomology groups $H^*(-, \mathcal{K}_*)$ (see [Qu2]) with the ring structure defined in [Gi] and consider the Chern character

$$ch : K_* \to H^*(-, \mathcal{K}_*) \otimes \mathbb{Q}$$

which is defined in [Gi] as well.

In the case of higher K-groups take the push-forwards defined in [Qu2]. The properties (1) to (5) from Definition 1.1.3 are verified in [Qu2]. Since Quilen's K_0 coincides with Grothendieck's K_0 and the push-forwards on K_0 defined in [Qu2] coincides with the push-forwards defined in [BS] the corresponding Euler structure is given again by $e^A(L) = [1] - [L^{-1}]$. The projective bundle theorem is proved in [Qu2]. Thus with this definitions Quillen's higher K-groups $K_*(-)$ is an oriented cohomology pretheory.

In the case of K-cohomology groups $H^*(-, \mathcal{K}_*)$ take the push-forwards defined in [Gi]. The properties (1) to (5) from Definition 1.1.3 are verified in [Gi]. Since the ring cohomology pretheories $\oplus_{p \geq 0} CH^p(-)$ and $\oplus_{p \geq 0} H^*(-, \mathcal{K}_*)$ are isomorphic (see [Qu2] and [Gi]) and moreover this isomorphism respects the integrations on these theories one conclude that the Euler structure on K-cohomology is given by $L(D) \mapsto [D] \in CH^1(-) = H^1(-, \mathcal{K}_1)$. The projective bundle theorem is proved in [Gi]. Thus with theses definitions K-cohomology $H^*(-, \mathcal{K}_*)$ is an oriented cohomology pretheory.

The Chern character $ch : K_* \to H^*(-, \mathcal{K}_*) \otimes \mathbb{Q}$ is a ring morphism of the ring cohomology pretheories as it is proved in [Gi]. Its Todd genus coincides with the Todd genus of the ring morphism $ch : K_0 \to CH^*_{\mathbb{Q}}$ above because the computations of the Todd genuses coincide tautologically in both cases. Thus the Todd genus of the Chern character $ch : K_* \to H^*(-, \mathcal{K}_*) \otimes \mathbb{Q}$ is the series $t/(1-e^{-t})$. This reproves the Riemann-Roch theorem of Gillet [Gi] for the Chern character.

One can replace K-cohomology by the etale cohomology [Mi] $\oplus_{j=-\infty}^{+\infty} H^*_{et}(-; \mathbb{Q}_l(j))$. The Todd genus is remained the same series $t/(1-e^{-t})$.

One can replace K-cohomology by the motivic cohomology by the motivic cohomology of Voevodsky $\oplus_{j=0}^{+\infty} H^*_{\mathcal{M}}(-, \mathbb{Z}(j))$ [SV]. The ring structure on the motivic cohomology is described in [SV]. The motivic cohomology is an oriented cohomology pretheory as it is explained in 2.6.15 below. It is explained there as well that the Euler structure in the motivic cohomology case is given by $e(L(D)) = [D] \in H^2(-, \mathbb{Z}(1)) = CH^1(-)$ where $[D]$ is again the class of the divisor.

The Chern character $ch : K_* \to \oplus_{j=0}^{+\infty} H^*_{\mathcal{M}}(-, \mathbb{Z}(j))$ is constructed in [Pu][Section 3]. The Todd genus is remained the same series $t/(1-e^{-t})$. Thus we recovered the Riemann-Roch theorem for this Chern character proved originally in [Pu][Section 3].

2.6.2. Adams operations.

Let

$$\Psi^n : K_*(-) \to K_*(-)$$

be the Adams operation on higher Quillen's K-theory (see [So] or [Gra]) considered on smooth varieties ($n \neq \pm 1$ and $n \neq 0$). The higher K-functor is an oriented cohomology theory as it is described in 2.6.1 above. The corresponding Euler structure is given by $e^K(L) = [\mathbf{1}] - [L^\vee]$.

The Adams operation $\Psi^n : K_* \to K_*$ is a ring morphism of the oriented cohomology theories. To compute its Todd genus consider a chain of relations

$$\varphi(e^K(\mathcal{O}(-1))) = \Psi^n([1] - [\mathcal{O}(1)]) = [\mathbf{1}] - [\mathcal{O}(1)]^n$$

and furthermore $t = e^K(\mathcal{O}(-1)) = [\mathbf{1}] - [\mathcal{O}(1)]$. Finally one gets

$$itd_\varphi(t) = ([\mathbf{1}] - [\mathcal{O}(1)]^n)/([\mathbf{1}] - [\mathcal{O}(1)]) = [\mathbf{1}] + [\mathcal{O}(1)] + \cdots + [\mathcal{O}(n-1)]$$

Thus for a line bundle L over a smooth variety X one has

$$itd_\varphi(L) = [\mathbf{1}] + [L^{-1}] + \cdots + [L^{-n+1}]$$

This class coincides with the element $\theta(L^{-1})$ but not with the element $\theta(L) = [\mathbf{1}] + [L] + \cdots + [L^{n-1}]$ which is used in the Riemann-Roch theorem for Ψ^n proved in [So].

There is no confusion because it is written in [So, p. 519] the following relation $[\mathbf{1}] - [L] = j_K(1)$ where $j : Y \to L$ is the zero section of a line bundle L over a smooth variety Y. But $j_K(1) = [\mathbf{1}] - [L^\vee]$ in that case.

Thus the Riemann-Roch formula for the operation Ψ^n holds exactly with the class $itd_\varphi(L)$ above.

2.6.3. The Adams operation Ψ^{-1}. Let $\Psi^{-1} : K-theory \to K-theory$ be the Adams operation considered on smooth varieties. It takes a vector bundle E to the class of its dual $E^{\vee}$. In this case for a rank n vector bundle E one has

$$td_{\varphi}(E) = (-1)^{rk(E)}[\wedge^n(E^{\vee})]$$

and thus for a projective morphism $f : Y \to X$ of smooth varieties and for an element $\alpha \in K(Y)$ one has the relation in $K(X)$

$$f_K(\Psi^{-1}\alpha \cdot K_Y) = (-1)^{dim(Y)-dim(X)}\Psi^{-1}(f_K(\alpha)) \cdot K_X,$$

where K_Y and K_X are the canonical classes of the varieties Y vector bundle E/Y)

$$f_K([K_Y \otimes E^{\vee}]) = (-1)^{\dim(Y)} f_K([E]).$$

Since for a vector bundle F/Y its push-forward $f_K([F])$ coincides with the Euler characteristic $\chi(F)$ the very last relations follows as well from the Serre duality for the vector bundles over a smooth projective varieties.

2.6.4. Total Steenrod square. (compare with [AH]). Let $k = \mathbb{R}$ and let $A(X) = H^*(X(\mathbb{R}); \mathbb{Z}/2)$ the usual singular cohomology of the underlying topological real point space. For a line bundle L over a smooth variety X consider the first Stiefel-Whitney class $w_1(L(\mathbb{R}))$ of the real vector bundle $L(\mathbb{R})$ over the space $X(\mathbb{R})$. The assignment $L \mapsto w_1(L(\mathbb{R}))$ is a Chern structure on A in the sence of [PS]. By [PS][Theorem 4.1.4] there exists a unique integration $f \mapsto f_A$ on A such that for every variety Y and every smooth divisor $i : D \hookrightarrow Y$ one has the relation $i_A(1) = w_1(L(D)(\mathbb{R}))$ in $A(Y)$. Now if X is a variety and $Y = L$ is a line bundle over X with the projection $p : L \to X$ and $D = X$ and $i : D = X \to L = Y$ is the zero section of L then

$$e^A(L) = s^A(s_A(1)) = s^A(w_1(L(D)(\mathbb{R}))) = s^A(w_1(p^*(L)(\mathbb{R}))) =$$
$$= w_1(s^*(p^*(L)(\mathbb{R}))) = w_1(L(\mathbb{R})).$$

Consider the total Steenrod square operation on the $\mathbb{Z}/2$-cohomology

$$Sq = 1 + Sq^1 + Sq^2 + \cdots : H^* \to H^*$$

Then one has a chain of relations in $H^*(pt)[[t]] = \mathbb{Z}/2[[t]]$

$$td_{\varphi}(t) = w_1(\xi)/Sq(w_1(\xi)) = t/(t+t^2) = 1/(1+t)$$

Thus $itd_{\varphi}(t) = 1 + t$ and $itd_{\varphi}(L) = 1 + w_1(L(\mathbb{R}))$ for a line bundle L and

$$itd(E) = w(E(\mathbb{R}))$$

(the total Stiefel-Whitney class of the real vector bundle $E(\mathbb{R})$ over the space $X(\mathbb{R})$). If $i : Y \hookrightarrow X$ is a closed imbedding of smooth varieties with the normal bundle N then for the element $\alpha = 1$ Theorem 2.5.3 states that

$$i_*(w(N(\mathbb{R}))) = Sq([Y])$$

where $[Y] = i_*(1)$ is the class of the variety Y. This is the known Wu formula.

2.6.5. Motivic Wu Formulae. In this example the characteristic of the ground field k is different of 2. Let $A(X) = \oplus_{q=0}^{\infty} H^{2p}_{\mathcal{M}}(X; \mathbb{Z}/2(p))$. The ring cohomology pretheory $X \mapsto A(X)$ is an oriented cohomology pretheory as it is explained in 2.6.15 below. Let

$$Sq : A \to A$$

be the operation $1 + Sq^1 + Sq^2 + \dots$ considered on varieties where Sq^i are the motivic Steenrod squares defined in [V2][Section 9, just above Theorem 9.3]. The Bockstein homomorphism β vanishes on the group $H^{2p}_{\mathcal{M}}(X; \mathbb{Z}/2(p))$ because it takes value in the group $H^{2p+1}_{\mathcal{M}}(X; \mathbb{Z}/2(p))$ which vanishes itself. This observation together with [V2][Proposition 9.6 and Theorem 9.4] shows that the operation $Sq : A \to A$ is a *ring morphism* of the ring cohomology pretheory A to itself.

Theorem 9.4 and Lemma 14.6 from [V2] shows that one has a chain of relations

$$itd_{Sq}(t) = Sq(c^H_1(\mathcal{O}(-1)))/c^H_1(\mathcal{O}(-1)) = (t + t^2)/t = 1 + t \in A(pt)[[t]] = \mathbb{Z}/2[[t]].$$

Thus $itd(E) = c^H(E) = 1 + c^H_1(E) + c^H_2(E) + \dots mod 2$ (the total Chern class mod 2). Now Theorem 2.5.3 states that

$$i_*(c^H(N)) = Sq([Y])$$

where $[Y] \in H^{2d}(X, \mathbb{Z}/2(d)) = CH^d(X)$ is the class of Y in the Chow group of X.

2.6.6. The total Segre class is the Todd genus of the total motivic Steenrod square [V2]. Let k and A be as in the previous example. Certain divisibility property of so called Segre numbers are deduced in this example from the Riemann-Roch theorem for the operation $Sq : A \to A$ from the previous example.

Consider n-dimensional smooth projective variety Y and its structural morphism to the point $f : Y \to pt$. Recall ([Fu, Sect.3.2]) that the Segre classes $s_d(E) \in H^{2d}_{\mathcal{M}}(Y, \mathbb{Z}(d)) = CH^d(Y)$ of a vector bundle E are described by the relations $c^H_t(E) = 1 + c^H_1(E) \cdot t + c^H_2(E) \cdot t^2 + \dots$ and $c^H_t(E)^{-1} = 1 + s_1(E) \cdot t + s_2(E) \cdot t^2 + \dots$ where t is a variable. The integer

$$s_n(Y) = deg(s_n(T_Y))$$

is called the Segre number of the variety Y. We claim that it always is divisible by 2.

To prove this claim consider the operation $Sq : A \to A$ from the previous example. Since for a vector bundle E one has $itd(E) = c(E)$ the Todd genus $td_{Sq}(E)$ of the vector bundle E is exactly the Segre class $s(E) = 1 + s_1(E) + s_2(E) + \dots$ modulo 2 of the vector bundle E. Thus one has the following relation in $H^0_{\mathcal{M}}((pt), \mathbb{Z}/2) = Z/2$

$$f_*(s(T_Y)) = Sq(f_*(1)) = 0$$

because already $f_*(1) = 0$. Hence $f_*(s_n(T_Y)) = 0 \in \mathbb{Z}/2$. Thus the Segre number $s_n(Y)$ is divisible by 2.

2.6.7. Total reduced power operation of Voevodsky [V2]. Let p be a prime different of the characteristic of the ground field k. Let $A(X) = \oplus_m \oplus_{n \geq 0} H^m_{\mathcal{M}}(X; \mathbb{Z}/p(n))$. The ring cohomology pretheory $X \mapsto A(X)$ is an oriented cohomology pretheory as it is explained in 2.6.15 below.

Consider an operator

$$P_{\mathcal{M}} : A \to A$$

where $P_{\mathcal{M}} = 1 + P^1 + P^2 + \ldots$ and P^i are the the motivic reduced power operators [V2][Section 9, just above Theorem 9.3]. As it follows from [V2][Proposition 9.6 and Theorem 9.3] the operator $P_{\mathcal{M}}$ is a ring transformation.

Theorem 9.4 and Lemma 14.6 from [V2] shows that one has a chain of relations in the ring $A(pt)[[t]]$

$$itd_{\varphi}(t) = P_{\mathcal{M}}(c_1^H(\mathcal{O}(-1)))/c_1^H(\mathcal{O}(-1)) = (t + t^p)/t = 1 + t^{p-1}.$$

Thus for a line bundle L its the inverse Todd genus is the class $1 + c_1^H(L)^{p-1}$ modulo p.

Let $itd(\sigma_1, \ldots, \sigma_n)$ be the inverse Todd genus let $td(\sigma_1, \ldots, \sigma_n)$ be the Todd genus and let the class $\bar{b}_i$ be the i-th homogeneous component of the series $td(\sigma_1, \ldots, \sigma_n)$. Now for a rank n vector bundle E over a smooth variety X set

$$\bar{b}_i(E) = \bar{b}_i(c_1^H(E), \ldots, c_n^H(E)) \in H^{2i}_{\mathcal{M}}(X, \mathbb{Z}/p(i)) = CH^i(X)/pCH^i(X).$$

Clearly the classes $\bar{b}_i$ vanish if i is not divisible by $p-1$. Finally for a smooth projective variety X of dimension d and its tangent bundle T_X set $\bar{b}_d(X) = deg(\bar{b}_d(T_X)) \in \mathbb{Z}/p\mathbb{Z}$.

We claim that for any smooth projective variety X its number $\bar{b}_d(X)$ always vanishes.

In fact by Corollary 2.5.5 one has a chain of relations in $H^0_{\mathcal{M}}(pt, \mathbb{Z}/p)$

$$deg(\bar{b}_d(T_X)) = f_*(\bar{b}_d(T_X)) = Sq_{\mathcal{M}}(f_*(1)) = 0$$

because already $f_*(1) = 0$. It's interesting to compare the computations presented in 2.6.5, 2.6.6 and 2.6.7 with the ones given in [Br].

2.6.8. The case of a finite etale morphism. Let A be as in the previous example. Let $f : Y \to X$ be a finite etale morphism. Then one has the relation

$$f_* \circ P = P \circ f_*.$$

This is just a particular case of Corollary 2.5.7.

2.6.9. A natural morphism $K_0^{alg} \to K^0_{top}$.

Let $k = \mathbb{C}$ and let

$$\varphi : K_0^{alg} \to K^0_{top} \tag{42}$$

be the morphism of the algebraic K_0-theory to the topological K^0-theory induced by the functor which takes an algebraic vector bundle to the underlying complex vector bundle. Recall that both functors K_0^{alg} and K^0_{top} are ring pretheories (the ring structures are induced by the tensor product of vector bundles). The morphism φ is a ring morphism of the pretheories because the forgetful functor commutes with the tensor product of vector bundles. Both pretheories are equipped with integrations: the pretheory K_0^{alg} is equipped with the integration defined in [BS][§5.d] (compare with the example 2.6.1), the functor K^0_{top} is equipped with an integration which can be defined as follows.

For a closed imbedding $i : Y \hookrightarrow X$ in $\mathcal{S}m$ a Thom-Gysin morphism is defined in [BFM] as the composition

$$i_{K^0_{top}} : K^0_{top}(Y(\mathbb{C})) \xrightarrow{\cup\lambda_N} K^0_{top,Y(\mathbb{C})}(N) \xrightarrow{exc^{-1}} K^0_{top,Y(\mathbb{C})}(X(\mathbb{C}))$$

where $N = N_{X(\mathbb{C})/Y(\mathbb{C})}$ is the normal bundle, the class $\lambda_N \in K^0top_X(N)$ is the Thom-Gysin class of the complex vector bundle N [BFM][Section 3.2] and the morphism exc^{-1} is the inverse to the excision isomorphism $exc : K^0_{top,Y(\mathbb{C})}(X(\mathbb{C})) \to K^0_{top,Y(\mathbb{C})}(N)$. For a variety $X \in \mathcal{S}m$ the projective bundle theorem [K][Chapter IV, 2.13] states that the cross-product map

$$K^0_{top}(\mathbf{P}^n) \otimes_{\mathbb{Z}} K^0_{top}(X) \to K^0_{top}(\mathbf{P}^n \times X)$$

is a ring isomorphism and moreover $K^0_{top}(pt)[t]/(t^{n+1}) \cong K^0_{top}(\mathbf{P}^n)$ is a ring isomorphism sending the variable t to the class $\zeta_n = [1] - [\mathcal{O}(-1)]$. For the projection $p : \mathbf{P}^n \times X \to X$ define a homomorphism

$$p_{K^0_{top}} : K^0_{top}(\mathbf{P}^n \times X) \to K^0_{top}(X)$$

as a unique $K^0_{top}(X)$-module map which for every integer $i = \{0, 1, \ldots, n\}$ takes the element $\zeta^i \cup 1 \in K^0_{top}(\mathbf{P}^n \times X)$ to the unit $1 \in K^0_{top}(X)$.

An arbitrary projective morphism $f : Y \to X$ can be decomposed as $f = p \circ i$, where $i : Y \hookrightarrow \mathbf{P}^n \times X$ is a closed imbedding and $p : \mathbf{P}^n \times X \to X$ is the projection. Set $f_{K^0_{top}} = p_{K^0_{top}} \circ i_{K^0_{top}}$. One can chech (following [PS]) that the map $f_{K^0_{top}}$ is well defined. Moreover the assignment $f \mapsto f_{K^0_{top}}$ is a perfect integration on the ring pretheory K^0_{top} (see again [PS]). By Corollary 1.1.11 this is a unique perfect integration on the pretheory K^0_{top} with the Euler structure given by $L/Y \mapsto [1] - [L^\vee] \in K^0_{top}(Y)$.

The Riemann-Roch theorem [BFM][Section 4.1] states that the morphism $\varphi : K_0^{alg} \to K^0_{top}$ commutes with the integrations.

This result is a pretty straighforward consequence of Corollary 1.1.10. To explain this observe that for a complex line bundle $p : L \to X$ over a variety $X \in \mathcal{S}m$ and its zero section $z : X \to L$ the pushforward $z_{K^0_{top}}$ in the ring pretheory K^0_{top} takes the unite $[1_Y] \in K^0_{top}(X)$ to the class $[1_L] - [p^*(L^\vee)] \in K^0_{top}(L)$. Thus the corresponding Euler class is given by

$$e^{top}(L) = [1_Y] - [L^\vee] \in K^0_{top}(Y).$$

For an algebraic line bundle L over an algebraic variety $Y \in \mathcal{S}m$ the Euler class $e^{alg}(L)$ of L is given by

$$e^{alg}(L) = [1_Y] - [L^\vee] \in K_0^{alg}(Y).$$

The projective bundle theorems of Grothendieck [Qu2][§8, Th.2.1] and [K][Chapter IV, 2.13] shows that both integrations are perfect and thus both pretheories are oriented in the sence of Definition 1.1.7. Obviously $\varphi(e^{alg}(L)) = e^{top}(L)$ and thus the morphism $\varphi : K_0^{alg} \to K^0_{top}$ is an oriented morphism of the oriented pretheories by Corollary 1.1.10. In the other the morphism φ commutes with the integrations on the oriented pretheories.

2.6.10. A natural ring morphism $K^{alg} \to K\mathbf{R}$ from [KW].

Let $k = \mathbb{R}$ be the field of reals. A Real version of the Riemann-Roch theorem from [BFM] is proved in [KW][Theorem 3.7]. Namely, for a variety V over the real numbers $\mathbb{R}$ the complex conjugation defines an involution on the underlying topological space $V(\mathbb{C})$ of complex points. Every algebraic vector bundle on V induces a complex vector bundle on $V(\mathbb{C})$, and conjugation gives E the structure of a Real vector bundle in the sence of Atiyah. Passing to Grothendieck groups, this induces a homomorphism $\alpha_0 : K_0^{alg} := K_0(V) \to K\mathbf{R}^0(V(\mathbb{C}))$, where $K\mathbf{R}^0$ is Atiyah's Real K-theory. The pretheory K_0^{alg} is equipped with the integration constructed in [BS]. The pretheory $K\mathbb{R}$ is equipped with an integration which can be defined as follows.

For a closed imbedding $i : Y \hookrightarrow X$ in $\mathcal{S}m$ a Thom-Gysin morphism is defined as the composition

$$i_{K\mathbf{R}^0} : K\mathbf{R}^0(Y(\mathbb{C})) \xrightarrow{\cup\lambda_N} K\mathbf{R}^0_{Y(\mathbb{C})}(N) \xrightarrow{exc^{-1}} K\mathbf{R}^0_{Y(\mathbb{C})}(X(\mathbb{C}))$$

where $N = N_{X(\mathbb{C})/Y(\mathbb{C})}$ is the normal bundle (it is a Real vector bundle over the Real space $Y(\mathbb{C})$), the class $\lambda_N \in K\mathbf{R}^0_{X(\mathbb{C})}(N)$ is the Thom-Gysin class of the Real vector bundle N [A][Theorem 2.4] and the morphism exc^{-1} is the inverse to the excision isomorphism $exc : K\mathbf{R}^0_{Y(\mathbb{C})}(X(\mathbb{C})) \to K\mathbf{R}^0_{Y(\mathbb{C})}(N)$. For a variety $X \in \mathcal{S}m$ the projective bundle theorem [A][The result stated just below Theorem 2.4] states that the cross-product map

$$K\mathbf{R}^0(\mathbf{P}^n(\mathbb{C})) \otimes_{\mathbb{Z}} K\mathbf{R}^0(X(\mathbb{C})) \to K\mathbf{R}^0(\mathbf{P}^n(\mathbb{C}) \times X(\mathbb{C}))$$

is a ring isomorphism and moreover $K\mathbf{R}^0(pt(\mathbb{C}))[t]/(t^{n+1}) \cong K\mathbf{R}^0(\mathbf{P}^n(\mathbb{C}))$ is a ring isomorphism sending the variable t to the class $\zeta_n = [1] - [\mathcal{O}(-1)]$. For the projection $p : \mathbf{P}^n \times X \to X$ define a homomorphism

$$p_{K\mathbf{R}^0} : K\mathbf{R}^0(\mathbf{P}^n(\mathbb{C}) \times X(\mathbb{C})) \to K\mathbf{R}^0(X(\mathbb{C}))$$

as a unique $K\mathbf{R}^0(X(\mathbb{C}))$-module map which for every integer $i = \{0, 1, \dots, n\}$ takes the element $\zeta^i \cup 1 \in K\mathbf{R}^0(\mathbf{P}^n(\mathbb{C}) \times X(\mathbb{C}))$ to the unit $1 \in K\mathbf{R}^0(X(\mathbb{C}))$.

An arbitrary projective morphism $f : Y \to X$ can be decomposed as $f = p \circ i$, where $i : Y \hookrightarrow \mathbf{P}^n \times X$ is a closed imbedding and $p : \mathbf{P}^n \times X \to X$ is the projection. Set $f_{K\mathbf{R}^0} = p_{K\mathbf{R}^0} \circ i_{K\mathbf{R}^0}$. One can chech (following [PS]) that the map $f_{K\mathbf{R}^0}$ is well defined. Moreover the assignment $f \mapsto f_{K\mathbf{R}^0}$ is a perfect integration on the ring pretheory $K\mathbf{R}^0$ (see again [PS]). By Corollary 1.1.11 this is a unique perfect integration on the pretheory $K\mathbf{R}^0$ with the Euler structure given by $L/Y \mapsto [1] - [L^\vee] \in K\mathbf{R}^0(Y(\mathbb{C}))$. The Riemann-Roch theorem [KW][Theorem 3.7] states that the ring morphism $\alpha : K^{alg} \to K\mathbf{R}^0$ commutes with these integrations.

This result is a pretty straighforward consequence of Corollary 1.1.10. To explain this observe that for variety $Y \in \mathcal{S}m$ and a Real line bundle $p : L \to Y(\mathbb{C})$ over the real space

$Y(\mathbb{C})$ and the zero section $z : Y(\mathbb{C}) \to L$ the pushforward $z_{K\mathbf{R}^0}$ in the ring pretheory $K\mathbf{R}^0$ takes the unite $[1_{Y(\mathbb{C})}] \in K\mathbf{R}^0(Y(\mathbb{C}))$ to the class $[1_L] - [p^*(L^\vee)] \in K\mathbf{R}^0(L)$. Thus the corresponding Euler class is given by $e^{K\mathbf{R}^0}(L) = [1_{Y(\mathbb{C})}] - [L^\vee] \in K\mathbf{R}^0(Y(\mathbb{C}))$. In particular, for an algebraic line bundle $p : L \to X$ the Euler class of the underlying Real line bundle $L(\mathbb{C})$ is equal to the class

$$e^{K\mathbf{R}^0}(L(\mathbb{C})) = [1_{Y(\mathbb{C})}] - [L^\vee(\mathbb{C})] \in K\mathbf{R}^0(Y(\mathbb{C})).$$

For an algebraic line bundle L over an algebraic variety $Y \in \mathcal{S}m$ the Euler class $e^{alg}(L)$ of L is given by

$$e^{alg}(L) = [1_Y] - [L^\vee(\mathbb{C})] \in K_0^{alg}(Y).$$

The projective bundle theorems of Grothendieck [Qu2][§8, Th.2.1] and [A][The result stated just below Theorem 2.4] shows that both integrations are perfect and thus both pretheories are oriented in the sence of Definition 1.1.7. Obviously $\varphi(e^{alg}(L)) = e^{K\mathbf{R}^0}(L)$ and thus the morphism $\varphi : K_0^{alg} \to K\mathbf{R}^0$ is an oriented morphism of the oriented pretheories by Corollary 1.1.10. In the other the morphism φ commutes with the integrations on the oriented pretheories.

2.6.11. Semi-topological complex and real K-theories [FW].
If the ground field k is the field $\mathbb{R}$ of reals then the semi-topological K-theory of real algebraic varieties $K\mathbb{R}^{semi}$ defined in[FW] is an oriented pretheory as it is proved in [FW]. For a real variety X it interpolates between the algebraic K-theory of X and Atiyah's Real K-theory of the associated Real space of complex points, $X(\mathbb{C})$. The resulting natural maps of spectra constructed in [FW]

$$K^{alg}(X) \to K\mathbb{R}^{semi}(X) \to K\mathbb{R}_{top}(X_{\mathbb{R}}(\mathbb{C}))$$

induce (passing to homotopies) ring morphisms of the oriented cohomology pretheories. Both morphisms are oriented morphisms of the oriented pretheories as it is shown in [FW]. This holds because the Euler structures on all three theories are given by the assignment $L \mapsto [\mathbf{1}] - [L^\vee]$.

In particular, the composition mappings $K^{alg}(X) \to K\mathbb{R}_{top}(X_{\mathbb{R}}(\mathbb{C}))$ induces (passing to homotopies) an oriented morphisms of the oriented cohomology pretheories. This gives another proof of the Real version of the Riemann-Roch theorem proved in [KW][Theorem 3.7] and discussed in the previous example.

2.6.12. Change of the Euler structure by a unite. Let A be an oriented cohomology pretheory and let $L \mapsto e^A(L)$ be the corresponding Euler structure. Let $a_0 \in A(pt)^{uc}$ be a unite. Then taking the series $r(t) = a_0 \in A(pt)[[t]]$ and replacing the given integration on A by the one defined in 2.3.2 one gets a new integration $f \mapsto f^{new}$ on A such that the corresponding Euler structure is given by $e^{new}(L) = e^A(L) \cup a_0$ for the unite $a_0 \in A(pt)^{uc}$. Then the Todd genus of the identity morphism $id : A \to A$ is the series $td_{id}(t) = a_0 \in A(pt)^{uc}[[t]]$ and for a rank n vector bundle E one has

$$td_{\varphi}(E) = a_0^n$$

2.6.13. Ring morphism $i : A \to A[\mathbf{b}]$ from [Me].

Let A be an oriented cohomology pretheory and let $\tilde{A} = A[\mathbf{b}]$ be a ring cohomology pretheory defined in [Me][Sect. 4] and given by $\tilde{A}(X) = A(X) \otimes \mathbb{Z}[\mathbf{b}]$, where $\mathbb{Z}[\mathbf{b}]$ is a polinomial ring in variables $\{b_n\}_{n=1}^{\infty}$. Let $i : A \hookrightarrow A[\mathbf{b}]$ be the inclusion.

The ring cohomology pretheory $\tilde{A} = A[\mathbf{b}]$ is equipped with a perfect integration given by $f \mapsto f_A \otimes \mathbb{Z}[\mathbf{b}]$ where $f \mapsto f_A$ is the perfect integration on A. Following [Me][Sect. 4] define a new integration on $A[\mathbf{b}]$. Namely by Theorem 2.3.2 there exists a unique new perfect integration on the ring cohomology theory $A[\mathbf{b}]$ such that the inverse Todd genus of the identity morphism $id_{A[\mathbf{b}]}$ is the formal power series $itd_{\varphi}(t) = \sum_{i=0}^{\infty} t^i \cdot b_i \in \tilde{A}[[t]]$. This new perfect integration is given by [Me][Sect. 4]

$$f_{\tilde{A}}(a) = f_A(a \cup \mathbf{P}_Y^A) \cup (\mathbf{P}_X^A)^{-1}. \tag{43}$$

Now consider the morphism

$$\varphi = i : A \to \tilde{A} = A[\mathbf{b}]$$

of the oriented cohomology pretheories (the pretheory $A[\mathbf{b}]$ is considered now with the new perfect integration). The inverse Todd genus of this morphism is the series $itd_{\varphi}(t) = \sum_{i=0}^{\infty} t^i \cdot b_i \in \tilde{A}[[t]]$. The inverse Todd genus of a vector bundle E over a smooth X is the class $P^A(E) \in \tilde{A}(X)$ defined in [Me][Sect. 3]. The Riemann-Roch theorem 2.5.4 says that for every element $a \in \tilde{A}(Y)$ the relation (43) holds in $\tilde{A}(X)$. In the other words the Riemann-Roch relation 2.5.4 holds in this case by the very definition of the new perfect integration on $A[\mathbf{b}]$.

If the characteristic of the ground field k is zero, then there is a ring cohomology pretheory Ω defined in [LM] which is equipped with a perfect integration defined in [LM]. By one of the main theorem of [LM] there is a unique morphism $\varphi : \Omega \to A$ which takes the Euler structure of Ω to the Euler structure of A. It is given by $[Y \xrightarrow{f} X] \mapsto f_A(1)$. By the same theorem there is a unique morphism $\varphi_{\tilde{A}} : \Omega \to \tilde{A}$ which takes the Euler structure of Ω to the Euler structure of $\tilde{A}$. It is given by $[Y \xrightarrow{f} X] \mapsto f_{\tilde{A}}(1)$. We stress here that the composition $i \circ \varphi_A$ does not coincide with $\varphi_{\tilde{A}}$.

The morphism $\varphi_{\tilde{A}}$ is used in [Me] to inject the cobordism theory Ω to the oriented theory $\tilde{A}$ (at least on fields) "preserving" the integration. The morphism $\varphi_{\tilde{A}}$ is an algebraic version of the generalized Hurewich homomorphism as it is mentioned in [Me][Intoduction].

2.6.14. Algebraic and complex cobordism.

Let $k = \mathbb{C}$ and let Ω^{top} be the restriction of the complex cobordism theory to the category $\mathcal{S}m$ (we use here the description of the complex cobordism theory given in [Qu1]). It is an oriented pretheory (with the usual integration described in [Qu1]). Let Ω be the algebraic cobordism pretheory defined in [LM] (with the perfect integration described in [LM]). Let

$$\varphi : \Omega \to \Omega^{top}$$

be a unique ring morphism of the algebraic cobordism theory Ω to the complex cobordism theory Ω^{top} which takes a generator $[Y \xrightarrow{f} X]$ to the class $Y(\mathbb{C}) \xrightarrow{f(\mathbb{C})} X(\mathbb{C})$. The ring morphism φ respects to the perfect integrations on the theories Ω and Ω^{top} [LM] (thus in this case we need not to use any Riemann-Roch type theorem). Thus φ is an oriented morphism in the sence of Definition 1.1.7.

2.6.15. A natural ring morphism $H^*_{\mathcal{M}}(-, \mathbf{Z}(*)) \to H^*(-, \mathcal{K}_*)$.

For every nonnegative integer n the complex $\mathbf{Z}(n)$ (see [SV]) vanishes over the interger n and the cohomology sheaf $\mathcal{H}^n(\mathbf{Z}(n))$ coincides with the sheaf $\mathcal{K}^M_n$. Moreover these identifications respect to the cup-products [SV]. Thus the compositions $\mathcal{H}^n(\mathbf{Z}(n)) \to \mathcal{K}^M_n \to \mathcal{K}_n$ respect to the cup-products where $\mathcal{K}_n$ is the sheaf associated with the presheaf of the Quillen K-groups. Thus we get a ring morphism of the cohomology pretheories

$$\varphi : H^*_{\mathcal{M}}(-, \mathbf{Z}(*)) \to H^*(-, \mathcal{K}_*[-*]).$$

The ring cohomology pretheory $H^*(-, \mathcal{K}_*[-*]$ is equipped with a perfect integration (see [Gi]). The corresponding Euler structure is given by $L \mapsto D(L) \in H^2(X, \mathcal{K}_1[-1]) = H^1(X, \mathcal{K}_1)$, where $D(L)$ is the divisor class associated with the line bundle L. With this definition $H^*(-, \mathcal{K}_*[-*]$ is an oriented cohomology pretheory.

To describe the motivic cohomology $H^*_{\mathcal{M}}(-, \mathbf{Z}(*))$ as an oriented cohomology pretheory recall [SV] that the morphism $H^2_{\mathcal{M}}(-, \mathbf{Z}(1)) \to H^2(X, \mathcal{K}_1[-1])$ is an isomorphism. Now the assignment $L \mapsto D(L) \in H^2_{\mathcal{M}}(-, \mathbf{Z}(1)) = H^2(X, \mathcal{K}_1[-1])$ is an Euler structure (see Examples in [PS]) for the ring cohomology pretheory $H^*_{\mathcal{M}}(-, \mathbf{Z}(*))$. By Theorem [PS][Theorem 4.1.4] there is a unique integration on the ring cohomology pretheory $H^*_{\mathcal{M}}(-, \mathbf{Z}(*))$ such that the corresponding Euler structure coincides with the Euler structure $L \mapsto D(L)$ (as it is mentioned just below 1.1.7 all the integrations constructed in [PS] are perfect). With this definitions $H^*_{\mathcal{M}}(-, \mathbf{Z}(*))$ is an oriented cohomology pretheory.

The ring morphism φ takes one Euler structure to the other one. Thus by Corollary 1.1.10 the ring morphism φ commutes with the perfect integrations.

2.6.16. A natural ring morphism $CH^* \to H^*_{\mathcal{M}}(-, \mathbf{Z}(*))$.

Recall that there are natural group isomorphisms $CH^n \to H^{2n}_{\mathcal{M}}(-, \mathbf{Z}(n))$. As it is proved in [W] these group morphisms respect to the cup-products. Thus we get a ring morphism

$$\varphi : CH^* \to H^*_{\mathcal{M}}(-, \mathbf{Z}(*))$$

of the ring cohomology pretheories.

The ring cohomology pretheory CH^* is equipped with the perfect integration (the classical one) as it is explained in the example 2.6.1. The ring cohomology pretheory $H^*_{\mathcal{M}}(-, \mathbf{Z}(*))$ is an oriented cohomology pretheory as it is explained in the previous example.

The corresponding Euler structures are given in both cases by the assignment $L \to D(L) \in CH^1(X) = H^2_{\mathcal{M}}(-, \mathbf{Z}(1))$. Thus by Corollary 1.1.10 the ring morphism φ respects to the integrations.

2.6.17. Bloch's Higher Chow Groups [Bl].

Bloch's higher Chow groups $CH^*(X, *)$ defined in [Bl] form a ring cohomology pretheory. It is equipped with an integration constructed in [Bl]. The corresponding Euler class is given by the assignment $L(D) \mapsto [D] \in CH^1(X) = CH^1(X,0)$. The projective bundle theorem from [Bl] shows that the integration is perfect. Thus the pretheory $X \mapsto CH^*(X,*)$ is oriented.

The motivic cohomology $H^*_{\mathcal{M}}(-,\mathbf{Z}(*))$ is an oriented cohomology pretheory as it is explained in the example 2.6.15.

It looks very reasonable to expect that the isomorphism of the motivic cohomology and higher Chow groups constructed in [V3] is in fact a ring isomorphism. Clearly that isomorphism takes the Euler structure on higher Chow groups to the Euler structure on the motivic cohomology. Thus by Corollary 1.1.10 this isomorphism is an oriented morphism of the oriented cohomology pretheories.

2.6.18. Comments concerning some examples which are missed for this text. There are examlples of oriented cohomology pretheories which we did not include in the text but which are definetely very interesting or could be quite interesting.

Among such examples are "motivic" cohomology defined in [L]. The Betti and etale realization functors are give the examples of oriented morphisms of the oriented cohomology pretheories because they take the Euler structure to the Euler one.

It's quite plausible that the topological Hochschild homology defined as in [GH] is an oriented cohomology pretheory and moreover it is quite plausible that the cyclotomic trace map $trc : K(-) \to TC(-,p)$ (which is a ring morphism by [GH]) is an oriented morphism of the oriented cohomology pretheories.

2.6.19. Balmer's Witt theory of schemes. The Witt pretheory $X \mapsto W^*(X) = \oplus_{-\infty}^{\infty} W^n(X)$ of Balmer [Ba] is ring pretheory by [GN]. However it is not an oriented pretheory because the Witt theory of a projective space is not isomorphic to the truncated polinomial ring (see [Wa]).

2.7 Generalized Chern character.

An oriented cohomology theory A gives rise via the corresponding Euler structure $L \to e^A(L)$ to a commutative formal group law $F_A(t_1,t_2)$ of dimension 1. This formal group law F_A is a formal power series in two variables (say t_1 and t_2) over the commutative ring $A(pt)^{uc}$. It is uniquelly defined by the relations $e^A(L_1 \otimes L_2) = F_A(e^A(L_1), e^A(L_2))$ which holds for every line bundles L_1 and L_2. The formal group law corresponding in this way to the pretheory K_0 (the K_0-functor) is the multiplicative formal group law $\mathbb{G}^f_m$. The main object of this subsection is to construct a ring morphism of the oriented pretheories

$$ch^{\varphi} : K_0 \to A$$

begining with a formal group law morphism $\varphi(t) : F_A \to \mathbb{G}^f_m \otimes_{\mathbb{Z}} A(pt)^{uc}$. The ring morphism ch^{φ} is called *generalized Chern character*. The inverse Todd genus $itd_{ch^{\varphi}}(t)$ of the generalized Chen character ch^{φ} is the series $\varphi(t)/t \in A(pt)^{uc}[[t]]$. If the coefficient at t in φ is invertible then the Todd genus $td_{ch^{\varphi}}(t)$ of the generalized Chern character is well defined and it coincides with the series $t/\varphi(t)$. These gives the corresponding Riemann-Roch theorems 2.5.3 and 2.5.5 for the ring morphism $ch^{\varphi} : K_0 \to A$. It is shown below in 2.7.4 that in the case $A = H^{ev}$ (see 2.6.1) and $\varphi = exp(t) - 1 : \mathbb{G}_a \to \mathbb{G}_{m,\mathbb{Q}}$ the generalized Chern character ch^φ coincides with the classical Chern character ch of Hirzebruch [H][Appendix 1, Section 23]. If $A = K_0$ and $\varphi = [p] : \mathbb{G}_m \to \mathbb{G}_m$ is the isogeny of the multiplication by p, then the generalized Chern character ch^φ coincides with the Adams operation $\psi^p : K_0 \to K_0$. In particular, if $p = 1$, then $ch^\varphi = id : K_0 \to K_0$.

Now let A be an oriented cohomology pretheory and let the assignment $L \mapsto e^A(L)$ be the corresponding Euler structure 1.1.4. Let

$$E_A^- : \bar{A}[[t]] \to A(\mathbf{P}^\infty) \tag{44}$$

be the isomorphism defined by $E_A^-(t) = \xi_A \in A(\mathbf{P}^\infty)$ with $\xi_A = e^A(\mathcal{O}(-1)) \in A(\mathbf{P}^\infty)$. Let

$$E_A^+ : \bar{A}[[t]] \to A(\mathbf{P}^\infty) \tag{45}$$

be the isomorphism defined by $E_A^+(t) = \zeta_A \in A(\mathbf{P}^\infty)$ with $\zeta_A = e^A(\mathcal{O}(1)) \in A(\mathbf{P}^\infty)$.

If we define $A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ as the projective limit $\varprojlim A(\mathbf{P}(V) \times \mathbf{P}(W))$, where V and W runs over all finite dimensional k-subspases of the k-vector space k^∞, then by projective bundle axiom 1.1.6 one has two ring isomorphisms

$$E_A^- \otimes E_A^- : A(pt)[[t_1, t_2]] \to A(\mathbf{P}^\infty \times \mathbf{P}^\infty), \tag{46}$$

$$E_A^+ \otimes E_A^+ : A(pt)[[t_1, t_2]] \to A(\mathbf{P}^\infty \times \mathbf{P}^\infty), \tag{47}$$

(here $t_1 = t \otimes 1$ and $t_2 = 1 \otimes t$ in $A(pt)[[t_1, t_2]] = A(pt)[[t]] \otimes A(pt)[[t]]$). The formal power series

$$F_A^-(t_1, t_2) = (E_A^- \otimes E_A^-)^{-1}[e(p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\mathcal{O}(-1)))] \in A(pt)^{uc}[[t_1, t_2]] \tag{48}$$

is the commutative formal group law F_A mentioned just above. The formal power series

$$F_A^+(t_1, t_2) = (E_A^+ \otimes E_A^+)^{-1}[e(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1)))] \in A(pt)^{uc}[[t_1, t_2]] \tag{49}$$

is as well the formal group law mensioned just above. Thus $F_A^-(t_1, t_2) = F_A^+(t_1, t_2)$ and we will use the shorter notation

$$F_A(t_1, t_2) \in A(pt)^{uc}[[t_1, t_2]] \tag{50}$$

for both of them.

In the case of the oriented ring pretheory K_0 the Euler class $e^K(L)$ of a line bundle L is the element $[\mathbf{1}] - [L^\vee]$. We will use in the case of K-functor the variables u_1, u_2 in this particular subsection. Thus the formal group law $F_K(u_1, u_2)$ is equal to $u_1 + u_2 - u_1 u_2$ This is a multiplicative formal group law which we denote $\mathbb{G}_m^f$.

2.7.1. Lemma. *Let $\varphi(t) \in A(pt)^{uc}[[t]]$. Then there is a unique additive morphism*

$$\alpha_\varphi : K_0 \to A \tag{51}$$

of the functors $A, K : \mathcal{S}m \to \mathcal{A}b$ satisfying the following properties

- *functoriality:* $\alpha^Y_\varphi \circ f^K = f^A \circ \alpha^X_\varphi$ *for every morphism* $f : Y \to X$*;*
- *normalization:* $\alpha^X_\varphi(L) = \varphi(e^A(L))$.

Proof. Begin with two different constructions of the required morphism $\alpha_\varphi : K_0 \to A$. This constructions are analogous to the constructions of the assignment $E \mapsto r(E)$ from the proof of Proposition 2.2.3.

First Construction. Let $\varphi^- = \varphi(e^A(\mathcal{O}(-1))) \in A(\mathbf{P}^\infty)$. For a variety X and a line bundle L over X there is a finite dimensional k-vector space V and a diagram $X \xleftarrow{p} X' \xrightarrow{f} \mathbf{P}(V)$ (see Claim 1.2.2) such that $p : X' \to X$ is a torsor over a vector bundle and the line bundles $p^*(L)$ and $f^*(\mathcal{O}_V(-1))$ are isomorphic. Set $\varphi(L) = (p^A)^{-1}(f^A(\varphi^-|_{\mathbf{P}(V)})) \in A(X)^{uc}$. One can check that the element $\varphi(L) \in A(X)^{uc}$ is well-defined, i.e. it does not depend on the particular choice of the diagram and moreover the element $\varphi(L)$ depends only of the isomorphism class of the line bundle L (we skip here a careful proof of the well-definess of the element $\varphi(L)$ because the second construction below is described in all the details). Clearly $\varphi(\mathcal{O}(-1)) = \varphi^- = \varphi(e^A(\mathcal{O}(-1))) \in A(\mathbf{P}^\infty)^{uc}$.

If E is a rank n vector bundle over X then set $\varphi_E = \varphi(\mathcal{O}_E(-1)) \in A(\mathbf{P}(E))^{uc}$. Consider an $A(X)^c$-module operator $A(\mathbf{P}(E))^c \xrightarrow{\varphi_E} A(\mathbf{P}(E))^c$ given by $a \mapsto a \cup \varphi_E$. By Theorem 2.1.2 the $A(X)^c$-module $A(\mathbf{P}(E))^c$ is free of rank n. Set

$$\alpha^X_\varphi(E) = trace(\varphi_E) \in A(X)^c \tag{52}$$

We claim that the assignment $E \mapsto \alpha^X_\varphi(E)$ is the required one. In fact, the functoriality of the class $\alpha^X_\varphi(E)$ is clear because of the functoriality of the class $\varphi(L)$ (on line bundles). The functoriality of the class $\alpha^X_\varphi(E)$ shows that the elements $\alpha^X_\varphi(E) \in A(X)^c$ are in fact in $A(X)^{uc}$. The normalization holds because for a line bundle L over X one has $\mathbf{P}(L) = X$ and $\mathcal{O}_E(-1) = L$ over X and $\alpha^X_\varphi(L) = \varphi_L = \varphi(e^A(L)) \in A(X)^{uc}$. To prove the relation $\alpha^X_\varphi(E) = \alpha^X_\varphi(E_1) + \alpha^X_\varphi(E_2)$ for a short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of vector bundles one may assume by Splitting principle 1.6.1 that $E = E_1 \oplus E_2$. In this case consider the closed imbeddings $i_s : \mathbf{P}(E_s) \hookrightarrow \mathbf{P}(E)$ for $s = 1, 2$ and recall that $A(\mathbf{P}(E_s))$ is a two-sided $A(\mathbf{P}(E))$-module by means of the pull-back operator i^A_s. Consider an endomorphism of the short exact sequence (29) given by the multiplication by ξ_E

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A(\mathbf{P}(E_1))^c & \xrightarrow{(i_1)_A} & A(\mathbf{P}(E))^c & \xrightarrow{i_2^A} & A(\mathbf{P}(E_2))^c & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \cup i_1^A(\varphi_E)} & & \uparrow{\scriptstyle \cup\varphi_E} & & \uparrow{\scriptstyle \cup i_2^A(\varphi_E)} & & \\
0 & \longrightarrow & A(\mathbf{P}(E_1))^c & \xrightarrow{(i_1)_A} & A(\mathbf{P}(E))^c & \xrightarrow{i_2^A} & A(\mathbf{P}(E_2))^c & \longrightarrow & 0
\end{array}$$

Obviously $i^A_s(\varphi_E) = \varphi_{E_s}$ in $A(\mathbf{P}(E_s))^c$ for $s = 1, 2$. The relation $\alpha^X_\varphi(E) = \alpha^X_\varphi(E_1) + \alpha^X_\varphi(E_2)$ follows. Thus the assignment $E/X \mapsto \varphi_X(E) \in A(X)$ gives rise to a group homomorphism

$$\alpha^X_\varphi : K_0(X) \to A(X)$$

which is functorial in X and normalised as it is required by the Lemma.

Second construction. Let $t_1, t_2, \ldots, t_n$ be independent variables and let $\sigma_1, \sigma_2, \ldots \sigma_n$ be the symmetric polynomial in these variables: $\sigma_1 = t_1 + t_2 + \cdots + t_n$, $\sigma_2 = t_1 t_2 + \cdots + t_{n-1} t_n$, ... , $\sigma_n = t_1 t_2 \ldots t_n$. Set

$$\varphi(\sigma_1, \sigma_2, \ldots, \sigma_n) = \Sigma_{i=1}^n \varphi(t_i) \in A(pt)^{uc}[[\sigma_1, \sigma_2, \ldots, \sigma_n]].$$

Define an invariant $\alpha_\varphi^X(E)$ of a vector bundle E with $\mathrm{rk} E = n$ over a smooth X as the evaluation of the series $\varphi(\sigma_1, \sigma_2, \ldots, \sigma_n)$ on the Chern classes of E

$$\alpha_\varphi^X(E) = \varphi(c_1^A(E), c_2^A(E), \ldots, c_n^A(E)) \in A^{uc}(X).$$

Here the right hand side is well-defined because the Chern classes are universally central and nilpotent (see 2.2.2).

Now check that for every variety $X \in \mathcal{S}m$ and every short exact sequence $0 \to E_1 \to E \to E_2 \to 0$ of vector bundles on X one has $\alpha_\varphi^X(E_1) + \alpha_\varphi^X(E_2) = \alpha_\varphi^X(E)$. By Splitting principle 1.6.1 one may assume that $E = E_1 \oplus E_2$ and as E_1 so E_2 is a direct sum of line bundles. In this case the relation $\alpha_\varphi^X(E_1) + \alpha_\varphi^X(E_2) = \alpha_\varphi^X(E)$ is obvious because for a line bundle L one has $\alpha_\varphi^X(L) = \varphi(c_1^A(L)) = \varphi(e^A(L))$ by the very construction of the assignment $E \mapsto \alpha_\varphi^X(E)$. These shows that for every $X \in \mathcal{S}m$ the assignment $E \to \alpha_\varphi^X(E)$ defines a group homomorphism $\alpha_\varphi^X : K_0(X) \to A(X)$.

We claim that the assignment $X \mapsto \alpha_\varphi^X$ is the required morphism $\alpha_\varphi : K_0 \to A$. In fact, for a morphism $f : Y \to X$ in $\mathcal{S}m$ the relation $\alpha_\varphi^Y \circ f^K = f^A \circ \alpha_\varphi^X$ holds by the functoriality of the Chern classes. The normalization property $\alpha_\varphi^X(L) = \varphi(e^A(L))$ holds by the very construction of α_φ.

To complete the proof of the Lemma it remains to verify the uniqueness assertion. For that let $\alpha_\varphi^{(1)}$ and $\alpha_\varphi^{(2)}$ be two functor morphisms satisfying the two properties from the Lemma. We must verify that for every vector bundle E the relation $\alpha_\varphi^{(1),X}(E) = \alpha_\varphi^{(2),X}(E)$ holds. By Splitting principle one may assume that E is a direct sum of line bundles. Since $\alpha_\varphi^{(1)}$ and $\alpha_\varphi^{(2)}$ are group homomorphisms it remains to check that for every line bundle L one has $\alpha_\varphi^{(1),X}(L) = \alpha_\varphi^{(1),X}(L)$. This relation holds because $\alpha_\varphi^{(1),X}(L) = \varphi(e^A(L)) = \alpha_\varphi^{(2),X}(L)$ by the normalization property. The Lemma follows.

$\square$

Set $I_A^-(t) = (E_A^-)^{-1}(e(\mathcal{O}(1))) \in A(pt)^{uc}[[t]]$ and $I_A^+(t) = (E_A^+)^{-1}(e(\mathcal{O}(-1))) \in A(pt)^{uc}[[t]]$. It's straightforward to check (compair with the arguments from 2.2.3) that for every variety X and every line bundle L over X one has $I_A^-(e^A(L)) = e^A(L^\vee) = I_A^+(e^A(L))$. Thus $I_A^-(t) = I_A^+(t)$ and we will use the notation

$$I_A(t) \in A(pt)^{uc}[[t]] \tag{53}$$

for both of them. The series $I_A(u)$ is the "inverse" element for the formal group law $F_A(t_1, t_2)$ in the sence that one has the relation $F_A(t, I_A(t)) = 0$. As was mentioned just above Lemma 2.7.1 the formal group law $F_K(u_1, u_2)$ (we use the variables u_1, u_2 in the case of K-theory in this particular subsection) corresponding to the oriented pretheory K_0 (the K_0-theory) is the multiplicative group law $u_1 + u_2 - u_1 u_2 \in K_0(pt)[[u_1, u_2]] = \mathbb{Z}[[u_1, u_2]]$, because $e^K(L) = [\mathbf{1}] - [L^\vee]$.

2.7.2. Theorem. *Let a series in one variable $\varphi(t) \in A(pt)^{uc}[[t]]$ be a formal group law morphism $F_A \to G_m^f \otimes_{\mathbb{Z}} A(pt)^{uc}$ and let $\varphi^\vee(t) = \varphi(I_A(t))$. Then the additive functor morphism*

$$ch^\varphi := \alpha_{1-\varphi^\vee} : K_0 \to A \tag{54}$$

from Lemma 2.7.1 is a ring functor morphism that is a ring morphism of the ring pretheories 1.1.7.

The ring morphism $ch^\varphi : K_0 \to A$ is called *the generalized Chern character*. In this Theorem the element $\varphi^\vee(t)$ is choosen (rather than the element $\varphi(t)$ itself) in order to get the following normalization property: if $A = K_0$ and $\varphi(t) = t$ then $ch^\varphi = id : K_0 \to K_0$ (see examples 2.7.4 below).

Proof. It's suffices to verify that for every variety X and every vector bundles E and F over X one has the relations

- $ch^\varphi(E)ch^\varphi(F) = ch^\varphi(E \otimes F) \in A(X)$;
- $ch^\varphi([\mathbf{1}]) = 1 \in A(X)$.

To prove the first relation observe that it is linear as in E so in F. By Splitting principle 1.6.1 one may assume that as E so F is a direct sum of line bundles. So by the bilinearity one may further assume that $E = L_1$ and $F = L_2$ are line bundles. The chain of relations

$$ch^\varphi(L_1 \otimes L_2) = 1 - \varphi(e^A(L_1^\vee \otimes L_2^\vee)) = 1 - \varphi(e^A(L_1^\vee +_{F^A} L_2^\vee)) =$$

$$= 1 - [\varphi(e^A(L_1^\vee)) +_{\mathbb{G}_m^f} \varphi(e^A(L_2^\vee))] = 1 - [\varphi(e^A(L_1^\vee)) + \varphi(e^A(L_2^\vee)) - \varphi(e^A(L_1^\vee))\varphi(e^A(L_2^\vee))] =$$

$$= (1 - \varphi(e^A(L_1^\vee)))(1 - \varphi(e^A(L_1^\vee))) = ch^\varphi(L_1)ch^\varphi(L_2).$$

To prove the second relation recall that $e^A([\mathbf{1}]) = 0 \in A(X)$. Since the free term of the series $\varphi(t)$ vanishes one gets the required relation $ch^\varphi([\mathbf{1}]) = 1$ in $A(X)$. The Theorem follows.

□

2.7.3. Theorem. *The inverse Todd genus $itd_{ch^\varphi}(t)$ of the ring morphism*

$$ch^\varphi : K_0 \to A$$

is the formal power series $\varphi(t)/t \in A(pt)^{uc}[[t]]$. If $\varphi(t) \in A(pt)^{uc}[[t]]$ is invertible, then the Todd genus of the ring morphism ch^φ is the formal power series $t/\varphi(t) \in A(pt)^{uc}[[t]]$.

Proof. In fact, the element $itd_{ch^\varphi}(\xi_A) \in A(\mathbf{P}^\infty)$ from Definition 2.5.1 is computed by the chain of relations

$$itd_{ch^\varphi}(\xi_A) = ch^\varphi(\xi_K)/\xi_A = [ch^\varphi(e^K(\mathcal{O}(-1)))]/e^A(\mathcal{O}(-1)) =$$

$$= [ch^{\varphi}([\mathbf{1}] - [\mathcal{O}(1)])]/e^A(\mathcal{O}(-1)) = [ch^{\varphi}([\mathbf{1}]) - ch^{\varphi}([\mathcal{O}(1)])] =$$
$$= [1 - (1 - \varphi(e^A(\mathcal{O}(-1))))]/e^A(\mathcal{O}(-1)) = [\varphi(e^A(\mathcal{O}(-1)))]/e^A(\mathcal{O}(-1)) = \varphi(\xi_K)/\xi_A.$$

Thus $itd_{ch^{\varphi}}(t) = \varphi(t)/t \in A(pt)^{uc}[[t]]$. The Theorem follows.

$\square$

2.7.4. Examples.

(1) If $k = \mathbb{C}$ and $A = H^{ev}$ from the example 2.6.1 then the formal group law F_A is the additive group law $t_1 + t_2$. Let $\varphi(t) = exp(t) - 1 = t + t^2/2! + t^3/3! + \dots$. Then φ is a formal group law morphism $F_A = \mathbb{G}_a \to \mathbb{G}_m^f$. Clearly the ring morphism ch_{φ} coincides with the usual Chern character $ch : K_0 \to CH^*$ 2.6.1.

(2) If $A = K_0$ then the Euler class is given by $L \mapsto [\mathbf{1}] - [L^{\vee}]$ and thus the formal group law F_A is the multiplicative formal group law $\mathbb{G}_m$ $t_1 + t_2 - t_1t_2$. Let p be a positive integer and let $\varphi(t) = 1 - (1 - t)^p$ is the isogeny $[p]$ of the multiplication by p of the formal group law $\mathbb{G}_m^f$ (say $[2] = 2t - t^2$). In this case the ring morphism $ch_{\varphi} : K_0 \to K_0$ is just the Adams operation $\psi^p : K_0 \to K_0$.

If the integer p is a negative integer and $\varphi = [p]$ is the isogeny of the multiplication by $[p]$ for the formal group law $\mathbb{G}_m^f$. Then the ring morphism $ch_{\varphi} : K_0 \to K_0$ coincides again with the Adams operation $\psi^p : K_0 \to K_0$.

In particular, if $p = -1$, then $[p] = -t - t^2 - t^3 - \dots$. And in this case the ring morphism $ch_{\varphi} : K_0 \to K_0$ coincides with the ring morphism $\psi^{-1} : K_0 \to K_0$ given by $[E] \mapsto [E^{\vee}]$. If $p = 1$ then $\varphi(t) = t$ and $ch_{\varphi} = id : K_0 \to K_0$. This relation is the motivation to use the element $\varphi^{\vee}$ (rather than the element φ itself) in the definition of the generalized Chern character $ch^{\varphi} : K_0 \to A$ given in Theorem 2.7.2.

2.7.5. A functor from the category of oriented cohomology pretheories and their ring morphisms to the category of commutative formal group laws of dimension 1.

It is convinient here to denote an oriented cohomology theory as a pair (A, ω) where A is a ring cohomology petheory and ω is a perfect integration on A. Every perfect integration ω on A defines the corresponding Euler structure (see (8)) which we denote here e^{ω} (in contrast with the symbol e^A which was repeatedly used above in the text). The formal power series

$$F_A(t_1, t_2) = (E_A^- \otimes E_A^-)^{-1}[e^{\omega}(p_1^*(\mathcal{O}(-1)) \otimes p_2^*(\mathcal{O}(-1)))] \in A(pt)^{uc}[[t_1, t_2]]$$

from (50) is a commutative formal group law which we denote in this subsubsection $F_{A,\omega}$.

One can check that for any ring morphism $\varphi : A \to B$ of oriented pretheories (A, ω_A) and (B, ω_B) the series $\Phi(t) \in \bar{B}[[t]]$ from the very begining of subsection 2.4 is in fact a formal group law morphism $F_{B,\omega_B} \to F_{A,\omega_A} \otimes_{\bar{A}^{uc}} \bar{B}^{uc}$ (if $(A, \omega_A) = (B, \omega_B)$ and $\varphi = id_A$ then $\Phi(t) = t$ is the identity morphism $F_A \to F_A$). If futhermore $\psi : B \to C$ is one more ring morphism of oriented pretheories (B, ω_B) and (C, ω_C) and

$$\Psi(v) = \psi_{\infty}(t) \in \bar{C}[[v]]$$

where v and t are the variables from the identifications $\bar{B}[[t]] = B(\mathbf{P}^\infty)$ and $\bar{C}[[v]] = C(\mathbf{P}^\infty)$ then the formal group law morphism $\Psi \circ \Phi : F_{C,\omega_C} \to F_{A,\omega_A} \otimes_{\bar{A}^{uc}} \bar{C}^{uc}$ corresponds in the mentioned way to the composition morphisms $\psi \circ \varphi : A \to C$ of the oriented theories (A, ω_A) and (C, ω_C). So the assignments

$(A, \omega) \mapsto F_{A,\omega}$ and $[\varphi : (A, \omega_A) \to (B, \omega_B)] \mapsto [\Phi(t) : F_{B,\omega_B} \to F_{A,\omega_A} \otimes_{\bar{A}^{uc}} \bar{B}^{uc}]$

form a functor

$$grp : OrientedTheories \to CommutatveFormalGroupLaws \tag{55}$$

2.7.6. One more example. Let $A(X) = CH^*(X)$ and let $B(X) = K_0(X) \otimes \mathbb{Q}$ and let

$$\varphi = ch^{-1} : CH^* \to K_0 \otimes \mathbb{Q}.$$

In this case

$$itd_\varphi = (-log(1-t))/t = \in \mathbb{Q}[[t]] = K_0(pt)_{\mathbb{Q}}[[t]]$$

where $log(1-t) = -t + t^2/2 - t^3/3 + \cdots \in \mathbb{Q}[[t]]$. This is the answer. It is proved for example as follows.

The formal group law associated with the oriented ring pretheory $CH*_{\mathbb{Q}}$ is the additive formal group law $t_1 + t_2 \in CH^*_{\mathbb{Q}}(pt)[[t_1, t_2]] = \mathbb{Q}[[t_1, t_2]$ which we denote here $\mathbb{G}_a$. The formal group law associated with the oriented ring pretheory $K_{0,\mathbb{Q}}$ is the multiplicative formal group law $t_1+t_2-t_1t_2 \in K_{0,\mathbb{Q}}(pt)[[t_1, t_2]] = \mathbb{Q}[[t_1, t_2]$ which we denote here $\mathbb{G}^f_M$. The Chern character $Ch : K_{0,\mathbb{Q}} \to CH^*_{\mathbb{Q}}$ induces the identity homomorphism on the coefficients $id : \mathbb{Q} = K_{0,\mathbb{Q}}(pt) \to CH^*_{\mathbb{Q}}(pt) = \mathbb{Q}$. The formal group law morphism $\mathbb{G}_a \to \mathbb{G}^f_m$ induced by the Chern character coincides with the series $\Phi(t) = 1 - e^{-t}$. If $\Psi(t) \in \mathbb{Q}[[t]]$ is the formal group law morphism induced by the ring morphism ch^{-1} then the relation $\Psi(\Phi(t)) = t$ holds in $\mathbb{Q}[[t]]$ because the composition

$$ch^{-1} \circ ch : K_{0,\mathbb{Q}} \to K_{0,\mathbb{Q}}$$

is the identity. There exists a unique solution of the equation $\Psi(\Phi(t)) = t$. Since the series $-log(1-t) \in \mathbb{Q}[[t]]$ satisfies the relation $-log(1-(1-e^{-t})) = t \in \mathbb{Q}[[t]]$ one gets finally $\Psi(t) = -log(1-t)$ and $itd_{ch^{-1}}(t) = Psi(t)/t = -log(1-t)/t$.

2.7.7. Remark about a work of J.F.Adams and G.Walker. Making the substitution $t = -u$ in the series $itd_{ch^{-1}}(t)$ one gets a new series $log(1+u)/u$. This is exactly the series from the main result of the article [AW]. So the main result of the article [AW] is formulated in terms of the *inverse Todd genus* $itd_{ch^{-1}}$ of the ring morphism

$$ch^{-1} : H^{ev} \to K_{0,\mathbb{Q}}.$$

It seems plausible that the reason why namely the series $log(1+u)/u$ is used in [AW] (rather than the series $-log(1-t)/t$) is the following one. The Thom-Gysin homomorphisms quite commonly used in the complex K-theory are normalized such that the corresponding Euler structure is given by the assignment $L \mapsto [L^\vee] - [1]$. However the integration of Grothendieck commonly used in the algebraic K_0-theory is such that the corresponding Euler structure is given by the assignment $L \mapsto [1]-[L^\vee]$. Thus the inverse Todd genus $itd^{top}_{ch^{-1}}(u) \in \mathbb{Q}[[u]]$ of the ring morphism

$$ch^{-1} : H^{ev} \to K^{top}_{\mathbb{Q}}$$

is obtained from the inverse Todd genus $td_{ch^{-1}}(t) \in \mathbb{Q}[[t]]$ of the morphism $ch^{-1} : CH^* \to K_{0,\mathbb{Q}}$ via the above mentioned substitution $t = -u$. Thus we get

$$itd^{top}_{ch^{-1}}(u) = log(1+u)/u \in \mathbb{Q}[[t]].$$

One could guess that the authors of the article [AW] knew this meaning of the series $log(1+u)/u$.

2.8 The relation $\{\mathbf{P}^n\}_A = l_A(\{\mathbb{C}P^n\}_{MU})$ in $A(pt)$.

If for a minute $A : \mathcal{S}m\mathcal{O}p \to Ab$ is an oriented cohomology theory in the sence of [PS], then for every integer n and the structural morphism $p : \mathbf{P}^n \to pt$ the push-forward $p_A : A(\mathbf{P}^n) \to A(pt)$ is defined such that the element $p_A(1) \in A(pt)$ is equal to the class $l_A(\{\mathbb{C}P^n\}_{MU})$, where $l_A : MU \to A(pt)$ is a ring homomorphism defined below in this subsection and $\{\mathbb{C}P^n\}_{MU} := (p_{\mathbb{C}})_{MU}(1) = [\mathbb{C}P^n]$. The main aim of the subsection is to prove that we are forced to make this choice for the definition of the pushforward $p_A : A(\mathbf{P}^n) \to A(pt)$ if we want to get finally a perfect integration on A. This is stated in Theorem 2.8.1 below in the subsection. So the classes $\{\mathbf{P}^n\}_A := p_A(1)$ and $[\mathbf{P}^n]_A := l_A([\mathbb{C}P^n]_{MU})$ coincides in $A(pt)$. Since this moment A is an oriented cohomology pretheory in the sence of this text 1.1.7.

Let $X \to MU^*(X)$ be the complex cobordism theory that is a generalized cohomology theory on CW-complexes represented by the ring spectra MU. We will often write below MU for the ring of coefficients of this cohomology theory that is for the value of this theory on the one point space $\{*\}$. The Thom-Gysin morphisms $f_{MU} : MU^*(Y) \to MU^*(X)$ (defined at least for every proper morphism $f : Y \to X$ of complex algebraic varieties [Qu1]) make the functor $MU : \mathcal{S}m/\mathbb{C} \to Rings$ an oriented cohomology pretheory in the sence of Definition 1.1.7. For a smooth projective complex algebraic variety $f : X \to pt$ set

$$\{X\} = f_{MU}(1) \in MU(\{*\}).$$

Let $L \mapsto e^{MU}(L)$ be the Euler structure on MU corresponding to the integration on MU (see (8)). The class $e^{MU}(L)$ is just the Conner-Floyd class $cf(L)$ [Qu1] and they both coincide with the first Chern class $c_1^{MU}(L)$ [CF]. Let $E^+_{MU} : MU(\{*\})[[t]] \to MU(\mathbb{C}P^\infty)$ be the isomorphism (45) and let $E^+_{MU} \otimes E^+_{MU} : MU(\{*\})[[t_1,t_2]] \to MU(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ be the isomorphism (47). Then the series

$$F_{MU}(t_1,t_2) = (E^+_{Mu} \otimes E^+_{MU})^{-1}[e(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1)))] \in MU(\{*\})[[t_1,t_2]] \qquad (56)$$

is the formal group law (50). This is the formal group law which Quillen considers in [Qu1]. The series

$$I_{MU}(t) = (E^+_{MU})^{-1}(e(\mathcal{O}(-1))) \in MU(\{*\})[[t]] \qquad (57)$$

is the "inverse" element of the formal group law $F_{MU}(t_1, t_2)$ (see (53)).

Let A be an oriented cohomology pretheory. For a smooth projective algebraic variety $g : T \to pt$ over the field k set

$$\{T\} = g_A(1) \in A(pt)^{uc}.$$

The element $\{T\}$ is universally central by Claim 2.1.4. To state the main result of this subsection we need to recall certain information concerning commutative formal group laws arrising from oriented cohomology pretheory.

Let $L \mapsto e^A(L)$ be the Euler structure on A corresponding to the integration on A (see (8)). Let $E_A^+ : \bar{A}[[t]] \to A(\mathbf{P}^\infty)$ be the isomorphism (45) and let $E_A^+ \otimes E_A^+ : A(pt)[[t_1, t_2]] \to A(\mathbf{P}^\infty \times \mathbf{P}^\infty)$ be the isomorphism (47). Then the series

$$F_A(t_1, t_2) = (E_A^+ \otimes E_A^+)^{-1}[e(p_1^*(\mathcal{O}(1)) \otimes p_2^*(\mathcal{O}(1)))] \in A(pt)^{uc}[[t_1, t_2]] \tag{58}$$

is the formal group law (50). The series

$$I_A(t) = (E_A^+)^{-1}(e(\mathcal{O}(-1))) \in A(pt)^{uc}[[t]] \tag{59}$$

is the "inverse" element of the formal group law $F_A(t_1, t_2)$ (see (53)).

Recall that $MU(\{*\}) = MU(\{*\})^{uc}$ because the ring $MU(\{*\})$ consists of even degree elements. (We will not use the fact that $MU(\{*\})$ is a polinomial ring but it is useful to know sometime that this is the case). By Quillen's theorem [Qu1] the formal group law F_{MU} is the universal one among all commutative formal group laws of dimension 1. Thus there is a unique ring homomorphism

$$l_A : MU(\{*\}) \to A(pt)^{uc}$$

such that $l_A(F_{MU}) = F_A \in A(pt)^{uc}[[t_1, t_2]]$.

2.8.1. Theorem. *For every integer n the relation $l_A(\{\mathbb{C}P^n\}) = \{\mathbf{P}^n\}$ holds in $A(pt)^{uc}$.*

Proof. Recall that we often will write in this proof MU for the coefficient ring $MU(\{*\})$ of the complex cobordism theory. We will write in this proof $\bar{A}^{uc}$ for the ring $A(pt)^{uc}$. For a variety $X \in \mathcal{S}m$ we will consider below in this proof the ring $A(X)$ as an MU-algebra through the homomorphism $l_A : MU \to \bar{A}^{uc}$. Let $l_A^n : \mathrm{MU}(\mathbb{C}P^n) \to A(\mathbf{P}^n)$ be the only MU-algebra homomorphism which takes the class $e^{\mathrm{MU}}(\mathcal{O}(1)) = c_1^{\mathrm{MU}}(\mathcal{O}(1))$ to the class $e^A(\mathcal{O}(1)) = c_1^A(\mathcal{O}(1)) \in A(\mathbf{P}^n)$.

The projective bundle theorem for complex cobordism shows that the cup-product identifies the MU-algebras $\mathrm{MU}(\mathbb{C}P^n) \otimes_{\mathrm{MU}} \mathrm{MU}(\mathbb{C}P^n)$ and $\mathrm{MU}(\mathbb{C}P^n \times \mathbb{C}P^n)$. The projective bundle theorem for the theory A shows that the cup-product identifies the MU-algebras $A(\mathbf{P}^n) \otimes_{\bar{A}} A(\mathbf{P}^n)$ and $A(\mathbf{P}^n \times \mathbf{P}^n)$. Set

$$l_A^{n,n} = l_A^n \otimes l_A^n : \mathrm{MU}(\mathbb{C}P^n) \otimes_{\mathrm{MU}} \mathrm{MU}(\mathbb{C}P^n) \to A(\mathbf{P}^n) \otimes_{\bar{A}} A(\mathbf{P}^n)$$

Let $\Delta : \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}P^n$ be the diagonal and let $\Delta : \mathbf{P}^n \to \mathbf{P}^n \times \mathbf{P}^n$ be the diagonal as well (the same symbal is used for both imbeddings). We claim that for every integer $n \geq 0$ the following relation holds in $A(\mathbf{P}^n \times \mathbf{P}^n)$

$$l_A^{n,n}(\Delta_{\mathrm{MU}}(1)) = \Delta_A(1). \tag{60}$$

For the proof of the Theorem we need as this relation so three Lemmas stated just below.

Let $\zeta_{MU,n} = c_1^{MU}(\mathcal{O}(1)) \in MU(\mathbb{C}P^n)$ and let $\zeta_{A,n} = c_1^A(\mathcal{O}(1)) \in A(\mathbf{P}^n)$. Let $p_n : \mathbb{C}P^n \to \{*\}$ be the projection. The same symbol p_n is used below for the projection $\mathbf{P}^n \to pt$.

2.8.2. Lemma. *One has $(p_n)_{MU}(\zeta^r_{MU,n}) = \{\mathbb{C}P^{n-r}\}$. In particular, $(p_n)_{MU}(\zeta^r_{MU,n}) = 1$.*

2.8.3. Lemma. *Let $\Delta : \mathbb{C}P^n \hookrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ be the diagonal imbedding. Identify $MU(\mathbb{C}P^n) \otimes_{MU} MU(\mathbb{C}P^n)$ with $MU(\mathbb{C}P^n \times \mathbb{C}P^n)$ by means of the cup-product. Then one has the relation in $MU(\mathbb{C}P^n \times \mathbb{C}P^n)$*

$$\Delta_{MU}(1) = \zeta^n_{MU,n} \otimes 1 + 1 \otimes \zeta^n_{MU,n} + \Sigma_{i>0,j>0} mu_{i,j} \zeta^i_{MU,n} \otimes \zeta^j_{MU,n}$$

for certain elements $mu(n)_{i,j}$ in $A(pt)$.

2.8.4. Lemma. *Let S and $T = \mathbb{C}P^n$ be smooth projective varieties and let $f : S \to \{*\}$, $g : T \to \{*\}$ be the structural morphisms. Let $p_S : S \times T \to S$ and $p_T : S \times T \to T$ be the projections. Then for any elements $a \in MU(S)$ and $b \in MU(T)$ one has the relation $(p_S)_{MU}((p_S)^{MU})(a) \cup (p_T)^{MU})(b)) = a \cup f_S^{MU}(g_{MU}(b))$. We will write $a \cup g_{MU}(b)$ for the element $a \cup f_S^{MU}(g_{MU}(b))$.*

These three Lemmas holds because the assignment $X \to MU(X)$ is a ring cohomology pretheory on the category $\mathcal{S}m/\mathbb{C}$ and the Thom-Gysin mappings make it an oriented cohomology pretheory such that the corresponding Euler structure is given by $L \mapsto cf(L)$ where (as was mentioned above in this subsection) for a line bundle L the element $cf(L) = c_1^{MU}(L)$ is the Conner-Floyd class of the line bundle L. Thus these three Lemmas follows from Lemmas 1.9.3, 1.9.4 and 1.9.5 respectively.

Assuming for a moment the relation (60) completes the proof of the Theorem as follows.

We proceed induction by m. One has $\{\mathbb{C}P^0\}_{MU} = id_{MU}(1) = 1$ where $id : \{*\} \to \{*\}$ is the identity map of the one point space and similarly $\{\mathbf{P}^0\}_A = id(1) = 1$. So the case $m = 0$ is obvious. Now assume that the relation $l_A(\{\mathbb{C}P^n m\}_{MU}) = \{\mathbf{P}^m\}_A$ holds for all $m < n$ and prove the relation $l_A(\{\mathbb{C}P^n\}_{MU}) = \{\mathbf{P}^n\}_A$.

Set for a minute $\zeta_{MU} = \zeta_{MU,n} \in MU(\mathbb{C}P^n)$ and $\zeta_A = \zeta_{A,n} \in A(\mathbf{P}^n)$. By Lemma 2.8.3 one has the relation in $MU(\mathbb{C}P^n \times \mathbb{C}P^n)$

$$\Delta_{MU}(1) = \zeta^n_{MU} \otimes 1 + 1 \otimes \zeta^n_{MU} + \Sigma_{i>0,j>0} mu^{(n)}_{i,j} \zeta^i_{MU} \otimes \zeta^j_{MU} \tag{61}$$

and one has the relation in $A(\mathbf{P}^n \times \mathbf{P}^n)$ by Lemma 1.9.4

$$\Delta_A(1) = \zeta^n_A \otimes 1 + 1 \otimes \zeta^n_A + \Sigma_{i>0,j>0} a^{(n)}_{i,j} \zeta^i_A \otimes \zeta^j_A \tag{62}$$

for certain elements $mu_{i,j}^{(n)}$ in $A(pt)$ and $a_{i,j}^{(n)}$ in $B(pt)$.

By the very definition of the homomorphism l_A^n one has $l_A^n(\zeta_{MU}) = \zeta_A \in A(\mathbf{P}^n)$ Now the relation 60 proves that

$$l_A(mu_{i,j}^{(n)}) = a_{i,j}^{(n)} \tag{63}$$

Let $pr_1 : \mathbb{C}P^n \times \mathbb{C}P^n \to \mathbb{C}P^n$ be the projection to the 1-st factor. Observe that $pr_1 \circ \Delta = id_{\mathbb{C}P^n}$. Thus applying the Thom-Gysin map $(pr_1)_{MU}$ (that is the push-forward) to the relations (61) and using the composition property of push-forwards in the MU-theory and Lemmas 2.8.2 and 2.8.4 one gets in $MU(\mathbb{C}P^n)$ the relation

$$1 = 1 + \{\mathbb{C}P^n\}_{MU}\zeta_{MU}^n + \Sigma_{i>0,j>0} mu_{i,j}^{(1)}\{\mathbb{C}P^{n-i}\}_{MU}\zeta_{MU}^j$$

The family $\{1, \zeta_{MU}, \dots, \zeta_{MU}^{n-1}\}$ is a free basis of the two-sided MU-module $MU(\mathbb{C}P^n)$. Thus equating coefficients at ζ_{MU}^n in the last relation one gets the following one in MU

$$\{\mathbb{C}P^n\}_{MU} = -mu_{1,n}^{(n)}\{\mathbb{C}P^{n-1}\}_{MU} - mu_{2,n}^{(n)}\{\mathbb{C}P^{n-2}\}_{MU} - \cdots - mu_{n,n}^{(1)}\{\mathbb{C}P^n 0\}_{MU} \tag{64}$$

Simultanously one gets the following relation in $\bar{A}uc$

$$\{\mathbf{P}^n\}_A = -a_{1,n}^{(n)}\{\mathbf{P}^{n-1}\}_A - a_{2,n}^{(n)}\{\mathbf{P}^{n-2}\}_A - \cdots - a_{n,n}^{(1)}\{\mathbf{P}^0\}_A \tag{65}$$

By the inductive hypothesis $l_A(\{\mathbb{C}P^n m\}_{MU}) = \{\mathbf{P}^m\}_A$ for $m < n$ and it was already mentioned that $l_A(mu_{i,j}^{(n)}) = a_{i,j}^{(n)}$ in $\bar{A}^{uc}$. Thus applying the homomorphism $l_A : MU \to \bar{A}^{uc}$ to the relation (64) one gets the required relation $l_A(\{\mathbb{C}P^n\}_{MU}) = \{\mathbf{P}^n\}_A$ in $\bar{A}^{uc}$.

To complete the proof of the Theorem it remains to prove the relation (60). For that we need in three other relation stated just below. Let Q be a rank n vector bundle over $\mathbf{P}^n$ defined by the short exact sequence

$$0 \to \mathcal{O}(-1) \to 1^{n+1} \to Q \to 0$$

and let $Q_{\mathbb{C}}$ be a rank n algebraic vector bundle over $\mathbb{C}P^n$ defined by the short exact sequence

$$0 \to \mathcal{O}(-1) \to \mathbb{C}^{n+1} \to Q_{\mathbb{C}} \to 0.$$

We claim the following three relations

$$\Delta_A(1) = c_n^A(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{66}$$

$$\Delta_{\mathrm{MU}}(1) = c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}})) \in \mathrm{MU}(\mathbb{C}P^n \times \mathbb{C}P^n). \tag{67}$$

$$l_A^{n,n}(c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = c_n^A(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) \in A(\mathbf{P}^n \times \mathbf{P}^n). \tag{68}$$

Assuming for a minute the last three relations one gets a chain of relations

$$\Delta_A(1) = c_n^A(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)) = l_A^{n,n}(c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = l_A^{n,n}(\Delta_{\mathrm{MU}}(1))$$

which proves the relation (60). It remains to prove relations (66), (67), (68).

We begin with the following lemma.

2.8.5. Lemma. *Let F/Y be a rank $m = n+1$ vector bundle and let $s : Y \to \mathbf{P}(F)$ be a section of the natural projection $p : \mathbf{P}(F) \to Y$. Consider the natural inclusion $\mathcal{O}_F(-1) \to p^*(F)$ and let Q be the factor-bundle $p^*(F)/\mathcal{O}_F(-1)$. In $A(\mathbf{P}(F))$ one has*

$$s_A(1) = c_n^A(\mathcal{O}_F(1) \otimes p^*s^*Q) \tag{69}$$

Note that Lemma 2.8.5 holds for the complex cobordism theory as well because as was already mentioned in this proof the assignment $X \mapsto MU(X)$ is the ring cohomology pretheory $MU : \mathcal{S}m/\mathbb{C} \to Rings$ and the Thom-Gysin mappings make it an oriented cohomology pretheory. The corresponding Euler structure is given by $L \mapsto cf(L) \in MU^2(X)$, where $cf(L) = c_1^{MU}(L)$ is the Conner-Floyd class of the line bundle L.

Relations (66) and (67) are particular cases ($Y = \mathbf{P}^n$, $E = 1^{n+1}$, $s = \Delta$, $p = p_1$) of the relation (69) stated in Lemma 2.8.5.

Proof of Lemma 2.8.5. Set $L = s^*(\mathcal{O}_F(-1))$. Then one has the obvious exact sequence of vector bundles on X

$$0 \to L \to F \to F' \to 0.$$

If M is a line bundle over X then replacing F by $F \otimes M$ we does not change as the variety $\mathbf{P}(F)$ so the vector bundle $\mathcal{O}_F(1) \otimes p^*(F')$. Thus one may assume that L is the trivial line bundle. In this case one has an isomorphism $s^*Q \cong F'$ of the vector bundles and it remains to verify the relation

$$s_A(1) = c_n^A(\mathcal{O}_F(1) \otimes p^*(E)), \tag{70}$$

where $E = F'$. By the slitting principle 1.6.1 one may assume further that $F = \mathbf{1} \oplus E$ and even that the vector bundle E is a direct sum of line bundles $E = \oplus_{i=1}^n L_i$.

Let $F_i = \mathbf{1} \oplus L_1 \oplus \cdots \oplus L_i$ and let $s_i : \mathbf{P}(F_{i-1}) \hookrightarrow \mathbf{P}(F_i)$ be the obvious closed imbedding. Then $s = s_n \circ \cdots \circ s_2 \circ s_1 : Y \to \mathbf{P}(F_n) = \mathbf{P}(\mathbf{1} \oplus E)$ and each s_i is a closed imbedding of a smooth divisor. Thus by the normalization formulae 1.1.8 for each integer $i = 1, \ldots, n$ one has the relation $(s_i)_A(1) = e^A(L(D_i)) \in A(\mathbf{P}(F_i))$, where $D_i = \mathbf{P}(F_{i-1})$ is considered as a divisor on $\mathbf{P}(F_i)$ (set $F_0 = \mathbf{1}$). If $p_i : \mathbf{P}(F_i) \to Y$ is the projection then the line bundle $L(D_i)$ over the variety $\mathbf{P}(F_i)$ is isomorphic to the line bundle $\mathcal{O}_{F_i}(1) \otimes p_i^*(L_i)$ which is in turn is isomorphic to the one $(s_n \circ \cdots \circ s_{i+1})^*(\mathcal{O}_E(1)) \otimes p^*(L_i)$. Now using the projection formulae and the Cartan formulae one gets the relation in $A(\mathbf{P}(\mathbf{1} \oplus E))$

$$s_A(1) = (s_n \circ \cdots \circ s_1)_A(1) = c_n^A(\mathcal{O}_E(1) \otimes (\oplus_{i=1}^n p^*(L_i))).$$

Since $E = \oplus_{i=1}^n L_i$ one gets the required relation $s_A(1) = c_n^A(\mathcal{O}_E(1) \otimes p^*(E))$. The Lemma follows.

$\square$

To prove the relation (68) consider the line bundle $\mathcal{O}(1)$ over $\mathbf{P}^\infty$ and consider on $\mathbf{P}^\infty \times \mathbf{P}^n$ the short exact sequence of vector bundles

$$0 \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1)) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}) \to q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q) \to 0.$$

where q_1 and q_2 are the projections of $\mathbf{P}^\infty \times \mathbf{P}^n$ to the first and the second factors respectively. This exact sequence and the Cartan formular for the total Chern class gives rise to the relation in $A(\mathbf{P}^\infty \times \mathbf{P}^n)$

$$c_1^A(q_1^*(\mathcal{O}(1)) \otimes q_2^*(\mathcal{O}(-1))) \cup c_n^A(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = c_{n+1}^A(q_1^*(\mathcal{O}(1)) \otimes q_2^*(1^{n+1}))$$

Now setting $t_1 = c_1(q_1^*(\mathcal{O}(1)))$ and $t_2 = c_1(q_2^*(\mathcal{O}(1)))$ one can rewrite the last relation in terms of the formal group law $F_A(t_1, t_2)$ (58) and its "inverse element" I_A (53) as follows

$$F_A(t_1, I_A(t_2)) \cup c_n^A(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)) = t_1^{n+1} \tag{71}$$

Similarly one gets the following relation in $\mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n)$

$$F_{\mathrm{MU}}(t_1^{\mathrm{MU}}, I_{\mathrm{MU}}(t_2^{\mathrm{MU}})) \cup c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}})) = (t_1^{\mathrm{MU}})^{n+1}$$

Here q_1 and q_2 are the projections of $\mathbb{C}P^\infty \times \mathbb{C}P^n$ to the first and to the second factors respectively, and F_{MU} and I_{MU} are the formal group law and its "inverse element" for the complex cobordism (57). Applying the homomorphism

$$l_A^{\infty,n} = l_A^\infty \otimes l_A^n : \mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n) \to A(\mathbf{P}^\infty \times \mathbf{P}^n)$$

to the last relation in $\mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n)$ one gets in the group $A(\mathbf{P}^\infty \times \mathbf{P}^n)$ the relation

$$F_A(t_1, I_A(t_2)) \cup l_A^{\infty,n}(c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}}))) = t_1^{n+1} \tag{72}$$

Observe that the element $F_A(t_1, I_A(t_2)) = t_1 -_{F_A} t_2$ is not a zero-divisor in the ring $A(\mathbf{P}^\infty \times \mathbf{P}^n) = A[t_2][[t_1]]/(t_2^{n+1})$. Thus the relations (71) and (72) prove the one

$$l_A^{\infty,n}(c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}}))) = c_n^A(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q)).$$

Now consider a commutative diagram

$$\begin{array}{ccc}
\mathrm{MU}(\mathbb{C}P^\infty \times \mathbb{C}P^n) & \xrightarrow{l_A^{\infty,n}} & A(\mathbf{P}^\infty \times \mathbf{P}^n) \\
{\scriptstyle (i\times id)^{\mathrm{MU}}}\downarrow & & \downarrow{\scriptstyle (i\times id)^A} \\
\mathrm{MU}(\mathbb{C}P^n \times \mathbb{C}P^n) & \xrightarrow{l_A^{n,n}} & A(\mathbf{P}^n \times \mathbf{P}^n),
\end{array}$$

in which the vertical arrows are the pull-backs induced by the obvious imbeddings. Since

$$(i \times id)^{\mathrm{MU}}(c_n^{\mathrm{MU}}(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q_{\mathbb{C}}))) = c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))$$

and

$$(i \times id)^A(c_n(q_1^*(\mathcal{O}(1)) \otimes q_2^*(Q))) = c_n^A(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q))$$

the commutativity of the last diagram proves the desired relation (68)

$$l_A^{n,n}(c_n^{\mathrm{MU}}(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q_{\mathbb{C}}))) = c_n^A(p_1^*(\mathcal{O}(1)) \otimes p_2^*(Q)).$$

The relation (60) is proved.

The proof of the Theorem 2.8.1 is completed.

$\square$

References

[A] *M.F. Atiyah.* K-theory and reality, The Quaterly Journal of Math., 17 (1966), 68, 367 - 386.

[Ad1] *J. F. Adams.* Algebraic topology in the last decade, in The selected works of J.Frank Adams, Ed. J.P.May and C.B.Thomas, Cambridge University Press, vol.I, (1992), 515 - 536.

[Ad2] *J. F. Adams.* Stable homotopy and generalised cohomology. University of Chicago Press, 1974.

[AH] *M.F. Atiyah, F. Hirzebruch.* Cohomologie-Operationen und charakteristische Klassen, Math.Zeitschr. 77 (1961), 149-187.

[AW] *J. F. Adams, G.Walker.* On complex Stiefel manifolds, Proc. Camb. Phil. Soc., 61 (1965), 81 - 103.

[Ba] *P. Balmer.* Triangular Witt groups I: The 12-term localization sequence, K-theory, 19 (2000), 311-363.

[Bl] *S. Bloch.* Algebraic cycles and higher K-theory, Adv. Math., 61 (1986), 267-304.

[BFM] *P.Baum, W.Fulton, R.MacPherson.* Riemann-Roch and topological K-theory for singular varieties. Acta Math., 143 (1979), 155-192.

[Br] *P. Brosnan.* Steenrod operations in Chow theory, http://www.math.uiuc.edu/K-theory/0370/, 1999.

[BS] *A. Borel, J.P. Serre.* Le theoreme de Riemann-Roch. BSMF, 86 (1958), 97-136.

[CF] *P.E. Conner, E.E. Floyd.* The relation of cobordism to K-theory. Lect. Notes Math., 28 (1966), Springer-Verlag, Berlin.

[Dy] *E. Dyer.* Relations between cohomology theories. Col. Alg. Topology(1962), 1-10.

[Fu] *W. Fulton.* Intersection theory. Springer-Verlag, 1984.

[FV] *E. Friedlander, V. Voevodsky.* Bivariant cycle cohomology, Cycles, Transfers, and Motivic homology theories, Annals of Math. Studies., 2000, 138-187.

[FW] *E. Friedlander, M. Walker.* Semi-topological K-theory of real varieties, http://www.math.uiuc.edu/K-theory/0453/, 2000.

[GH] *T. Geisser, L.Hesselholt.* Topological cyclic homology of schemes, http://www.math.uiuc.edu/K-theory/0530/, 2001.

[Gi] *H. Gillet.* Riemann-Roch Theorems for Higher Algebraic K-Theory. Advances in Math. 40 (1981), 203-289.

[GN] *S. Gille, A.Nenashev.* Pairings in triangular Witt theory, http://www.math.uiuc.edu/K-theory/0530/, 2001.

[Gra] *D.Grayson.* Adams operations on higher K-theory. *K*-theory 6 (1992), no.2, 97-111.

[Gr] *A. Grothendieck.* Bull.Soc.Math.France, 86 (1958), 136-154.

[Har] *R. Hartshorne.* Algebraic geometry.Graduated texts in Mathematics 52,1977.

[H] *F. Hirzebruch.* Topological methods in algebraic geometry. Die Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, b. 131, 1966.

[J] *J. Jouanolou.* Une suite exacte de Mayer-Vietoris en K-the'orie algebrique, Lect. Notes Math., 341 (1973), 293-316.

[K] *M. Karoubi.* *K*-theory. Grundlehren Der Mathematischen Wissenschaften 226, 1978.

[KW] *M.Karoubi, C.Weibel.*Algebraic and real *K*-theory of real varieties. http://www.math.uiuc.edu/K-theory/0473/, 2001 .

[L] *M. Levine.* Motivic cohomology and algebraic cycles a categorical approach, http://www.math.uiuc.edu/K-theory/0107/, 1996.

[LM] *M. Levine, F. Morel.* Cobordism algebrique I. C.R. Acad. Sci. Paris Se'r. I, Math. 332 (2001).

[Me] *A. Merkurjev.* Algebraic oriented cohomology theories. http://www.math.ucla.edu/K-theory/0535.

[Mi] *J. S. Milne.* Etale Cohomology. Princeton University Press, 1980.

[Mo] *F. Morel.* Theorie homotopique des schemas, Asterisque 256, (1999).

[PS] *I. Panin, A. Smirnov.*Push-forwards in oriented cohomology theories of algebraic varieties. http://www.math.uiuc.edu/K-theory/0459.

[Pu] *O. Pushin.* Higher Chern classes and Steenrod operations in motivic cohomology. http://www.math.uiuc.edu/K-theory/0536/2002.

[PY] *I. Panin, S. Yagunov.* Rigidity for Orientable Functors. Preprint MPI (Bonn) #69, 2000 (to appear in J.Pure and Appl.Algebra).

[Qu1] *D. Quillen.* On the formal group laws of unoriented and complex cobordism theory. Bull. AMS 75(1969), 1293-1298.

[Qu2] *D.Quillen.* Higher algebraic *K*-theory:I. Lect.Notes Math. 341 (1973), 85-147.

[S] *A. Suslin.* Higher Chow groups and etale cohomology, in Cycles, Transfers, and Motivic homology theories, Annals of Math. Studies, 143 (2000), 239-254.

[So] *Ch. Soule.* Operations en K-theory algebrique. Canad. J. Math. 37 (1985), no.3, 488-550.

[SV] *A. Suslin, V. Voevodsky.* Bloch-Kato conjecture and motivic cohomology with finite coefficients. The arithmetics and geometry of algebraic cycles, 117-189, NATO Sci.Ser.C Math.Phys.Sci., 548, Kluver Acad.Publ., Dortrecht, 2000.

[V] *V. Voevodsky.* Triangulated Categories of Motives Over a Field, in Cycles, Transfers, and Motivic homology theories, Annals of Math. Studies., 2000, 188-238.

[V1] *V. Voevodsky.* $\mathbf{A}^1$-Homotopy theory. Doc. Math., Extra Vol. ICM 1998(I), 417-442.

[V2] *V. Voevodsky.* Reduced power operation in motivic cohomology, http://www.math.uiuc.edu/K-theory/0487

[V3] *V. Voevodsky.* Motivic cohomology are isomorphic to higher Chow groups, http://www.math.uiuc.edu/K-theory/0378

[Wa] *C. Walter.* Witt theory of a projective bundle {lecture on mini-course in Muenster, March 2001}.

[W] *C. Weibel.* Products in Higher Chow Groups and Motivic Cohomology. http://www.math.uiuc.edu/K-theory/0263/, 1998.