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VOLUME ENTROPY OF HYPERBOLIC GRAPH SURFACES

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Abstract.

A graph surface P is a 2-dimensional polyhedron having a simplest kind of nontrivial singularities which result from gluing surfaces with compact boundaries along boundary components. We study the behavior of the volume entropy $h(g)$ of hyperbolic metrics g on a closed graph surface P depending on the lenghts of singular geodesics $Q \subset P$. We show that always $h(g) > 1$ and $h(g) \to \infty$ as $L_g(Q) \to \infty$ for at least one singular geodesic Q.

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1. Introduction

A *graph surface* P is a 2-dimensional polyhedron built out of compact surfaces $\{P_v\}_{v\in V}$ with nonempty boundaries glued along boundary components. A loop $Q \subset P$ which results from gluing at least three boundary components is said to be *singular*. The surfaces P_v, $v \in V$ are called the *blocks* of P. We assume that P is not a surface, i.e., it contains at least one singular loop, and that every block is maximal, i.e., each its boundary component considered as a curve in P is singular. Furthermore, we always assume that the Euler characteristic $\chi(P_v) < 0$ for each block P_v. A compact graph surface has only a finite number of singular loops. A compact graph surface without boundary is said to be *closed*.

All metrics on P we consider are assumed to have geodesic singularities, i.e., every singular loop $Q \subset P$ is geodesic. A metric g on P is said to be *hyperbolic* if its Gaussian curvature $K_g \equiv -1$ outside of singularities. It follows from our definitions that any graph surface P admits a hyperbolic metric g with prescribed lengths $L_g(Q) = l_Q$ of the singular loops $Q \subset P$.

We study the volume entropy of nonpositively curved (NPC for shortness) metrics on P. Given a NPC-metric g on P, we consider the metric universal covering X of P and put

$$h(g) = \lim_{R\to\infty} \frac{1}{R} \ln \operatorname{vol} B_R(x),$$

where $B_R(x)$ is a (closed) ball of radius R centered at $x \in X$. It is well known [5] that this limit exists and is independent of the choice of $x \in X$.

Theorem 1.1. *Let P be a closed graph surface. Then*
(1) *$h(g) > 1$ for each hyperbolic metric g on P;*
(2) *if $L_{g_t}(Q) \to \infty$ for a singular loop $Q \subset P$ and a family of hyperbolic metrics g_t on P then $h(g_t) \to \infty$;*
(3) *$h(g_t) \to 1$ for a family of hyperbolic metrics g_t on P if and only if all singular loops $Q \subset P$ collaps, i.e., $L_{g_t}(Q) \to 0$.*

The invariant $h(g)$ coincides with other important quantities such as the Hausdorff dimension of the boundary at infinity $\partial_\infty X$ and the infimal dimension of invariant densities on $\partial_\infty X$. It plays a fundamental role in many questions, however, its precise value is known only in exceptional cases, cp. [7].

Our main tool is a modified Poincaré series $\mathcal{P}_S(t)$ introduced in Subsec. 2.2. This is a version of the standard Poincaré series $\mathcal{P}(t)$ motivated by [4] where instead of a group orbit the summation is taken over the singular lines. The critical exponent of $\mathcal{P}_S(t)$ is still the one of $\mathcal{P}(t)$ and thus it coincides with $h(g)$. We prove Theorem 1.1(1) in Sect. 3. The central point of the paper is Sec. 5, where (2) is proven by using Interval Exchange Transformations. The "if" condition from (3) is proven in Sec. 4 referring to the initial definition of $h(g)$. The "only if" condition from (3) is proven in the last section with help (1), (2) and semicontinuity of h w.r.t. the geometric convergence.

2. More about graph surfaces

Here we collect some properties of graph surfaces which we need for the proof of our results.

2.1. The building associated with a graph surface.

The universal covering X of a closed graph surface P has a natural structure of a building $\mathcal{X}$ whose apartaments are homeomorphic to $\mathbb{R}^2$ and every chamber covers a block of P. It is convenient to describe the combinatorial structure of $\mathcal{X}$ via a graph $G_{\mathcal{X}}$ which is defined as follows. The set of its vertices is decomposed as $S \cup V$. The vertices from S are (one-to-one) associated with the edges of $\mathcal{X}$ and the vertices from V with the chambers of $\mathcal{X}$. Every edge of $G_{\mathcal{X}}$ connects vertices of different types and two such vertices $s \in S$, $v \in V$ are connected by an edge if and only if the edge of $\mathcal{X}$ corresponding to s is a boundary component of the chamber corresponding to v. Therefore, the sets S, V are infinite, each vertex $s \in S$ has a finite degree at least 3, and the degree of each vertex $v \in V$ is infinite. Obviously, the graph $G_{\mathcal{X}}$ is a tree.

The action of the group $\Gamma = \pi_1(P)$ on X preserves the building structure $\mathcal{X}$ and hence defines an action of Γ on $G_{\mathcal{X}}$. The quotient $G_{\mathcal{X}}/\Gamma = G_P$ is a finite graph associated with G_P.

We define the *combinatorial* distance on $G_{\mathcal{X}}$ by the condition that the length of every edge of $G_{\mathcal{X}}$ equals $1/2$. Therefore, for $s \in S$, $v \in V$ the distance $d(s,v)$ is a half-integer and $|sv| := d(s,v) - 1/2$ is equal to the number of vertices from S lying in the interior of the segment $sv \subset G_{\mathcal{X}}$.

2.2. Poincaré series.

We fix a reference point $x_0 \in X$ and assuming that a NPC-metric on P is given, we consider the Poincaré series

$$\mathcal{P}(t) = \sum_{\gamma \in \Gamma} \exp\left(-t \operatorname{dist}(x_0, \gamma x_0)\right).$$

Its critical exponent δ is defined as

$$\delta = \sup\{t \in \mathbb{R} \colon \mathcal{P}(t) \text{ diverges}\} = \inf\{t \in \mathbb{R} \colon \mathcal{P}(t) \text{ converges}\}.$$

It is easy to see that δ is independent of the choice of the reference point x_0. The following fact is well known and we give its proof for convenience of the reader.

Lemma 2.1.. *Let g be a NPC-metric on P. Then the critical exponent of $\mathcal{P}$ coincides with $h(g)$.*

Proof. We fix $R > 0$ and represent

$$\mathcal{P}(t) = \sum_{k=0}^{\infty} \sum_{kR \le |x_0 \gamma x_0| < (k+1)R} e^{-t|x_0 \gamma x_0|},$$

where we abbreviate $|xx'| := \operatorname{dist}(x, x')$. Thus

$$e^{-R} \sum_{k=0}^{\infty} s_k e^{-tkR} \le \mathcal{P}(t) \le \sum_{k=0}^{\infty} s_k e^{-tkR},$$

where $s_k = \#\{\gamma \in \Gamma \colon kR \le |x_0 \gamma x_0| < (k+1)R\}$.

Next we show that the critical exponent for $\mathcal{P}$ coincides with

$$\delta = \limsup_{k \to \infty} \frac{1}{kR} \ln s_k.$$

To this end we fix ε, $0 < \varepsilon < \delta$. Then $s_k \le e^{(\delta+\varepsilon)kR}$ for all sufficiently large k, say for $k \ge k_0$. Then

$$\sum_{k \ge k_0} s_k e^{-(\delta+2\varepsilon)kR} \le \sum_{k \ge k_0} e^{(\delta+\varepsilon)kR} e^{-(\delta+2\varepsilon)kR} = \sum_{k \ge k_0} e^{-\varepsilon kR} < \infty.$$

Hence, $\mathcal{P}(t)$ converges for all $t > \delta$. Furthermore, there are infinitely many k with $s_k \geq e^{(\delta-\varepsilon)kR}$. Thus

$$\mathcal{P}(\delta-\varepsilon) \geq e^{-R}\sum_{k=0}^{\infty} s_k e^{-(\delta-\varepsilon)kR} \geq e^{-R}\sum_{k} e^{(\delta-\varepsilon)kR}e^{-(\delta-\varepsilon)kR} = \infty,$$

and $\mathcal{P}(t)$ diverges for all $t < \delta$.

Now we put $n_k := \sum_{i=0}^{k} s_i$ be the number of points of the orbit Γx_0 in the ball $B_{kR}(x_0)$. We claim that $\delta = \delta'$, where

$$\delta' = \limsup_{k\to\infty} \frac{1}{kR}\ln n_k.$$

Obviously, $\delta' \geq \delta$. Let us prove the opposite inequality. One can assume that $\delta' > 0$. Assume that $s_k \leq e^{(\delta'-\varepsilon)kR}$ for some $0 < \varepsilon < \delta'$ and all $k \geq k_1$. Then

$$n_k = n_{k_1-1} + \sum_{i=k_1}^{k} s_i \leq n_{k_1-1} + ce^{(\delta'-\varepsilon)kR},$$

where c is independent of k. It follows that $\delta' \leq \delta' - \varepsilon$, a contradiction. Hence, $\delta = \delta'$. Finally, $\operatorname{vol} B_{kR}(x_0) \asymp n_k$ and consequently $h(g) = \delta'$. □

Every connected component of the inverse image $\pi^{-1}(Q)$ of a singular loop $Q \subset P$ under the universal cover $\pi : X \to P$ is an infinite geodesic with respect to the metric lifted to X. Such geodesics are called *singular lines*. Every singular line is adjacent to at least three chambers of $\mathcal{X}$.

For $s \in S$ we put $\mathrm{d}_{\mathrm{geo}}(s) = \operatorname{dist}(x_0, s)$ and define a modified Poincaré series

$$\mathcal{P}_S(t) := \sum_{s\in S} \exp\left(-t\,\mathrm{d}_{\mathrm{geo}}(s)\right).$$

Lemma 2.2. *Assume that a NPC-metric g on P is fixed. Then the critical exponents of $\mathcal{P}$ and $\mathcal{P}_S$ coincide and their common value is equal to the volume entropy $h(g)$ of g.*

Proof. We fix $s_0 \in S$, put $S_0 = \Gamma s_0 \subset S$ and define $\mathcal{P}_{S_0}(t) = \sum_{s\in S_0} \exp\left(-t\,\mathrm{d}_{\mathrm{geo}}(s)\right)$. It suffices to show that $\mathcal{P}$ and $\mathcal{P}_{S_0}$ have a common critical exponent since S is a finite union of the Γ-orbits of its points and thus $\mathcal{P}_S$ is a finite sum of series of type $\mathcal{P}_{S_0}$.

We choose a reference point x_0 lying on the singular line s_0. The Γ-orbit of x_0 can be represented as the union over all $s \in S_0$ of the orbits of infinite cyclic groups conjugated to the stabilizer in Γ of s_0. For $s \in S_0$ we consider the point $x_s \in s$ closest to x_0. Let γ_s be a generator of the stabilizer $\mathrm{St}(s) \subset \Gamma$ of s, y_0 be a closest to x_0 point from $s \cap \Gamma x_0$, $L = \operatorname{dist}(y_0, \gamma_s y_0)$. Clearly, L is independent of the choice $s \in S_0$ since it is equal to the length of the singular geodesic $Q_0 = \pi(s_0) \subset P$. For $n \in \mathbb{Z}$ we put $y_n = \gamma_s^n y_0$. One can assume that $x_s \in y_0y_1$. Then

$$\operatorname{dist}(x_0, y_n) \leq \operatorname{dist}(x_0, y_0) + |n|L \leq \mathrm{d}_{\mathrm{geo}}(s) + |n+1|L.$$

This implies

$$\mathcal{P}(t) \geq c(t)\mathcal{P}_{S_0}(t),$$

where $c(t) = \sum_{n\in\mathbb{Z}} \exp\left(-t|n|L\right) = (\exp(tL)+1)\,/\,(\exp(tL)-1)$.

On the other hand, since P admits a hyperbolic metric, X is Gromov-hyperbolic with respect to any metric lifted from P. In particular, the metric on X is Gromov-hyperbolic. Then

$$\operatorname{dist}(x_0, y_n) \geq \mathrm{d}_{\mathrm{geo}}(s) + |n|L - \sigma$$

for some constant $\sigma > 0$ independent of s and n because x_0x_s is orthogonal to s and thus the angle of the triangle $x_0x_sy_n$ at x_s is separated from zero independently of $s \in S$ and $n \in \mathbb{Z}$. This implies

$$\mathcal{P}(t) \le C(t)\mathcal{P}_{S_0}(t),$$

where $C(t) = \exp(\sigma t)c(t)$. The assertion immediately follows from the obtained estimates. $\square$

3. Estimating the entropy from below

In this section we prove the estimate (1) from Theorem 1.1. Let X be the universal covering of a closed graph surface P endowed with a hyperbolic metric g. We use the same notation g for its lift to X. The group $\Gamma = \pi_1(P)$ acts on the set S of singular lines in X, and we fix a (finite) set $\Sigma \subset S$ which represents the Γ-orbits of this action. In other words, Σ is actually the set of all singular loops in P.

Given $\sigma \in \Sigma$ we let $\mathcal{K}_\sigma$ be the set of all chambers in $\mathcal{X}$ adjacent to σ, $K_\sigma = \cup\{K\colon K \in \mathcal{K}_\sigma\}$, $S_\sigma \subset S$ the set of all boundary singular lines of K_σ, $\partial K_\sigma = \cup\{s\colon s \in S_\sigma\}$. Then the finite number $k_\sigma = |\mathcal{K}_\sigma| \ge 3$ is the degree of the singular line σ.

Next we fix $x_\sigma \in \sigma$ and for each $s \in S_\sigma$ we put $l_s(x_\sigma) = \mathrm{dist}(x_\sigma, s)$. Finally, for every chamber $K \in \mathcal{K}_\sigma$ we let $S_{K,\sigma}$ be the subset of S_σ consisting of all singular lines which are not boundary components of K, $m_K(x_\sigma) = \sum_{s\in S_{K,\sigma}} \exp\left(-l_s(x_\sigma)\right)$.

Lemma 3.1.. *For every $\sigma \in \Sigma$, $x_\sigma \in \sigma$, $K \in \mathcal{K}_\sigma$ we have*

$$\frac{(k_\sigma - 1)\pi}{4} \le m_K(x_\sigma) \le k_\sigma - 1.$$

Proof. The estimate from below is based on the well known fact that the limit set of every fuchsian group of the second kind has the Hausdorff dimension strictly less than 1 (see [1], [2], [9]). Thus the boundary at infinity $\partial_\infty K \subset S^1$ has zero Lebesgue measure for every chamber K of $\mathcal{X}$, where K is considered as a subset of H^2 and $S^1 = \partial_\infty \mathrm{H}^2$.

From the hyperbolic geometry we know that $\exp\left(-l_s(x_\sigma)\right) = \tan(\psi_{s,\sigma}/4)$ for every $s \in S_\sigma$, where $\psi_{s,\sigma}$ is the angle at which the line s is observed from x_σ. Using that $\alpha \le \tan\alpha \le \frac{4}{\pi}\alpha$ for every $\alpha \in [0, \pi/4]$ and that $0 < \psi_{s,\sigma} \le \pi$, we obtain

$$\frac{\psi_{s,\sigma}}{4} \le \exp\left(-l_s(x_\sigma)\right) \le \frac{1}{\pi}\psi_{s,\sigma}.$$

By the remark made at the beginning of the proof we have $\sum_{s\in S_{K,\sigma}} \psi_{s,\sigma} = (k_\sigma - 1)\pi$ for every chamber $K \in \mathcal{K}_\sigma$. Hence

$$\frac{(k_\sigma - 1)\pi}{4} \le m_K(x_\sigma) \le k_\sigma - 1.$$

$\square$

For $\sigma \in \Sigma$, $K \in \mathcal{K}_\sigma$ we consider the function

$$m_K(x_\sigma, t) = \sum_{s\in S_{K,\sigma}} \exp\left(-tl_s(x_\sigma)\right),$$

$x_\sigma \in \sigma$, $t \ge 1$. By Lemma 3, $m_K(x_\sigma, t) \le k_\sigma - 1$ for every $x_\sigma \in \sigma$, $t \ge 1$.

Every isometry from Γ which leaves invariant the singular line σ also leaves invariant each chamber adjacent to σ. Thus m_K is L_σ-periodic in x_σ, where L_σ is the length of the singular closed geodesic $\pi(\sigma) \subset P$. Furthermore, m_K is obviously continuous both in x_σ, t and monotonically decreases to 0 as $t \to \infty$. Since $(k_\sigma - 1)\pi/4 > 1$, it follows from Lemma 3.1 that for each $x_\sigma \in \sigma$ there exists $p(x_\sigma, K) > 1$ such that $m_K(x_\sigma, p(x_\sigma, K)) = 1$. Furthermore, due to L_σ-periodicity we have $p_{\sigma,K} := \inf_{x_\sigma \in \sigma} p(x_\sigma, K) > 1$. Thus

$$p := \min \{p_{\sigma,K} \colon \sigma \in \Sigma, K \in \mathcal{K}_\sigma\} > 1.$$

We take a reference point $x_0 = x_{\sigma_0}$ with $\sigma_0 \in \Sigma$ and put $\mathcal{K}_0 := \mathcal{K}_{\sigma_0}$, $K_0 := K_{\sigma_0}$, $S_0 := S_{\sigma_0}$. Then for each $s \in S_0$ we define the number $l_s := l_s(x_0)$ and the point $x_s \in s$ with $\mathrm{dist}(x_0, x_s) = l_s$.

Consider a chamber $K \notin \mathcal{K}_0$ adjacent to K_0 along $s \in S_0$. Now for a singular line $s' \subset \partial K$ we put $l_{s'} := l_s + \mathrm{dist}(x_s, s')$, $x_{s'} \in s'$ with $\mathrm{dist}(x_s, x_{s'}) = \mathrm{dist}(x_s, s')$. By iteration of this construction, for every singular line $s \in S$ we obtain a point $x_s \in s$ and a number $l_s > 0$ (the points x_s play an auxiliary role and they are needed only to define l_s).

We put $k(P) := \min_{\sigma \in \Sigma} k_\sigma$ be the minimal degree of singularities. Then, we recall, $k(P) \geq 3$.

Lemma 3.2. *For each $n \in \mathbb{N}$ we have*

$$\sum_{d(s_0,s)=n} \exp(-pl_s) \geq \frac{k(P)}{k(P)-1},$$

where $d(s_0, s)$ is the combinatorial distance between s_0, $s \in G_{\mathcal{X}}$.

Proof. It follows from the definition of p that $m_K(x_\sigma, p) \geq 1$ for each $\sigma \in \Sigma$, $x_\sigma \in \sigma$, $K \in \mathcal{K}_\sigma$. Hence $\sum_{s \in S_0} \exp(-pl_s) \geq \frac{k(P)}{k(P)-1}$, i.e., the assertion is true for $n = 1$. Let $n \geq 2$. For $s \in S$ with $d(s_0, s) = n$ there is a unique $s' \in S$ with $d(s_0, s') = n-1$, $d(s', s) = 1$. Then $l_s = l_{s'} + \mathrm{dist}(x_{s'}, s)$. Furthermore, there exists an isometry $\phi \in \Gamma$ with $\sigma := \phi(s') \in \Sigma$. Then

$$\sum_{s''} e^{-p \,\mathrm{dist}(x_{s'}, s'')} = m_K(\phi(x_{s'}), p) \geq 1,$$

where $K \in \mathcal{K}_\sigma$ is the chamber having a nontrivial intersection with the geodesic segment $\phi(x_0 x_{s'})$ and the summation is over $s'' \in S$ with $d(s', s'') = 1$, $d(s_0, s'') = n$. Thus

$$\sum_{d(s_0,s)=n} e^{-pl_s} \geq \sum_{d(s_0,s')=n-1} e^{-pl_{s'}},$$

which implies the required estimate. ☐

Proposition 3.3. *Let P be a closed graph surface endowed with a hyperbolic metric g. Then $h(g) > 1$.*

Proof. We have $\mathrm{d}_{\mathrm{geo}}(s) = \mathrm{dist}(x_0, s) \leq l_s$ for all $s \in S$ by the definition of l_s. Using Lemma 3.2 we obtain

$$\mathcal{P}_S(p) = \sum_{s \in S} e^{-p \,\mathrm{d}_{\mathrm{geo}}(s)} \geq 1 + \sum_{n \geq 1} \sum_{d(s_0,s)=n} e^{-pl_s} = \infty.$$

Thus $h(g) \geq p > 1$. ☐

4. Estimating the entropy from above

Here we prove that $h(g_t) \to 1$ if all singular loops $Q \subset P$ collaps for a family of hyperbolic metrics g_t. This implies, in particular, that the estimate $h(g_t) > 1$ is optimal.

Proposition 4.1. *If $L_{g_t}(Q) \to 0$ for all singular loops $Q \subset P$ and a family of hyperbolic metrics g_t on P, then $h(g_t) \to 1$.*

Proof. We use the initial definition of $h(g)$. Let $\Lambda_t(Q)$ be the radius of the maximal regular tubular neighborhood of a singular geodesic $Q \subset P$ with respect to g_t, $\Lambda_t = \min_Q \Lambda_t(Q)$. Then $\Lambda_t \to \infty$ since $L_{g_t}(Q) \to 0$ for all Q. Let $\overline{k} = \max_{\sigma\in\Sigma} k_\sigma$ be the maximal degree of singularities. For each t, we consider the following space Y_Λ which depends only on $\Lambda = \Lambda_t$ and $\overline{k}$ (since $\overline{k}$ is fixed in our arguments we drop it from the notation).

The space Y_Λ is a 2-dimensional polyhedron having a S^1-rotational symmetry around a distinguished point y_0 which is built of a countable set of copies of the hyperbolic plane H^2 glued along concentric disks and annuli. Each geodesic section of Y_Λ starting at y_0 is an infinite tree rooted at y_0 all of whose vertices except y_0 have the degree $\overline{k}$ and the length of each edge is equal to Λ (the degree of y_0 equals 1).

Let X be the universal covering of P, $x_0 \in X$ a point with $\mathrm{dist}(x_0, s) \geq \Lambda$ for each singular line $s \subset X$. Then, obviously, the volume of each ball in X centered at x_0 is less or equal to the volume of the ball of the same radius in Y_Λ centered at y_0. Hence, $h(g_t) \leq h(Y_{\Lambda_t})$.

An easy straightforward computation shows that the ball in Y_Λ of the radius $n\Lambda$ has the volume $\asymp \overline{k}^{n-1} e^{n\Lambda}$. Thus

$$h(Y_\Lambda) = \lim_{n\to\infty} \frac{n\Lambda + (n-1)\ln \overline{k}}{n\Lambda} = 1 + \frac{\ln \overline{k}}{\Lambda}.$$

Finally, we obtain

$$1 < h(g_t) \leq 1 + \frac{\ln \overline{k}}{\Lambda_t} \to 1$$

as $L_{g_t}(Q) \to 0$ for all singular loops $Q \subset P$. □

5. The entropy of a graph surface with long singular geodesic

Here we show that the entropy $h(g)$ of a hyperbolic metric g goes to infinity as the length at least one singular geodesic, $L_g(Q)$, becomes arbitrarily large. This effect is due to the Gauss-Bonnet formula

$$\mathrm{vol}(g) = 2\pi \sum_{v\in V} |\chi(P_v)|.$$

To indicate the idea of the approach we start with a simplest case having additional symmetries. Namely, we suppose that a graph surface P consists of three blocks each of which is a torus with a hole. Furthermore, we consider a hyperbolic metric g on P such that the blocks are pairwise isometric by reflections with respect to the singular loop $Q \subset P$. Then

$$h(g) \geq \frac{\ln 2}{2\Lambda(g)},$$

where $\Lambda(g)$ is the radius of the maximal regular tubular neighborhood of Q (note that $\Lambda(g) \to 0$ as $L(Q) \to \infty$ by Gauss-Bonnet). To prove this estimate, we note that each block contains a geodesic segment of length $a = 2\Lambda$ orthogonal to Q at its ends. Due to the symmetry, these segments form a geodesic bipartite graph with two vertices and three edges between them. This graph is lifted into the universal covering X of P as an infinite homogeneous tree T. Choosing a reference point at its vertex and taking into account only singular lines intersected by T, we easily obtain

$$\mathcal{P}_S(t) \geq \sum_{n=0}^{\infty} e^{-tna} 2^n.$$

Thus $h(g) \geq \ln 2/a \to \infty$ as $L(Q) \to \infty$.

However, even in this simplest case with three pairwise isometric blocks, there is no hope to find that sort of embedded graphs with short edges if we destroy the symmetry by rotating the blocks while gluing them along Q. Moreover, several additional effects arise in general case and to treat them we consider the following construction.

Bridges and interval exchange transformations. Consider a δ-neighborhood U_δ of a singular geodesic $Q \subset P$ with δ less than the maximal radius of disks sitting in the ideal triangle in H^2. To simplify the situation, we assume that no other singular geodesic visits U_δ. Then for each $x \in U_\delta$ there is at most two minimizers connecting x with points of Q closest to it.

To show that $h(g) \to \infty$ as $L(Q) \to \infty$ it suffices to study only U_δ with arbitrarily small fixed δ. We also may assume for simplicity that the degree of Q equals three and thus $U_\delta \setminus Q$ consists of three connected components labeled by letters a_1, a_2, a_3.

The *cut locus* of U_δ consists of all $x \in U_\delta$ having two minimizers connecting x with Q. Let $B \subset U_\delta$ be the union of the cut locus and the corresponding minimizers. We define a *bridge* as the closure (in P) of a connected component of $B \setminus Q$. By Gauss-Bonnet the relative part of Q sitting in the bridges is arbitrarily close to 1 as $L(Q) \to \infty$ (while δ is fixed). Moreover, the number of bridges is bounded depending only on the topology of P because otherwise (the absolute value of) the Euler characteristic of $\overline{U}_\delta$ would be huge and a huge number of its boundary components must be filled up by disks to obtain P. The boundary ∂D of any such disk D consists of pieces of equidistants to Q approaching to Q and intersecting at small angles for sufficiently small δ. The number k of the equidistant pieces of ∂D must be greater than two by their convexity. The area of D is close to $(k-2)\pi$ by the Gauss-Bonnet formula. Hence, the number of disks is effectively bounded by the area of P, a contradiction.

We fix a reference point $x_0 \in Q$ and consider piecewise geodesic paths associated with admissible words of letters of the alphabet $A = \{a_1, a_2, a_3\}$ (a word $\alpha \in A^N$ is *admissible* if no consecutive letters coincide). Every such path p_α starts at x_0 and each its segment corresponding to a letter a_i connects points x, $x' \in Q$ as follows. We go from x along Q to a closest bridge sitting in (the closure of) a_i (at this point there is some ambiguity in the construction if there are two such bridges). Then we cross the bridge along the geodesic orthogonal to Q until we reach the cut locus. From the cut locus point we go to Q along the other minimizer and finally go to x' along Q in the same direction and by the same distance as at the first step (we assume that an orientation of Q is fixed). The path p_α is also called *admissible.*

For $i = 1, 2, 3$ we consider a disjoint decomposition $Q = B_i \cup C_i$, where B_i is the part of the bridges corresponding to the component $a_i \subset U_\delta$. Its complement C_i is the union of a bounded number of open intervals called *dangerous* (intersecting dangerous intervals by paths p_α described above makes them longer and causes difficulties for the estimate $h(g) \to \infty$). Let D_i be the set of the midpoints of the dangerous intervals. The procedure above actually defines a map $\overline{a}_i : Q \setminus D_i \to Q$, $\overline{a}_i(x) = x'$, which is isometric on each connected component (maximal interval) of the domain.

In this subsection we study the case where the images of these intervals do not overlap for each i and, hence, $\overline{a}_i : Q \setminus D_i \to Q \setminus D_i$ is an involution leaving invariant the Lebesgue measure on Q. Without loss of generality we may assume that each block of P is orientable. Then each $\overline{a}_i$ reverses the orientation on each maximal interval, thus every composition $\overline{a}_i \circ \overline{a}_j$ preserves the orientation. Hence, we can extend $\overline{a}_i \circ \overline{a}_j$ to Q by the condition this map be continuous from the right. This extension is nothing else as a well known interval exchange transformation (IET for brevity) of a circle [8]. For example, this is the case for $P = T_1 \cup T_2 \cup T_3$ where each T_i is a torus with one hole as at the beginning of this section (and without the symmetry assumption). The following Proposition in particular shows how to handle this case.

Proposition 5.1. *Given an arbitrarily small $\delta > 0$, we have $h(g) \geq \ln 2/4\delta$ for all hyperbolic metrics g on P with sufficiently large $L_g(Q)$ such that the involutions $\overline{a}_i$, $i = 1, 2, 3$ preserve the Lebesgue measure (recall that $\overline{a}_i$ depend on δ).*

Proof. Let p_α, $\alpha \in A^N$ be an admissible path of combinatorial length N starting at $x_0 \in Q$. It consists of segments corresponding to the letters of the word α. Such an segment is called *long* if its initial point lies in a dangerous interval of Q; otherwise the segment is called *short*. Every short segment has length $\leq 2\delta$ and the length of each long segment is $> 2\delta$. Let $A(\delta)$ be the maximal length of long segments. Clearly, the main contribution in $A(\delta)$ is made by the lengths of the dangerous intervals and thus $A(\delta) \leq c/\delta$ where the constant c depends only on topology of P (by Gauss–Bonnet).

For $N \in \mathbb{N}$ we put $N = N' + N''$, where $N' = N'(p_\alpha)$ is the number of short segments of the path p_α, $\alpha \in A^N$. It will be shown that for sufficiently large $L_g(Q)$, every $N \in \mathbb{N}$ and every admissible path p_α, $\alpha \in A^N$ we have

(1) $1 \geq N'/N \geq 1 - 2\delta/A(\delta)$.

Assuming this we first prove the required estimate $h(g) \geq \ln 2/4\delta$. We estimate the Poincaré series $\mathcal{P}_S(t)$ from below using the admissible paths. The length of p_α is at most $2\delta N' + A(\delta)N''$. Taking into account all admissible $\alpha \in A^N$ we see that there is at most 2^N (homotopically) different admissible paths of (combinatorial) length N. Hence, we have

$$\mathcal{P}_S(t) \geq \sum_N e^{-t(2\delta N' + A(\delta)N'')} 2^N.$$

Every summand of the right hand side sum is at least 1 for $t \leq N \ln 2/\left(2\delta N' + A(\delta)N''\right)$. Using (1) we obtain that this condition is fullfilled for each $t \leq \ln 2/4\delta$. Thus $\mathcal{P}_S(t)$ diverges for such t and hence $h(g) \geq \ln 2/4\delta$.

It remains to prove (1). Given an admissible $\alpha \in A^N$, we consider the path p_α starting at $x \in Q$. Let $N''(x, \alpha)$ be the number of long segments of p_α, i.e., $N''(x, \alpha)$ is the number of dangerous intervals of Q hit by p_α. Let $T = T(\delta)$ be the total length of dangerous intervals. We actually noticed above that $T \leq c/\delta$, where c depends only on (topology of) P.

Since the involutions $\overline{a}_i$, $i = 1, 2, 3$ preserve the Lebesgue measure on Q, we have $\int_Q N''(x, \alpha)\, \mathrm{d}\, x \leq N \cdot T$. Thus the mean value

$$\frac{1}{L_g(Q)} \int_Q \frac{N''(x, \alpha)}{N} \mathrm{d}x \leq \frac{T}{L_g(Q)}$$

can be made arbitrarily small for sufficiently large $L_g(Q)$ and we can find $x \in Q$ with $N'/N \geq 1 - 2\delta/A(\delta)$, where $N' = N - N''(x, \alpha)$. Connecting x_0 with x along Q we slightly modify the construction of p_α which results to a small factor of order $e^{-tL_g(Q)}$ in front of the sum estimating $\mathcal{P}_S(t)$ from below. However, the divergence of $\mathcal{P}_S(t)$ is independent of this factor. $\square$

5.2. Measure concentration. It is not difficult to construct examples of graph surfaces P with hyperbolic metrics having arbitrarily long singular loops for which the condition of Proposition 5.1 is not fulfilled. Moreover, one can arrange an example such that for some (periodic) sequence $\alpha \in A^\infty$ the mean value estimate above fails. This is because measures induced from the Lebesgue one by sequences of maps $\overline{a}_i$ may concentrate on dangerous intervals. We rescue the result by showing that still the mean value estimate holds for *most* of words $\alpha \in A^N$.

Now we consider a general case admitting that $\overline{a}_i : Q \setminus D_i \to Q$ may not preserve the Lebesgue measure. Recall that $\overline{a}_i$ maps each maximal interval $J \subset Q \setminus D_i$ isometrically and the sum of the lengths of all such J's (corresponding to a fixed i) equals $L_g(Q)$. In other words, we admit that images of some maximal intervals may overlap. In this case, we slightly modify $\overline{a}_i$ by affine shrinking and stretching on appropriate intersections $J \cap C_i$ with dangerous intervals in such a way that $\overline{a}_i : Q \setminus D_i \to Q \setminus D_i$ be an involution (we use the same notation $\overline{a}_i$ for a modified map). Note that each modified $\overline{a}_i$ is still isometric on each $J \cap B_i$. The concrete construction of the modification plays no role. It is only important that the induced measure $(\overline{a}_i)^*\mu_0$ is close to the Lebesgue measure μ_0 on Q. More precisely, we rescale the metric on Q by the factor $1/L_g(Q)$ so that $\mu_0(Q) = 1$. Then we have $(\overline{a}_i)^*\mu_0(Z) := \mu_0 \circ \overline{a}_i^{-1}(Z)$ for each Borel subset $Z \subset Q$, $(\overline{a}_i)^*\mu_0(Q) = \mu_0(Q) = 1$ and

$$d\left((\overline{a}_i)^*\mu_0, \mu_0\right) \le \varepsilon$$

where $\varepsilon = \varepsilon(\delta, g)$ estimates the total length of the dangerous intervals in the rescaled metric, $\varepsilon \le c/\left(\delta L_g(Q)\right)$. Here we use the metric $d(\mu, \mu') = \sup_Z |\mu(Z) - \mu'(Z)|$ on the space $\mathfrak{M}$ of Borel probability measures on Q.

Recall that for $\alpha \in A^N$ with N even, $\alpha = (a_{i_N}, \dots, a_{i_1})$ the composition $\overline{\alpha} = \overline{a}_{i_N} \circ \dots \circ \overline{a}_{i_1}$ is well defined as a map $Q \to Q$ which is (topologically conjugate to) an IET. We put $\alpha_2 = (a_2, a_1)$, $\alpha_3 = (a_3, a_1)$. Then every word $\alpha \in \hat{A}^N$ in the alphabet $\hat{A} = \{\alpha_2, \alpha_3\}$ is admissible as a word in the alphabet A and its length, $2N$, in A is even. Thus the corresponding map $\overline{\alpha}$ is an generalized (i.e., not necessary preserving the Lebesgue measure) IET, $\overline{\alpha} : Q \to Q$. For $\alpha \in \hat{A}^N$, $\alpha = (\alpha_{i_N}, \dots, \alpha_{i_1})$ we let

$$\mu_\alpha = \frac{1}{N+1}\left(\mu_0 + (\overline{\alpha}_{i_1})^*\mu_0 + (\overline{\alpha}_{i_2} \circ \overline{\alpha}_{i_1})^*\mu_0 + \dots + (\overline{\alpha})^*\mu_0\right)$$

be the mean value of induced measures along the sequence α. Next, we define

$$\mu_N = \frac{1}{|\hat{A}^N|} \sum_{\alpha \in \hat{A}^N} \mu_\alpha$$

(the measure $\mu_N \in \mathfrak{M}$ as all our constructions depends on the choice of $\delta > 0$ and the hyperbolic metric g on P).

For the union $C = C_1 \cup C_2 \cup C_3$, $C = C(\delta, g)$ of the dangerous intervals we have $\mu_0(C) < \varepsilon$ with $\varepsilon \le c/\left(\delta L_g(Q)\right)$. Thus in the case where $\overline{a}_i$ preserve the measure μ_0, we have $\mu_N = \mu_0$, in particular, $\mu_N(C) < \varepsilon$. The proof of the following Lemma is based on the same ideas as the proof of Proposition 5.1.

Lemma 5.2. *Assume that for every ε, $\delta > 0$ and every sequence $\{g_k\}$ of hyperbolic metrics on P with $L_{g_k}(Q) \to \infty$ there exists a tail of $\{g_k\}$ such that for every g from that tail there exists a sequence $N_m \to \infty$ with $\mu_{N_m}(C) < \varepsilon$. Then $h(g_k) \to \infty$ as $k \to \infty$.*

Proof. Since $\mu_{N_m}(C) < \varepsilon$, there exist at least $|\hat{A}^{N_m}|/2$ words $\alpha \in \hat{A}^{N_m}$ with $\mu_\alpha(C) < 2\varepsilon$. Thus

$$\frac{1}{(N_m+1)L_g(Q)} \int_Q N''(x, \alpha)\, \mathrm{d}x = \mu_\alpha(C) < 2\varepsilon,$$

where $N_m + 1 = N' + N''$ and $N'' = N''(x, \alpha)$ is defined in the proof of Proposition 5.1. Note that $|\hat{A}^N| = 2^N$ and recall that the maximal length $A(\delta)$ of long segments satisfies $A(\delta) \leq c/\delta$. Assuming that $\varepsilon \leq \delta^2/c$, we obtain a mean value estimate required to prove the condition (1) from the proof of Proposition 5.1 for infinitely many $N = N_m$ and thus the estimate $h(g) \geq \mathrm{const}/\delta$. Hence, $h(g_k) \to \infty$ as $k \to \infty$. □

It remains to consider the case where the condition of Lemma 5.2. is not satisfied. Then there exist ε, $\delta > 0$ and a sequence $\{g_k\}$ of hyperbolic metrics on P with $L_{g_k}(Q) \to \infty$ such that for every g_k and all sufficiently large (depending on g_k) N we have $\mu_{N,g_k}(C) \geq \varepsilon$ (the measure μ_{N,g_k} depends also on δ). Since δ is fixed and $L_{g_k}(Q) \to \infty$, we have $\mu_0(C) \to 0$ as $k \to \infty$, and we can assume that the exists a limit set $C_\infty = \lim_{k\to\infty} C$ (w.r.t. the Hausdorff topology) which is automatically finite and $|C_\infty|$ is bounded only by the topology of P. Thus we can also assume that for each $\alpha \in \hat{A}^N$ the generalized IET $\overline{\alpha}_k$ converge to an IET $\overline{\alpha} : Q \to Q$ as $k \to \infty$ in the sense that $\overline{\alpha}$, $\overline{\alpha}_k$ are conjugated by a homeomorphism $f_{\alpha,k} : Q \to Q$ and $f_{\alpha,k} \to \mathrm{id}_Q$ as $k \to \infty$. Finally, each $\overline{\alpha}$ preserves μ_0.

Lemma 5.3. *There exists a measure $\mu_\infty \in \mathfrak{M}$ with the following properties*
(i) $((\overline{\alpha}_2)^*\mu_\infty + (\overline{\alpha}_3)^*\mu_\infty) = 2\mu_\infty$;
(ii) $\mu_\infty(C_\infty) \geq \varepsilon$.

Proof. We note that $\hat{A}^{N+1} = \alpha_2\hat{A}^N \cup \alpha_3\hat{A}^N$. From the definition of $\mu_N = \mu_{N,g_k}$ we obtain

$$\frac{1}{2}\left((\overline{\alpha}_{2,k})^*\mu_{N,g_k} + (\overline{\alpha}_{3,k})^*\mu_{N,g_k}\right) + \frac{1}{N+1}\mu_0 = \frac{N+2}{N+1}\mu_{N+1,g_k}$$

and $d(\mu_{N+1,g_k}, \mu_{N,g_k}) = O(\frac{1}{N})$. Hence, the condition (i) is satisfied for each accumulation point $\mu_{\infty,g_k} \in \mathfrak{M}$ of μ_{N,g_k} as $N \to \infty$ in the weak topology. Since $\mathfrak{M}$ is compact in the weak topology such an accumulation point exists for each k.

It follows from the assumption $\mu_{N,g_k}(C) \geq \varepsilon$ that $\mu_{\infty,g_k}(C) \geq \varepsilon$. Thus any accumulation point μ_∞ for $\mu_{\infty,k}$ as $k \to \infty$ is a required measure. □

The condition (ii) means that the measure μ_∞ has atoms sitting in the finite C_∞. Together with the condition (i) this implies a strong restriction on the IET $\overline{\alpha}_2$, $\overline{\alpha}_3$. Recall that by the construction, the break point sets (i.e., the ends of the exchanged intervals) of these IET are contained in C_∞.

Lemma 5.4. *There exists $n = n(\varepsilon) \in \mathbb{N}$ such that $\overline{\alpha}_2^n$, $\overline{\alpha}_3^n$ are identities on an (half-open) interval $J \subset Q$, $J = [x, x')$ with $x \in C_\infty$.*

Proof. By properties of IET, it suffices to show that $\overline{\alpha}_2^n(x) = x = \overline{\alpha}_3^n(x)$. The group G generated by $\overline{\alpha}_2$, $\overline{\alpha}_3$ acts on Q by IET. Since $\mu_\infty(Q) = 1$, there exists $x \in G(C_\infty)$ with $\mu_\infty(x) = \max\{\mu_\infty(x') \colon x' \in G(C_\infty)\}$. Then $\mu_\infty(x) \geq \varepsilon/|C_\infty|$. By Lemma 5.3, we have $\mu_\infty(x_2) + \mu_\infty(x_3) = 2\mu_\infty(x)$, where $\overline{\alpha}_2(x_2) = x = \overline{\alpha}_3(x_3)$. By maximality of $\mu_\infty(x)$ we obtain $\mu_\infty(x_2) = \mu_\infty(x_3) = \mu_\infty(x)$. Thus the orbit $G(x)$ is finite, $\mu_\infty(x') = \mu_\infty(x)$ for each $x' \in G(x)$, consequently $|G(x)| \leq |C_\infty|/\varepsilon$, and we can assume that $x \in C_\infty$. The claim follows. □

For $i = 2, 3$ we put $\beta_i = \alpha_i^n \in A^{2n}$ and $\overline{\beta}_i = \overline{\alpha}_i^n : Q \to Q$. For every sufficiently large k, the corresponding generalized IET $\overline{\beta}_{i,k}$ leaves invariant an interval $J_{i,k} \subset Q$ and $\overline{\beta}_{i,k} \to \overline{\beta}_i$ $J_{i,k} \to J$ as $k \to \infty$. In particular, $\overline{\beta}_{i,k} : J_{i,k} \to J_{i,k}$ is a homeomorphism converging to the identity as $k \to \infty$ for $i = 2, 3$.

Let $\mathcal{B} = \{\beta_2, \beta_3\}$, $\mathcal{B}_0^\infty = \cup_{N\in\mathbb{N}}\mathcal{B}^N$ be the set of the words in the alphabet $\mathcal{B}$. For $\beta \in \mathcal{B}_0^\infty$ we denote by $\overline{\beta}_k : Q \to Q$ the corresponding generalized IET w.r.t. the metric g_k. The set

$B_k \subset \mathcal{B}_0^\infty$ consists of all β with $\overline{\beta}'_k(z) \in J'$ for each initial segment β' of β, where $z \in J$ is the midpoint and $J' \subset J$ is the middle third subinterval (we identify Q with the standard circle of length 1). We say that a subset $A \subset \mathcal{B}_0^\infty$ has an exponential growth if there exists $b > 1$ such that $|A \cap \mathcal{B}^N| \geq b^N$ for each sufficiently large N.

Lemma 5.5. *The set B_k has an exponential growth for every sufficiently large k.*

Proof. This is almost obvious. We can assume that $J' \subset J_{2,k} \cap J_{3,k}$ for all k. Since $\overline{\beta}_{2,k}$, $\overline{\beta}_{3,k}$ are almost identities on $J_{2,k}$, $J_{3,k}$ respectively and the sequence $\left\{\overline{\beta}_{2,k}^m(z)\right\}_{m\in\mathbb{Z}} \subset J_{2,k}$ is monotone, we can combine words $w \in B_k$ of arbitrarily large length from the pieces of kind β_2^{-p}, β_2^m, $m, p \in \mathbb{N}$ with β_3 inserted between them so that $w'(z) \in J'$ for each initial segment w' of w. Clearly, there are sufficiently many such words to prove an exponential growth rate of B_k. □

Corollary 5.6. *The set B_k has also an exponential growth w.r.t. the alphabet A.*

Proof. This follows from Lemma 5.4 according to which the letters β_2, β_3 of $\mathcal{B}$ are words of the length $2n$ in the alphabet A and $n = n(\varepsilon)$ is independent of k. □

Now we conclude that $h(g_k) \to \infty$ as $k \to \infty$ as follows. For each $\alpha \in B_k$ we consider the piecewise geodesic path p_α defined above which starts at z. It follows from the definition of B_k that p_α misses dangerous intervals and moreover p_α crosses bridges over segments with the length at most $\delta'_k \to 0$ as $k \to \infty$ because the interval J' and all its images under the action of the group G (see the proof of Lemma 5.4) are far away from the dangerous set C_∞. We estimate $\mathcal{P}_S(t)$ from below using the paths p_α, $\alpha \in B_k$ (cp. the introduction to this section and the proof of Proposition 5.1). Since there are exponentialy many such paths by Corollary 5.6, we obtain $h(g_k) \geq \mathrm{const}/\delta'_k \to \infty$ as $k \to \infty$.

To complete the proof of Theorem 1.1(2) we note that the assumption that other singular loops miss the neighborhood U_δ is inessential. We have never used that Q is a (topological) circle because IET are typically discontinuous. Thus we can from the very beginning define U_δ as the δ-neighborhood of the union of all singular loops in P.

6. Graph surfaces with the entropy close to 1

Here we complete the proof of Theorem 1.1 by showing that $L_{g_t}(Q) \to 0$ for all singular loops $Q \subset P$ and a family of hyperbolic metrics g_t on Q such that $h(g_t) \to 1$. The crucial ingredient is semicontinuity of h from below. To formulate it we need to extend the class hyperbolic graph surfaces. Namely, we admit that some blocks of P may be noncompact. However, we always assume that each singular curve $Q \subset P$ is compact and a hyperbolic metric g on P is complete with a finite volume. The estimate $h(g) > 1$ holds for g and its proof is the same as in the compact case.

A hyperbolic metric g on P is called *baseframed* if a standard baseframe ω is chosen at a nonsingular point of P. One says that a sequence of hyperbolic baseframed metrics (g_n, ω_n) on a graph surface P converges *geometrically* to a hyperbolic baseframed metric (g, ω) on a graph surface P' if, for each compact subpolyhedron $K \subset P'$ containing ω, there are smooth embeddings $\phi_n : K \to P$ (i.e., smooth on each subsurface of K) for every sufficiently large n such that ϕ_n sends ω to ω_n and ϕ_n tends to an isometry in the C^∞ topology (see [6]).

Proposition 6.1. *Assume that that a sequence of hyperbolic baseframed metrics (g_n, ω_n) on a graph surface P converges geometrically to (g, ω) on a graph surface P'. Then $h(g) \leq \liminf h(g_n)$.*

Proof. This follows from known results and approaches. The graph surface P' is obtained by cutting open P along the finite system of simple closed geodesics whose lengths $\to 0$ as $n \to \infty$. Taking the metric completion (w.r.t. g_n) one easily sees that this operation does not increase h because it does not increase the volume of balls in the universal covering. Thus we can assume that $P = P'$ and, using hyperbolicity of the metrics, canonically identify the boundaries at infinity $\partial_\infty X$ of the universal coverings acted upon by (isomorphic) fundamental groups Γ_n, Γ of (P, g_n), (P, g) respectively.

Next, one uses the fact that $h(g_n)$ equals the infimum (which is achieved) of the dimensions of Γ_n-invariant densities on $\partial_\infty X$. This is proven in [3], Lemma 6.7 for CAT(-1)-spaces (which are hyperbolic graph surfaces) following the classical argument of [10], Cor. 4 for Kleinian groups.

Now, [6], Theorem 7.7 applies: by the last result, for each n there is a Γ_n-invariant density of dimension $h(g_n)$. Taking the weak limit of a subsequence, one obtains a Γ-invariant density of dimension $\liminf h(g_n)$. Thus $h(g) \leq \liminf h(g_n)$ again the result above. $\square$

Proof of Theorem 1.1. The property (1) is proven in Sec. 3, (2) in Sec. 5. Taking into account the result of Sec. 4 it remains to show that $L_{g_n}(Q) \to 0$ for each singular loop $Q \subset P$ if $h(g_n) \to 1$ for a sequence of hyperbolic metrics g_n on P.

Assume that it is not the case. By Theorem 1.1 (2), the lengths $L_{g_n}(Q)$ of all singular loops are uniformly bounded above. Choosing baseframes ω_n appropriately, we can assume that (g_n, ω_n) converge geometrically to a hyperbolic baseframed metric (g, ω) on a graph surface P' having a singular loop Q with $0 < L_g(Q) < \infty$. Then $h(g) > 1$ by Theorem 1.1 (1). On the other hand, we have $h(g) \leq \liminf_n h(g_n) = 1$ by Proposition 6.1. This contradiction completes the proof of Theorem 1.1. $\square$

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