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EXISTENCE OF LIMITING DISTRIBUTION FOR THE THETA SERIES ON THE SIEGEL HALF SPACE

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ABSTRACT

wLet $m \geq 1$ be an integer. For any Z from Siegel upper half space set

$$\Theta(Z) = \sum_{\overline{n} \in \mathbb{Z}^m} exp(\pi i\, {}^t\overline{n} Z \overline{n}).$$

The function Θ is unchanged under every substitution $X \longmapsto X + P$ where P is a symmetric matrix with integral entries and even diagonal. Therefore, for any $Y > 0$ the function $\Theta_Y(\cdot) = (\det Y)^{1/4}\Theta(\cdot + iY)$ may be viewed as a complex-valued random variable on the torus $\mathbb{T}^{m(m+1)/2}$ with the Haar measure. It is asserted in the main theorem of this note that there exists a weak limit of the distribution of $\Theta_{\tau Y}$ as $\tau \to \infty$ which does not depend on the choice of Y. This theorem is an extension to higher dimension of some known results for $m = 1$. We also establish the rotational invariance of the limiting distribution. The proof of the main theorem makes use of Dani–Margulis' and Ratner's results on dynamics of unipotent flows.

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§1. Introduction

Let $m \geq 1$ be an integer. For every rectangular matrix M we denote by ${}^tM$ the transpose of M. Set

$$\Theta(Z) = \sum_{\overline{n} \in \mathbb{Z}^m} \exp(\pi i\, {}^t\overline{n} Z \overline{n}), \tag{1.1}$$

where $Z = \operatorname{Re} Z + i \operatorname{Im} Z = X + iY$ is an $m \times m$ complex symmetric matrix whose imaginary part Y is positively definite (what is denoted by $Y > 0$). The set $\mathcal{H}_m$ of all such matrices Z is known as the *Siegel upper half space.* The function Θ is clearly unchanged under every substitution $X \longmapsto X + P$ where P is a symmetric matrix with integral entries and even diagonal. Such matrices P form a discrete uniform subgroup of the additive group of real symmetric $m \times m$-matrices with a quotient group isomorphic to the torus $\mathbb{T}^{m(m+1)/2}$. Therefore, for any $Y > 0$ the function $\Theta_Y(\cdot) = (\det Y)^{1/4}\Theta(\cdot + iY)$ may be viewed as a function on the torus and, with respect to the Haar probability measure, Θ_Y may be considered as a random variable taking values in $\mathbb{C}$. The main claim in this note is the following result.

Theorem 4.1. *There exists a weak limit of the distribution of $\Theta_{\tau Y}$ as $\tau \to 0$ which does not depend on the choice of Y. Convergence is uniform in Y from every compact subset of the set of all $m \times m$ positively definite matrices.*

In fact Theorem 4.1 gives also a description of the limiting distribution as a distribution of some random variable on a certain probability space. Our observation (Corollary 4.1) deduced from this description is that the limiting distribution is invariant with respect to rotations.

Theorem 4.1 generalizes some earlier papers where the case $m = 1$ was investigated. In [JVH2] (see also [JVH1] and [JVH3]) existence of a limiting distribution was established by means of the careful study of the asymptotic behavior of $\Theta(X + iY)$ as $Y \to 0$. The diophantine nature of X plays here the cruicial role. The corresponding mathematical technique goes back to Hardy and Littlewood [HL1, HL2].

An alternative approach to the proof can be found in the papers [Ma1, Ma2] by J. Marklof. The conclusion that the limiting distribution exists was deduced in these papers from the fact that a closed orbit of the horocycle flow on the unit tangent bundle of a noncompact finite area surface of constant negative curvature is becoming uniformly distributed with respect to the natural probability measure on this bundle under the action of geodisic flow which carries this closed horocycle out of a cusp. The latter fact is proved in [Ma1] and [Ma2] in the spirit of Sarnak's paper [Sa] where the result is derived from some analytic properties of the appropriate Eisenstein series' (for which properties the reader is refered in [Sa] to a paper of A. Selberg). In a sense this method allows to retell, in terms of the asymptotical behavior of the uniform measure on a moving closed horocycle, the results about the appropriate Eisenstein series'. In particular, more precise asymptotics established in such way can be found in ([Ma1, Ma2]).

In our paper we deal with the case of arbitrary $m \geq 1$ and restrict ourself only with the first term of the asymptotics. The most general plan is the same as in Marklof's papers: roughly, we deduce the main result from the fact that a compact orbit of some $m(m+1)/2$-dimensional unipotent subgroup of $\mathbf{Sp}(2m, \mathbb{R})$ is becoming uniformly distributed under the action of a certain natural flow on $\Gamma \backslash \mathbf{Sp}(2m, \mathbb{R})$ where Γ is a subgroup of finite index in $\mathbf{Sp}(2m, \mathbb{Z})$ (in fact, we deal with the metaplectic group $\mathbf{Mp}(2m, \mathbb{R})$ and its discrete subgroup rather than with $\mathbf{Sp}(2m, \mathbb{R})$ and Γ.) So, as in [Ma1, Ma2], the first step in the proof is a reduction to a certain problem of uniform distribution ("lifting of the theta function"); the second one is to establish this uniform distribution property. However, realization of both steps in the present paper is quite different from the approach in ([Ma1, Ma2]).

In the present paper lifting is done in a straightforward way and applies to any function with some neccessary automorphicity properties, while the corresponding procedure in Marklof's papers is based on the Schrödinger equation and operates with much more special class of automorphic functions. The latter approach seems to be not transportable immediately to higher dimension.

We tried to describe the reduction in rather general (mostly algebraic) terms. Though such reduction seems to be well known to the specialists long ago we are not able to indicate a reference to a source which would give a clear and compact exposition of the procedure described in Section 2 of the present paper. Remark that despite the fact that a lot is known about the multipliers in the "functional equation" (subsection 2.3), we avoid explicit expressions (for example, in terms of the Gauss sums) whenever they are useless in the present context.

The second step is a proof of the asymptotic uniform distribution property. At least two different approaches to this problem are possible.

Firstly, this conclusion may be deduced from rather general mixing properties of the group actions on finite volume homogeneous spaces known in the literature. As these mixing properties belong to functions on the homogeneous space in question, this method suggests some smoothening of a compact horocycle to substitute it by a function. Such approach in the case $m = 1$ is sketched in ([EMM], Theorem 7.1). The corresponding smoothening procedure is contained in the "wave front lemma" (Theorem 3.1 in [EMM]). As to the results on mixing the reader is addressed there to some earlier known results substantially based on the group representation theory. The arguments used in [EMM] for $m = 1$ can be extended to cover arbitrary m.

Secondly (and this is our choice in the present paper), the fact which we need in this paper may be deduced from general results on unipotent flows. Such deduction is contained in Section 3 of this paper. The main reference here is a paper [DM] by Dani and Margulis, also M. Ratner's famous results on Raghunathan conjecture and its measure-theoretical version are used.

The authors expect that for the rather specific situation of the present paper some relatively self-contained proof of this result should be reachable, together with some explicit estimates of the rate of convergence.

Despite we are mostly interested in the action of the metaplectic group $\mathbf{Mp}(2m, \mathbb{R})$ on its homogeneous spaces, we usually start in Section 2 with consideration of the analogous action of $\mathbf{Sp}(2m, \mathbb{R})$. One reason for this is that the latter case can be handled in terms of matrices and is more intuitive. After that we can easily pass to the case of the group $\mathbf{Mp}(2m, \mathbb{R})$. Another reason is that while Θ-series gives us an example of an automorphic form of half integer weight related to $\mathbf{Mp}(2m, \mathbb{R})$, we can be also interested in forms of integer weight related to $\mathbf{Sp}(2m, \mathbb{R})$ to which the exploited approach also can be applied.

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§2. Theta-lift

Our goal in this section is to reimplant the function Θ from $\mathcal{H}_m$ to a new domain where exists a dynamics responsible for the chaotic behavior of Θ on $\mathcal{H}_m$ as the imaginary part of $Z \in \mathcal{H}_m$ goes to 0. This procedure is called here theta-lift. As we shall see, the function Θ is not just passively "lifted" but also serves as a tool when one "lifts" a discrete subgroup of $\mathbf{Sp}(2m, \mathbb{R})$.

We need to settle some terminology which will be used in the sequel. Let G_1 and G_2 be groups acting (for definiteness, on the left) on sets X_1 and X_2, correspondingly, and let $h : G_1 \to G_2$, $\varphi : X_1 \to X_2$ be, respectively, a homomorphism and be a map. We say that these actions are *(h, φ)-compatible* (also *h-compatible*, *φ-compatible*, etc) if for every $x \in X_1$ and $g \in G_1$ we have that $\varphi(gx) = h(g)\varphi(x)$. In particular, if $G_1 = G_2 = G$, $h = \mathrm{Id}_G$, the identical isomorphism of G, and $\varphi : X_1 \to X_2$ is such that the actions of G on X_1 and X_2 are φ-compatible, then the map φ is called *equivariant.* The same terminology will be applied to left actions and even to actions of two group on two sets from different sides (in the latter case an antihomomorphism should be taken as h) and to the situation when φ is a map between certain sets $Y_1 \supset X_1$ and $Y_2 \supset X_2$ such that $\varphi(X_1) \subset X_2$. We call *G-space* a set with an action of a group G.

2.1. The Siegel half space and the group $\mathbf{Sp}(2m, \mathbb{R})$.

By definition, the Siegel half space $\mathcal{H}_m$ consists of all $m \times m$ complex symmetric matrices $Z = \mathrm{Re}\, Z + i\, \mathrm{Im}\, Z$ with $\mathrm{Im}\, Z$, the imaginary part of Z, positively definite. $\mathcal{H}_m$ posses a number of remarkable structures, in partricular, it is a $m(m+1)$-dimensional Riemannian manifold and $m(m+1)/2$- dimensional complex variety on which the symplectic group $\mathbf{Sp}(2m, \mathbb{R})$ acts transitivly respecting these structures. More precisely, every $g \in \mathbf{Sp}(2m, \mathbb{R})$ may be represented in the form of a 2×2 block matrix

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{2.1}$$

with $m \times m$ real entries satisfying

$${}^t\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}. \tag{2.2}$$

The relation (2.2) is equivalent to every of the following two conditions imposed on the entries A, B, C, D of the above block matrix ([Kli], p.1):

$${}^tCA = {}^tAC,\ {}^tBD = {}^tDB,\ {}^tAD - {}^tCB = 1, \tag{2.2$'$}$$

or

$$A\,{}^tB = B\,{}^tA, C\,{}^tD = D\,{}^tC, A\,{}^tD - B\,{}^tC = 1. \tag{2.2$''$}$$

The inverse of the matrix g in (2.1) is given by

$$g^{-1} = \begin{pmatrix} {}^tD & -\,{}^tB \\ -\,{}^tC & tA \end{pmatrix}. \tag{2.3}$$

Then the matrix g defined by (2.1) acts on $\mathcal{H}_m$ by a fractional linear transformation

$$Z \mapsto (AZ + B)(CZ + D)^{-1}. \tag{2.4}$$

2.2. 1-cocycles and the group $\mathbf{Mp}(2m, \mathbb{R})$.

Let G be a group acting on the left on a set H. Recall that a function J on $G \times H$ taking values in a certain group is called a cocycle (more precisely, a 1-cocycle) if for any $g_1, g_2 \in G, Z \in H$ we have

$$J(g_1g_2, Z) = J(g_1, g_2Z)J(g_2, Z).$$

For

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix},\ Z \in \mathcal{H}_m,$$

we set

$$j(g, Z) = \det(CZ + D).$$

A straightforward calculation shows that j is a cocycle. Moreover, it is continuous in $g \in \mathbf{Sp}(2m,\mathbb{R})$ and holomorphic in $Z \in \mathcal{H}_m$ In description of the transformational properties of Θ with respect to certain discrete subgroups of $\mathbf{Sp}(2m,\mathbb{R})$ an important role is played by a cocycle ρ satisfying $\rho^2(g,Z) = j(g,Z)$. Such a contitnuous in g cocycle does not exist in the precise sense. The reason is that the group $\mathbf{Sp}(2m,\mathbb{R})$ is not simply connected (its fundamental group is isomorphic to $\mathbb{Z}$). One needs to pass to a double cover of $\mathbf{Sp}(2m,\mathbb{R})$ in order to obtain an appropriate domain for ρ. The group $\mathbf{Sp}(2m,\mathbb{R})$ has a unique nonsplitting central extension by the group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$. The corresponding exact sequence is

$$0 \to \mathbb{Z}_2 \to \mathbf{Mp}(2m,\mathbb{R}) \to \mathbf{Sp}(2m,\mathbb{R}) \to 1$$

with the metaplectic group $\mathbf{Mp}(2m,\mathbb{R})$ and the covering homomorphism $\pi : \mathbf{Mp}(2m,\mathbb{R}) \to \mathbf{Sp}(2m,\mathbb{R})$. There exists a unique continuous $\mathbf{Mp}(2m,\mathbb{R})$-cocycle ρ such that

$$\rho^2(g,Z) = j(\pi g, Z), \;\; g \in \mathbf{Mp}(2m,\mathbb{R}), Z \in \mathcal{H}_m. \tag{2.5}$$

We describe now a convinient realization of the group $\mathbf{Mp}(2m,\mathbb{R})$, the related extension of $\mathbf{Sp}(2m,\mathbb{R})$ and the cocyle ρ. The case when $m=1$ is considered in detail in ([LM], 1.8.25); the general case exhibits no distinction. $\mathbf{Sp}(2m,\mathbb{R})$ is defined as a set of all pairs (g,d) where d is a holomorphic function on $\mathcal{H}_m$ such that $d^2(Z) = j(g,Z)$. The composition law in $\mathbf{Mp}(2m,\mathbb{R})$ is defined by the relation: $(g_1,d_1)\cdot(g_2,d_2) = (g_1g_2,d)$, where $d(Z) = d_1(g_2Z)d_2(Z)$. We set also $\pi(g,d) = g$. $\mathbf{Mp}(2m,\mathbb{R})$ is a connected Lie group locally isomorphic to $\mathbf{Sp}(2m,\mathbb{R})$ via π. The kernel of π is a cyclic group of order 2. It is contained in the center of $\mathbf{Mp}(2m,\mathbb{R})$ which is a cyclic group of order 4. The cocycle ρ is defined by $\rho((g,d),Z) = d(Z)$. It follows from the definition of the multiplication in $\mathbf{Mp}(2m,\mathbb{R})$ that ρ is a cocycle for $\mathbf{Mp}(2m,\mathbb{R})$, the relation (2.5) follows from the explicit construction of $\mathbf{Mp}(2m,\mathbb{R})$ described above.

2.3. Discrete subgroups related to Θ and the functional equation.

Let (V_{2m},Ω) be a standard symplectic space of dimension $2m$. This means that a symplectic basis $P_1, P_2, \ldots, P_m, Q_1, Q_2, \ldots, Q_m$ is chosen for V so that the skew-symmetric bilinear form Ω satisfies for $i,j = 1,\ldots,m$ the relations $\Omega(P_i,P_j) = 0$, $\Omega(Q_i,Q_j) = 0$, $\Omega(P_i,Q_j) = -\Omega(Q_i,P_j) = \delta_{i,j}$. Let $r = \mathbb{Z}P_1 \oplus \mathbb{Z}P_2 \oplus \cdots \oplus \mathbb{Z}P_m \oplus \mathbb{Z}Q_1 \oplus \mathbb{Z}Q_2 \oplus \cdots \oplus \mathbb{Z}Q_m$ be a lattice in V_{2m}, and let a function $\chi : r \to \{1,-1\}$ is defined by the relation

$$\chi\Big(\sum k_iP_i + \sum l_iQ_i\Big) = (-1)^{\sum k_il_i}.$$

The group $\mathbf{Sp}(2m,\mathbb{R})$ acts on (V_{2m},Ω) by linear symplectic transformations, and its discrete subgroup $\mathbf{Sp}(2m,\mathbb{Z})$ is exactly the group leaving the lattice r stable. Let us consider a subgroup $\Gamma_{1,2}$ of $\mathbf{Sp}(2m,\mathbb{Z})$ leaving the function χ stable. This notation agrees with ([Mum], Sect. 2.5) and goes back to Igusa. The group $\Gamma_{1,2} \subset \mathbf{Sp}(2m,\mathbb{R})$ is a particular case of a subgroup $\Gamma(r,\chi) \in \mathbf{Sp}(2m,\mathbb{R})$ defined in [L,V] for some class of pairs (r,χ). It is easy to verify (see [LV], 2.2.19, or [Mum], Chap.2, Sect.5) that if $g \in \mathbf{Sp}(2m,\mathbb{Z})$ is given by (2.1) then g falls into Γ if and only if the symmetric matrices ${}^tAB$ and ${}^tDC$ have even diagonal entries (or, equivalently, $A\,{}^tB$ and $D\,{}^tC$ have even diagonal entries). It is $\Gamma_{1,2}$ which is involved into the functional equation for Θ as defined by (1.1). We need to pass to some subgroup of $\Gamma_{1,2}$ by two reasons: firstly, we would prefer to deal with a normal subgroup of $\mathbf{Sp}(2m,\mathbb{Z})$ (that is not the case for $\Gamma_{1,2}$) because in this case one of the corresponding homogeneous spaces covers one another one properly. Secondly, we need such subgroup of $\mathbf{Sp}(2m,\mathbb{Z})$ that Θ^2 is an automorphic function of weight 1 with respect to this subgroup (see (2.6) below for the precise formulation). One of the possible choices is the subgroup Γ_4 we define now.

Let us denote by Γ_k (or by $\Gamma_k(m)$ to emphasize the dependence on m) a subgroup of $\mathbf{Sp}(2m,\mathbb{Z})$ consisting of all matrices g such that $g \equiv I_{2n}(\mod k)$, where I_{2m} is the identity matrix (such subgroups are known as *principal congruence subgroups* of $\mathbf{Sp}(2m,\mathbb{Z})$).

We are particularly interested in Γ_4 which posses the following two properties

i) $\Gamma_4 \subset \Gamma_{1,2} \subset \mathbf{Sp}(2m,\mathbb{Z})$ and Γ_4 is a normal subgroup of $\mathbf{Sp}(2m,\mathbb{Z})$ of a finite index;

ii) for any $\gamma \in \Gamma_4, Z \in \mathcal{H}_m$ we have that

$$\Theta^2(\gamma Z) = j(\gamma, Z)\Theta^2(Z). \tag{2.6}$$

Concerning ii) see ([LV], 2.2.24), or ([Mum], Chap. 2, (5.9)), i) follows immediately from the definition of Γ_4.

For convenience of the reader we remind in the rest of the section 2.3 some facts (despite we do not need them in the present paper) about the behavior of Θ under the action of $\Gamma_{1,2}$ on $\mathcal{H}_m$, including the role of "a rather tricky 8^{th} root of 1" ([Mum], p. 189).

The identity (2.6) is a consequence of the "functional equation" for the function Θ ([LV], 2.2.37; [Mum], Chap. 2, (5.1)) The latter equation asserts, in its full version, that for every $Z \in \mathcal{H}_m$ and $\gamma \in \Gamma_{1,2}$ we have that

$$\Theta(\gamma Z) = \xi(\gamma) j(\gamma, Z)^{1/2}\Theta(Z), \tag{2.7}$$

where $\xi(\gamma)$ is a certain 8^{th} root of 1. The latter formula determines $\xi(\gamma)$ up to a sign so that there exists such a character k of the group $\Gamma_{1,2}$ taking values in 4^{th} roots of 1 that

$$\Theta^2(\gamma Z) = k(\gamma) j(\gamma, Z)\Theta^2(Z). \tag{2.8}$$

One can explicitly calculate the character k on a certain subgroup of $\Gamma_{1,2}$ (see [LV], 2.2.24 and further references there). Furthermore, the character k equals identically 1 on the group $\Gamma_4 \subset \Gamma_{1,2}$ (see [LV], 2.2.24) or ([Mum], Chap.2, (5.9)). The latter fact is equivalent to (2.6).

2.4. Theta lifts Γ_4.

The epimorphic homomorphism $\pi : \mathbf{Mp}(2m,\mathbb{R}) \to \mathbf{Sp}(2m,\mathbb{R})$ can not be splitted in the sense that there is no continuous homomrphism $q : \mathbf{Sp}(2m,\mathbb{R}) \to \mathbf{Mp}(2m,\mathbb{R})$ such that $\pi q = \mathrm{Id}_{\mathbf{Sp}(2m,\mathbb{R})}$. Nevertheless, for some subgroups of $\mathbf{Sp}(2m,\mathbb{R})$ a homomorphism with the latter property can be constructed. We consider in this subsection a discrete subgroup possessing this property. Some examples of connected Lie subgroups sharing this feature will occur in the subsection 2.7.

For the discrete subgroup $\Gamma_4 \in \mathbf{Sp}(2m,\mathbb{R})$ we shall describe now a homomrphism $\lambda : \Gamma_4 \to \mathbf{Mp}(2m,\mathbb{R})$ such that $\pi\lambda = \mathrm{Id}_{\Gamma_4}$. Set for $g \in \mathbf{Sp}(2m,\mathbb{R}),\ Z \in \mathcal{H}_m$

$$\theta_g(Z) = \Theta(gZ)/\Theta(Z).$$

It is clear that the map $(g, Z) \mapsto \theta_g(Z)$ is a (trivial) cocycle. Let us define now a map $\lambda : \Gamma_4 \to \mathbf{Mp}(2m,\mathbb{R})$ by the relation

$$\lambda(\gamma) = (\gamma, \theta_\gamma), \gamma \in \Gamma_4. \tag{2.9}$$

It follows from (2.6) that this definition really provides an element of $\mathbf{Mp}(2m,\mathbb{R})$. Furthermore, λ is a homomorphism because the map $(\gamma, Z) \mapsto \theta_\gamma(Z)$ is a Γ_4-cocycle (in fact, even a coboundary). Indeed, for every $\gamma_1,\ \gamma_2 \in \Gamma_4$ we have that

$$\lambda(\gamma_1\gamma_2) = (\gamma_1\gamma_2, \theta_{\gamma_1}(\gamma_2\cdot)\theta_{\gamma_2}(\cdot)) = (\gamma_1, \theta_{\gamma_1})(\gamma_2, \theta_{\gamma_2}) = \lambda(\gamma_1)\lambda(\gamma_2).$$

The relation $\pi\lambda = \mathrm{Id}_{\Gamma_4}$ is an obvious consequence of (2.9), which, in particular, implies that the kernel of λ is trivial. Denote $\lambda(\Gamma_4)$ by $\widehat{\Gamma}_4$. Thus, λ is an isomorphism of Γ_4 onto $\widehat{\Gamma}_4$. Again, it follows from (2.9) that

$$\rho(\lambda(\gamma), Z) = \theta_\gamma(Z)$$

for every $\gamma \in \Gamma_4, Z \in \mathcal{H}_m$. The latter relation implies the following transformation formula for Θ refining (2.6): for every $Z \in \mathcal{H}_m$, $\gamma \in \widehat{G}_4$ we have that

$$\Theta(\pi(\gamma)Z) = \rho(\gamma, Z)\Theta(Z). \tag{2.10}$$

Indeed, we see that

$$\Theta(\pi(\gamma)Z)/\Theta(Z) = \theta_{\pi(\gamma)}(Z) = \rho((\pi(\gamma), \theta_{\pi(g)}), Z) = \rho(\lambda(\pi(\gamma)), Z) = \rho(\gamma, Z).$$

2.5. $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ covers $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$.

The action of the symplectic group $\mathbf{Sp}(2m,\mathbb{R})$ on the Siegel half space $\mathcal{H}_m$ by the formula (2.4) is transitive. The stabilizer of the point $Z_0 = iI_m \in \mathcal{H}_m$ is a compact subgroup of $\mathcal{H}_m$. It is isomorphic to the unitary group $\mathbf{U}(m)$ and consists of all matrices of the form

$$\begin{pmatrix} \operatorname{Re} Q & \operatorname{Im} Q \\ -\operatorname{Im} Q & \operatorname{Re} Q \end{pmatrix}$$

with $Q \in \mathbf{U}(m)$. In view of this embedding we may identify $\mathcal{H}_m$ and $\mathbf{Sp}(2m,\mathbb{R})/\mathbf{U}(m)$ as left $\mathbf{Sp}(2m,\mathbb{R})$-spaces.

The Siegel modular group $\mathbf{Sp}(2m,\mathbb{Z})$ is a discrete subgroup of $\mathbf{Sp}(2m,\mathbb{R})$. It is a *lattice* (that is the homogeneous space $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ has a finite volume induced by the Haar measure on $\mathbf{Sp}(2m,\mathbb{R})$) but not a *uniform* lattice (or subgroup) in $\mathbf{Sp}(2m,\mathbb{R})$ because $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ is noncompact. Both of these assertions may be deduced from the analysis of the (left) action of $\mathbf{Sp}(2m,\mathbb{Z})$ on $\mathcal{H}_m = \mathbf{Sp}(2m,\mathbb{R})/\mathbf{U}(m)$. Indeed, $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ is (a total space of) a principal fiber bundle with the fiber $\mathbf{U}(m)$ over $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathcal{H}_m = \mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})/\mathbf{U}(m)$. As to the quotient space $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathcal{H}_m$, the classical Siegel fundamental domain of finite volume for the action of $\mathbf{Sp}(2m,\mathbb{Z})$ on $\mathcal{H}_m$ is known to give rise to a noncompact quotient space because it has limiting points on the boundary of $\mathcal{H}_m$. More detail, including its explicit description and calculation of the volume (in assumption that $\mathbf{Sp}(2m,\mathbb{Z})$ is supplied with the counting Haar measure), can be found in [Kli]. As $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ is fibred over $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathcal{H}_m = \mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})/\mathbf{U}(m)$ with a compact fiber $\mathbf{U}(m)$, this implies that $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ is a homogeneous space (actually a manifold) of finite invariant measure.

According to 2.2 $\pi\colon \mathbf{Mp}(2m,\mathbb{R}) \to \mathbf{Sp}(2m,\mathbb{R})$ is a two-to-one covering homorphism. Also the homogeneous space $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ is a double cover of the homogeneous space $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ via a map induced by π. More precisely, we define a covering map $\pi_4 : \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R}) \to \Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ as follows. Let $\widehat{\Gamma}_4 g \in \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ be a coset corresponding to a certain $g \in \mathbf{Mp}(2m,\mathbb{R})$. Then we set $\pi_4(\widehat{\Gamma}_4 g) = \Gamma_4\pi(g)$. It is clear that an $\mathbf{Mp}(2m,\mathbb{R})$-space $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ and an $\mathbf{Sp}(2m,\mathbb{R})$-space $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ are (π,π_4)-compatible.

Moreover, as follows from 2.4, $\pi|_{\widehat{\Gamma}_4}$ is an isomorphism onto Γ_4 with inverse isomorphism λ defined by (2.7). Consequently, $\widehat{\Gamma}_4$ has only trivial intersection with the kernel of π. Recall also that the kernel of π is isomorphic to $\mathbb{Z}_2$ and belongs to the center of $\mathbf{Mp}(2m,\mathbb{R})$. Thus, every Γ_4-coset in $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$ is the image under π_4 of exactly two $\widehat{\Gamma}_4$-cosets in $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$ which are interchanged by the action of a nontrivial element in the kernel of π.

As Γ_4 is a normal subgroup of $\mathbf{Sp}(2m,\mathbb{Z})$, the homogeneous space $\Gamma_4\backslash\mathbf{Sp}(2m,\mathbb{R})$, in its turn, covers $\mathbf{Sp}(2m,\mathbb{Z})\backslash\mathbf{Sp}(2m,\mathbb{R})$ with multplicity equal to the index of Γ_4 in $\mathbf{Sp}(2m,\mathbb{Z})$.

2.6. Implementation of theta on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$.

Recall that $Z_0 = iI_m \in \mathcal{H}_m$. Set for $g \in \mathbf{Mp}(2m,\mathbb{R})$

$$\widehat{\Theta}(g) = \Theta((\pi g)Z_0)/\rho(g, Z_0).$$

According to the formula (2.10) for any $\gamma \in \widehat{\Gamma}_4$ and $g \in \mathbf{Mp}(2m,\mathbb{R})$ we have that

$$\begin{aligned}\widehat{\Theta}(\gamma g) &= \Theta(\pi(\gamma g)Z_0)/\rho(\gamma g, Z_0) = \Theta(\pi(\gamma)\pi(g)Z_0)/\rho(\gamma g, Z_0)\\ &= \big(\rho(\gamma,\pi(g)Z_0)\Theta(\pi(g)Z_0)\big)/\rho(\gamma g, Z_0) = \Theta(\pi(g)Z_0)/\rho(g, Z_0)\\ &= \widehat{\Theta}(g).\end{aligned}$$

Thus, $\widehat{\Theta}$ may be considered as a function on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m,\mathbb{R})$.

2.7. Lifting subgroups $\mathbf{B}_+$ and $\mathbf{B}_-$.

Let $\mathbb{GL}^+(m,\mathbb{R})$ be the mutiplicative group of $m\times m$ real matrices with positive determinant, and let $\mathbf{S}_m(\mathbb{R})$ be the additive group of $m\times m$ real symmetric matrices. Let $\mathbf{B}_+$ and $\mathbf{B}_-$ be subgroups of $\mathbf{Sp}(2m,\mathbb{R})$ defined by the relations

$$\mathbf{B}_+ = \left\{\begin{pmatrix} A & B\\ 0 & {}^tA^{-1}\end{pmatrix} : A \in \mathbb{GL}^+(m,\mathbb{R}), B\,{}^tA \in \mathbf{S}_m(\mathbb{R})\right\}, \tag{2.11}$$

$$\mathbf{B}_- = \left\{\begin{pmatrix} A & 0\\ C & {}^tA^{-1}\end{pmatrix} : A \in \mathbb{GL}^+(m,\mathbb{R}), {}^tAC \in \mathbf{S}_m(\mathbb{R})\right\}. \tag{2.12}$$

The groups $\mathbf{B}_+$ and $\mathbf{B}_-$ are connected Lie subgroups in $\mathbf{Sp}(2m,\mathbb{R})$. We shall describe here homomorphic embeddings of subgroups $\mathbf{B}_+$ and $\mathbf{B}_-$ into $\mathbf{Mp}(2m,\mathbb{R})$ which are right inverse of π over these subgroups.

The cocycle j introduced in 2.2 reduces on $\mathbf{B}_+$ to the map

$$j_+\colon\ (g, Z) = \left(\begin{pmatrix} A & B\\ 0 & {}^tA^{-1}\end{pmatrix}, Z\right) \mapsto (\det A)^{-1},$$

which is essentially a positive character of $\mathbf{B}_+$. This character admits a positive square root $g \mapsto (\det A)^{-1/2}$. Denote the corresponding cocycle by $j_+^{1/2}$. Thus, in view of 2.2, we can embed $\mathbf{B}_+$ into $\mathbf{Mp}(2m,\mathbb{R})$ by means of a homomorphism μ_+ defined by the relation

$$\mu_+(g) = (g, j_+^{1/2}(g)),\quad g \in \mathbf{B}_+.$$

We set $\widehat{\mathbf{B}}_+ = \mu_+(\mathbf{B}_+)$. It is clear that $\pi\mu_+ = \mathrm{Id}_{\mathbf{B}_+}$.

An analogous embedding exists also for $\mathbf{B}_-$. The cocycle j introduced in 2.2 reduces on $\mathbf{B}_-$ to the function

$$j_-\colon\ (g, Z) = \left(\begin{pmatrix} A & 0\\ C & {}^tA^{-1}\end{pmatrix}, Z\right) \mapsto \det(CZ + {}^tA^{-1}).$$

For every $g \in \mathbf{B}_-$ $\lim_{Z\to 0} j_-(g, Z) = \det({}^tA^{-1})$, where A is the upper left block of the matrix g. As $\mathcal{H}_m$ is simply connected, for every $g \in \mathbf{B}_-$ there exists $j_-^{1/2}(g,\cdot)$, a unique holomorphic square root of $j_-(g,\cdot)$ with the property that $\lim_{Z\to 0} j_-^{1/2}(g, Z) = (\det A)^{-1/2}$, where $(\det A)^{-1/2}$ is the positive value of the square root of $(\det A)^{-1}$. It is clear that $j_-^{1/2} : (g, Z) \mapsto j_-^{1/2}(g, Z)$

satisfy the cocycle identity up to a sign. That the identity is satisfied completely follows from two facts: $Z = 0$ is a fixed point for the group $\mathbf{B}_-$ acting by (2.4) on the closed upper Siegel half space $\overline{\mathbf{H}}_m = \{Z = X + iY : X, Y$ are $m \times m$ symmetric real matrices, $Y \geqslant 0\}$; the map $g \mapsto \lim_{Z \to 0} j_-^{1/2}(g, Z)$ is a group homomorphism. Thus, we can embed $\mathbf{B}_-$ into $\mathbf{Mp}(2m, \mathbb{R})$ by means of a homomorphism μ_- defined as follows:

$$\mu_-(g) = (g, j_-^{1/2}(g, \cdot)), \;\; g \in \mathbf{B}_+.$$

Again we set $\widehat{\mathbf{B}}_- = \mu_-(\mathbf{B}_-)$ and see that $\pi\mu_- = I_{\mathbf{B}_-}$.

Let us consider now a Lie subgroup $\mathbf{D}$ of $\mathbf{Sp}(2m, \mathbb{R})$ defined by the relation

$$\mathbf{D} = \mathbf{B}_+ \cap \mathbf{B}_-.$$

It consists of matrices having a form

$$d(A) = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix},$$

where $A \in \mathbb{GL}^+(m, \mathbb{R})$. Thus, $d : \mathbb{GL}^+(m, \mathbb{R}) \to \mathbf{Sp}(2m, \mathbb{R})$ is a Lie groups embedding. There are two seemingly different embeddings of $\mathbf{D}$ to $\mathbf{Mp}(2m, \mathbb{R})$: by means of μ_+, and by means of μ_-. In fact, the restrictions of them to $\mathbf{D}$ agree because so are the cocycles $j_+^{1/2}$ and $j_-^{1/2}$. Thus, we can set

$$\mu = \mu_+|_{\mathbf{D}} = \mu_-|_{\mathbf{D}}, \quad \widehat{d} = \mu d, \quad \widehat{\mathbf{D}} = \mu(\mathbf{D}) = \widehat{d}(\mathbb{GL}^+(m, \mathbb{R})).$$

2.8. Horocyclic subgroups in $\mathbf{Sp}(2m, \mathbb{R})$ and $\mathbf{Mp}(2m, \mathbb{R})$.

Let $\mathbf{H}_+$ be a Lie subgroup of $\mathbf{B}_+$ which consists of matrices having a form

$$h_+(B) = \begin{pmatrix} I_m & B \\ 0 & I_m \end{pmatrix},$$

where $B \in \mathbf{S}_m(\mathbb{R})$ (observe that $h_+ : \mathbf{S}_m(\mathbb{R}) \to \mathbf{Sp}(2m, \mathbb{R})$ is a Lie groups embedding). It follows from the relation

$$\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} I_m & F \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} I_m & AF\,{}^tA \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \tag{2.13}$$

(with $A \in \mathbb{GL}^+(m, \mathbb{R}), B\,{}^tA, F \in \mathbf{S}_m(\mathbb{R})$) that $\mathbf{H}_+$ is a normal subgroup in $\mathbf{B}_+$. A special case of (2.13) is

$$\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} I_m & F \\ 0 & I_m \end{pmatrix} = \begin{pmatrix} I_m & AF\,{}^tA \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}, \tag{2.14}$$

where $A \in \mathbb{GL}^+(m, \mathbb{R}), F \in \mathbf{S}_m(\mathbb{R})$ ((2.13), in its turn, may be deduced from (2.14)). Every element of $\mathbf{B}_+$ is a product of uniquely determined elements of $\mathbf{D}$ and $\mathbf{H}_+$ so that

$$\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} \tag{2.15}$$

$$= \begin{pmatrix} I_m & B\,{}^tA \\ 0 & I_m \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} I_m & A^{-1}B \\ 0 & I_m \end{pmatrix}$$

with some $A \in \mathbb{GL}^+(m,\mathbb{R}), B\,{}^tA \in \mathbf{S}_m(\mathbb{R})$. Thus, $\mathbf{B}_+ = \mathbf{D}\mathbf{H}_+ = \mathbf{H}_+\mathbf{D}$. Henceforth, $\mathbf{B}_+$ is a semidirect product of $\mathbf{H}_+$ by $\mathbf{D}$. In particular, $\mathbf{D}$ is a quotient of $\mathbf{B}_+$ by a normal subgroup $\mathbf{H}_+$. It is convinient to rewrite the relations (2.14) and (2.15) in terms of the maps h_+ and d. We obtain that

$$d(A)h_+(F) = h_+(AF\,{}^tA)d(A) \tag{2.16}$$

and

$$\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix} = h_+(B\,{}^tA)d(A) = d(A)h_+(A^{-1}B), \tag{2.17}$$

where $A \in \mathbb{GL}^+(m,\mathbb{R}), B\,{}^tA \in \mathbf{S}_m(\mathbb{R})$. In particular, we see that

$$\mathbf{B}_+ = d(\mathbb{GL}^+(m,\mathbb{R}))h_+(\mathbf{S}_m(\mathbb{R})) = h_+(\mathbf{S}_m(\mathbb{R}))d(\mathbb{GL}^+(m,\mathbb{R})). \tag{2.18}$$

Denote the images of $\mathbf{D}$ and $\mathbf{H}_+$ under μ_+ by $\widehat{\mathbf{D}}$ and $\widehat{\mathbf{H}}_+$, correspondingly. Let us fix denotations $\widehat{h}_+ = \mu_+h_+$ and $\widehat{d} = \mu d$ $(= \mu_+|_{\mathbf{D}}\, d)$ for Lie embeddings of, respectively, $\mathbf{S}_m(\mathbb{R})$ and $\mathbb{GL}^+(m,\mathbb{R})$ into $\mathbf{Mp}(2m,\mathbb{R})$. As μ_+ is a Lie groups isomorphism of $\mathbf{B}_+$ onto $\widehat{\mathbf{B}}_+$ the subgroup $\widehat{\mathbf{H}}_+ = \mu_+(\mathbf{H}_+)$ is normal in $\widehat{\mathbf{B}}_+$. Also any relation in the group $\mathbf{B}_+$ written in terms of h_+ and d transforms automatically into similar relation in the group $\widehat{\mathbf{B}}_+$ written in terms of $\widehat{h}_+$ and $\widehat{d}$. In particular, it follows from (2.16)–(2.18) that for every $A \in \mathbb{GL}^+(m,\mathbb{R}), B\,{}^tA \in \mathbf{S}_m(\mathbb{R})$ we have

$$\widehat{d}(A)\widehat{h}_+(F) = \widehat{h}_+(AF\,{}^tA)\widehat{d}(A), \tag{2.19}$$

$$\mu_+\left(\begin{pmatrix} A & B \\ 0 & {}^tA^{-1} \end{pmatrix}\right) = \widehat{h}_+(B\,{}^tA)\widehat{d}(A) = \widehat{d}(A)\widehat{h}_+(A^{-1}B) \tag{2.20}$$

and

$$\widehat{\mathbf{B}}_+ = \widehat{d}(\mathbb{GL}^+(m,\mathbb{R}))\widehat{h}_+(\mathbf{S}_m(\mathbb{R})) = \widehat{h}_+(\mathbf{S}_m(\mathbb{R}))\widehat{d}(\mathbb{GL}^+(m,\mathbb{R})). \tag{2.21}$$

Thus, we see that $\widehat{\mathbf{B}}_+ = \widehat{\mathbf{D}}\widehat{\mathbf{H}}_+ = \widehat{\mathbf{H}}_+\widehat{\mathbf{D}}$ and that $\widehat{\mathbf{B}}_+$ is a semidirect product of $\widehat{\mathbf{H}}_+$ by $\widehat{\mathbf{D}}$. As above, $\widehat{\mathbf{D}}$ is a quotient of $\widehat{\mathbf{B}}_+$ by a normal subgroup $\widehat{\mathbf{H}}_+$.

All what was written above about $\mathbf{H}_+$, $\mathbf{D}$ and $\mathbf{B}_+$, belongs, with minor changes, to $\mathbf{H}_-$, $\mathbf{D}$ and $\mathbf{B}_-$. Let $\mathbf{H}_-$ be a Lie subgroup of $\mathbf{B}_-$ which consists of matrices, having a form

$$h_-(C) = \begin{pmatrix} I_m & 0 \\ C & I_m \end{pmatrix}, \; C \in \mathbf{S}_m(\mathbb{R}).$$

We set $\widehat{\mathbf{H}}_- = \mu_-(\mathbf{H}_-)$. Again, $\mathbf{H}_-$ $(\widehat{\mathbf{H}}_-)$ is a normal subgroup of $\mathbf{B}_-$ (respectively, of $(\widehat{\mathbf{B}}_-)$), $\mathbf{B}_- = \mathbf{D}\mathbf{H}_- = \mathbf{H}_-\mathbf{D}$ (respectively, $\widehat{\mathbf{B}}_- = \widehat{\mathbf{D}}\widehat{\mathbf{H}}_- = \widehat{\mathbf{H}}_-\widehat{\mathbf{D}}$), and $\mathbf{B}_-$ $(\widehat{\mathbf{B}}_-)$ is a semidirect product of $\mathbf{H}_-$ by $\mathbf{D}$ (respectively, of $\widehat{\mathbf{H}}_-$ by $\widehat{\mathbf{D}}$). Let us set $\widehat{h}_- = \mu_-h_-$. Each of the relations (2.13)–(2.18) have their natural analogs in this situation. We only write down few of them. For every $A \in \mathbb{GL}^+(m,\mathbb{R}), G \in \mathbf{S}_m(\mathbb{R})$ we have that

$$\begin{pmatrix} I_m & 0 \\ G & I_m \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}\begin{pmatrix} I_m & 0 \\ {}^tAGA & I_m \end{pmatrix}, \tag{2.14'}$$

$$h_-(G)d(A) = d(A)h_-(AG\,{}^tA) \tag{2.16'}$$

and

$$\widehat{h}_-(G)\widehat{d}(A) = \widehat{d}(A)\widehat{h}_-(AG\,{}^tA). \tag{2.19'}$$

We also need $(g(s))_{s\in\mathbb{R}}$, a one-parameter subgroup of $\mathbf{D}$, which is defined by

$$g(s) = \begin{pmatrix} I_m \exp(-s) & 0 \\ 0 & I_m \exp s \end{pmatrix},$$

and the commutation relations

$$h_+(B)\ g(s) = g(s)\ h_+(B \exp(2s)), \quad h_-(B) g(s) = g(s)\ h_-(B \exp(-2s)) \tag{2.22}$$

which hold for every $B \in \mathbf{S}_m(\mathbb{R})$ and $s \in \mathbb{R}$. The relations (2.22) are lifted to $\mathbf{Mp}(2m, \mathbb{R})$ in the form

$$\widehat{h}_+(B)\ \widehat{g}(s) = \widehat{g}(s)\ \widehat{h}_+(B \exp(2s)), \quad \widehat{h}_-(B)\widehat{g}(s) = \widehat{g}(s)\ \widehat{h}_-(B \exp(-2s)) \tag{2.23}$$

where $\widehat{g}(s) = \mu(g(s))$, $s \in \mathbb{R}$. It follows from (2.22) that for every $f_+ \in \mathbf{H}_+$ and $f_- \in \mathbf{H}_-$ we have that $g_s\ f_+\ g_{-s} \to I_{2m}$ and $g_{-s}\ f_-\ g(s) \to I_{2m}$ as $s \to \infty$ (I_{2m} stands here for the $2m \times 2m$ identity matrix, the identity of the group $\mathbf{Sp}(2m, \mathbb{R})$). By this reason the groups $\mathbf{H}_+$ and $\mathbf{H}_-$ are called, respectively, the expanding and the contracting *horocyclic subgroups* for the subgroup $(g(s))_{s\in\mathbb{R}}$. Analogously, it follows from (2.16) that $\widehat{\mathbf{H}}_+, \widehat{\mathbf{H}}_- \subset \mathbf{Mp}(2m, \mathbb{R})$ are, respectively, the expanding and the contracting horocyclic subgroups for $(\widehat{g}(s))_{s\in\mathbb{R}}$ converging under conjugation by $\widehat{g}(s)$ to $\widehat{I}_{2m}$, the identity of $\mathbf{Mp}(2m, \mathbb{R})$, as s tends, respectively, to $+\infty$ or to $-\infty$.

2.9. Dynamics on the groups $\mathbf{Sp}(2m, \mathbb{R})$ and $\mathbf{Mp}(2m, \mathbb{R})$ and their homogeneous spaces.

Sending $(g, s) \in \mathbf{Sp}(2m, \mathbb{R}) \times \mathbb{R}$ to $gg(s)$, we define a flow (that is an action of $\mathbb{R}$) on $\mathbf{Sp}(2m, \mathbb{R})$. This flow can be pushed down from $\mathbf{Sp}(2m, \mathbb{R})$ to its factor group by its center $\{I_{2m}, -I_{2m}\}$. The latter flow can be identified with the flow of holomorphic orthonormal frames over the Siegel half space $\mathcal{H}_m$, an extension of the geodesic flow on the unit tangent bundle of $\mathcal{H}_m$. Investigation of frame flows on Riemannian manifolds starts with a paper [Ar] of V. A. Arnold and was continued by M. Brin, M. Gromov, W. Ballmann and others; we do not make use of these results in the present paper. Analogously, the relation $(g, s) \mapsto g\widehat{g}(s)$ for $(g, s) \in \mathbf{Mp}(2m, \mathbb{R}) \times \mathbb{R}$ defines a flow on $\mathbf{Mp}(2m, \mathbb{R})$ generated by the one-parameter subgroup $(\widehat{g}(s))_{s\in\mathbb{R}}$ of $\mathbf{Mp}(2m, \mathbb{R})$. These flows on $\mathbf{Sp}(2m, \mathbb{R})$ and on $\mathbf{Mp}(2m, \mathbb{R})$ agree in the sense that for every $g \in \mathbf{Mp}(2m, \mathbb{R})$ and $s \in \mathbb{R}$ we have that $\pi(g\widehat{g}(s)) = \pi(g)g(s)$. Furthermore, they give rise to flows on right homogeneous spaces of the groups $\mathbf{Sp}(2m, \mathbb{R})$ and $\mathbf{Mp}(2m, \mathbb{R})$. We denote the action of corresponding flows on every homogeneous spaces of the form $\Gamma\backslash\mathbf{Sp}(2m, \mathbb{R})$ or $\widehat{\Gamma}\backslash\mathbf{Mp}(2m, \mathbb{R})$ (where $\Gamma \subset \mathbf{Sp}(2m, \mathbb{R})$ and $\widehat{\Gamma} \subset \mathbf{Mp}(2m, \mathbb{R})$ are some discrete subgroups) as the right multiplication by the corresponding element of $(g(s))_{s\in\mathbb{R}}$ and $(\widehat{g}(s))_{s\in\mathbb{R}}$. For example, for $x \in \Gamma\backslash\mathbf{Sp}(2m, \mathbb{R})$ $xg(s)$ denotes the coset $\Gamma gg(s)$ if $x = \Gamma g$ with obvious changes for $(\widehat{g}(s))_{s\in\mathbb{R}}$. We shall call any of these actions a *quasigeodesic flow*. From the point of view of the differentiable dynamics, all these flows belong (for $m > 1$) to the class of *partially hyperbolic dynamical systems*.

As was explained in 2.5, the isomorphisms $\lambda : \Gamma_4 \to \widehat{\Gamma}_4$ allows us to construct from π a covering map $\pi_4 : \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R}) \to \Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R})$ of mutiplicity 2. Because the actions by the left and right mutiplications commute, we have for every $s \in \mathbb{R}$ and $x \in \widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R})$ that

$$\pi_4(x\widehat{g}(s)) = \pi_4(x)g(s).$$

Exactly as in the case of $(g(s))_{s\in\mathbb{R}}$ and $(\widehat{g}(s))_{s\in\mathbb{R}}$, we introduce an action of the group $\mathbf{S}_m(\mathbb{R})$ on the homogeneous space $\Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R})$ by the relation $(x, F) \to xh_+(F)$ for $(x, F) \in \Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R}) \times \mathbf{S}_m(\mathbb{R})$. Analogously, we introduce a similar action of $\mathbf{S}_m(\mathbb{R})$ on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R})$

by the relation $(x, F) \to x\widehat{h}_+(F)$ for $(x, F) \in \widehat{\Gamma}_4 \backslash \mathbf{Mp}(2m, \mathbb{R}) \times \mathbf{S}_m(\mathbb{R})$. Each of these actions will be called the *expanding horocycle* action. These actions are connected by the relation

$$\pi_4(x\widehat{h_+}(F)) = \pi_4(x)h_+(F), \; x \in \widehat{\Gamma}_4 \backslash \mathbf{Mp}(2m, \mathbb{R}), \; F \in \mathbf{S}_m(\mathbb{R}).$$

We skip completely analogous denotations and properties for *contracting horocycle* actions. From now on we set $\mathcal{M}_4 = \Gamma_4 \backslash \mathbf{Sp}(2m, \mathbb{R})$ and $\widehat{\mathcal{M}_4} = \widehat{\Gamma}_4 \backslash \mathbf{Mp}(2m, \mathbb{R})$. We also supply the homogeneous spaces $\mathcal{M}_4$ and $\widehat{\mathcal{M}_4}$ with probability measures P_4 and $\widehat{P}_4$, produced from the Haar measures on $\mathbf{Sp}(2m, \mathbb{R})$ and $\mathbf{Mp}(2m, \mathbb{R})$ by norming. The measures P_4 and $\widehat{P}_4$ are invariant under the action of $\mathbf{Sp}(2m, \mathbb{R})$ and $\mathbf{Mp}(2m, \mathbb{R})$, respectively, and π_4 sends $\widehat{P}_4$ to P_4.

Remark 2.1. The flow induced by the one-peremeter subgroup $g(s)$ on $\mathbf{Sp}(2m, \mathbb{R})$ can be pushed down to the factor of $\mathbf{Sp}(2m, \mathbb{R})$ by its center $\{I_{2m}, -I_{2m}\}$. The latter flow can be identified with the flow of holomorphic orthonormal frames over the Siegel half space $\mathcal{H}_m$, an extension of the geodesic flow on the unit tangent bundle of $\mathcal{H}_m$. Investigation of frame flows on Riemannian manifolds starts with a paper [Ar] of V. A. Arnold and was continued by M. Brin, M. Gromov, W. Ballmann and others; we do not make use of these results in the present paper.

2.10. Action of $\mathbf{B}_+$ on $\mathcal{H}_m$ and compact horocycles in Siegel cylinders ("left horocycles").

Let us note that the action of $\mathbf{B}_+$ on $\mathcal{H}_m$ by the formula (2.4) is transitive. To establish this we consider the group $\mathbf{H}_+$ operating on $\mathcal{H}_m$ by the same formula, as a subgroup of $\mathbf{B}_+$. This action is free and reduces to translations of $\mathcal{H}_m$ parallel to the "real direction." The $\mathbf{H}_+$-orbit of a point $X + iY \in \mathcal{H}_m$ consists of all matrices $X' + iY$ with X' real symmetric. Next we observe that the subgroup $\mathbf{H}_+$ is normal in $\mathbf{B}_+$ with the quotient group $\mathbf{D}$. This implies that every element of $\mathbf{D}$ moves every $\mathbf{H}_+$-orbit onto another $\mathbf{H}_+$-orbit. Thus, to see transitivity of $\mathbf{B}_+$ on $\mathcal{H}_m$ it suffices to prove that $\mathbf{D}$ acts transitively on the set of $\mathbf{H}_+$-orbits in $\mathcal{H}_m$, or even to establish that there exists such a $\mathbf{D}$-orbit which intersects every $\mathbf{H}_+$-orbit. The latter fact follows from considering a $\mathbf{D}$-orbit of the point $Z_0 = iI_m$. Actually, we see that for every $A \in \mathbb{GL}^+(m, \mathbb{R})$ $d(A)Z_0 = iA\,{}^tA$ which implies that the $\mathbf{D}$-orbit of Z_0 consists of all $m \times m$ matrices iY with $Y > 0$, and we are done.

Remark 2.2. The action of the group $\mathbf{B}_+$ on $\mathcal{H}_m$ is not free for $m > 1$. Indeed, the stabilizer of the point Z_0 for this action consists of all matrices of the form $d(A)$ with $A \in \mathbf{SO}(m, \mathbb{R})$, the group of $m \times m$ real orthogonal matrices with determinant 1. $\square$

If we are given an action of $\mathbf{Sp}(2m, \mathbb{R})$ or $\mathbf{Mp}(2m, \mathbb{R})$ on a certain space, a *horocycle* is, by definition, an orbit of the group $\mathbf{H}_+$ with respect to this action. More precisely, we should speak in this case of an *expanding horocycle*, calling an oribt of $\mathbf{H}_-$ a *contracting horocycle*. However, omitting the adjective in the rest of the present section should not lead to any ambiguity because we concentrate now on orbits of $\mathbf{H}_+$.

It is somewhat more convinient for us to consider, instead of actions of $\mathbf{H}_+$ and $\widehat{\mathbf{H}}_+$, equivalent actions of the group $\mathbf{S}_m(\mathbb{R})$ as defined in 2.9 by means of the group isomorhisms h_+ and $\widehat{h}_+$. These actions of $\mathbf{S}_m(\mathbb{R})$ are tacitly ment by "$\mathbf{S}_m(\mathbb{R})$-actions" in the rest of this section. In particular, an $\mathbf{S}_m(\mathbb{R})$-orbit in $\mathcal{M}_4$ (in $\widehat{\mathcal{M}_4}$) is the same as an $\mathbf{H}_+$-orbit (respectively, an $\widehat{\mathbf{H}}_+$-orbit). For every $A \in \mathbb{GL}^+(m, \mathbb{R})$ we define an automorphism of the group $\mathbf{S}_m(\mathbb{R})$ by the relation $i_A(F) = A^{-1}F\,{}^tA^{-1}$, $F \in \mathbf{S}_m(\mathbb{R})$.

Compact left horocycles may be associated with arbitrary uniform subgroup N of the group $\mathbf{H}_+$. As $\mathbf{H}_+$ is abelian and $N \subset \mathbf{H}_+$, the action of $\mathbf{H}_+$ on $\mathcal{H}_m$ gives rise to a left action of $\mathbf{H}_+$

on the space $N\backslash\mathcal{H}_m$, the *Siegel cylinder* of N. The stationary subgroup of every point in $N\backslash\mathcal{H}_m$ is N. Hence, any $\mathbf{H}_+$-orbit in $N\backslash\mathcal{H}_m$, as a homogeneous $\mathbf{H}_+$-space, is isomorphic to $N\backslash\mathbf{H}_+$, or, equivalently, to $N'\backslash\mathbf{S}_m(\mathbb{R})$, where $N' = h_+^{-1}(N)$, that is, to the torus $\mathbb{T}^{m(m+1)/2}$ considered as an $\mathbb{R}^{m(m+1)/2}$-space. In particular, the only $\mathbf{H}_+$-invariant probability measure on an $\mathbf{H}_+$-orbit in $N\backslash\mathcal{H}_m$ is that one which comes, in view of this isomorphism, from the probability Haar measure on the torus $\mathbb{T}^{m(m+1)/2}$. Thus, the Siegel cylinder $N\backslash\mathcal{H}_m$ is a union of compact horocycles, and each of them is supplied with a unique $\mathbf{H}_+$-invariant probability measure.

As $N \subset \mathbf{H}_+$, the space of $\mathbf{H}_+$-orbits in the Siegel cylinder $N\backslash\mathcal{H}_m$ is the same as the space of $\mathbf{H}_+$-orbits in $\mathcal{H}_m$, on which the group $\mathbf{D}$ acts transitively. Thus we have a transitive action of $\mathbf{D}$ on the space of $\mathbf{H}_+$-orbits in $N\backslash\mathcal{H}_m$, despite that there is no natural projection of the action of $\mathbf{D}$ on $\mathcal{H}_m$ by the formula (2.4) to the analogous action on $N\backslash\mathcal{H}_m$. Nevertheless, the action of $\mathbf{D}$ on $\mathbf{H}_+$-orbits $N\backslash\mathcal{H}_m$ can be refined to a certain action of $\mathbf{D}$ on $N\backslash\mathcal{H}_m$ respecting the $\mathbf{H}_+$-orbit structure. For example, set for every $A \in \mathbb{GL}^+(m,\mathbb{R})$ and $Z = X + iY \in \mathcal{H}_m$

$$\varkappa_{d(A)}(Z) = X + iAY\,{}^tA. \tag{2.25}$$

This formula defines an action of $\mathbf{D}$ on $N\backslash\mathcal{H}_m$ (or on $\mathcal{H}_m$) which sends $\mathbf{H}_+$-orbits onto $\mathbf{H}_+$-orbits. Such action has nothing to do with the formula (2.4) and does not preserve Riemannian and complex structures of the manifolds under consideration. However, the map $\varkappa_{d(A)}$ sends the $\mathbf{H}_+$-orbit of $Z = X + iY \in \mathcal{H}_m$ onto the $\mathbf{H}_+$-orbit of $\varkappa_{d(A)}(Z)$ so that the two corresponding $\mathbf{S}_m(\mathbb{R})$-spaces are $(i_A, \varkappa_{d(A)})$-compatible where i_A is an automorphism of the group $\mathbf{S}_m(\mathbb{R})$ defined above. Moreover, the map $\varkappa_{d(A)}$ transports the $\mathbf{H}_+$-invariant probability measures on the first orbit to that on the second one. The latter conclusion follows from unicity of such measure and the fact that for φ-compatible spaces φ carries an invariant measure to an invariant one.

We are going to take as N one of the following two discrete subgroups of $\mathbf{H}_+$ connected with discrete subgroups of $\mathbf{Sp}(2m,\mathbb{R})$:

$$N_4 = \Gamma_4 \cap \mathbf{H}_+, \quad N_{1,2} = \Gamma_{1,2} \cap \mathbf{H}_+.$$

Both of N_4 and $N_{1,2}$ are uniform subgroups of $\mathbf{H}_+$. These groups and their position in $\mathbf{H}_+$ are completely determined, up to an isomorphism, by describing the groups $N_4' = h_+^{-1}(N_4)$ and $N_{1,2}' = h_+^{-1}(N_{1,2})$, as subgroups of $\mathbf{S}_m(\mathbb{R})$. It is clear that N_4' is the additive group of $m \times m$ integral symmetric matrices with entries congruent to $0 \mod 4$, and $N_{1,2}'$ is the additive group of integral symmetric matrices with even diagonal entries. The groupes N_4 and N_4' play an important role in setting and describing correspondence between compact horocycles related to $\mathcal{H}_m$ ("left horocycles") which were just discussed and those in $\mathcal{M}_4$ or in $\widehat{\mathcal{M}}_4$ ("right horocycles") which will be considered in 2.11.

Comparing the $\mathbf{H}_+$-spaces $N_4\backslash\mathcal{H}_m$ and $N_{1,2}\backslash\mathcal{H}_m$ we see that the latter space is covered by the former one with multplicity equal to the index of N_4 as a subgroup of $N_{1,2}$. The latter index is equal to that of N_4' as a subgroup of $N_{1,2}'$, that is to $2m^2$. This covering map is compatible with the action of $\mathbf{H}_+$, sends every $\mathbf{H}_+$-orbit in $N_4\backslash\mathcal{H}_m$ to an $\mathbf{H}_+$-orbit in $N_{1,2}\backslash\mathcal{H}_m$, and, in view of the above remark on invariant measures on compact $\mathbf{H}_+$-orbits in the Siegel cylinder, also sends the invariant probability measure on every $\mathbf{H}_+$-orbit in $N_4\backslash\mathcal{H}_m$ to the analogous measure on the image of this orbit. Moreover, actions of the group $\mathbf{D}$ by the formula (2.25) on two Siegel cylinders $N_4\backslash\mathcal{H}_m$ and $N_{1,2}\backslash\mathcal{H}_m$ are compatible with this covering map.

2.11. Compact horocycles in $\mathcal{M}_4$ and $\widehat{\mathcal{M}}_4$ ("right horocycles").

Every point of the space $\mathcal{M}_4$ is a coset of the form $\Gamma_4 g$ for some $g \in \mathbf{Sp}(2m,\mathbb{R})$. We shall see that the set $x_0\mathbf{B}_+ \subset \mathcal{M}_4$, the $\mathbf{B}_+$-orbit of the point $x_0 = \Gamma_4$, is a union of compact orbits of the group $\mathbf{H}_+$. Analogously, in the space $\widehat{\mathcal{M}}_4$ we set $\widehat{x}_0 = \widehat{\Gamma}_4$ and shall consider the $\widehat{\mathbf{B}}_+$-orbit of the point $\widehat{x}_0$ which is also a union of compact orbits of $\widehat{\mathbf{H}}_+$. Remark that $\pi_4(\widehat{x}_0) = x_0$ and that π_4 maps $\widehat{x}_0\widehat{\mathbf{B}}_+$ onto $x_0\mathbf{B}_+$ so that the actions of $\widehat{\mathbf{B}}_+$ on $\widehat{x}_0\widehat{\mathbf{B}}_+$ and the action of $\mathbf{B}_+$ on $x_0\mathbf{B}_+$ are $(\mu_+^{-1}, \pi_4|_{\widehat{\mathbf{B}}_+})$-compatible.

For every $A \in \mathbb{GL}^+(m,\mathbb{R})$ we define maps $\varphi_A : x_0\mathbf{H}_+ \to x_0 d(A)\mathbf{H}_+$ and $\widehat{\varphi}_A : \widehat{x}_0\widehat{\mathbf{H}}_+ \to \widehat{x}_0\widehat{d}(A)\widehat{\mathbf{H}}_+$ by the relations $\varphi_A(x_0h(F)) = x_0h(F)d(A)$ and $\widehat{\varphi}_A(\widehat{x}_0\widehat{h}(F)) = \widehat{x}_0\widehat{h}(F)\widehat{d}(A)$, respectively. As in 2.10, for every $A \in \mathbb{GL}^+(m,\mathbb{R})$ an automorphism i_A of the group $\mathbf{S}_m(\mathbb{R})$ is defined by the relation $i_A(F) = A^{-1}F\,{}^tA^{-1}, F \in \mathbf{S}_m(\mathbb{R})$.

Proposition 2.1.

i) *The set $x_0\mathbf{B}_+$ is a union of $\mathbf{S}_m(\mathbb{R})$-orbits; the action of the group $\mathbf{D}$ on $x_0\mathbf{B}_+$ preserves the partition of $x_0\mathbf{B}_+$ into $\mathbf{S}_m(\mathbb{R})$-orbits; $\mathbf{D}$ acts transitively on the set of $\mathbf{S}_m(\mathbb{R})$-orbits in $x_0\mathbf{B}_+$.*

ii) *Every $\mathbf{S}_m(\mathbb{R})$-orbit in $x_0\mathbf{B}_+$ is of the form $x_0d(A)\mathbf{H}_+(= x_0\mathbf{H}_+d(A))$ with some $A \in \mathbb{GL}^+(m,\mathbb{R})$; stabilizer of every point from this orbit is the group $i_A(N_4') = A^{-1}N_4'\,{}^tA^{-1} \subset \mathbf{S}_m(\mathbb{R})$; as an $\mathbf{S}_m(\mathbb{R})$-space, this orbit is isomorphic to the $\mathbf{S}_m(\mathbb{R})$-space $A^{-1}N_4'\,{}^tA^{-1}\backslash \mathbf{S}_m(\mathbb{R})$; both of these spaces are compact.*

iii) *For every $A \in \mathbb{GL}^+(m,\mathbb{R})$ the $\mathbf{S}_m(\mathbb{R})$-spaces $x_0\mathbf{H}_+$ and $x_0d(A)\mathbf{H}_+$ are (i_A,φ_A)-compatible.*

iv) *Every $\mathbf{S}_m(\mathbb{R})$-orbit in $x_0\mathbf{B}_+$ carries a unique $\mathbf{S}_m(\mathbb{R})$-unvariant probability measure; the map φ_A transports such measure from the $\mathbf{S}_m(\mathbb{R})$-orbit $x_0\mathbf{H}_+$ to the analogous measure on $\mathbf{S}_m(\mathbb{R})$-orbit $x_0d(A)\mathbf{H}_+$.*

Moreover, for the homogeneous space $\widehat{\mathcal{M}}_4$ the assertions $\widehat{\text{i}})$–$\widehat{\text{iv}})$ are valid which are completely analogous to those i)–iv) where $\mathbf{B}_+$, $\mathbf{H}_+$, $\mathbf{D}$, x_0, d, φ_A and the $\mathbf{S}_m(\mathbb{R})$-action via h_+ are substituted by $\widehat{\mathbf{B}}_+$, $\widehat{\mathbf{H}}_+$, $\widehat{\mathbf{D}}$, $\widehat{x}_0$, $\widehat{d}$, $\widehat{\varphi}_A$ and the $\mathbf{S}_m(\mathbb{R})$-action via $\widehat{h}_+$, respectively.

Finally, the $\widehat{\mathbf{B}}_+$-space $\widehat{x}_0\widehat{\mathbf{B}}_+$ and the $\mathbf{B}_+$-space $x_0\mathbf{B}_+$ are $(\mu_+^{-1}, \pi_4|_{\widehat{x}_0\widehat{\mathbf{B}}_+})$-compatible, and the map $\pi_4|_{\widehat{x}_0\widehat{\mathbf{B}}_+}$ sends every $\mathbf{S}_m(\mathbb{R})$-orbit in $\widehat{x}_0\widehat{\mathbf{B}}_+$ to an $\mathbf{S}_m(\mathbb{R})$-orbit in $x_0\mathbf{B}_+$ in an equivariant and measure preserving way.

Proof. We start with proving the assertions $\widehat{\text{i}})$–$\widehat{\text{iv}})$. Let us introduce a discrete subgroup of $\mathbf{Mp}(2m,\mathbb{R})$:

$$\widehat{N}_4 = \widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+.$$

Remind that $N_4 = \Gamma_4 \cap \mathbf{H}_+$ and $h_+^{-1}(N_4) = N_4'$.

Lemma 2.1. *The following relations between μ_+, λ and subgroups N_4, $\widehat{N}_4$ and N_4' take place:*

$$\mu_+|_{N_4} = \lambda|_{N_4} \tag{2.26}$$

and

$$\widehat{N}_4 = \mu_+(N_4), \quad \widehat{h}_+^{-1}(\widehat{N}_4) = N_4'. \tag{2.27}$$

Proof of the lemma. The map π is one-to-one having been restricted either to $\widehat{\Gamma}_4$ or to $\widehat{\mathbf{H}}_+$, with $\pi(\widehat{\Gamma}_4) = \Gamma_4$, $\pi(\widehat{\mathbf{H}}_+) = \mathbf{H}_+$ and inverse maps λ and μ_+, correspondingly. This implies that $\widehat{N}_4 \subset \mu_+(N_4)$ because $\widehat{N}_4 = \widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+ = \mu_+(\pi(\widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+)) \subset \mu_+(\pi(\widehat{\Gamma}_4) \cap \pi(\widehat{\mathbf{H}}_+)) = \mu_+(\Gamma_4 \cap \mathbf{H}_+) = \mu_+(N_4)$.

Next we verify that λ restricted to $\Gamma_4 \bigcap \mathbf{H}_+$ agrees with μ_+. This is equivalent to the assertion that the cocycles involved into the definitions of the maps λ and μ_+ coincide having been restricted to $\Gamma_4 \cap \mathbf{H}_+$. In fact both of them are equal to 1. This is clearly seen for μ_+ from its definition in 2.7. As to λ, the corresponding cocycle was defined in 2.4 as a map $(g, Z) \to \Theta(gZ)/\Theta(Z)$. Remark that $h_+(B)Z = Z + B, B \in \mathbf{S}_m(\mathbb{R})$, and that $\Theta(Z + P) = \Theta(Z)$ if P is a symmetric integral matrix with even diagonal entries that is the case if $P \in h_+^{-1}(\Gamma_4 \cap \mathbf{H}_+) = N_4'$. Thus, (2.26) is proved.

Now we see that $\mu_+(\Gamma_4 \bigcap \mathbf{H}_+) \subset \widehat{\Gamma}_4 \bigcap \widehat{\mathbf{H}}_+$ because the maps $\mu_+ : \mathbf{H}_+ \to \widehat{\mathbf{H}}_+$ and $\lambda : \Gamma_4 \to \widehat{\Gamma}_4$ are ono-to-one and agree on $\Gamma_4 \bigcap \mathbf{H}_+$. This establishes the first of the relations (2.27). We also have that $\widehat{h}_+^{-1}(\widehat{N}_4) = h_+^{-1}(\mu_+^{-1}(\widehat{N}_4)) = h_+^{-1}(N_4) = N_4'$. $\square$

As $\mathbf{S}_m(\mathbb{R})$-orbit in $\widehat{x}_0 \widehat{\mathbf{B}}_+$ is the same as $\widehat{\mathbf{H}}_+$-orbit, $\widehat{\text{i}}$) follows from the facts that $\widehat{\mathbf{B}}_+$ is a semidirect product of $\widehat{\mathbf{H}}_+$ by $\widehat{\mathbf{D}}$ and from transitivity of action of $\widehat{\mathbf{B}}_+$ on $\widehat{x}_0 \widehat{\mathbf{B}}_+$.

In $\widehat{\text{ii}}$), the first assertion is a consequence of normality of the inclusion $\widehat{\mathbf{H}}_+ \subset \widehat{\mathbf{B}}_+$ and transitivity in $\widehat{\text{i}}$). It follows from abelianity of $\mathbf{S}_m(\mathbb{R})$ and transitivity of its action on its orbit that the stabilizer is the same for every point within an orbit of $\mathbf{S}_m(\mathbb{R})$. By the first assertion in $\widehat{\text{ii}}$), every $\mathbf{S}_m(\mathbb{R})$-orbit neccessarily contains a point of the form $\widehat{x}_0 \widehat{d}(A)$, and a matrix $F \in \mathbf{S}_m(\mathbb{R})$ belongs to the stabilizer of $\widehat{x}_0 \widehat{d}(A)$ if and only if there exists such $k \in \widehat{\Gamma}_4$ that

$$\widehat{d}(A)\widehat{h}_+(F) = k\widehat{d}(A).$$

We see that by (2.19)

$$k = \widehat{d}(A)\widehat{h}_+(F)(\widehat{d}(A))^{-1} = \widehat{h}_+(AF\,{}^t\!A),$$

which implies that $k \in \widehat{h}_+(\mathbf{S}_m(\mathbb{R})) = \widehat{\mathbf{H}}_+$. Thus F stabilizes $\widehat{x}_0\widehat{d}(A)$ if and only if $\widehat{h}_+(AF\,{}^t\!A) \in \widehat{\Gamma}_4 \cap \widehat{\mathbf{H}}_+ = \widehat{N}_4$, or, equivalently, if and only if $AF\,{}^t\!A \in \widehat{h}_+^{-1}(\widehat{N}_4) = N_4'$, or, finally, if and only if $F \in A^{-1}\widehat{h}_+^{-1}(\widehat{N}_4)\,{}^t\!A^{-1} = A^{-1}N_4'\,{}^t\!A^{-1}$, and description of the stationary subgroup in $\widehat{\text{ii}}$) is established. Transitive actions of every abelian group are classified by their stationary subgroups, hence the isomorphicity of two $\mathbf{S}_m(\mathbb{R})$-spaces in $\widehat{\text{ii}}$) is proved. Compactness follows from the facts that N_4' is a uniform subgroup in $\mathbf{S}_m(\mathbb{R})$ and that the latter property is preserved by the automorphism i_A, which completes the proof of $\widehat{\text{ii}}$).

To prove $\widehat{\text{iii}}$) it suffices to see that for every F_0, $F \in \mathbf{S}_m(\mathbb{R})$ we obtain from (2.19) that

$$\widehat{\varphi}_A(\widehat{x}_0\widehat{h}_+(F_0)\widehat{h}_+(F)) = \widehat{x}_0\widehat{h}_+(F_0)\widehat{h}_+(F)\widehat{d}(A)$$

$$= \widehat{x}_0\widehat{h}_+(F_0)\widehat{d}(A)\widehat{h}_+(A^{-1}F\,{}^t\!A^{-1}) = \widehat{\varphi}_A(\widehat{x}_0\widehat{h}_+(F_0))\widehat{h}_+(i_A(F)).$$

Unicity of invariant probability measure on every $\widehat{\mathbf{H}}_+$-orbit in $\widehat{x}_0\widehat{\mathbf{B}}_+$ is a consequence of $\widehat{\text{ii}}$) and unicity of the Haar measure on the torus $A^{-1}N_4'\,{}^t\!A^{-1}\backslash\mathbf{S}_m(\mathbb{R})$.

The last assertion of the proposition is a direct consequence of $(\pi, \widehat{\pi})$-compatibility of the $\mathbf{Mp}(2m, \mathbb{R})$-space $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R})$ and the $\mathbf{Sp}(2m, \mathbb{R})$-space $\Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R})$ established in 2.5. As a consequence, we get compatibility of the actions of $\mathbf{S}_m(\mathbb{R})$ and $\mathbb{GL}^+(m, \mathbb{R})$ on $\widehat{\Gamma}_4\backslash\mathbf{Mp}(2m, \mathbb{R})$ and on $\Gamma_4\backslash\mathbf{Sp}(2m, \mathbb{R})$. Now the latter fact together with the assertions $\widehat{\text{i}}$) - $\widehat{\text{iv}}$) imply i)–iv). $\square$

2.12. Correspondence between left and right horocycles.

We have seen in 2.11 that the set $x_0\mathbf{B}_+ \subset \mathcal{M}_4$, the $\mathbf{B}_+$-orbit of the point $x_0 \in \mathcal{M}_4$, is a union of compact $\mathbf{S}_m(\mathbb{R})$-orbits; analogously, the $\widehat{\mathbf{B}}_+$-orbit of the point $\widehat{x}_0$ in the space $\widehat{\mathcal{M}}_4$ is a union of compact $\mathbf{S}_m(\mathbb{R})$-orbits; the group $\mathbb{GL}^+(m,\mathbb{R})$ acts transitively on the set of $\mathbf{S}_m(\mathbb{R})$-orbits of both of these $\mathbf{S}_m(\mathbb{R})$-actions; actions of $\widehat{\mathbf{B}}_+$ and $\mathbf{B}_+$ on the orbits of $\widehat{x_0}$ and x_0, respectively, are $(\mu_+^{-1}, \pi_4|_{\widehat{\mathbf{B}}_+})$-compatible. Let $Z = X + iY \in \mathcal{H}_m$ and let $Y^{1/2}$ be a unique positive definite square root of Y. Define a map $\psi : \mathcal{H}_m \to \mathbf{B}_+$ by the relation

$$\psi(Z) = h(X)d(Y^{1/2}).$$

The map ψ satisfies the relation

$$\psi(Z)Z_0 = Z, \quad Z \in \mathcal{H}_m,$$

(remind that $Z_0 = iI_m \in \mathcal{H}_m$). Another important property of ψ is that it is $\mathbf{S}_m(\mathbb{R})$-equivariant respective to the action of $\mathbf{S}_m(\mathbb{R})$ on the left on $\mathcal{H}_m$ and on $\mathbf{B}_+$. In particular, if $Z' = X' + iY$ is such an element of $\mathcal{H}_m$ that $Z' - Z \in N_4'$ then $\psi(Z') = h(X')d(Y^{1/2}) = h(X'-X)h(X)d(Y^{1/2}) \in \Gamma_4\psi(Z)$. Thus, ψ defines a map of the Siegel cylinder $N_4\backslash\mathcal{H}_m$ to the $\mathbf{B}_+$-orbit of $x_0 \in \mathcal{M}_4$. Denote this map by ψ_4. Completely analogously we define maps

$$\widehat{\psi}(\cdot) = \mu_+(\psi(\cdot))$$

and

$$\widehat{\psi}_4(\cdot) = \widehat{x_0}\widehat{\psi}(\cdot)$$

from $\mathcal{H}_m$ to $\widehat{\mathbf{B}}_+$ and from $N_4\backslash\mathcal{H}_m$ to $\widehat{x}_0\widehat{\mathbf{B}}_+ \subset \widehat{\mathcal{M}}_4$, respectively. For one-parameter groups $(g(s))_{s\in\mathbb{R}}$ and $(\widehat{g}(s))_{s\in\mathbb{R}}$ we call $\varkappa$-actions the actions on $N_4\backslash\mathcal{H}_m$ defined by $(s, Z) \mapsto \varkappa_{g(s)}(Z)$ and $(s, Z) \mapsto \varkappa_{\pi(\widehat{g}(s))}(Z)$, correspondingly.

Proposition 2.2.

i) *Each $\mathbf{S}_m(\mathbb{R})$-orbit in $N_4\backslash\mathcal{H}_m$ is compatible with its ψ_4-image in $\mathcal{M}_4$. More precisely, the $\mathbf{S}_m(\mathbb{R})$-orbit of a point of the form $X + iY + N_4' \in N_4\backslash\mathcal{H}_m$ is $(i_{Y^{1/2}}, \psi_4)$-compatible, as a $\mathbf{S}_m(\mathbb{R})$-space, with the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $x_0 g(Y^{1/2}) \in \mathcal{M}_4$. Invariant probability measures on these $\mathbf{S}_m(\mathbb{R})$-orbits are transported by ψ_4 one to another.*

ii) *The map ψ_4 is equvariant relative to the $\varkappa$-action of the one-parameter group $(g(s))_{s\in\mathbb{R}}$ on $N_4\backslash\mathcal{H}_m$ and the action of $(g(s))_{s\in\mathbb{R}}$ on $\mathcal{M}_4$.*

iii) *For every $s \in R$ and $Z = X + iY + N_4' \in N_4\backslash\mathcal{H}_m$ the map ψ_4 sends the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $\varkappa_{g(s)}(Z) \in N_4\backslash\mathcal{H}_m$ to the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $x_0 d(Y^{1/2})g(s)$.*

Quite analogously, the similar assertions $\widehat{\text{i}})$–$\widehat{\text{iii}})$ hold true for $\mathbf{Mp}(2m,\mathbb{R})$ and related groups, actions and homogeneuos spaces.

Proof. First we verify compatibility in i). For every $Z = X + iY + N_4' \in N_4\backslash\mathcal{H}_m$ and $F \in \mathbf{S}_m(\mathbb{R})$ we have that

$$\begin{aligned}
&\psi_4(h(F)(Z)) \\
&= \psi_4(X + F + iY + N_4') = x_0 h(X + F)d(Y^{1/2}) = x_0 h(X)d(Y^{1/2})h(Y^{-1/2}F\,{}^tY^{-1/2}) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \psi_4(Z)h(i_{Y^{1/2}}(F)).
\end{aligned}$$

Both $\mathbf{S}_m(\mathbb{R})$-orbits are compact transitive spaces with unique probability invariant measures, and ψ_4 must transport one such measure to another because it preserves invariance.

Now we establish ii). If $Z = X + iY + N_4' \in N_4\backslash\mathcal{H}_m$ then we see that

$$\psi_4(\varkappa_{g(s)}(Z)) = \psi_4(X + i\exp(-2s)Y + N_4')$$

$$= x_0 h(X) d(\exp(-s)Y^{1/2}) = x_0 h(X) d(Y^{1/2}) g(s) = \psi_4(Z) g(s).$$

Thus we proved that for every $s \in \mathbb{R}$, $Z \in N_4\backslash\mathcal{H}_m$ the relation

$$\psi_4(\varkappa_{g(s)}(Z)) = \psi_4(Z) g(s)$$

holds, that is ψ_4 is an equivariant map relative to the actions under consideration.

Finally, the $\mathbf{S}_m(\mathbb{R})$-orbit of $Z = X + iY + N_4'$ contains the point $iY + N_4'$ which image under ψ_4 is $x_0 d(Y^{1/2})$. Thus iii) is a direct consequence of i) and ii). We omit completely analogous proof of the assertions $\widehat{\text{i}}$)–$\widehat{\text{iii}}$). $\square$

2.13. Comparing Θ and $\widehat{\Theta}$.

For any $m \times m$ matrix $Y > 0$ we define a function Θ_Y on $\mathbf{S}_m(\mathbb{R})$ by the relation

$$\Theta_Y(X) = (\det Y)^{1/4} \Theta(\cdot + iY), \quad X \in \mathbf{S}_m(\mathbb{R}).$$

As was remarked in the Introduction the function Θ_Y is invariant with respect to the group $N_{1,2}' \subset \mathbf{S}_m(\mathbb{R})$ and may be considered as a function on the torus $N_{1,2}'\backslash\mathbf{S}_m(\mathbb{R})$.

Proposition 2.3. *Let $Z = X + iY \in \mathcal{H}_m$. Then the distribution of the function $\Theta_Y(\cdot)$ with respect to the Haar probability measure on $N_{1,2}'\backslash\mathbf{S}_m(\mathbb{R})$ is the same as the distribution of the function $\widehat{\Theta}$ on the right horocycle in $\widehat{\mathcal{M}}_4$ containing $\widehat{\psi}_4(Z + N_4')$ and supplied with the $\mathbf{S}_m(\mathbb{R})$-invariant probability measure.*

Proof. As $N_4' \subset N_{1,2}' \subset \mathbf{S}_m(\mathbb{R})$, Θ_Y may be also considered as a function on the torus $N_4'\backslash\mathbf{S}_m(\mathbb{R})$ which covers $N_{1,2}'\backslash\mathbf{S}_m(\mathbb{R})$. It is clear that the distribution of Θ_Y with respect to the probability Haar measures on these tori is the same because the covering map is measure preserving. Thus, to establish the proposition we may substitute $N_{1,2}' \subset \mathbf{S}_m(\mathbb{R})$ by $N_4'\backslash\mathbf{S}_m(\mathbb{R})$ in the statement.

The function Θ_Y on $N_4'\backslash\mathbf{S}_m(\mathbb{R})$ is just a restriction to the left horocycle containing $iY + N_4'$ of the function

$$Z = X + iY + N_4' \mapsto (\det Y)^{1/4} \Theta(X + iY),$$

defined on the Siegel cylinder $N_4\backslash\mathcal{H}_m$. To complete the proof, we shall show that this function can be identified with $\widehat{\Theta}(\widehat{\psi}_4(\cdot))$. According to the subsection 2.6, for $g \in \mathbf{Mp}(2m, \mathbb{R})$

$$\widehat{\Theta}(\widehat{x}_0 g) = \Theta((\pi g) Z_0)/\rho(g, Z_0).$$

Then, in view of the definition of the functions $\widehat{\psi}_4$ and $\widehat{\psi}$, we see that for $Z = X + iY + N_4'$

$$\widehat{\Theta}(\widehat{\psi}_4(Z)) = \Theta(\pi(\widehat{\psi}(Z))Z_0)/\rho(\widehat{\psi}(Z), Z_0) = \Theta(Z)/\rho(\widehat{\psi}(Z), Z_0).$$

Taking in view, together with definition of the map μ_+ in 2.7, the relations $\widehat{\psi}(Z) = \mu_+(\psi(Z)) \in \widehat{\mathbf{B}}_+$ and $\psi(Z) = h(X) d(Y^{1/2})$, we conclude that

$$\rho(g, Z_0) = (\det Y)^{-1/4}.$$

Thus,

$$\widehat{\Theta}(\widehat{\psi}_4(X+iY+N_4')) = (\det Y)^{1/4}\Theta(X+iY),$$

which proves the proposition. $\square$

Remark 2.3. Let us notice that we can take instead of Γ_4 and $\widehat{\Gamma}_4$ any pair of lattices $\Gamma \subset \mathbf{Sp}(2m,\mathbb{R})$ and $\widehat{\Gamma} \subset \mathbf{Mp}(2m,\mathbb{R})$ such that $\pi|_{\widehat{\Gamma}}$ is an one-to-one map of $\widehat{\Gamma}$ onto Γ and that Γ is a normal subgroup of finite index in $\mathbf{Sp}(2m,\mathbb{Z})$. Then $\widehat{\Gamma}\backslash\mathbf{Mp}(2m,\mathbb{R})$ is a double covering $\Gamma\backslash\mathbf{Sp}(2m,\mathbb{R})$ via a map induced by π which we denote by $\pi_{\widehat{\Gamma}}$. All the considerations in sections 2.5, 2.9, and 2.10 (except for the last part involving the subgroup $\Gamma_{1,2}$), 2.11 and 2.12 applies to this more general case without any changes. $\square$

§3. Uniform distribution of closed horocycles

In the following theorem we deal with the family of closed orbits for the action of the group $\widehat{\mathbf{H}}_+$ on the space $\widehat{\mathcal{M}_4}$ considered in Proposition 2.1. For every $A \in \mathbb{GL}^+(m,\mathbb{R})$ let us denote by $\mathcal{O}(A)$ the compact $\widehat{\mathbf{H}}_+$-orbit $\widehat{x}_0\widehat{d}(A)\widehat{\mathbf{H}}_+ \subset \widehat{\mathcal{M}_4}$.

Theorem 3.1. *Let us denote by $\widehat{P}_4$ the $\mathbf{Mp}(2m,\mathbb{R})$-invariant probability measure on the homogeneous space $\widehat{\mathcal{M}_4}$ and, for every $A \in \mathbb{GL}^+(m,\mathbb{R})$, by $\widehat{P}_4(A)$ the $\widehat{\mathbf{H}}_+$-invariant probability measure on the compact $\widehat{\mathbf{H}}_+$-orbit $\mathcal{O}(A)$. Then $\widehat{P}_4(\exp(-s)A)$ weakly converges to $\widehat{P}_4$ as $s \to +\infty$. This convergence is uniform in A over compact subsets of $\mathbb{GL}^+(m,\mathbb{R})$.*

Let G be a connected Lie group and let Γ be a discrete subgroup in G. For any closed subgroup W of G let $\mathcal{S}(W)$ denotes the set of all $x \in \Gamma\backslash G$ for which there exists a proper closed subgroup H of G containing W such that Hx admits a finite invariant measure (under this condition Hx is automatically a proper closed subset of $\Gamma\backslash G$). Set $\mathcal{G}(W) = \Gamma\backslash G - \mathcal{S}(W)$.

We need some more concepts considered in [DM].Because we consider in this section group actions on the right rather on the left as in [DM], some realtions look differently in the present paper.

For any closed subgroups H and W og G let $X(H,W) = \{g \in G : gW \subset Hg\}$. For any proper closed subgroup H such that $H \cap \Gamma$ is a lattice in H, $\Gamma\backslash\Gamma X(H,W)$ is contained in $\mathcal{S}(W)$. Let us denote by $\mathcal{H}$ the calss of all proper closed connected subgroups H of G such that $H \cap \Gamma$ is a lattice in H and $\mathrm{Ad}(H \cap \Gamma$ is Zarisski-dense in $\mathrm{Ad}\, H$ where Ad denotes the adjoint representation og G.

One of the main tools in the proof of the Theorem 3.1 is the following result by Dani and Margulis ([DM], Theorem 2).

Proposition 3.1. *Let G be a connected Lie group and let Γ be a lattice in G. Let μ be the G-invariant probability measure on $\Gamma\backslash G$. Let $\left(u^{(n)}(\cdot)\right)_{n\geq 1}$ be a sequence of* Ad*-unipotent one-parameter subgroups converging to an* Ad*-uinipotent one-parameter subgroup $u(\cdot)$: that is, $u^{(n)}(t) \to u(t)$ for all $t \in \mathbb{R}$. Let $\{x_n\}$ be a sequence in $\Gamma\backslash G$ converging to a point in $\mathcal{G}(\{u(t)\})$ and let $\{T_n\}$ be a sequence in $\mathbb{R}^+$, tending to infinity. Then for any bounded continuous function φ on $\Gamma\backslash G$,*

$$\frac{1}{T_n}\int_0^{T_n} \varphi(u^{(n)}(t)\ x_n)dt \to \int_{\Gamma\backslash G} \varphi d\mu.$$

Lemma 3.1. *There exists such $p > 0$ that for every $A \in \mathbb{GL}^+(m,\mathbb{R})$ there exist points x^A, $z^A \in \widehat{\mathcal{M}_4}$ such that*

1) $x^A \in \mathcal{O}(A\widehat{g}(0))$,

2) z^A *is a p-periodic point for the quasigeodesic flow (that is* $z^A\widehat{g}(p) = z^A$,
3) $x^A\widehat{g}(np) \to z^A$ *as* $n \to +\infty$.

Proof of Lemma 3.1. Choose a matrix

$$\gamma_0 = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix} \in \Gamma_4(1)$$

with eigenvalues $0 < \lambda^{-1} < 1 < \lambda$ (for example, we may set $\gamma_{11} = 13, \gamma_{12} = 8, \gamma_{21} = 8, \gamma_{22} = 5$). Then the matrix

$$\gamma = \gamma_0 \otimes I_m = \begin{pmatrix} \gamma_{11}I_m & \gamma_{12}I_m \\ \gamma_{21}I_m & \gamma_{22}I_m \end{pmatrix}$$

is an element of Γ_4. Let the matrix $r \in \mathbf{Sl}(2, \mathbb{R})$ diagonalizes γ_0 so that we have

$$\gamma_0 r = rd$$

where

$$d = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

The matrix r can be represented as follows:

$$r = \begin{pmatrix} r_{11} & r_{12} \\ r_{13} & r_{22} \end{pmatrix} = r^{(+)}r^{(0)}r^{(-)} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}.$$

Set

$$R = r \otimes I_m, D = d \otimes I_m,$$

$$R^{(+)} = r^{(+)} \otimes I_m, R^{(0)} = r^{(0)} \otimes I_m, R^{(-)} = r^{(-)} \otimes I_m.$$

Then we have

$$RD = \gamma R$$

and

$$R = R^{(+)}R^{(0)}R^{(-)}.$$

In denotations of the Section 2 these relation can be rewritten as

$$Rd(\lambda^{-1}) = \gamma R$$

and

$$R = h_+(uI_m)d(kI_m)h_-(vI_m).$$

In view of sections 2.4 and 2.7 the first of the latter relations implies that

$$\widehat{R}\widehat{d}(\lambda^{-1}) = ql(\gamma)\widehat{R}$$

with $\widehat{R}$ defined by the relation

$$\widehat{R} = \widehat{h}_+(uI_m)\widehat{d}(kI_m)\widehat{h}_-(vI_m).$$

and with q lying in the center of $\mathbf{Mp}(2m,\mathbb{R})$ and satisfying $q^2=1$. Then we see that

$$\widehat{R}\widehat{d}(\lambda^{-2})=l^2(\gamma)\widehat{R}.$$

It follows that

$$\widehat{h}_+(uI_m)\widehat{g}(2n\log\lambda)=\widehat{h}_+(uI_m)\widehat{d}(kI_m)\widehat{h}_-(vI_m)\widehat{h}_-(-vI_m)\widehat{d}(-kI_m)\widehat{g}(2n\log\lambda)$$

$$=\widehat{R}\widehat{g}(\log k)\widehat{g}(2n\log\lambda))\widehat{h}_-(-v\lambda^{-2n}k^{-1}I_m)=l^{2n}(\gamma)\widehat{R}\widehat{g}(\log k)\widehat{h}_-(-v\lambda^{-2n}k^{-1}I_m).$$

Multiplying on the right by $\widehat{d}(A)$, we obtain by (2.16′) that

$$\widehat{h}_+(uI_m)\widehat{d}(A)\widehat{g}(2n\log\lambda)=\gamma^{2n}\widehat{R}\widehat{d}(k^{-1}A)\widehat{h}_-(-v\lambda^{-2n}k^{-1}A\,{}^tA).$$

Passing to $\widehat{\Gamma}_4$-cosets we conclude that

$$\widehat{\Gamma}_4\widehat{h}_+(uI_m)\widehat{d}(A)\widehat{g}(2n\log\lambda)=\widehat{\Gamma}_4\widehat{R}\widehat{d}(k^{-1}A)\widehat{h}_-(-v\lambda^{-2n}k^{-1}A\,{}^tA)\to\widehat{\Gamma}_4\widehat{R}d(k^{-1}A)$$

as $n\to\infty$.

Thus we can set $p=2\log\lambda$, $z^A=\widehat{\Gamma}_4\widehat{R}d(k^{-1}A)$ and $x^A=\widehat{h}_+(uI_m)\widehat{d}(A)\in\mathcal{O}(A)$. □

Lemma 3.2. *Let $z\in\widehat{\mathcal{M}}_4$ be a periodic point for the quasigeodesic flow on $\widehat{\mathcal{M}}_4$. Then the set $z\widehat{\mathbf{H}}_+$ is dense in $\widehat{\mathcal{M}}_4$.*

Proof of Lemma 3.2. The subgroup $\widehat{\mathbf{H}}_+$ of $\mathbf{Mp}(2m,\mathbb{R})$ is *horosherical* relative to $\widehat{g}(\cdot)$ that is

$$\widehat{\mathbf{H}}_+=\{g\in\mathbf{Mp}(2m,\mathbb{R}):\widehat{g}(t)g\widehat{g}(-t)\to I_{\mathbf{Mp}(2m,\mathbb{R})}\text{ as }t\to+\infty\}.$$

As Z is periodic the semiorbit $\{z\widehat{g}(t),t\to 0\}$ is bounded. The group $\widehat{\mathbf{H}}_+$ acts on $\widehat{\mathcal{M}}_4$ ergodically (this follows, for example, from Theorem 7 [Mo2]), and the assertion of the lemma follows from theorem 7 in [Da]. □

Lemma 3.3. *Let $z\in\widehat{\mathcal{M}}_4$ be a periodic point for the quasigeodesic flow on $\widehat{\mathcal{M}}_4$. Then there exists such an one-parameter unipotent subgroup $\{u_o(\cdot)\}$ in $\widehat{\mathbf{H}}_+$ that $z\in\mathcal{G}(W)$.*

Proof of Lemma 3.3. We shall use denotations introduced in this section after the statement of Theorem 3.1 meaning $\mathbf{Mp}(2m,\mathbb{R})$ by G and $\widehat{\Gamma}_4$ by Γ.

Choose $g\in G$ such that $z=\widehat{\Gamma}_4 g$. Suppose that an one-parameter unipotent group $u\in H_+$ is such that $z\in\mathcal{S}(u)$. By proposition 2.3 in [DM], for some $H\in\mathcal{H}$ $z\in\widehat{\Gamma}_4\backslash\widehat{\Gamma}_4X(H,u)$ or, equivalently, $u\subset g^{-1}Hg$.

Thus, $u\subset g^{-1}Hg\cap\widehat{\mathbf{H}}_+$. We assert that $g^{-1}Hg\cap\widehat{\mathbf{H}}_+$ is a proper subgroup of $\widehat{\mathbf{H}}_+$. If not it follows from Lemma 3.2 that the orbit $zg^{-1}Hg$ is dense in $\widehat{\mathcal{M}}_4$. On the other hand, $zg^{-1}Hg=\Gamma_4Hg$ is a proper subset of $\widehat{\mathcal{M}}_4$: up to a right translation it is a closed orbit Γ_4H of H, a proper subgroup of $\mathbf{Mp}(2m,\mathbb{R})$, and we arrive to a contradiction.

Let us note now that $\mathcal{H}$ is at most countble by the Proposition 2.1 in [DM]. Thus $\{u:x\in\mathcal{S}(u)\}$ is contained in a countable union of proper subgroups of $\widehat{\mathbf{H}}_+$, and we choose any one-parameter subgroup from its complement in this group. □

Proof of Theorem 3.1. Let us fix a compact subset $K \in \mathbb{GL}^+(m,\mathbb{R})$. We need to prove that for every bounded continuous function φ on $\widehat{\mathcal{M}}_4$ the relation

$$\int_{\mathcal{O}(\exp(-s)A)} \varphi \; d\widehat{P}_4(A) \to \int_{\widehat{\mathcal{M}}_4} \varphi \; d\widehat{P}_4$$

holds uniformly in $A \in K$ as $s \to +\infty$. If it is not so there exists a bounded continuous function φ, a sequence $(A_n)_{n\geq 1}$ of matrices from K and an increasing sequence $(s_n)_{n\geq 1}$ of positive numbers tending to infinity such that for some $\delta > 0$ the relation

$$\left| \int_{\mathcal{O}(\exp(-s_n)A_n)} \varphi \; d\widehat{P}_4(\exp(-s_n)A_n) - \int_{\widehat{\mathcal{M}}_4} \varphi \; d\widehat{P}_4 \right| \geq \delta$$

holds for all $n \geq 1$. We may and shall assume (passing to a subsequence if necessary) that for some $A \in K$ $A_n \to A$.

According to Lemma 3.1 and Lemma 3.3 there exist a sequence $(x_n)_{n\geq 1}$, a point $z \in \widehat{\mathcal{M}}_4$ and an one-parameter subgroup $(u(t))_{t\in\mathbb{R}}$ of the group $\widehat{\mathbf{H}}_+$ such that $x_n \in \mathcal{O}(\exp(-s_n)A)$ for every $n \geq 1$, $x_n \to z$ and $z \in \mathcal{G}(u(\cdot))$. Each x_n has a form $x_n = \widehat{\Gamma}\widehat{g}(s_n)h_n A$ where $h_n \in \widehat{\mathbf{H}}_+$. For every $n \geq 1$ set $x_n' = \widehat{\Gamma}\widehat{g}(s_n)h_n A_n$. Then

$$x_n' = \widehat{\Gamma}\widehat{g}(s_n)h_n A_n = \widehat{\Gamma}\widehat{g}(s_n)h_n A(\widehat{g}(s_n)h_n A)^{-1}\widehat{g}(s_n)h_n A_n$$

$$= x_n A^{-1} A_n \to z.$$

It follows from the points $\widehat{\text{ii}}$), $\widehat{\text{iv}}$) of Proposition 2.1 that the action of $(u(t))_{t\in\mathbb{R}}$ on $\mathcal{O}(\exp(-s_n)A_n)$ with the invarinat measure $\widehat{P}_4(\exp(-s)A_n)$ is just flow of translations on the torus with Haar measure on which the whole group $\widehat{\mathbf{H}}_+$ acts transitively by translations. A generic one-parameter subgroup of $\widehat{\mathbf{H}}_+$ acts on $\mathcal{O}(\exp(-s_n)A_n)$ ergodically, and we can find such a sequence $(u_n(\cdot))_{n\geq 1}$ of one-parameter subgroups of $\widehat{\mathbf{H}}_+$ that for every $n \geq 1$ u_n acts on $\mathcal{O}(\exp(-s_n)A_n)$ ergodically and $u_n \to u$ as $n \to \infty$. Let us choose a positive real sequence $(\epsilon_n)_{n\geq 1}$ tending to 0. Applying ergodic theorem to every dynamical system $(\mathcal{O}(\exp(-s_n)A_n),\ u_n(\cdot), \widehat{P}_4(\exp(-s)A_n)$ we may construct a sequence $(T_n)_{n\geq 1}$ such that $T_n \to \infty$ and for every $n \geq 1$ we have

$$\left| \int_0^{T_n} \varphi(u^{(n)}(t)x_n') \; dt - \int_{\mathcal{O}(\exp(-s_n)A_n)} \varphi \; d\widehat{P}_4(\exp(-s_n)A_n) \right| < \epsilon_n.$$

On the other hand, from Proposition 3.1 (with (x_n') instead of (x_n)) we obtain that

$$\int_0^{T_n} \varphi(u^{(n)}(t)x_n') \; dt \to \int_{\widehat{\mathcal{M}}_4} \varphi \; d\widehat{P}_4.$$

From the latter two relations we conclude that

$$\int_{\mathcal{O}(\exp(-s_n)A_n)} \varphi \; d\widehat{P}_4(\exp(-s_n)A_n) \to \int_{\widehat{\mathcal{M}}_4} \varphi \; d\widehat{P}_4$$

as $n \to \infty$, and we arrived to a contradiction. $\square$

§4. Existence of limiting distribution

The main result of this section and the whole paper is the following theorem.

Theorem 4.1. *Let the function Θ_Y be defined, as in 2.12, by the relation*

$$\Theta_Y(X) = (\det Y)^{1/4}\Theta(\cdot + iY), \quad X \in \mathbf{S}_m(\mathbb{R}).$$

Then the distribution of $\Theta_{\tau Y}(\cdot)$ with respect to the Haar probability measure on $N'_{1,2}\backslash\mathbf{S}_m(\mathbb{R})$ converges weakly as $\tau \to 0$ to the distribution of the function $\widehat{\Theta}$ with respect to the $\mathbf{Mp}(2m,\mathbb{R})$-invariant probability measure $\widehat{P}$ on the homogeneous space $\widehat{\mathcal{M}}_4$. The convergence is uniform in Y on compact subsets of the set of all $m \times m$ positive definite matrices.

Not so much is known about the properties of the limiting distribution. The following corollary reduces the problem of the study of the limiting distribution to the study of its radial part.

Corollary 4.1. *The limiting distribution in Theorem 3.1 is rotation invariant.*

Proof. According to the section 2.6

$$\widehat{\Theta}(g) = \Theta((\pi g)Z_0)/\rho(g, Z_0)$$

for any $g \in \mathbf{Mp}(2m,\mathbb{R})$. Let us take now $u \in \pi^{-1}(\mathbf{U}(m))$ where the unitary group $\mathbf{U}(m)$ is supposed to be embedded into $\mathbf{Sp}(2m,\mathbb{R})$ as a stabilizer of the point $Z_0 = iI_m \in \mathcal{H}_m$ (see section 2.5). Notice that for any $u \in \pi^{-1}(\mathbf{U}(m))$ the function $\widehat{\Theta}(\cdot u)$ has the same distribution as $\widehat{\Theta}$ with respect to the invariant probability measure $\widehat{P}$. On the other hand in view of the cocycle property of ρ we see that

$$\rho(gu, Z_0) = \rho(g, \pi(u)Z_0)\rho(u, Z_0) = \rho(g, Z_0)\rho(u, Z_0)$$

and

$$\widehat{\Theta}(gu) = \widehat{\Theta}(g)/\rho(u, Z_0).$$

By (2.5) and definition of cocycle j we have

$$\rho^2(u, Z_0) = j(\pi u, Z_0) = \det(i\,\mathrm{Im}\,\pi(u) + \mathrm{Re}\,\pi(u)) = \det(\pi(u)).$$

As the determinant of an unitary matrix may take arbitrary complex value of modulus 1 it that the same holds for the family $\{\rho(u, Z_0), u \in \pi^{-1}(\mathbf{U}(m))\}$ which implies the rotational invariance of the distribution of $\widehat{\Theta}$. $\square$

Remark 4.1. It is possible to show that for any $m \times m$ positive-definite matrix Y the integrals of $|\Theta_{\tau Y}|^2$ over the left compact horocycle containing $i\tau Y$ are bounded in τ if $0 < \tau < \tau_0$. This implies that the distributions of the family of functions $(\Theta_{\tau Y})_{0<\tau<\tau_0}$ is tight. In view of this fact the main contents of theorem 4.1 is that all partial limits of the distribution of $\Theta_{\tau Y}$ as $\tau \to 0$ are in fact the same. Thus, the limiting distribution should have finite second moment. It follows from results in [Ma2] that for $m = 1$ the moment of order α is finite if $\alpha < 4$ and infinite if $\alpha > 4$.

Proof of Theorem 4.1. Set $\tau = \exp(-2s)$ and consider the limiting behavior of $\Theta_{\exp(-2s)Y}$ as $s \to \infty$. According to Proposition 2.3 with $Z = Z_s = i\exp(-2s)Y + N'_{1,2}$, the distribution of the function $\Theta_{\exp(-2s)Y}$ with respect to the Haar probability measure on $N'_{1,2}\backslash\mathbf{S}_m(\mathbb{R})$ agrees with the distribution of the function $\widehat{\Theta}$ on the closed $\mathbf{S}_m(\mathbb{R})$-orbit of the point $\widehat{\psi}_4(Z_s) \in \widehat{\mathcal{M}}_4$, supplied with the $\mathbf{S}_m(\mathbb{R})$-invariant probability measure. Remark that $Z_s = \widehat{\varkappa}_{\widehat{g}(s)}(Z_0)$, and by Proposition 2.2 the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $\widehat{\psi}_4(Z_s)$ is the right translate by $\widehat{g}(s)$ of the $\mathbf{S}_m(\mathbb{R})$-orbit of the point $\widehat{\psi}_4(Z_0) = \widehat{x}_0 d(Y^{1/2})$. Thus, the distribution of the function $\Theta_{\exp(-2s)Y}$ agrees with the distribution of the function $\widehat{\Theta}$ on the right translate by $\widehat{g}(s)$ of the closed expanding horocycle $\widehat{x}_0 Y^{1/2}\widehat{\mathbf{H}}_+$. It follows now from Theorem 3.1 that the $\mathbf{S}_m(\mathbb{R})$-invariant probability measure on the horocycle $\widehat{x}_0 Y^{1/2}\widehat{\mathbf{H}}_+\widehat{g}(s)$ tends weakly to the $\mathbf{Mp}(2m,\mathbb{R})$-invariant probability measure $\widehat{P}$ on $\widehat{\mathcal{M}}_4$ and this convergence is uniform in Y from a compact set. □

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