PDMI PREPRINT — 1/2001

ON THE BIQUADRATIC METAPLECTIC HECKE OPERATORS

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February 1, 2001

ABSTRACT

This paper is devoted to the study of the biquaratic metaplectic Hecke operators on the space of biquadratic metaplectic forms. One has found explicit formula for the action of these operators.

1. Preliminaries.

Let $\mathbb{X} = \mathbb{C}^3 \times \mathbb{R}_+^{*2}$, Γ be the discrete subgroup of $\mathrm{SL}(3,\mathbb{C})$ and $\chi : \Gamma \to \mathbb{C}^*$ be an unitary character. We say $F : \mathbb{X} \to \mathbb{C}$ is an automorphic form under Γ with multiplier system χ if F is a $\mathbb{C}^\infty$–function satisfying the conditions

(a) $F(\gamma w) = \chi(\gamma) F(w)$ for $\gamma \in \Gamma, w \in \mathbb{X}$;

(b) F is an eigenfunction of all the invariant differential operators on $\mathbb{X}$;

(c) There is $c \in \mathbb{R}$ such that $|F(w)| < N(w)^c$ for all $w \in \mathbb{X}$.

By $N(w)$ we mean $\mathrm{Norm}(n(w)a(w))$ as it is defined in the Sec. 1.3.3 in [4]. Also we say that F is a biquadratic metaplectic form if F is an automorphic form under $\Gamma(4)$ with multiplier system $\chi = \kappa_3$. Here $\Gamma(4)$ is a principal congruence subgroup mod 4 in $\mathrm{SL}(3,\mathbb{Z}[i])$ and κ_3 is so-called Bass–Milnor–Serre homorphism of the group $\Gamma(4)$ attached to the biquadratic residue symbol (see [1, 3]).

Let $\lambda = 1 + i$ and $\|\cdot\|$ be the norm in $\mathbb{Z}[i]$. For $c, d \in \mathbb{Z}[i]$, $\gcd(4c, d) = 1$ we denote by $\left(\frac{c}{d}\right)$ the biquadratic residue symbol. This symbol has the following properties (see [2]):

(a) If $a \equiv b \pmod{d}$ then $\left(\frac{a}{d}\right) = \left(\frac{b}{d}\right)$;

(b) $\left(\frac{c}{ab}\right) = \left(\frac{c}{a}\right)\left(\frac{c}{b}\right)$; (c) $\left(\frac{ab}{d}\right) = \left(\frac{a}{d}\right)\left(\frac{b}{d}\right)$;

(d) If $a, b \equiv 1 \pmod{\lambda^3}$, $\gcd(a,b) = 1$ then $\left(\frac{b}{a}\right)\left(\frac{a}{b}\right)^{-1} = (-1)^{(\|a\|-1)(\|b\|-1)/16}$;

(e) If $\delta = 1 + \alpha_4\lambda^4 + \alpha_5\lambda^5 + \alpha_6\lambda^6 + \ldots$ $(\alpha_k = 0, 1)$ then

$$\left(\frac{\lambda}{\delta}\right) = i^{-\alpha_4 + 2\alpha_6}, \quad \left(\frac{i}{\delta}\right) = (-1)^{\alpha_4 + \alpha_5};$$

(f) If $\delta = 1 + \lambda^3 + \alpha_4\lambda^4 + \alpha_5\lambda^5 + \alpha_6\lambda^6 + \ldots$ $(\alpha_k = 0, 1)$ then

$$\left(\frac{\lambda}{\delta}\right) = -i^{-\alpha_4 + 2\alpha_6}, \quad \left(\frac{i}{\delta}\right) = i(-1)^{\alpha_4 + \alpha_5};$$

(g) If $b,\ k \equiv 0 \pmod{4}$, $a \equiv 1 \pmod{\lambda^3}$, $\gcd(a, b) = 1$ then $\left(\frac{b}{a + kb}\right) = \left(\frac{b}{a}\right)$.

Also we assume that $\left(\frac{0}{d}\right) = 1$ for $d \in \mathbb{Z}[i]$, $\gcd(4, d) = 1$.

Proposition 1. *Let* $\gamma = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in \Gamma(4)$ *and* $\gcd(b_3, c_3) = 1$. *Then*

$$\kappa_3(\gamma) = \left(\frac{-b_3}{c_3}\right)^{-1} \left(\frac{b_1c_3 - b_3c_1}{b_2c_3 - b_3c_2}\right).$$

Proof. The proof is essentially that of the Theorem 0.2.4 in [4]. $\square$

2. The definition of the Hecke operators

In what follows we set $G = \Gamma(4)$, $\chi = \kappa_3$, $p \in \mathbb{Z}[i]$ is prime, $p \equiv 1 \pmod 4$ and

$$\alpha = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p^4 \end{pmatrix}.$$

Definition 1. *Let $G\alpha G = \bigcup_{i=1}^{d} G\alpha\delta_i$ where $\{\delta_i\}_{i=1}^{d}$ is the set of all matrices $\delta_i \in G$ such that $G\alpha\delta_i \cap G\alpha\delta_j = \varnothing$ for all $i \neq j$. We define the Hecke operator T_α as*

$$T_\alpha F(w) = \sum_{i=1}^{d} \bar{\chi}(\delta_i) F(\alpha\delta_i(w)),$$

F is an biquadratic metaplectic form, $w \in \mathbb{X}$.

Remark 1. Obviously the set $\{\delta_i\}_{i=1}^{d}$ is a representative system of $G \cap \alpha^{-1}G\alpha\backslash G$ and one has $d = [G : G \cap \alpha^{-1}G\alpha]$.

To prove the correctness of this definition let us choose another representative system $\{\delta_i'\}_{i=1}^{d} \subset G$ such that $G\alpha G = \bigcup_{i=1}^{d} G\alpha\delta_i'$ and $G\alpha\delta_i' \cap G\alpha\delta_j' = \varnothing$ for all $i \neq j$. For each i there exists some j such that $G\alpha\delta_i = G\alpha\delta_j'$, therefore $\alpha\delta_i(\alpha\delta_j')^{-1} = h_{ij} \in G \cap \alpha G\alpha^{-1}$. If we assume that $\chi(\alpha^{-1}\theta\alpha) = \chi(\theta)$ for all $\theta \in G \cap \alpha G\alpha^{-1}$ then we have:

$$\sum_{j=1}^{d} \bar{\chi}(\delta_j') F(\alpha\delta_j'(w)) = \sum_{j=1}^{d} \bar{\chi}(\alpha^{-1}h_{ij}^{-1}\alpha\delta_i) F(h_{ij}^{-1}\alpha\delta_i(w)) =$$

$$\sum_{j=1}^{d} \bar{\chi}(\alpha^{-1}h_{ij}^{-1}\alpha)\chi(h_{ij}^{-1})\bar{\chi}(\delta_i) F(\alpha\delta_i(w)) = \sum_{i=1}^{d} \bar{\chi}(\delta_i) F(\alpha\delta_i(w)).$$

Thus the correctness of the Definition 1 follows from the proposition below.

Proposition 2. *$\chi(\alpha^{-1}\theta\alpha) = \chi(\theta)$ for all $\theta \in G \cap \alpha G\alpha^{-1}$.*

Proof. Let $\theta = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in G \cap \alpha G\alpha^{-1}$. It is easy to see that

$$\alpha^{-1}\theta\alpha = \begin{pmatrix} a_1 & a_2 & a_3p^4 \\ b_1 & b_2 & b_3p^4 \\ c_1p^{-4} & c_2p^{-4} & c_3 \end{pmatrix}$$

and $c_1,\ c_2 \equiv 0 \pmod{p^4}$, $\gcd(c_3, p) = 1$.

We can find such γ that $\gcd(b_3 + \gamma a_3, c_3) = 1$ and

$$\theta' = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G \cap \alpha G\alpha^{-1}.$$

Hence

$$\theta'\theta = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1+\gamma a_1 & b_2+\gamma a_2 & b_3+\gamma a_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$

$$\alpha^{-1}\theta'\theta\alpha = \begin{pmatrix} a_1 & a_2 & a_3p^4 \\ b_1+\gamma a_1 & b_2+\gamma a_2 & (b_3+\gamma a_3)p^4 \\ c_1p^{-4} & c_2p^{-4} & c_3 \end{pmatrix},$$

matrices $\theta'\theta$, $\alpha^{-1}\theta'\theta\alpha$ satisfy the conditions of the Proposition 1 and $\chi(\alpha^{-1}\theta'\alpha) = \chi(\theta') = 1$.

So we have $\chi(\alpha^{-1}\theta\alpha) = \chi(\alpha^{-1}\theta'\alpha)^{-1}\chi(\alpha^{-1}\theta'\theta\alpha) = \chi(\alpha^{-1}\theta'\theta\alpha)$. By using the Proposition 1 we get that

$$\chi(\alpha^{-1}\theta'\theta\alpha) = \left(\frac{-(b_3+\gamma a_3)p^4}{c_3}\right)^{-1}\left(\frac{(b_1+\gamma a_1)c_3-(b_3+\gamma a_3)c_1}{(b_2+\gamma a_2)c_3-(b_3+\gamma a_3)c_2}\right) = \chi(\theta'\theta).$$

Since $\chi(\theta'\theta) = \chi(\theta')\chi(\theta) = \chi(\theta)$ we obtain that $\chi(\alpha^{-1}\theta\alpha) = \chi(\theta)$. $\square$

Let us prove that $T_\alpha F$ is a biquadratic metaplectic form.

1) If $\gamma \in G$ then

$$(T_\alpha F)(\gamma w) = \sum_{i=1}^{d} \bar{\chi}(\delta_i)F(\alpha\delta_i\gamma w) = \chi(\gamma)\sum_{i=1}^{d}\bar{\chi}(\delta_i\gamma)F(\alpha\delta_i\gamma w).$$

As it was noted in the Remark 1 the set $\{\delta_i\}_{i=1}^d$ (and therefore the set $\{\delta_i\gamma\}_{i=1}^d$) is a representative system of $G\cap\alpha^{-1}G\alpha\backslash G$. Hence we obtain that $(T_\alpha F)(\gamma w) = \chi(\gamma)(T_\alpha F)(w)$.

2) Obviously, if $DF = \lambda_F(D)F$, where D is an invariant differential operator on $\mathbb{X}$ then $D(T_\alpha F) = \lambda_F(D)(T_\alpha F)$.

3) Since $T_\alpha F$ is a finite sum, this function satisfies the growth condition.

3. Explicit formulae of the representative system.

Proposition 3. *Let $\mathbb{F}_q$ be the finite field which consists of $q = p^m$ elements. Then*

(1) $\operatorname{ord}\mathrm{GL}(n,\mathbb{F}_q) = (q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$;

(2) $\operatorname{ord}\mathrm{SL}(n,\mathbb{F}_q) = (q^n-1)(q^n-q)\cdots(q^n-q^{n-2})q^{n-1}$.

Proof. Obviously, we have the bijective mapping of the group $\mathrm{GL}(n,\mathbb{F}_q)$ into the set of all bases in the linear space $V_n = \mathbb{F}_q^n$. If we fix $k-1$ entries in some basis then we can choose k^{th} entry from $q^n - q^{k-1}$ elements of V_n. Hence we obtain the first part of the proposition. Since $[\mathrm{GL}(n,\mathbb{F}_q) : \mathrm{SL}(n,\mathbb{F}_q)] = q-1$ it yields second equality. $\square$

Proposition 4. *The set $\{\delta_i\}_{i=1}^d$ can be chosen as*

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \qquad (*)$$

$$\begin{pmatrix} 2-p & (2-p)f+pt(p-1) & (1-p)^2(p^2-5p+7)/4+pt \\ 4 & 4f+p+pl(p-1) & (1-p)(2-p)^2+pl \\ 0 & p-1 & 1 \end{pmatrix}, \qquad (**)$$

$$\begin{pmatrix} (2-p)p+pt(p-1) & 0 & 1-p+pt \\ pl(p-1) & 1 & pl \\ p-1 & 0 & 1 \end{pmatrix}, \qquad (***)$$

where $b, c \bmod p^4$, $t, l \bmod p^3$, $f \bmod p$ *and* $d = \|p\|^8 + \|p\|^7 + \|p\|^6$.

Proof. Let us put

$$\alpha' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p \end{pmatrix}, \quad \gamma = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

The group $G \cap \alpha^{-1} G \alpha$ consists of all matrices $\gamma \in G$ such that $a_3 \equiv b_3 \equiv 0 \pmod{p^4}$ and the group $G \cap \alpha'^{-1} G \alpha'$ consists of all matrices $\gamma \in G$ such that $a_3 \equiv b_3 \equiv 0 \pmod{p}$. If M_1 and M_2 is a representative system of $(G \cap \alpha'^{-1} G \alpha') \backslash G$ and $(G \cap \alpha^{-1} G \alpha) \backslash (G \cap \alpha'^{-1} G \alpha')$, respectively, then we can represent the system $\{\delta_i\}_{i=1}^{d}$ as $\delta_i = \delta'' \delta'$ with $\delta' \in M_1$, $\delta'' \in M_2$ (see Remark 1).

Let $d = d_1 d_2$ where $d_1 = [G : G \cap \alpha'^{-1} G \alpha']$, $d_2 = [G \cap \alpha'^{-1} G \alpha' : G \cap \alpha^{-1} G \alpha]$. For the sake of brevity we set $\hat{G} = \mathrm{SL}(3, \mathbb{Z}[i])$. It is easy to prove that $d_1 = [\hat{G} : \hat{G} \cap \alpha'^{-1} \hat{G} \alpha']$. If $\{\sigma_i\}_{i=1}^{d_1}$ is a representative system $\hat{G} \cap \alpha'^{-1} \hat{G} \alpha' \backslash \hat{G}$ then one can check that $\{\sigma_i \bmod p\}_{i=1}^{d_1}$ is a representative system $(\hat{G} \cap \alpha'^{-1} G \alpha') \bmod p \backslash \hat{G} \bmod p$. By using the Proposition 3 we get that

$$d_1 = \frac{\mathrm{ord}(\hat{G} \bmod p)}{\mathrm{ord}((\hat{G} \cap \alpha'^{-1} \hat{G} \alpha') \bmod p)} = \frac{\mathrm{ord}\,\mathrm{SL}(3, \mathbb{F}_{\|p\|})}{\|p\|^2 \,\mathrm{ord}\,\mathrm{GL}(2, \mathbb{F}_{\|p\|})} = \frac{(\|p\|^3 - 1)(\|p\|^3 - \|p\|)\|p\|^2}{\|p\|^2(\|p\|^2 - 1)(\|p\|^2 - \|p\|)} = \|p\|^2 + \|p\| + 1.$$

Thus the set M_1 can be chosen as

$$\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2-p & (2-p)f & (1-p)^2(p^2-5p+7)/4 \\ 4 & 4f+p & (1-p)(2-p)^2 \\ 0 & p-1 & 1 \end{pmatrix}, \quad \begin{pmatrix} (2-p)p & 0 & 1-p \\ 0 & 1 & 0 \\ p-1 & 0 & 1 \end{pmatrix},$$

where b, c, $f \bmod p$.

Obviously, the set M_2 can be represented as

$$\begin{pmatrix} 1 & 0 & pt \\ 0 & 1 & pl \\ 0 & 0 & 1 \end{pmatrix},$$

where $t, l \bmod p^3$ and it concludes the proof. □

4. The reduction to the upper triangular form.

We fix for any $k > 0$ the residue system $\bmod\, p^k$ which satisfies the following conditions:

1) all elements of this system are divisable by 4,

2) this residue system contains 0.

All residue systems $\bmod\, p^k$ in this section will satisfy conditions above.

Definition 2. *We denote by N the set of all matrices*

$$n = \begin{pmatrix} p^{k_1} & a & b \\ 0 & p^{k_2} & c \\ 0 & 0 & p^{k_3} \end{pmatrix}$$

such that $k_1 + k_2 + k_3 = 4$, $a \bmod p^{k_2}$, b, $c \bmod p^{k_3}$ and one has the following:
(1) *if $k_1 = 0$, $k_2 \neq 0$, $k_3 \neq 0$ then $\gcd(c, p) = 1$;*
(2) *if $k_1 \neq 0$, $k_2 = 0$, $k_3 \neq 0$ then $\gcd(b, p) = 1$;*
(3) *if $k_1 \neq 0$, $k_2 \neq 0$, $k_3 = 0$ then $\gcd(a, p) = 1$;*
(4) *if $k_1 \neq 0$, $k_2 \neq 0$, $k_3 \neq 0$ then $\gcd(ac, p) = 1$.*

It is easy to check that

$$n^{-1} = \begin{pmatrix} p^{-k_1} & -ap^{-k_1-k_2} & acp^{-4} - bp^{-k_1-k_3} \\ 0 & p^{-k_2} & -cp^{-k_2-k_3} \\ 0 & 0 & p^{-k_3} \end{pmatrix},$$

where $n \in N$.

Proposition 5. *$G\alpha G = \bigcup_{n \in N} Gn$.*

Proof. 1) Let us prove that $N \subset G\alpha G$. As it follows from the Definition 1 and Remark 1 it is sufficient to find for each $n \in N$ such g, $h \in G$ that $n = g\alpha h$. For $k_i = 4$, $i = 1 \dots 3$ and $n \in N$ we see that $n \in G\alpha G$ if and only if $\mathrm{diag}(p^{k_1}, p^{k_2}, p^{k_3}) \in G\alpha G$. So we can assume that if $n \in N$ and $k_i = 4$, $i = 1 \dots 3$ then $a = b = c = 0$. Analogously if $n \in N$, $k_1 = 0$, $k_2 \neq 0$, $k_3 \neq 0$ then we can put $a = 0$. With this changes in mind we are able to prove that

$$\gcd(4p^{k_2+k_3}, 4(p^{k_1+k_3} - ap^{k_3}), p^{k_1+k_2} - 4cp^{k_1} + 4ac - 4bp^{k_2}) = 1.$$

Therefore one can choose such t_i, l_i that

$$m = \begin{pmatrix} t_1 & t_2 & t_3 \\ l_1 & l_2 & l_3 \\ 4p^{k_2+k_3} & 4(p^{k_1+k_3} - ap^{k_3}) & p^{k_1+k_2} - 4cp^{k_1} + 4ac - 4bp^{k_2} \end{pmatrix} \in G.$$

The existence of the matrix m follows from the Proposition 2.2 in [3]. Therefore we obtain that $(p^{k_1+k_2} - 4cp^{k_1} + 4ac - 4bp^{k_2})d_1 - 4(p^{k_1+k_3} - ap^{k_3})d_2 + 4p^{k_2+k_3}d_3 = 1$ where $d_1 = t_1 l_2 - t_2 l_1$, $d_2 = t_1 l_3 - t_3 l_1$, $d_3 = t_2 l_3 - t_3 l_2$ and

$$g = \begin{pmatrix} t_1 p^{k_1} & at_1 + t_2 p^{k_2} & bt_1 + ct_2 + t_3 p^{k_3} \\ l_1 p^{k_1} & al_1 + l_2 p^{k_2} & bl_1 + cl_2 + l_3 p^{k_3} \\ 4 & 4 & 1 \end{pmatrix} \in G.$$

Direct multiplying of matrices shows that $\alpha g n^{-1} = m \in G$, hence for each $n \in N$ we have that $n \in G\alpha G$.

2) We have to prove that $Gn' \cap Gn'' = \varnothing$ for $n' \neq n''$, $n', n'' \in N$. If $gn' = n''$, where $g \in G$ then g is the upper triangular matrix. Let

$$g = \begin{pmatrix} 1 & a_2 & a_3 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{pmatrix}, \quad n' = \begin{pmatrix} p^{k_1'} & a' & b' \\ 0 & p^{k_2'} & c' \\ 0 & 0 & p^{k_3'} \end{pmatrix}, \quad n'' = \begin{pmatrix} p^{k_1''} & a'' & b'' \\ 0 & p^{k_2''} & c'' \\ 0 & 0 & p^{k_3''} \end{pmatrix}.$$

Then $k_i' = k_i'' = k_i$, $a'' = a' + a_2 p^{k_2}$, $c'' = c' + b_3 p^{k_3}$. By using the Definition 2 we get that $a' = a''$, $a_2 = 0$, $c' = c''$, $b_3 = 0$. Moreover, we have $b'' = b' + a_3 p^{k_3}$ and $b'' = b'$, $a_3 = 0$ hence $n' = n''$. Thus if $Gn' \cap Gn'' \neq \varnothing$ then $n' = n''$.

3) One can find that $\operatorname{ord} N = \|p\|^8 + \|p\|^7 + \|p\|^6$. Since $\operatorname{ord} N = d$ (see Proposition 4) we get this proposition. $\square$

The Proposition 5 yields that for all δ_i, $i = 1 \dots d$ one can choose such matrices $n_i \in N$, $h_i \in G$ that $h_i n_i = \alpha \delta_i$. Then we have

$$T_\alpha F(w) = \sum_{i=1}^{d} \bar{\chi}(\delta_i)\chi(h_i)F(n_i(w)).$$

It follows from the Proposition 1.4 that $\chi(\delta_i) = 1$ and it remains to evaluate $\chi(h_i)$. Let us write δ, n instead of δ_i, n_i.

1) If δ is the matrix of the form $(*)$ in the Proposition 4 then $\alpha\delta \in N$ and $h = 1$, $\chi(h) = 1$.

2) Let δ be the matrix of the form $(**)$ in the Proposition 4. We have that

$$\alpha\delta = \begin{pmatrix} 2-p & (2-p)f + pt(p-1) & (1-p)^2(p^2 - 5p + 7)/4 + pt \\ 4 & 4f + p + pl(p-1) & (1-p)(2-p)^2 + pl \\ 0 & p^4(p-1) & p^4 \end{pmatrix}.$$

If $h = \alpha\delta n^{-1} \in G$, $n \in N$ then $k_1 = 0$. Let $k_2 = 0$. Then $\alpha\delta \in G\alpha\delta'$ where δ' as in $(*)$ hence we have $k_2 \neq 0$. By direct multiplying of matrices we obtain:

$$h = \begin{pmatrix} * & * & * \\ 4 & p^{-k_2}(4f + p + lp(p-1) - 4a) & * \\ 0 & p^{k_3}(p-1) & p^{k_2} - c(p-1) \end{pmatrix} = \begin{pmatrix} * & * & * \\ b_1 & b_2 & b_3 \\ 0 & c_2 & c_3 \end{pmatrix}.$$

It is easy to check that $\gcd(b_1, b_3, c_3) = 1$. Therefore we can choose the matrix

$$\theta = \begin{pmatrix} 1 & 0 & \beta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in G$$

in such a way that $\gcd(b_3 + b_1\beta, c_3) = 1$. Hence the matrix $h\theta$ satisfies the condition of the Proposition 1 and $\chi(h\theta) = \chi(h)$. By using the biquadratic reciprocity law and the Proposition 1 we obtain following:

$$\chi(h\theta) = \left(\frac{-b_3 - b_1\beta}{c_3}\right)^{-1}\left(\frac{b_1 c_3}{b_2 c_3 - (b_3 + b_1\beta)c_2}\right) =$$
$$= \left(\frac{b_3 + b_1\beta}{c_3}\right)^{-1}\left(\frac{b_2 c_3 - (b_3 + b_1\beta)c_2}{c_3}\right)\left(\frac{b_1}{b_2 c_3 - (b_3 + b_1\beta)c_2}\right) =$$
$$= \left(\frac{b_3 + b_1\beta}{c_3}\right)^{-1}\left(\frac{b_3 + b_1\beta}{c_3}\right)\left(\frac{c_2}{c_3}\right)\left(\frac{b_1}{b_2 c_3 - (b_3 + b_1\beta)c_2}\right) =$$
$$= \left(\frac{p^{k_3}(p-1)}{p^{k_2} - c(p-1)}\right)\left(\frac{4}{b_2 c_3 - (b_3 + b_1\beta)c_2}\right) = \left(\frac{p}{p^{k_2} - c(p-1)}\right)^{k_3}\left(\frac{p-1}{p^{k_2} - c(p-1)}\right) =$$
$$= \left(\frac{-c(p-1)}{p}\right)^{k_3}\left(\frac{p-1}{p^{k_2}}\right) = \left(\frac{c}{p}\right)^{k_3}.$$

Thus we get that $\chi(h) = \left(\frac{c}{p}\right)^{k_3}$.

3) Let δ be the matrix of the form $(***)$ in the Proposition 4. If $h = \alpha\delta n^{-1} \in G$, $n \in N$ then $k_1 \neq 0$ and one has

$$h = \begin{pmatrix} \frac{(2-p)+t(p-1)}{p^{k_1-1}} & * & * \\ \frac{l(p-1)}{p^{k_1-1}} & \frac{1}{p^{k_2}} - \frac{al(p-1)}{p^{k_1+k_2-1}} & \begin{matrix} lp(p-1)\left(\frac{ac}{p^4} - \frac{b}{p^{k_1+k_3}}\right) - \\ -\frac{c}{p^{k_2+k_3}} + \frac{l}{p^{k_3-1}} \end{matrix} \\ p^{k_2+k_3}(p-1) & -ap^{k_3}(p-1) & p^{k_1+k_2} + (p-1)(ac-bp^{k_2}) \end{pmatrix}.$$

To simplify notations we take (b_1, b_2, b_3) and (c_1, c_2, c_3) to be the second and the third row of the matrix h.

a) Let $\gcd(b_3, c_3) = 1$. Then $b_1c_3 - b_3c_1 = c(p-1)$ and $b_2c_3 - b_3c_2 = p^{k_1} - b(p-1)$. By using the biquadratic reciprocity law we obtain:

$$\left(\frac{b_1c_3 - b_3c_1}{b_2c_3 - b_3c_2}\right) = \left(\frac{c(p-1)}{p^{k_1} - b(p-1)}\right) = \left(\frac{c}{p^{k_1} - b(p-1)}\right)\left(\frac{p-1}{p^{k_1}}\right) = \left(\frac{c}{p^{k_1} - b(p-1)}\right).$$

$$\left(\frac{-b_3}{c_3}\right) = \left(\frac{p^4}{c_3}\right)\left(\frac{b_3}{c_3}\right) = \left(\frac{lp(p-1)(ac-bp^{k_2}) + lp^{1+k_1+k_2} - cp^{k_1}}{(p-1)(ac-bp^{k_2}) + p^{k_1+k_2}}\right) =$$
$$\left(\frac{-cp^{k_1}}{(p-1)(ac-bp^{k_2}) + p^{k_1+k_2}}\right) =$$
$$\left(\frac{(p-1)(ac-bp^{k_2}) + p^{k_1+k_2}}{p}\right)^{k_1}\left(\frac{c}{ac(p-1) + p^{k_2}(p^{k_1} - b(p-1))}\right) =$$
$$\left(\frac{bp^{k_2} - ac}{p}\right)^{k_1}\left(\frac{c}{p^{k_2}(p^{k_1} - b(p-1))}\right).$$

The Proposition 1 yields that

$$\chi(h) = \left(\frac{bp^{k_2} - ac}{p}\right)^{-k_1}\left(\frac{c}{p}\right)^{-k_2}\left(\frac{c}{p^{k_1} - b(p-1)}\right)^{-1}\left(\frac{c}{p^{k_1} - b(p-1)}\right) =$$
$$\left(\frac{bp^{k_2} - ac}{p}\right)^{-k_1}\left(\frac{c}{p}\right)^{-k_2}.$$

b) Let $\gcd(b_3, c_3) \neq 1$ and $k_3 \neq 0$. By the Definition 2 we get that

$$\gcd((p-1)(ac-bp^{k_2}) + p^{k_1+k_2}, p) = 1$$

and the following conditions are equivalent:

1) $\gcd(b_3, c_3) \neq 1$;

2) $\gcd(lp(p-1)(ac-bp^{k_2}) - cp^{k_1} + lp^{1+k_1+k_2}, (p-1)(ac-bp^{k_2}) + p^{k_1+k_2}) \neq 1$;

3) $\gcd(-cp^{k_1}, (p-1)(ac-bp^{k_2}) + p^{k_1+k_2}) \neq 1$;

4) $\gcd(c, (p-1)(ac-bp^{k_2}) + p^{k_1+k_2}) \neq 1$;

5) $\gcd(c, p^{k_1+k_2} - bp^{k_2}(p-1)) \neq 1$;

6) $\gcd(c, p^{k_1} - b(p-1)) \neq 1$.

One exists such $c_1 \equiv c \pmod{4p^{k_3}}$ that $\gcd(c_1, p^{k_1} - b(p-1)) = 1$. If $c_1 = c + tp^{k_3}$, $t \equiv 0 \pmod 4$ and

$$\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \in G$$

then we have

$$\theta n = \begin{pmatrix} p^{k_1} & a & b \\ 0 & p^{k_2} & c_1 \\ 0 & 0 & p^{k_3} \end{pmatrix},$$

$h\theta^{-1} = \alpha\delta(\theta n)^{-1} \in G$ and the matrix $h\theta^{-1}$ satisfies conditions of the Proposition 1. Thus $\chi(h\theta^{-1}) = \chi(h)$ and the point (a) yields

$$\chi(h) = \chi(h\theta^{-1}) = \left(\frac{bp^{k_2} - ac_1}{p}\right)^{-k_1} \left(\frac{c_1}{p}\right)^{-k_2} = \left(\frac{bp^{k_2} - ac}{p}\right)^{-k_1} \left(\frac{c}{p}\right)^{-k_2}.$$

Let $k_3 = 0$. Then $k_1 + k_2 = 4$, $k_1 \neq 0$, $\gcd(a, p) = 1$ and

$$n = \begin{pmatrix} p^{k_1} & a & 0 \\ 0 & p^{k_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Also we have that

$$h = \begin{pmatrix} * & * & * \\ lp^{1-k_1}(p-1) & p^{-k_2} - alp^{-3}(p-1) & lp \\ p^{k_2}(p-1) & -a(p-1) & p^4 \end{pmatrix}.$$

Put

$$\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By using properties of Bass–Milnor–Serre homomorphisms we have $\chi(\lambda h \lambda^{-1}) = \chi(h)$. Since

$$\lambda h \lambda^{-1} = \begin{pmatrix} * & * & * \\ -p^{k_2}(p-1) & p^4 & a(p-1) \\ lp^{1-k_1}(p-1) & -lp & p^{-k_2} - alp^{-3}(p-1) \end{pmatrix}$$

and $\gcd(a(p-1), p^{3-k_2} - al(p-1)) = 1$ the matrix $\lambda h \lambda^{-1}$ satisfies conditions of the Proposition 1 and

$$\chi(h) = \chi(\lambda h \lambda^{-1}) = \left(\frac{-a(p-1)}{p^{-k_2} - alp^{-3}(p-1)}\right)^{-1} \left(\frac{1-p}{p^{k_1}}\right) =$$
$$\left(\frac{a(p-1)}{p^{4-k_2} - alp(p-1)}\right)^{-1} = \left(\frac{a}{p^{k_1}}\right)^{-1} = \left(\frac{a}{p}\right)^{k_2}.$$

Finally we have that

$$\chi(h) = \begin{cases} \left(\dfrac{a}{p}\right)^{k_2} & \text{if } k_1 \neq 0, k_3 = 0, \\ \left(\dfrac{bp^{k_2} - ac}{p}\right)^{-k_1} \left(\dfrac{c}{p}\right)^{-k_2}, & \text{if } k_1 \neq 0, k_3 \neq 0. \end{cases}$$

Thus we have proved following theorem.

Theorem. *Let $F(w)$ be the biquadratic metaplectic form, $w \in \mathbb{X}$ and T_α be the Hecke operator. Also we denote by (p^k) and $(p^k)^*$ the residue system* mod p^k *and the reduced residue system* mod p^k *respectively. Then*

$$T_\alpha F(w) = \sum_{b(p^4),c(p^4)} F(\begin{pmatrix} 1 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & p^4 \end{pmatrix} w) + \sum_{a(p),b(p^3),c(p^3)^*} \left(\frac{c}{p}\right)^3 F(\begin{pmatrix} 1 & a & b \\ 0 & p & c \\ 0 & 0 & p^3 \end{pmatrix} w) +$$

$$\sum_{a(p^2),b(p^2),c(p^2)^*} \left(\frac{c}{p}\right)^2 F(\begin{pmatrix} 1 & a & b \\ 0 & p^2 & c \\ 0 & 0 & p^2 \end{pmatrix} w) + \sum_{a(p^3),b(p),c(p)^*} \left(\frac{c}{p}\right) F(\begin{pmatrix} 1 & a & b \\ 0 & p^3 & c \\ 0 & 0 & p \end{pmatrix} w) +$$

$$\sum_{a(p^4)} F(\begin{pmatrix} 1 & a & 0 \\ 0 & p^4 & 0 \\ 0 & 0 & 1 \end{pmatrix} w) + \sum_{a(p^3)^*} \left(\frac{a}{p}\right)^3 F(\begin{pmatrix} p & a & 0 \\ 0 & p^3 & 0 \\ 0 & 0 & 1 \end{pmatrix} w) +$$

$$\sum_{a(p^2)^*} \left(\frac{a}{p}\right)^2 F(\begin{pmatrix} p^2 & a & 0 \\ 0 & p^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} w) + \sum_{a(p)^*} \left(\frac{a}{p}\right) F(\begin{pmatrix} p^3 & a & 0 \\ 0 & p & 0 \\ 0 & 0 & 1 \end{pmatrix} w) +$$

$$F(\begin{pmatrix} p^4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} w) + \sum_{b(p^3)^*,c(p^3)} \left(\frac{b}{p}\right)^3 F(\begin{pmatrix} p & 0 & b \\ 0 & 1 & c \\ 0 & 0 & p^3 \end{pmatrix} w) +$$

$$\sum_{a(p)^*,b(p^2),c(p^2)^*} \left(\frac{a}{p}\right)^3 \left(\frac{c}{p}\right)^2 F(\begin{pmatrix} p & a & b \\ 0 & p & c \\ 0 & 0 & p^2 \end{pmatrix} w) +$$

$$\sum_{a(p^2)^*,b(p),c(p)^*} \left(\frac{a}{p}\right)^3 \left(\frac{c}{p}\right) F(\begin{pmatrix} p & a & b \\ 0 & p^2 & c \\ 0 & 0 & p \end{pmatrix} w) + \sum_{b(p^2)^*,c(p^2)} \left(\frac{b}{p}\right)^2 F(\begin{pmatrix} p^2 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & p^2 \end{pmatrix} w) +$$

$$\sum_{a(p)^*,b(p),c(p)^*} \left(\frac{a}{p}\right)^2 \left(\frac{c}{p}\right) F(\begin{pmatrix} p^2 & a & b \\ 0 & p & c \\ 0 & 0 & p \end{pmatrix} w) + \sum_{b(p)^*,c(p)} \left(\frac{b}{p}\right) F(\begin{pmatrix} p^3 & 0 & b \\ 0 & 1 & c \\ 0 & 0 & p \end{pmatrix} w).$$

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