PDMI PREPRINT — 15/2000

SPACES OF CURVATURE

BOUNDED ABOVE

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191011, St.-Petersburg, Fontanka, 27
September, 2000

Abstract.

A short survey on Curvature Bounded Above (CBA) spaces. Local properties, constructions and results are discussed.

Contents

1. Motivation

To motivate the study of Curvature Bounded Above (CBA for brevity) spaces let me list some results which essentially use them.

1. p-adic superrigidity of lattices in $\mathrm{Sp}(n,1)$ and F_4 was proved by developping the theory of harmonic maps into singular Non Positively Curved (NPC) spaces, e.g. Euclidean Bruhat-Tits buildings (M. Gromov, R. Schoen, [GS]).

2. For each $n \geq 5$, there are examples of closed topological n-manifolds M^n with piecewise flat NPC metrics whose universal coverings $\widetilde{M}^n$ are not homeomorphic to $\mathbb{R}^n$ (M. Davis, T. Januczkiewicz, [DJ], reflection groups and hyperbolization). Furthermore, the interior of every compact contractible PL-manifold C^n, again $n \geq 5$, supports a complete metric g of strictly negative curvature (F. Ancel, C. Guilbault, [AG], spines and V. Berestovskii's metrization of polyhedra, [Be]). When ∂C^n is not simply connected, there are geodesics of g which are wild curves in C^n.

3. Solution of an old-standing problem concerning the existence of uniform estimates on the number of collisions in semi-dispersing billiards (D. Burago, S. Ferleger, A. Kononenko, [BFK]).

4. Examples of metric spaces with decent calculus, e.g. admitting Poincaré inequalities, for which all quasi-conformal automorphisms are quasi-symmetric, with Hausdorff dimension not an *integer*. These spaces come out as ideal boundaries of some hyperbolic buildings (M. Bourdon, H. Pajot, [BP]).

2. Defining CBA

For $\kappa \in \mathbb{R}$, let M_κ be the model 2-space of constant curvature κ, $D_\kappa = \operatorname{diam} M_\kappa \in (0,\infty]$.

We say that a metric space U is CAT(κ) if for every triple $S = \{\overline{x}, \overline{y}, \overline{z}\} \subset M_\kappa$ with perimeter $< 2D_\kappa$ and every isometric $f : S \to U$, there is a 1-Lip extension $\{\overline{y}, \overline{t}, \overline{z}\} \to U$ of $f|\{\overline{y}, \overline{z}\}$, where $\overline{t} \in \overline{yz}$ is the midpoint, and every such extension defines a 1-Lip extension $\overline{f} : S \cup \{\overline{t}\} \to U$ of f. In this case, a triple $f(S) = \{x, y, z\}$ is said to be *κ-thin*. A metric space X is CBA(κ) if it is CAT(κ) locally.

This definition is actually the same as the standard local comparison of medians.

Remark 2.1. Comparing the midlines instead of the medians leads to Busemann's condition of nonpositive curvature (for $\kappa = 0$). The corresponding class of spaces is much larger than CBA(0) and includes, in particular, linear normed spaces. The basic distinction is that CBA(κ) implies the existence of angles whereas Busemann's NPC not.

Remark 2.2. Stabilizing CBA(κ) one can redefine the CAT(κ)-condition in an equivalent way saying that for each $\varepsilon > 0$, $0 < P < 2D_\kappa$ there exists $\delta > 0$ such that for every $(1,\delta)$-quasi-isometric $f : S \to U$ with perimeter of S bounded by P, every $(1,\delta)$-quasi-isometric extension $\{\overline{y}, \overline{t}, \overline{z}\} \to U$ of $f|\{\overline{y}, \overline{z}\}$ with the midpoint $\overline{t} \in \overline{yz}$ defines a $(1,\varepsilon)$-quasi-isometric extension $\overline{f} : S \cup \{\overline{t}\} \to U$ of f. The stabilization is useful for proving that CBA-condition behaves well under limiting operations.

It is very much surprising how far one can go starting with the CBA(κ) condition. For a complete X, this includes the local existence and uniqueness of geodesics, existence and comparison of angles, propagation from local to global comparison, the infinitesimal theory etc. The basic device for extracting of various properties from the definition is this.

Assume that xyt, xtz are κ-thin triangles, where $t \in yz$. Then, if the perimeter of xyz is less than $2D_\kappa$, then xyz is κ-thin.

We distinguish three major cases of CAT(κ)-spaces with quite different flavor, results and approaches: $\kappa < 0$, $\kappa = 0$ and $\kappa > 0$. CAT(κ)-spaces with $\kappa < 0$ are Gromov hyperbolic and, conversely, by a recent result of M. Bonk and O. Schramm [BS], every Gromov hyperbolic space (with a mild restriction) is roughly quasi-isometric to a convex subset in H^n. The most important examles are hyperbolic groups. All Hadamard spaces are CAT(0) and the tangent spaces of every CBA-space are Hadamard. Significance of CAT(κ)-spaces with $\kappa > 0$ is that the space of directions of every CBA-space is CAT(1) and the Tits boundary at infinity $\partial_T X$ of every Hadamard space is CAT(1).

3. Propagation from local to global

A remarkable property of the comparison conditions, CBA as well as CBB, is that they propagate from local to global. However, there is a fundamental distinction between CBA and CBB cases: in any CBB(κ) space every geodesic triangle satisfies the angle comparison with M_κ (Toponogov's theorem), whereas for CBA spaces this is not the case. This is a major source of problems for proving or cheking the CBA-condition.

A geodesic $\gamma : [0,1] \to X$ has no conjugate points, if for some its neighborhood U (in the compact-open topology) the map $p : U \to X \times X$, $p(\sigma) = (\sigma(0), \sigma(1))$ is a homeomorphism on the neighborhood $V = p(U)$ of $(\gamma(0), \gamma(1))$. Now, we have

Every geodesic γ of length $L(\gamma) < D_\kappa$ in a complete CBA(κ)-space X has no conjugate points. Moreover, every narrow triangle with two sides sufficiently close to γ is κ-thin (S. Alexander, R. Bishop, [AB1]).

The main ingredient of the proof is the following middle-third construction. For geodesics γ, σ, γ', $\sigma' : [0,1] \to X$ such that $\gamma'(0) = \gamma(0)$, $\gamma'(1) = \sigma(1/2)$, $\sigma'(0) = \gamma(1/2)$, $\sigma'(1) = \sigma(1)$ we denote $(\gamma', \sigma') = \Lambda(\gamma, \sigma)$ and put $\mu(\gamma, \sigma) = \max\{|\gamma(1/2)\sigma(0)|, |\gamma(1)\sigma(1/2)|\}$.

For every $0 < P < \frac{2}{3}D_\kappa$ there exist $\delta = \delta(P)$, $\lambda = \lambda(P)$, $\delta > 0$, $0 < \lambda < 1$ such that if for pairs of geodesics (γ, σ), $(\gamma', \sigma') = \Lambda(\gamma, \sigma)$ in a metric space X the triangles Δ with sides γ, γ', $\gamma(1)\gamma'(1)$ and Δ' with sides σ, σ', $\sigma(0)\sigma'(0)$ are κ-thin and $P(\Delta)$, $P(\Delta') \le P$, $\mu(\gamma, \sigma) < \delta$ then

$$\mu(\gamma', \sigma') \le \lambda \mu(\gamma, \sigma).$$

Applied to the case $\kappa \le 0$ this leads to an extension of the classical Hadamard-Cartan theorem to the Hadamard spaces:

For every $\kappa \le 0$, every complete simply connected CBA(κ)-space satisfies CAT(κ).

There is a standard trap in the proof: if one has a space in which a geodesic between any two points is unique then it is naturally to suppose that the geodesics vary continuously with their ends. This is indeed the case for the *locally compact* CBA(κ)-spaces. However, in general, this is not anymore true. One can observe this effect looking at the cartweel. As an example (due to W. Ballmann) one can take the metric completion X of $B = \cup_n B_n$ where B_1 is the circle of length 3, and B_{n+1} is obtained from B_n by connecting every x, $y \in B_n$ with $\mathrm{dist}(x,y) > 1$ by a segment of length 1. Then X is a complete CBA(κ)-space for each $\kappa \in \mathbb{R}$ such that every two points from the dense subset B are connected by a unique geodesic. However, geodesics in X do not vary continuously with their ends.

4. Recognizing CBA-spaces

Let me give a fundamental example of a space which is not CBA. This is the Euclidean cone X over a circle of length $< 2\pi$. Clearly, X contains arbitrarily short geodesic bigons (which live near the vertex), thus it cannot be a CBA-space.

Basic examples of CBA(κ)-spaces are simplicial complexes obtained by gluing together simplices of constant curvature κ. In such a complex, the link of each simplex has itself the structure of a complex built out of spherical simplices (the case $\kappa = 1$). It turns out that

The original complex will be CBA(κ) if and only if the link of every simplex is (globally) CAT(1). This is equivalent to saying that the link of each simplex should contain no closed geodesic of length strictly less than 2π [Ba].

Closely related is the following result due to V. Berestovskii [Be].

Any simplicial complex admits a piecewise spherical CAT(1)-metric.

For the proof, one should take the barycentric subdivision and introduce the metric in which every simplex of the subdivision is isometric to the standard spherical one of the same dimension. This result has several important applications, among them are examples of CBA(1)-spaces homeomorphic to n-manifolds ($n \geq 5$) which contains points with the direction space not homeomorphic to a manifold (this settles in negative a question of A. Aleksandrov): take a homological Poincaré sphere Σ^3 and put $X = S^1(S^1(\Sigma^3))$ be its double suspension. Its known that X is homeomorphic to S^5, whereas $Y = S^1(\Sigma^3)$ is not a manifold. Starting with a triangulation of Σ^3 one introduces a CBA(1)-metric on X such that the direction space at some point is Y. In dimesion $n = 3, 4$ the problem seems to be open (and related to the Poincaré conjecture for $n = 4$).

5. Different types of convergence

In general, CBA-condition does not survive the Hausdorff–Gromov convergence. The reason is that the size of CAT(κ)-neighborhoods may become arbitrarily small. The standard example is this: the hyperboloids $X_\varepsilon = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 - z^2 = \varepsilon^2\}$ with the induced intrinsic metrics are CBA(0) and they HG-converge to the double cone $X_0 = \{(x, y, z) \in \mathbb{R}^3 \colon x^2 + y^2 - z^2 = 0\}$ as $\varepsilon \to 0$ which is not CBA. However, we easily have

if $X_n \to X$ by Hausdorff-Gromov and X_n are CBA(κ) with convexity radii uniformly separated from 0, then X is CBA(κ)

(the convexity radius $r(x)$ at $x \in X$ is the radius of the maximal ball centered at x with CAT(κ) induced metric).

One can define the HG-convergence as follows. The *distortion* of a map $f : X \to Y$ between metric spaces is

$$\operatorname{dis}(f) = \sup_{x,x' \in X} |d_Y(f(x), f(x')) - d_X(x, x')| \,.$$

Let A be the class of *all* maps $X \to Y$. Putting $\delta(X, Y) = \inf_{f \in A} \operatorname{dis}(f)$, one defines

$$|XY|_{HG} = \max\{\delta(X, Y), \delta(Y, X)\} \,.$$

The convergence with respect to this metric is equivalent to the Hausdorff-Gromov convergence.

Replacing the class A of all maps by the class of all *homeomorphisms*, we arrive at the *uniform* metric and the *uniform* convergence respectively. We say that a sequence of metrics $\{d_k\}$ on X has no local blow-ups if for every $x \in X$ there are $n \in \mathbb{N}$ and $\varepsilon > 0$ such that the restriction $\mathrm{id}^\varepsilon_{n,n+k}$ of the identity map $\mathrm{id}_{n,n+k} : (X, d_n) \to (X, d_{n+k})$ on the ball $B^{d_n}_\varepsilon(x)$ has dilatation $\operatorname{dil}(\mathrm{id}^\varepsilon_{n,n+k}) \leq C_n < \infty$ for all $k \in \mathbb{N}$.

Assume that a sequence $\{d_n\}$ of metrics without local blow-ups on X uniformly converges to a metric d. If d_n is complete, CBA(κ) and its metric topology is locally compact for every $n \geq 1$, then d is CBA(κ).

The condition that d is a metric is essential as the example above $X_\varepsilon \to X_0$ shows. Here we have the uniform convergence without blow-ups, however, the limiting d is only *pseudo-metric.*

Finally, we introduce the *homotopy* metric and the *homotopy* convergence which is well adapted to the CBA(0)-condition. Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse

$g: Y \to X$. For compact metric spaces X, Y we put

$$|XY|_h = \inf_{f,g} \max\{\mathrm{dis}(f), \mathrm{dis}(g)\},$$

where the infimum is taken over all homotopy equivalences $(f, g) : X \leftrightarrow Y$. This homotopy distance defines a metric on the classes of isometric compact metric spaces.

Assume that $|X_n X|_h \to 0$ as $n \to \infty$, where all X_n are CBA(κ) with $\kappa \le 0$. Then X is CBA(κ).

This is proved in [ArB]. For $\kappa > 0$, the CBA(κ)-condition does not survive, in general, even the homotopy convergence as the following example due to S. Ivanov shows.

For every $\varepsilon > 0$ there exists a contractible closed 2-polyhedron X with CBA(1)-metric having the diameter less than ε.

This polyhedron X consists of huge number of blocks X_i (depending on ε), each of which is obtained as follows. Fix a tiny $\delta > 0$ (depending on ε) and take a unit sphere S^2 with an open ball B removed, where the boundary curve σ of B has length δ. Next, take a tree $T \subset S^2 \setminus B$ with the root vertex on σ such that the length of every edge of T is δ and whose vertices form a sufficiently dense subset in S^2. Finally, identify all vertices of T getting X_i. Different blocks are glued together in such a way that the vertices $v_i \in X_i$ are identified into the unique vertex $v \in X$ and the boundary curve σ_i of every block X_i is identified with a curve of another block X_j which originates from an edge of T_j. This gives the contractibility of X (the similar effect provides the contractibility of the *dunce* cap). To ensure the CBA(1)-condition one should solve a combinatorial problem to guarantee the 2π-systole condition for the link of the vertex $v \in X$.

The question, is it possible to find similar ε-small CBA(1)-metrics on a *fixed* closed contractible 2-polyhedron, remains open.

6. Infinitesimal properties of CBA-spaces

6.1. Tangent cone.

Let Σ be a metric space with $\operatorname{diam}\Sigma \le \pi$. The *euclidean cone $C(\Sigma)$ over Σ* is defined as follows. The underlying set will be $\Sigma \times [0, \infty)/\Sigma \times \{0\}$. Given σ_1, $\sigma_2 \in \Sigma$, we consider embeddings $\rho : \{\sigma_1, \sigma_2\} \times [0, \infty) \to \mathbb{R}^2$ such that $|\rho(\sigma_i, t)| = t$ and $\angle_0(\rho(\sigma_1, t_1), \rho(\sigma_2, t_2)) = |\sigma_1 \sigma_2|$, and we equip $C(\Sigma)$ with the unique metric for which these embeddings are isometric. The space $C(\Sigma)$ is CAT(0) if and only if Σ is CAT(1).

Given a CBA(κ)-space X, $x \in X$ we let $\Sigma_x X$ be the *direction space* at x, which is the (metric completion of the) set of equivalence classes of geodesic segments with initial point x (two such segments are equivalent if they have zero angle at x). The *tangent cone* of X at x, denoted by $T_x X$, is the euclidean cone $C(\Sigma_x X)$.

If X is locally compact then $T_x X$ is a HG-limit of blow-ups $\frac{1}{\varepsilon}(X, x)$ as $\varepsilon \to 0$. In general case, CBA($\varepsilon\kappa$)-spaces $\frac{1}{\varepsilon}(X, x)$ converge to $T_x X$ on finite subsets. Thus the tangent cone $T_x X$ of a CBA-space X is a CAT(0)-space at every $x \in X$; consequently the direction space $\Sigma_x X$ is a CAT(1)-space. (The last fact is due to I. Nikolaev, [Ni]).

The proof is straightforward, based on the existence of angles and uses the stabilized definition of the CAT(κ)-condition.

6.2. Scalar product and its concavity. We use notation $v = r\sigma$ for $v = (\sigma, r) \in T_x X$. Given $v = r\sigma$, $v' = r'\sigma' \in T_x X$ one defines their *scalar product*

$$\langle v, v' \rangle := \frac{1}{2}(r^2 + {r'}^2 - d_x(v, v')^2).$$

The CAT(0)-property of T_xX implies the concavity of the scalar product: whenever $\gamma : [0,1] \to T_xX$ is a geodesic and $w \in T_xX$, then

$$\langle \gamma(t), w\rangle \geq (1-t)\langle \gamma(0), w\rangle + t\langle \gamma(1), w\rangle$$

for all $t \in [0,1]$. By iterating this inequality one obtains

Let $C \subset T_xX$ be the convex hull of a finite set $\{u_1, \dots, u_k\}$. Then for every $v \in C$ there exist $\mu_1, \dots, \mu_k \geq 0$ with $\sum_i \mu_i = 1$ such that

$$\sum_i \mu_i \langle u_i, w\rangle \leq \langle v, w\rangle$$

for all $w \in T_xX$. This is due to U. Lang and V. Schroeder, [LS], who together with B. Pavlović have used it to establish a remarkable Lipschitz extension property of Hadamard spaces, [LPS].

6.3. Geometric dimension.

One defines the *geometric dimension* of CBA spaces to be the smallest function (taking values in $\mathbb{N} \cup \infty$) on the class of CBA spaces such that

(1) $\mathrm{GeomDim}\,(X) = 0$ if X is discrete;
(2) $\mathrm{GeomDim}\,(X) \geq 1 + \mathrm{GeomDim}\,(\Sigma_x X)$ for every $x \in X$.

In other words, to find the geometric dimension of a CBA-space we look for the largest number of times that we can pass to spaces of directions without getting the empty set. This notion and the related results are due to B. Kleiner, [Kl].

For every CBA-space X we have $\mathrm{GeomDim}\,(X) = \sup\{\mathrm{TopDim}\,(K) \colon K \subset X$ *is compact*$\}$, *where* TopDim *is the topological dimension.*

Let X be a CBA space with $\mathrm{GeomDim}\,(X) = n < \infty$. *Then* $\sup\{k \colon \forall \varepsilon > 0\, \exists (1+\varepsilon)$ *– bilipschitz embedding $U \to X$ of an open $U \subset \mathbb{R}^k$*$\} = n$.

Let X be a locally compact Hadamard space on which Isom(X) acts cocompactly. Then $\sup\{k \colon$ *There is an isometric embedding* $\mathbb{R}^k \to X\} = 1 + \mathrm{GeomDim}\,(\partial_T X)$.

Here $\partial_T X$ is the ideal boundary of X equipped with the Tits metric. Actually, a number of other properties related to $\mathrm{GeomDim}\,(X)$ are proved in [Kl], and here I have listed only most important ones.

A key ingredient in the proofs is the notion of a barycentric simplex. For $z = (z_0, \dots, z_n) \subset X$ (with sufficitently small diameter if $\kappa > 0$), the barycentric simplex determined by z is the singular simplex $\sigma_z : \Delta_n \to X$ which maps each $\alpha = (\alpha_0, \dots, \alpha_n) \in \Delta_n$ to the unique minimum of the uniformly convex function $\phi_\alpha = \sum \alpha_i \,\mathrm{dist}(z_i, \cdot)^2$. Barycentric simplices are Lipschitz and possess the following remarkable property.

If $x \in \sigma_z(\Delta_n) \setminus \sigma_z(\partial\Delta_n)$, then $\mathrm{GeomDim}\,(\Sigma_x) \geq n-1$.

For the proof, one considers the differential $g_\alpha = \mathrm{d}\,\phi_\alpha = \sum \alpha_i f_i : T_xX \to \mathbb{R}$. It turns out that for $\alpha \in \partial\Delta_n$, its restriction to the unit sphere $\Sigma_x X \subset T_xX$ possesses a unique minimum and thus defines a Lipschitz $\sigma_f : \partial\Delta_n \to \Sigma_x X$ which is a nondegenerate $(n-1)$-chain of barycentric simplices. Argue by induction, one concludes that $\mathrm{GeomDim}\,(\Sigma_x X) \geq n-1$.

This leads to the estimate $\mathrm{TopDim}\,(K) \leq \mathrm{GeomDim}\,(X)$ for compact subsets $K \subset X$, and using the nondegenerate part $\sigma_z(\Delta_n) \setminus \sigma_z(\partial\Delta_n)$, one obtains bilipschitz embeddings of open sets $U \subset \mathbb{R}^n$ into X.

Though it is not stated explicitly, the existence of bilipschitz embeddings $U \subset \mathbb{R}^n \to X$ should imply (an extension of) the Rademacher–Stepanoff theorem on the differentiability of Lipschitz functions to CBA-spaces, cp. [KS, Ch].

7. Gluing CBA-spaces

A simple but very useful tool is the following Reshetnyak's gluing theorem [Re1].

Let X_1, X_2 be complete locally compact CBA(κ)-spaces. Suppose that there are convex sets $C_i \subset X_i$ and an isometry $f : C_1 \to C_2$. Attach these spaces together along f. Then the resulting space X is CBA(κ).

The proof is more or less straightforward and uses the comparison of angles. However, there is a standard trap while checking the condition of the theorem. $\mathbb{R}^2$ with an open disk removed is CBA(0) with convex boundary $C = \partial X$. Gluing two copies of X along the boundary, we obtain a CBA(0)-space Y^2. However, taking $\mathbb{R}^n$ with $n \geq 3$ instead of $\mathbb{R}^2$ we obtain Y^n which is not CBA(0). The reason is that $\mathbb{R}^n$ with an open ball removed is only CBA(κ) where $\kappa = r^{-2}$, r is the radius of the ball. Thus Y^n is only CBA(r^{-2}). Similarly, the hyperbolic space H^n with an open horoball removed is only CAT(0) for $n \geq 3$, but not CAT(-1).

On the other hand, if we glue two copies of $\mathbb{R}^n$ along closed isometric balls, then the resulting space Z^n is CAT(0) for every $n \geq 1$. These additional pieces which save the CAT(0)-property are called *fins* and they were succesfully used in the proofs of quite different results.

Warped product of Hadamard spaces: if B and F are Hadamard spaces and $f : B \to \mathbb{R}$ is convex, then $B \times_f F$ is a Hadamard space.

This result is due to S. Alexander and R. Bishop [AB2]. The proof proceeds by reduction to the case $F = \mathbb{R}$ and approximation of $B \times_f \mathbb{R}$ by subspaces W_ε of Hadamard spaces W_ε^* which are constructed as follows. We decompose $B \times \mathbb{R}$ into three regions $W_\varepsilon^{(0)} = \{(p,u)\colon -\varepsilon f(p) \leq u \leq \varepsilon f(p)\}$, $U_\varepsilon^{(0)} = \{(p,u)\colon \varepsilon f(p) \leq u\}$, $L_\varepsilon^{(0)} = \{(p,u)\colon u \leq -\varepsilon f(p)\}$, where two last are closed convex subset because f is convex. Taking isometric copies $W_\varepsilon^{(n)}$, $U_\varepsilon^{(n)}$, $L_\varepsilon^{(n)}$ of these, we construct W_ε^* identifying isometric pairs $U_\varepsilon^{(n)}$ with $L_\varepsilon^{(n+1)}$ for each $n \in \mathbb{Z}$. To summarize, the space W_ε^* consists of the mutually isometric strips $W_\varepsilon^{(n)}$ and the mutually isometric fins $U_\varepsilon^{(n)} = L_\varepsilon^{(n+1)}$ with the appropriate boundary components identified. Although the space W_ε obtained by gluing together the strips $W_\varepsilon^{(n)}$ typically has positive infinite curvature, we recover nonpositive curvature by gluing on the fins.

The same sort of construction which takes into account fins have been used in [BFK] to prove a uniform estimate on the number of collisions in semi-dispersing billiards. The idea is to develop a billiard trajectory into a geodesic in a CBA-space obtained by gluing step by step convex walls hitted by the trajectory. This translates a difficult dynamical problem into a geometric one which can be solved by geometric methods.

Trying to generalize the Reshetnyak's gluing theorem, one can ask about conditions which would guarantee that gluing two smooth Riemannian manifolds M_1, M_2 of the same dimension $n \geq 3$ along isometric boundaries gives a CBA(κ)-space, if both M_1, M_2 are CBA(κ). It is naturally to conjecture that such a condition must be $L_1 + L_2 \leq 0$, where L_i is the second fundamental form of ∂M_i. This was recently proved by N. Kosovskii, [Ko], in a weaker form (without κ-estimate).

Assume that $L := L_1 + L_2 \leq 0$ at the corresponding points of ∂M_1, ∂M_2. Then $M = M_1 \cup M_2$ is locally a CBA-space.

Let $\Gamma \subset M$ be the singular hypersurface obtained from ∂M_1, ∂M_2 while gluing M. The approach is to extend appropriately the Riemannian metric, say, of $M_1 \subset M$ to a neighborhood of Γ in M and then using L to perturb the metric on M_1 smoothing Γ and pumping its singularity into the curvature of the perturbed metric. The main issue is to obtain a uniform (in the perturbation parameter δ) curvature estimate from above. This is achieved via tremendous analytic calculations. The perturbed metrics form a sequence converging without local blow-ups to the initial one, which shows that the gluing gives a CBA-space.

8. 2-DIMENSIONAL POLYHEDRA

In the case of 2-polyhedra, we have at our disposal a more or less complete description of general CBA-metrics. Several new effects, which are absent for surfaces, arise for polyhedra having topological singularities.

8.1. Singular edges are curves with bounded variation of turn. Let Y be the union of three rays with the common vertex v, $X = Y \times \mathbb{R}$. Then $l = v \times \mathbb{R} \subset X$ is the (topologically) singular edge. The important fact is that for every CBA-metric on X, the edge l has the bounded turn variation. It was formulated as a question in [BaB] and then proved by B. Kleiner (unpublished).

The explanation (not the proof!) is very simple. Assume that we have a CBA-metric on Y, in which l is piecewise geodesic. The link of every point $x \in l$ is the bipartite graph L with two vertices and three edges between them. The CBA-condition implies that the systole of L is at least 2π. Now, if x is a (metric) vertex with a small angle from one of the faces, then the length of the corresponding edge of L is short. Hence, the lengths of *both* other edges must be large to satisfy the systole condition. Thus we have a large negative curvature at x. This is mechanism which translates turns of the singular edge into negative curvature of the polyhedron. If l would have the unbounded turn variation, then it would imply a huge accumulation of negative curvature along l which should destroy the topology of the polyhedron. This explains why singular edges are curves with the bounded turn variation $|\tau|$ w.r.t. a *tame* CBA-metric, i.e., which is the uniform limit of piecewise smooth CBA-metrics. An efficient estimate of $|\tau|$ follows from the Gauss-Bonnet formula.

8.2. Signed curvature measure and the Gauss-Bonnet formula for tame metrics. Let X be a 2-polyhedron with piecewise smooth metric d. For a face $f \subset X$, let K denote the Gaussian curvature of f. The curvature of a Borel subset $B \subset f$ is defined by

$$\omega(B) = \int_B K \, d\sigma,$$

where σ is the area measure of f.

For an edge e and a face f adjacent to e, we denote by τ_f the turn of e from the side of f, that is, for a Borel subset $B \subset e$, $\tau_f(B) = \int_B k_f \, ds$, k_f being the geodesic curvature of e with respect to f. The sign of k_f is chosen in such a way that k_f be positive for a convex f. By definition, the curvature of a Borel subset $B \subset e$ is the sum of the turns from the side of all faces adjacent to e,

$$\omega(B) = \sum_{f|e} \tau_f(B).$$

For a vertex v, let $\chi(v) = \chi(\Lambda_v)$ be the Euler characteristic of the link of v in X. On Λ_v, the metric d induces the angle (pseudo-)metric α_d in which the length of an edge corresponding to a face $f \subset X$ is the angle $\alpha(v, f)$ of this face at v, $0 \le \alpha(v, f) \le 2\pi$. Let $\alpha(v)$ be the length of the link Λ_v with respect to α_d, i.e., the sum of the lengths of all edges of Λ_v. By definition, the curvature of v is

$$\omega(v) = (2 - \chi(v))\pi - \alpha(v).$$

By additivity, these definitions extend to the Borel subsets $B \subset X$. This defines the *signed curvature measure* ω of a piecewise smooth metric on X. Now, we have the Gauss-Bonnet formula

$$\omega(X) = 2\pi\chi(X)$$

for d, where $\chi(X)$ is the Euler characteristic of X, [BaB]. The notion of the signed curvature measure ω and the Gauss-Bonnet formula have been extended to the tame CBA-metrics on X in [ArB]. Using it, the following estimate of turn was obtained there.

Let X be a closed 2-polyhedron. For the turn variation of the essential 1-skeleton of X with respect to a CBA(κ)-metric d we have

$$|\tau|(\mathrm{esk}_1 X) \leq c_1 + c_2 \cdot \omega(X \setminus \mathrm{esk}_1 X),$$

where c_1, c_2 are constants depending only on the topology of X. In particular, $|\tau|(\mathrm{esk}_1 X) \leq c_1 + c_2 \cdot \kappa \,\mathrm{Area}\,(X, d)$.

8.3. Gluing condition, characterization and approximation theorems.

We denote by $\mathcal{R}_\kappa$ the class of locally compact 2-polyhedra with a CBA(κ)-metric, all *boundary* edges of which are curves of the finite turn variation. Let M_i be a domain on a surface of class $\mathcal{R}_\kappa$ having the compact closure $\overline{M}_i$ and bounded by finitely many curves of finite turn variation (some of these curves may degenerate to points). We glue a polyhedron X from a collection of such domains M_i requiring that the following two conditions be fulfilled.

(i) For any Borel subset B of an arbitrary edge $e \subset X$ and any domains M_i, M_j adjacent to e, we have

$$\tau_i(B) + \tau_j(B) \leq 0,$$

where τ_i, τ_j are the turns of e from the (different!) sides of M_i, M_j respectively.

(ii) For any vertex $x \in X$, the length of each noncontractible loop in the link Λ_x is at least 2π.

For CBA-metrics on 2-polyhedrons we have a much more general Gluing Theorem than the Reshetnyak's theorem.

A polyhedron X glued from surfaces $M_i \in \mathcal{R}_\kappa$ is of class $\mathcal{R}_\kappa$ if and only if the conditions (i), (ii) are fulfilled.

The proof is based on the following Limit Metric Theorem.

Assume that a metric d on a compact 2-polyhedron X is the uniform limit of a sequence $d_n \in \mathcal{R}_\kappa$ such that the positive curvature parts ω_n^+ of these metrics are uniformly bounded on $X \setminus \mathrm{esk}_1 X$ and the turn variations of the boundary edges are uniformly bounded. Then $d \in \mathcal{R}_\kappa$.

This was proved in [BuB] under an additional condition that the lengths of $\mathrm{esk}_1 X$ are uniformly bounded. In [Iv], it was shown that this condition follows from the others. The main issue in the proof of the Limit Metric Theorem is to obtain a uniform separation from zero of the convexity radius for metrics d_n.

Furthermore, we have a Characterization Theorem of CBA-metrics on 2-polyhedra:

Each polyhedron $X \in \mathcal{R}_\kappa$ can be glued from surfaces $M_i \in \mathcal{R}_\kappa$ in such a way that the conditions (i), (ii) are fulfilled.

The proof is based on the following Approximation Theorem [BuB].

Every metric $d \in \mathcal{R}_\kappa$ on a locally compact 2-polyhedron X is the uniform limit of a sequence of piecewise smooth metrics $d_n \in \mathcal{R}_\kappa$ on X such that the curvature variations $|\omega_n|$ of d_n are (locally) uniformly bounded on $X \setminus \mathrm{esk}_1 X$, the essential 1-skeleton $\mathrm{esk}_1 X$ has (locally) uniformly bounded lengths, and the boundary edges have (locally) uniformly bounded turn variations.

The proof essentially uses the infinitesimal theory of locally compact CBA-spaces due to B. Kleiner.

It would be much desirable to replace the condition in the Limit Metric Theorem that the metric d is a certain uniform limit by a weaker condition that d is the homotopy limit of metrics d_n on the corresponding 2-polyhedra X_n. However, this is still an open question.

8.4. Rigidity due to Gauss-Bonnet.

There is another remarkable geometric effect in the class of 2-polyhedra with a NPC-metric which is completely invisible for surfaces. To start with we consider the following invariant of a finite graph Λ,

$$\sigma(\Lambda) := \inf_{\alpha} \frac{\alpha(\Lambda)}{\mathrm{sys}\,(\alpha)},$$

the *size* of Λ, where α is a length (pseudo)-metric on Λ, $\alpha(\Lambda)$ the length of Λ, $\mathrm{sys}\,(\alpha)$ the length of a shortest essential loop in Λ. Next, we put

$$\omega(\Lambda) := (2 - \chi(\Lambda))\pi - 2\pi\sigma(\Lambda)$$

be the *maximal total curvature* of Λ. Then $\omega_\alpha(\Lambda) := (2-\chi(\Lambda))\pi - \alpha(\Lambda) \leq \omega(\Lambda)$ for every length metric α on Λ with $\mathrm{sys}\,(\alpha) \geq 2\pi$. A length metric α is called *minimal* if $\omega_\alpha(\Lambda) = \omega(\Lambda)$, which is equivalent to $\alpha(\Lambda) = 2\pi\sigma(\Lambda)$.

A NPC-metric d on a closed 2-polyhedron X is *tight* if all maximal surfaces of X are flat, all maximal essential edges of X are geodesics and for any essential vertex $v \in X$ the induced angle metric α_d on the link Λ_v is minimal.

Assume that a closed 2-polyhedron X admits a NPC-metric. Then $\sum_{v\in V_e} \omega(\Lambda_v) \geq 2\pi\chi(X)$ and equality holds if and only if one, and hence any, NPC-metric of X is tight.

This easily follows from Gauss-Bonnet. The reason is that for the summands of Gauss-Bonnet

$$2\pi\chi(X) = \sum_v \omega_\alpha(\Lambda_v) + \sum_f \omega(f) + \sum_e \omega(e)$$

we have $\omega(f)$, $\omega(e) \leq 0$, $\omega_\alpha(\Lambda_v) \leq \omega(\Lambda_v)$ for any NPC-metric on X, and the equalities imply that the metric is tight.

The examples of polyhedra with tight metrics include branched coverings of degree 2 over 2-skeleton of n-simplex, $n \geq 4$, factors Y/Γ, where Y is a thick Euclidean 2-building and Γ is a properly discontinuous and cocompact group of automorphisms of X, etc. Furthermore, there are closed 2-polyhedra with Gromov hyperbolic fundamental group carrying tight NPC-metrics. Such a polyhedron admits no CAT(-1)-metric, [BaB], and, moreover, its fundamental group admits no discrete, cocompact action on any CAT(-1) 2-polyhedron, [Ka].

8.5. Metrics with bounded total curvature.

For more than forty years, there is a theory of metrics on surfaces which includes CBA- and CBB-metrics as rather particular cases. These are the metrics with *bounded total curvature* (BTC-metrics) synthetically defined by the requirement that total excess of any system of nonoverlapping geodesic triangles be (locally) uniformly bounded, [AZ]. From an equivalent analytical point of view, such a metric can locally be given by

$$ds^2 = \lambda(x,y)(dx^2 + dy^2),$$

where $\ln\lambda(x,y)$ is the difference of two subharmonic functions, [Re2]. This class of metrics is closed in the topology of the uniform convergence, and curves with the bounded turn variation play a key role for a gluing theorem and the Gauss-Bonnet formula. Gluing two BTC-surfaces along boundaries which are curves with the bounded turn variation gives a BTC-surface.

All this together with the Characterization Theorem for CBA-metrics on 2-polyhedra allows to suggest that it should exist a theory of BTC-metrics on 2-polyhedra. However, it is even unclear how to define such a metric: any straightforward generalization of the surface case fails.

A particular related question is this. Let Y be again the union of three rays with the common vertex v, $X = Y \times \mathbb{R}^2$. Assume the a CBA-metric on X is given. Is it true that the metric induced on the singular edge $v \times \mathbb{R}^2$ has the bounded total curvature?

Finally, I mention a recent result by V. Slavskii, [Sl], which estimates the deviation of a BTC-metric from a model one via the variation of the curvature measure. Let ω and Area be the curvature and the area measures respectively of a BTC-metric on a surface P homeomorphic to the closed disc. For $\kappa \in \mathbb{R}$, we let $\omega_\kappa = \omega - \kappa\,\mathrm{Area}$ be a signed measure, ω_κ^+, ω_κ^- its positive and negative parts, $|\omega_\kappa| = \omega_\kappa^+ + \omega_\kappa^-$.

If $\omega_\kappa^+(P) + \kappa^+\,\mathrm{Area}\,(P) < \pi$, *then there exists a locally bilipschitz map* $F : P \to M_\kappa$ *with the local Lipschitz constant* $\mathrm{Lip}\,(F) = \exp\left(\frac{1}{\pi}|\omega_\kappa|(P)\right)$.

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