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# On a family of Schur rings over a finite cyclic group

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**Abstract**

We disprove the following long-time hypothesis: any Schur ring over a finite cyclic group is determined by a suitable permutation group having this cyclic group as a regular subgroup. In addition, we give a negative solution to the problem whether any weak isomorphism of Schur rings over a finite cyclic group is induced by a strong isomorphism.

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# 1 Introduction

In 1933 I. Schur showed that any primitive permutation group containing a regular cyclic subgroup of composite order is 2-transitive. In contrast to W. Burnside who using the character theory proved a similar statement for cyclic groups of prime power order, the idea of Schur was to reduce the problem to studying special subrings of a group ring. Namely he showed that given a permutation group $G$ containing a regular subgroup $H$, the $\mathbb{Z}$-module spanned by the orbits of a one-point stabilizer of $G$ (treated after the choice of the point in a natural way as subsets of $H$) form a subring of the group ring of $H$. This subring is obviously closed under the componentwise multiplication and the involution induced by taking inverse elements in $H$. Since then any subring of the group ring of $H$ satisfying the last property is called a Schur ring (or briefly S-ring) over $H$. H. Wielandt wrote in [12, p.54] that "Schur had conjectured for a long time that every S-ring over $H$ is determined by a suitable permutation group $G$". However, this proved to be false and first counterexamples were found by Wielandt in [11, Theorem 25.7]. In honour of this Schur's fallacy the S-rings arising from permutation groups were called Schurian. It should be mentioned that all the counterexamples found by Wielandt are S-rings over noncyclic groups. Taking into account that Schur himself mainly worked with S-rings over a cyclic group and the fact that such S-rings are Schurian whenever the order of the group is a prime power [9, 3] or the product of two distinct primes [5], one can conjecture (and this really has been a long-time hypothesis communicated by M. Klin to the authors) that any S-ring over a finite cyclic group is Schurian. Some specialists believed that this hypothesis is surely true in the case when the order of the group is squarefree. In this paper we show that the hypothesis is false even in this case (statement (1) of Theorem 1.1).

One more problem about S-rings over a group $H$ concerns the isomorphism problem for Cayley graphs over $H$. Any such graph $\Gamma$ is given by a subset $X_\Gamma$ of $H$ (the neibourhood of 1) and generates an S-ring $\mathcal{A}_\Gamma$ which is the smallest S-ring over $H$ containing the element $\xi_\Gamma = \sum_{x \in X_\Gamma} x$. Moreover, any isomorphism from $\Gamma$ to another Cayley graph $\Gamma'$ induces a week isomorphism $\varphi : \mathcal{A}_\Gamma \to \mathcal{A}_{\Gamma'}$ such that $\varphi(\xi_\Gamma) = \xi_{\Gamma'}$. (By a week isomorphism of S-rings we mean a $\mathbb{Z}$-module isomorphism preserving the ordinary and the componentwise multiplications.) On the other hand, given Cayley graphs $\Gamma$ and $\Gamma'$ it is a routine task to determine whether there exists a weak isomorphism from $\mathcal{A}_\Gamma$ to $\mathcal{A}_{\Gamma'}$ satisfying the last property. Thus the isomorphism problem for Cayley graphs is reduced to the following problem: is a given weak isomorphism of S-rings induced by a strong isomorphism of them. (As to the definition of a strong isomorphism see Subsection 2.2; any such isomorphism inducing the above $\varphi : \mathcal{A}_\Gamma \to \mathcal{A}_{\Gamma'}$ is nothing else than an isomorphism from $\Gamma$ to $\Gamma'$.) We call an S-ring separable if any weak isomorphism from it to another S-ring over the same group is induced by a strong isomorphism. If any S-ring was separable, the isomorphism problem for Cayley graphs would become trivial. In this paper we show that there exist S-rings (even over a cyclic group of squarefree order) which are not separable (statement (2) of Theorem 1.1). This explains to some extent why the isomorphism problem for circulant graphs is still open (see [8]).

The following statement is the main result of the paper.

**Theorem 1.1** *Let $n = p_1p_2p_3p_4n'$ where $p_1, p_2, p_3, p_4$ are primes such that $\{p_1, p_2\} \cap \{p_3, p_4\} = \emptyset$ and $n'$ is a positive integer, and $D = \mathrm{GCD}(p_1 - 1, p_2 - 1, p_3 - 1, p_4 - 1)$. Then*

(1) *if $D > 2$, then there exists a non-Schurian S-ring over $\mathbb{Z}_n$;*

(2) *if $D$ is not squarefree, then there exists a non-separable S-ring over $\mathbb{Z}_n$.*

The smallest examples of non-Schurian as well as non-separable S-rings guaranteed by Theorem 1.1 arise for $n = 5^2 \cdot 13^2$ and for $n = 5 \cdot 13 \cdot 17 \cdot 29$ if $n$ is supposed to be squarefree. The explicit constructions are given in Section 4. In fact, the proof of the theorem shows that each S-ring over $\mathbb{Z}_{n'}$ yields at least one non-Schurian (resp. non-separable) S-ring over $\mathbb{Z}_n$ whenever $D > 2$ (resp. $D$ is not squarefree).

It should be noted that the problems considered in this paper are similar to those arising in the theory of separability and Schurity of coherent configurations developed by the authors in a series of papers (see [1] and references of it). In particular, we introduced in [1] the Schurity number and the separability number of a coherent configuration showing to what extent it can be non-Schurian or non-separable. Moreover, it was proved that these numbers can be arbitrarily large. It would be interesting to define and study similar numbers for S-rings.

The proof of Theorem 1.1 is based on the construction of the generalized wreath product of two S-rings (roughly speaking the resulting S-ring is obtained by identifying a factor S-ring of the first operand with a sub-S-ring of the second one) defined and studied in Section 3. In fact, the significance of this construction is much wider. As it is proved in [2] any S-ring over a finite cyclic group can be obtained by taking tensor products and generalized wreath products from S-rings of $\mathbb{Z}$-dimension 2 and orbit S-rings $\mathcal{O}(K, G)$ defined in the end of Subsection 2.1.

In Section 2 the definitions and notations concerning S-rings over Abelian groups (including those of Schurity and separability) are given. Unfortunately there is no generally accepted terminology about S-rings (see [11, 9, 5, 7, 8]). The terminology of this paper is compatible with one of [10, 1] and reflects the fact that each S-ring can be considered in a natural way as a cellular ring (see [4]). For example, algebraic and combinatorial isomorphisms from [8] are the same as isomorphisms from [7] and S-isomorphism from [5] and called here weak and strong isomorphisms respectively.

**Notation.** As usual we denote by $\mathbb{Z}$, $\mathbb{Z}_n$ and $\mathbb{Z}_n^*$ ($n$ is a positive integer) the ring of integers, the ring of integers modulo $n$ and the group of invertible elements of $\mathbb{Z}_n$. We save notation $\mathbb{Z}_n$ also for the additive group of the ring $\mathbb{Z}_n$.

Given a set $A$ the free $\mathbb{Z}$-module spanned by $A$ will be denoted by $\mathbb{Z}[A]$. For $X \subset A$ we set $\xi(X) = \sum_{x \in X} x$. If $f : A \to B$ is a mapping, then we denote by the same letter $f$ the induced $\mathbb{Z}$-module homomorphism from $\mathbb{Z}[A]$ to $\mathbb{Z}[B]$.

The cardinality of a finite set $X$ is denoted by $|X|$.

## 2 Schur rings over Abelian groups

**2.1.** Let $G$ be a finite Abelian group. On the $\mathbb{Z}$-module $\mathbb{Z}[G]$ one can define the involution, the ordinary multiplication and the Hadamard multiplication by

$$\xi^{(-1)}=\sum_{g\in G}a_g g^{-1},\quad \xi\cdot\eta=\sum_{g,h\in G}a_g b_h gh,\quad \xi\circ\eta=\sum_{g\in G}a_g b_g g$$

where $\xi=\sum_{g\in G}a_g g$ and $\eta=\sum_{g\in G}b_g g$. Obviously, $\xi(X^{-1})=\xi(X)^{(-1)}$ and $\xi(X\cap Y)=\xi(X)\circ\xi(Y)$ for any $X,Y\subset G$.

According to [11] a submodule $\mathcal{A}$ of $\mathbb{Z}[G]$ is called a *Schur ring* (briefly *S-ring*) over $G$ if it is closed under the involution, the ordinary and Hadamard multiplications and contains the unity elements (1 and $\xi(G)$) with respect to the multiplications. It follows that any S-ring $\mathcal{A}$ has a uniquely determined $\mathbb{Z}$-base consisting of the elements $\xi(X)$ where $X$ runs over a family $\mathcal{S}=\mathcal{S}(\mathcal{A})$ (also uniquely determined by $\mathcal{A}$) of pairwise disjoint nonempty subsets of $G$ such that

$$\{1\}\in\mathcal{S},\quad \bigcup_{X\in\mathcal{S}}X=G,\quad \text{and}\quad X\in\mathcal{S}\ \Rightarrow\ X^{-1}\in\mathcal{S}.$$

We call the elements of $\mathcal{S}(\mathcal{A})$ *basic sets* of $\mathcal{A}$ and denote by $\mathcal{S}^*(\mathcal{A})$ the set of all unions of them. The basic set of $\mathcal{A}$ containing $x\in G$ is denoted by $[x]$.

The set of all S-rings over a group $G$ is ordered by inclusion. The largest and the smallest elements of this set are $\mathbb{Z}[G]$ and $\mathbb{Z}1+\mathbb{Z}\xi$ where $\xi=\xi(G\setminus\{1\})$ respectively. Further, let $K$ be a subgroup of $\mathrm{Aut}(G)$ and $G\cdot K$ a semidirect product of $G$ and $K$ acting on $G$ in a natural way. Then by the Schur theorem (see [11, Theorem 24.1]) the $\mathbb{Z}$-module $\mathcal{A}$ spanned by the elements $\xi(X)$ where $X$ runs over the orbits of the one-point stabilizer $(G\cdot K)_1$ is an S-ring over $G$. Since $(G\cdot K)_1=K$, we see that $\mathcal{S}(\mathcal{A})$ coincides with the set of all orbits of $K$ on $G$. The S-ring $\mathcal{A}$ will be denoted by $\mathcal{O}(K,G)$.

**2.2.** There are two natural types of isomorphisms of S-rings. Namely, we say that S-rings $\mathcal{A}$ over $G$ and $\mathcal{A}'$ over $G'$ are *Cayley isomorphic* if there exists a group isomorphism $f:G\to G'$ such that $f(\mathcal{A})=\mathcal{A}'$ (see Notation). Any such $f$ is called a *Cayley isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$. On the other hand, $\mathcal{A}$ and $\mathcal{A}'$ are called *weakly isomorphic* if there exists a $\mathbb{Z}$-module isomorphism $\varphi:\mathcal{A}\to\mathcal{A}'$ (called a *weak isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$) such that

$$\varphi(\xi\cdot\eta)=\varphi(\xi)\cdot\varphi(\eta)\tag{1}$$

$$\varphi(\xi\circ\eta)=\varphi(\xi)\circ\varphi(\eta)\tag{2}$$

for all $\xi,\eta\in\mathcal{A}$. It follows from (2) that $\varphi$ induces a bijection $X\mapsto X^\varphi$ from $\mathcal{S}^*(\mathcal{A})$ onto $\mathcal{S}^*(\mathcal{A}')$ (taking $\mathcal{S}(\mathcal{A})$ to $\mathcal{S}(\mathcal{A}')$) such that $\varphi(\xi(X))=\xi(X^\varphi)$. Clearly, $\{1\}^\varphi=\{1'\}$ and $G^\varphi=G'$. It can also be proved that $(X^{-1})^\varphi=(X^\varphi)^{-1}$ for all $X\in\mathcal{S}^*(\mathcal{A})$.

One more type of isomorphisms of S-rings comes from isomorphisms of coherent configurations (see [5]). Namely, a bijection $f:G\to G'$ is called a *strong isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ if

$$f([x]y)=[f(x)]f(y),\quad x,y\in G.\tag{3}$$

It immediately follows from the definition that $f(1) = 1'$ and the mapping $X \mapsto f(X)$ is a bijection from $\mathcal{S}(\mathcal{A})$ onto $\mathcal{S}(\mathcal{A}')$. So $f$ induces a $\mathbb{Z}$-module isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ satisfying (2). It is easy to see that it also satisfies (1)[1] and hence is a weak isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. Certainly, any Cayley isomorphism is a strong one.

Given a weak isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ denote by $\mathrm{Iso}(\mathcal{A}, \mathcal{A}', \varphi)$ the set of all strong isomorphisms from $\mathcal{A}$ to $\mathcal{A}'$ inducing $\varphi$. We say that an S-ring $\mathcal{A}$ over $G$ is *separable* if $\mathrm{Iso}(\mathcal{A}, \mathcal{A}', \varphi) \neq \emptyset$ for any weak isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ where $\mathcal{A}'$ is an S-ring over $G$. Further, the set $\mathrm{Iso}(\mathcal{A}, \mathcal{A}, \mathrm{id}_\mathcal{A})$ is obviously a permutation group on $G$. We denote it by $\mathrm{Aut}(\mathcal{A})$ and call the *automorphism group* of $\mathcal{A}$. The S-ring $\mathcal{A}$ is called *Schurian* if the set $\mathcal{S}(\mathcal{A})$ equals the set of all orbits of the group $\mathrm{Aut}(\mathcal{A})$ on $G$. We note that the properties of an S-ring to be separable or Schurian are obviously preserved by strong isomorphisms.

**2.3.** Let $\mathcal{A}$ be an S-ring over a group $G$, $H$ a subgroup of $G$ belonging to $\mathcal{S}^*(\mathcal{A})$ and $i : H \to G$ and $\pi : G \to G/H$ natural injection and surjection. Then obviously the $\mathbb{Z}$-module $\mathcal{A}_H = i^{-1}(\mathcal{A})$ is an S-ring over $H$ and also

$$\mathcal{S}(\mathcal{A}_H) = \{X \in \mathcal{S}(\mathcal{A}) : \ X \subset H\}.$$

Similarly, the $\mathbb{Z}$-module $\mathcal{A}_{G/H} = \pi(\mathcal{A})$ is an S-ring over the group $G/H$,

$$\mathcal{S}(\mathcal{A}_{G/H}) = \{\pi(X) : \ X \in \mathcal{S}(\mathcal{A})\}$$

and moreover $\pi(\xi(X))$ is a positive integer multiple of $\xi(\pi(X))$ for all $X \in \mathcal{S}(\mathcal{A})$ (see [8, Proposition 1.6]). The last means that given $X \in \mathcal{S}(\mathcal{A})$ the number $|Hx \cap X|$ does not depend on the choice of $x \in X$. It is also easy to see that $\pi^{-1}(\xi) \in \mathcal{A}$ for all $\xi \in \mathcal{A}_{G/H}$. It follows from the definitions that if $\mathcal{A}$ is a Schurian S-ring, then both $\mathcal{A}_H$ and $\mathcal{A}_{G/H}$ are also Schurian.

All said in the above paragraph can be completely generalized to the case when $i : G_1 \to G$ and $\pi : G \to G_2$ are an injection and a surjection such that $\mathrm{Im}(i)$ and $\ker(\pi)$ belong to $\mathcal{S}^*(\mathcal{A})$ where $G_1$ and $G_2$ are arbitrary Abelian groups. In particular, the $\mathbb{Z}$-modules $i^{-1}(\mathcal{A})$ and $\pi(\mathcal{A})$ are S-rings over the groups $G_1$ and $G_2$ respectively.

Let $\mathcal{A}$ and $\mathcal{A}'$ be S-rings over $G$ and $G'$ and $\varphi : \mathcal{A} \to \mathcal{A}'$ a weak isomorphism. If $H$ is a subgroup of $G$ belonging to $\mathcal{S}^*(\mathcal{A})$, then $H' = H^\varphi$ is a subgroup of $G'$ belonging to $\mathcal{S}^*(\mathcal{A}')$ and also $|H| = |H'|$. Obviously, $\varphi(\mathcal{A}_H) = \mathcal{A}'_{H'}$. So $\varphi$ induces a weak isomorphism

$$\varphi_H : \mathcal{A}_H \to \mathcal{A}'_{H'}.$$

Similarly, $\varphi$ induces a weak isomorphism

$$\varphi_{G/H} : \mathcal{A}_{G/H} \to \mathcal{A}'_{G'/H'}$$

such that $\varphi_{G/H}(\pi(\xi)) = \pi'(\varphi(\xi))$, $\xi \in \mathcal{A}$, where $\pi : G \to G/H$ and $\pi' : G' \to G'/H'$ are natural projections (see [7, Lemma 1.7]).

[1] This is trivial for cellular rings and hence holds also for S-rings, because the natural correspondence between them takes weak (resp. strong) isomorphisms of S-rings to weak (resp. strong) isomorphisms of cellular rings; see also [8, Proposition 1.7].

Let now $f \in \mathrm{Iso}(\mathcal{A}, \mathcal{A}', \varphi)$. Then $f(H) = H'$. So the restriction of $f$ to $H$ belongs to $\mathrm{Iso}(\mathcal{A}_H, \mathcal{A}'_{H'}, \varphi_H)$. Denote it by $f_H$. Further, it follows from (3) and the last equality that $f(Hx) = H'f(x)$ for all $x \in G$. So $f$ induces a bijection $f_{G/H} : G/H \to G'/H'$ obviously belonging to $\mathrm{Iso}(\mathcal{A}_{G/H}, \mathcal{A}'_{G'/H'}, \varphi_{G/H})$.

# 3 Generalized wreath products

Let $\mathcal{A}_l$ be an S-ring over an Abelian group $G_l$ $(l = 1, 2)$, $M$ be a group and $\pi_0 : G_1 \to M$ and $i_0 : M \to G_2$ be a surjection and an injection respectively for which

$$\pi_0(\mathcal{A}_1) = (i_0)^{-1}(\mathcal{A}_2). \tag{4}$$

Let further $G$ be an Abelian group and $i : G_1 \to G$ and $\pi : G \to G_2$ be an injection and a surjection such that

$$\mathrm{Im}(i) \geq \ker(\pi) \quad \text{and} \quad \pi_0 \circ i_0 = i \circ \pi. \tag{5}$$

Denote by $\mathcal{A}$ the smallest S-ring over $G$ containing the set $i(\mathcal{A}_1) \cup \pi^{-1}(\mathcal{A}_2)$.

**Theorem 3.1** *In the above notation $\mathcal{S}(\mathcal{A}) = \mathcal{S}_1(\mathcal{A}) \cup \mathcal{S}_2(\mathcal{A})$ where $\mathcal{S}_1(\mathcal{A}) = \{i(X) : X \in \mathcal{S}(\mathcal{A}_1)\}$ and $\mathcal{S}_2(\mathcal{A}) = \{\pi^{-1}(X) : X \in \mathcal{S}(\mathcal{A}_2),\ X \subset G_2 \setminus \mathrm{Im}(i \circ \pi)\}$.*

**Proof**. Denote by $\mathcal{A}'$ the submodule of $\mathbb{Z}[G]$ spanned by the set $\Xi = \{\xi(X') : X' \in \mathcal{S}_1(\mathcal{A}) \cup \mathcal{S}_2(\mathcal{A})\}$. Since $\mathrm{Im}(i) \geq \ker(\pi)$, the elements of $\mathcal{S}_1(\mathcal{A}) \cup \mathcal{S}_2(\mathcal{A})$ form a partition of $G$. So the set $\Xi$ is a $\mathbb{Z}$-base of $\mathcal{A}'$. Clearly, $\mathcal{A}' \subset \mathcal{A}$. Thus, to complete the proof we need to verify that $\mathcal{A} \subset \mathcal{A}'$. To do this we prove first that $i(\mathcal{A}_1) \cup \pi^{-1}(\mathcal{A}_2) \subset \mathcal{A}'$ and then that $\mathcal{A}'$ is closed with respect to the multiplication in $\mathbb{Z}[G]$.

The nontrivial part of the former statement is to prove that

$$\xi(\pi^{-1}(X)) \in i(\mathcal{A}_1) \quad \text{for all} \quad X \in \mathcal{S}(\mathcal{A}_2),\ X \subset \mathrm{Im}(i \circ \pi). \tag{6}$$

Let $X = (i \circ \pi)(Y)$ for some $Y \subset G_1$. Then without loss of generality we can assume that $Y$ is a union of cosets by the group $i^{-1}(\ker(\pi))$. On the other hand, since $i \circ \pi = \pi_0 \circ i_0$, we have $X = (\pi_0 \circ i_0)(Y) = i_0(\pi_0(Y))$. Since $X \in \mathcal{S}(\mathcal{A}_2)$, the set $\pi_0(Y)$ belongs to $\mathcal{S}((i_0)^{-1}(\mathcal{A}_2))$. Now the equality (4) implies that $\pi_0(Y) \in \mathcal{S}(\pi_0(\mathcal{A}_1))$. It follows from the assumption on $Y$ that $Y \in \mathcal{S}^*(\mathcal{A}_1)$ (see Subsection 2.3). Thus

$$\xi(\pi^{-1}(X)) = \xi(\pi^{-1}((i \circ \pi)(Y))) = \xi(i(Y))$$

and hence $\xi(\pi^{-1}(X)) \in i(\mathcal{A}_1)$.

Let now $X', Y' \in \mathcal{S}_1(\mathcal{A}) \cup \mathcal{S}_2(\mathcal{A})$. We will prove that $\xi(X')\xi(Y') \in \mathcal{A}'$. If $X', Y' \in \mathcal{S}_1(\mathcal{A})$, this is the consequence of the fact that $\mathcal{A}_1$ is closed with respect to the multiplication in $\mathbb{Z}[G_1]$. If $X', Y' \in \mathcal{S}_2(\mathcal{A})$, then this follows from the straightforward equality

$$\pi^{-1}(\xi_1)\pi^{-1}(\xi_2) = a\pi^{-1}(\xi_1\xi_2), \quad \xi_1, \xi_2 \in \mathbb{Z}[G_2] \tag{7}$$

where $a = |\ker(\pi)|$, the closedness of $\mathcal{A}_2$ with respect to the multiplication in $\mathbb{Z}[G_2]$ and (6). Finally, let $X' \in \mathcal{S}_1(\mathcal{A})$ and $Y' \in \mathcal{S}_2(\mathcal{A})$. Then $X' = i(X)$ for $X \in \mathcal{S}(\mathcal{A}_1)$, $Y' = \pi^{-1}(Y)$ for $Y \in \mathcal{S}(\mathcal{A}_2)$, $Y \subset G_2 \setminus \mathrm{Im}(i \circ \pi)$, and by (4) we have $\pi_0(X) = (i_0)^{-1}(Z)$ for some $Z \in \mathcal{S}(\mathcal{A}_2)$. It suffices to prove that

$$\xi(X')\xi(Y') = b\pi^{-1}(\xi(Z)\xi(Y)) \tag{8}$$

for some integer $b$. To do this we observe that $X' \subset \pi^{-1}(Z)$ and the positive integer $|X' \cap Lg|$ does not depend on $Lg \subset \pi^{-1}(Z)$ where $L = \ker(\pi)$ (see Subsection 2.3). Denote this number by $b$. Then obviously $\xi(X' \cap Lg)\xi(Lh) = b\xi(Lgh)$ for all $h \in G$. On the other hand, $\xi(Lg)\xi(Lh) = a\xi(Lgh)$. Thus

$$\xi(X')\xi(Lh) = b \sum_{Lg, Lg \cap X' \neq \emptyset} \xi(Lgh) = b \sum_{Lg, Lg \subset \pi^{-1}(Z)} \xi(Lgh) = (b/a)\xi(\pi^{-1}(Z))\xi(Lh).$$

Now observing that $\xi(Lh) = \xi(\pi^{-1}(\pi(Lh))$, summing up over all $Lh \subset Y'$ (then $\{Lh\}$ runs over $Y$) and applying (7) for $\xi_1 = \xi(Z)$ and $\xi_2 = \xi(Y)$ we obtain (8).∎

We say that the S-ring $\mathcal{A}$ is the *generalized wreath product* of the S-rings $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to $\mathcal{M} = (M, i_0, \pi_0)$ and $\mathcal{G} = (G, i, \pi)$ and denote it by $\mathcal{A}_1 \wr_{(\mathcal{M},\mathcal{G})} \mathcal{A}_2$. If $G_1$ is a subgroup of $G$, $G_2 = G/\ker(\pi)$, $M = G_1/\ker(\pi)$ and $i, i_0$ and $\pi, \pi_0$ are natural injections and projections, then the generalized wreath product is called *standard*. It is easy to see that any generalized wreath product is Cayley isomorphic to a standard one.

For example, given positive integers $p, q, r$ let $G_1 = \mathbb{Z}_{pq}$, $G_2 = \mathbb{Z}_{qr}$, $\mathcal{A}_1 = \mathbb{Z}[G_1]$, $\mathcal{A}_2 = \mathbb{Z}[G_2]$, $M = \mathbb{Z}_q$, $G = \mathbb{Z}_{pqr}$ and $i_0, \pi_0, i, \pi$ be the homomorphisms taking 1 to 1 for $\pi_0$ and $\pi$ and 1 to $r$ for $i_0$ and $i$. Then the S-ring $\mathcal{A}$ has $pq$ one-element basic sets $\{rx\}$, $x = 0, 1, \ldots, pq-1$, and $q(r-1)$ $p$-element basic sets $\{x + qry :\ y = 0, 1, \ldots, p-1\}$, $x = 0, 1, \ldots, qr-1$, $x \not\equiv 0 \,(\mathrm{mod}\, r)$.

The important special case of our construction explaining its name arises for $|M| = 1$. In this case if the generalized wreath product is a standard one, then each element of $\mathcal{S}(\mathcal{A})$ either coincides with some element of $\mathcal{S}(\mathcal{A}_1)$ or equals the union of cosets by $G_1$ belonging to some $X \in \mathcal{S}(\mathcal{A}_2)$, $X \neq \{1_{G_2}\}$; the reader familiar with the correspondence between S-rings and cellular rings (see [4]) will easily observe that the cellular ring corresponding to $\mathcal{A}$ is the wreath product in sense of [10] of the cellular rings corresponding to $\mathcal{A}_1$ and $\mathcal{A}_2$. Further, the underlying group $G$ of $\mathcal{A}$ is an extension of $G_1$ with the help of $G_2$. It is well-known (see [6]) that the classes of such extensions modulo isomorphisms identical on $G_1$ and $G_2$ form with respect to the addition of extensions a finite Abelian group $\mathrm{Ext}_{\mathbb{Z}}(G_2, G_1)$. For example, if the groups $G_1$ and $G_2$ are cyclic, then $\mathrm{Ext}_{\mathbb{Z}}(G_2, G_1)$ is a cyclic group of order $\mathrm{GCD}(|G_1|, |G_2|)$. Moreover the generators of this group correspond to cyclic groups $G$.

For an arbitrary $\mathcal{M}$ one can easily find from $G_1$ and $G_2$ a group $G$ satisfying (5). Moreover, if the groups $G_1$ and $G_2$ are cyclic, then $G$ can also be chosen cyclic. Certainly, in general, the group $G$ is not uniquely determined up to isomorphism. However, even if such groups are isomorphic, the corresponding generalized wreath products are not necessarily Cayley isomorphic (even for $|M| = 1$). Nevertheless the following statement holds.

**Theorem 3.2** *For fixed $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{M}$ the S-rings $\mathcal{A}_1 \wr_{(\mathcal{M},\mathcal{G})} \mathcal{A}_2$ are pairwise strongly isomorphic for all admissible $\mathcal{G}$.*

**Proof**. Let $\mathcal{A} = \mathcal{A}_1 \wr_{(\mathcal{M},\mathcal{G})} \mathcal{A}_2$ and $\mathcal{A}' = \mathcal{A}_1 \wr_{(\mathcal{M},\mathcal{G}')} \mathcal{A}_2$. Without loss of generality we assume that these generalized wreath products are standard ones. Then $G = G'$ (as sets), the group $G_1$ is a subgroup of both $G$ and $G'$, and $G/L = G'/L = G_2$ (as groups) where $L = \ker(\pi) = \ker(\pi')$. In particular, $G/G_1 = G'/G_1$. Denote this group by $H$. Let us define a bijection $f : G \to G'$ as follows. If $x \in G_1$, then set $f(x) = x$. Further, for each $T \in H \setminus \{1\}$ arbitrarily choose $g_T, g'_T \in T$ such that $Lg_T = Lg'_T$. For a fixed $T$ set by definition

$$f(xg_T) = xg'_T, \quad x \in G_1$$

(here and below the multiplications in the left and right sides of formulas are in $G$ and $G'$ respectively). Thus, $f$ is a correctly defined bijection from $G$ onto $G$ preserving any coset by $L$. Let us prove that $f$ is a strong isomorphism from $\mathcal{A}$ to $\mathcal{A}'$. To do this we observe that by the definition of $f$ we have $f([x]) = [f(x)]$ for all $x \in G$. So equality (3) is equivalent to the inclusion $f([x]y) \subset [f(x)]f(y)$, $x, y \in G$, and the latter is equivalent to the correlation

$$f(xy) \in [f(x)]f(y), \quad x, y \in G. \tag{9}$$

Let us prove it. Indeed, this is trivial for $x, y \in G_1$. If $x \in G \setminus G_1$, $y \in G$, then it is so because $[f(x)] \supset Lx$ (see Theorem 3.1) and $Lxy = LxLy$. Finally, let $x \in G_1$ and $y \in G \setminus G_1$. Then $y \in T$ for some $T \in H \setminus \{1\}$ and by the definition of $f$ we have

$$f(xy) = f(xzg_T) = xzg'_T = xf(y) = f(x)f(y)$$

where $y = zg_T$ with $z \in G_1$. Thus (9) follows and we are done.■

Below we will study weak isomorphisms of generalized wreath products. Without loss of generality we confine ourselves to those of standard ones.

**Theorem 3.3** *Let $\mathcal{A} = \mathcal{A}_1 \wr_{(\mathcal{M},\mathcal{G})} \mathcal{A}_2$ and $\mathcal{A}' = \mathcal{A}'_1 \wr_{(\mathcal{M}',\mathcal{G}')} \mathcal{A}'_2$ be standard generalized wreath products. Then the mapping*

$$\varphi \mapsto (\varphi_{G_1}, \varphi_{G_2}) \tag{10}$$

*defines a bijection between the set of all weak isomorphisms $\varphi : \mathcal{A} \to \mathcal{A}'$ such that $\varphi(i(\mathcal{A}_1)) = i'(\mathcal{A}'_1)$ and $\varphi(\pi^{-1}(\mathcal{A}_2)) = (\pi')^{-1}(\mathcal{A}'_2)$, and the set of all pairs $(\varphi_1, \varphi_2)$ of weak isomorphisms $\varphi_1 : \mathcal{A}_1 \to \mathcal{A}'_1$, $\varphi_2 : \mathcal{A}_2 \to \mathcal{A}'_2$ such that $(\varphi_1)_M = (\varphi_2)_M$.*

**Proof**. The correctness of the definition of the mapping (10) is the consequence of the obvious equalities $(\varphi_{G_1})_M = (\varphi_{G_2})_M = \varphi_M$. The injectivity of (10) follows from Theorem 3.1 because $(i(X))^\varphi = i(X^{\varphi_{G_1}})$ for $X \in \mathcal{S}(\mathcal{A}_1)$ and $(\pi^{-1}(X))^\varphi = \pi^{-1}(X^{\varphi_{G_2}})$ for $X \in \mathcal{S}(\mathcal{A}_2)$. To prove the surjectivity let $\varphi_1 : \mathcal{A}_1 \to \mathcal{A}'_1$ and $\varphi_2 : \mathcal{A}_2 \to \mathcal{A}'_2$ be weak isomorphisms such that $(\varphi_1)_M = (\varphi_2)_M$. Let us define a bijection from the set $\mathcal{S}(\mathcal{A}) = \mathcal{S}_1(\mathcal{A}) \cup \mathcal{S}_2(\mathcal{A})$ to the set $\mathcal{S}(\mathcal{A}') = \mathcal{S}_1(\mathcal{A}') \cup \mathcal{S}_2(\mathcal{A}')$ (see Theorem 3.1) as follows:

$$(X')^\varphi = \begin{cases} i(X^{\varphi_1}), & \text{if } X' = i(X),\ X \in \mathcal{S}(\mathcal{A}_1), \\ \pi^{-1}(X^{\varphi_2}), & \text{if } X' = \pi^{-1}(X),\ X \in \mathcal{S}(\mathcal{A}_2),\ X \subset G_2 \setminus M. \end{cases} \tag{11}$$

This bijection produces in a natural way a $\mathbb{Z}$-module isomorphism from $\mathcal{A}$ to $\mathcal{A}'$ denoted also by $\varphi$. Clearly, $\varphi$ satisfies (2). Let us prove that it satisfies (1). If $\xi_1, \xi_2 \in i(\mathcal{A}_1)$, then this follows from the fact that $\varphi_1$ satisfies (1). Further, since $(\varphi_1)_M = (\varphi_2)_M$, the equality $(\pi^{-1}(X))^{\varphi} = \pi^{-1}(X^{\varphi_2})$ holds for all $X \in \mathcal{S}(\mathcal{A}_2)$ (cf. (11)). So due to (7) if $\xi_1, \xi_2 \in \pi^{-1}(\mathcal{A}_2)$, then $\varphi$ satisfies (1) because $\varphi_2$ does. Finally, if $\xi_1 \in i(\mathcal{A}_1)$ and $\xi_2 \in \pi^{-1}(\mathcal{A}_2)$, then the required statement follows from (8) and above. Thus $\varphi : \mathcal{A} \to \mathcal{A}'$ is a weak isomorphism. Since obviously $\varphi_{G_1} = \varphi_1$ and $\varphi_{G_2} = \varphi_2$, we are done.$\blacksquare$

# 4 Explicit constructions

Below we deal with S-rings over cyclic groups $\mathbb{Z}_n$ for different $n$. If $m, n$ are positive integers, $m$ divides $n$, then denote by $i_{m,n} : \mathbb{Z}_m \to \mathbb{Z}_n$ and $\pi_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ the group homomorphisms taking 1 to $n/m$ and to 1 respectively. Using them we identify both the subgroup $i_{m,n}(\mathbb{Z}_m)$ and the factorgroup $\mathbb{Z}_n/\ker(\pi_{n,m})$ of $\mathbb{Z}_n$ with $\mathbb{Z}_m$. If $\mathcal{A}$ is an S-ring over $\mathbb{Z}_n$ and $\mathbb{Z}_m$ belongs to $\mathcal{S}^*(\mathcal{A})$, then $\mathcal{A}_{\mathbb{Z}_m}$ will be denoted by $\mathcal{A}_m$ and $\mathcal{A}_{\mathbb{Z}_n/\mathbb{Z}_m}$ by $\mathcal{A}^{n/m}$. Similarly, if $\varphi$ and $f$ are a weak isomorphism and a strong isomorphism from $\mathcal{A}$ to another S-ring, we write $\varphi_m$ and $f_m$ instead of $\varphi_{\mathbb{Z}_m}$ and $f_{\mathbb{Z}_m}$ and $\varphi^{n/m}$ and $f^{n/m}$ instead of $\varphi_{\mathbb{Z}_n/\mathbb{Z}_m}$ and $f_{\mathbb{Z}_n/\mathbb{Z}_m}$. Let finally $\mathcal{A}_l$ be an S-ring over $\mathbb{Z}_{n_l}$, $l = 1, 2$, and $(\mathcal{A}_1)^m = (\mathcal{A}_2)_m$ for some $m$ dividing both $n_1$ and $n_2$. Then for $\mathcal{M} = (\mathbb{Z}_m, i_{m,n_2}, \pi_{n_1,m})$ and $\mathcal{G} = (\mathbb{Z}_n, i_{n_1,n}, \pi_{n,n_2})$ where $n = n_1 n_2/m$ conditions (4) and (5) are satisfied. In this case the generalized wreath product $\mathcal{A}_1 \wr_{(\mathcal{M},\mathcal{G})} \mathcal{A}_2$ will be denoted by $\mathcal{A}_1 \wr_m \mathcal{A}_2$. We omit $m$ if $m = 1$.

Let $p_1, p_2, p_3, p_4$ be primes with $\{p_1, p_2\} \cap \{p_3, p_4\} = \emptyset$ and $d$ a positive integer dividing $p_i - 1$ for all $i = 1, \ldots, 4$. Denote by $K_i$ the subgroup of $\mathbb{Z}_{p_i}^*$ of order $d$ and by $\mathcal{A}_i$ the S-ring $\mathcal{O}(K_i, \mathbb{Z}_{p_i})$, $i = 1, \ldots, 4$ (see Subsection 2.1). For $i \in \{1, 2\}$, $j \in \{3, 4\}$ fix an isomorphism $f_{i,j} : K_i \to K_j$ and set

$$K_{i,j} = \{(x, y) \in \mathbb{Z}_{p_i p_j}^* : \ f_{i,j}(x) = y\} \tag{12}$$

(we note that since $p_i \neq p_j$, we have $\mathbb{Z}_{p_i p_j}^* = \mathbb{Z}_{p_i}^* \times \mathbb{Z}_{p_j}^*$). Then $K_{i,j}$ is a subgroup of $\mathbb{Z}_{p_i p_j}^*$ of order $d$. Set $\mathcal{A}_{i,j} = \mathcal{O}(K_{i,j}, \mathbb{Z}_{p_i p_j})$. It is easy to see that the subgroups $\mathbb{Z}_{p_i}$ and $\mathbb{Z}_{p_j}$ of $\mathbb{Z}_{p_i p_j}$ are unions of basis sets of the S-ring $\mathcal{A}_{i,j}$ and $[1] = K_{i,j}$. A straightforward check shows that

$$(\mathcal{A}_{1,3})^{p_3} = \mathcal{A}_3 = (\mathcal{A}_{2,3})_{p_3}, \quad (\mathcal{A}_{1,4})^{p_4} = \mathcal{A}_4 = (\mathcal{A}_{2,4})_{p_4}.$$

So the S-rings $\mathcal{A}_{1,2,3} = \mathcal{A}_{1,3} \wr_{p_3} \mathcal{A}_{2,3}$ over $\mathbb{Z}_{p_1 p_2 p_3}$ and $\mathcal{A}_{1,2,4} = \mathcal{A}_{1,4} \wr_{p_4} \mathcal{A}_{2,4}$ over $\mathbb{Z}_{p_1 p_2 p_4}$ can be considered. It follows from the definition of $\mathcal{A}_{1,2,3}$ that $\mathbb{Z}_{p_3}$ and $\mathbb{Z}_{p_1}$ belong to $\mathcal{S}^*(\mathcal{A}_{1,2,3})$ and $(\mathcal{A}_{1,2,3})_{p_1 p_3} = \mathcal{A}_{1,3}$, $(\mathcal{A}_{1,2,3})^{p_2 p_3} = \mathcal{A}_{2,3}$. So $\mathbb{Z}_{p_3}$, $\mathbb{Z}_{p_1 p_2} \in \mathcal{S}^*(\mathcal{A}_{1,2,3})$. A straightforward check using Theorem 3.1 shows that $(\mathcal{A}_{1,2,3})^{p_1 p_2} = (\mathcal{A}_{1,2,3})_{p_1 p_2} = \mathcal{A}_1 \wr \mathcal{A}_2$. Similarly, $\mathbb{Z}_{p_4}$, $\mathbb{Z}_{p_1 p_2} \in \mathcal{S}^*(\mathcal{A}_{1,2,4})$ and $(\mathcal{A}_{1,2,4})^{p_1 p_2} = (\mathcal{A}_{1,2,4})_{p_1 p_2} = \mathcal{A}_1 \wr \mathcal{A}_2$. In particular,

$$(\mathcal{A}_{1,2,3})^{p_1 p_2} = \mathcal{A}_1 \wr \mathcal{A}_2 = (\mathcal{A}_{1,2,4})_{p_1 p_2}.$$

Thus we can define an S-ring $\mathcal{A} = \mathcal{A}(\{p_i\}, d, \{f_{i,j}\})$ over $\mathbb{Z}_{p_1 p_2 p_3 p_4}$ by $\mathcal{A} = \mathcal{A}_{1,2,3} \wr_{p_1 p_2} \mathcal{A}_{1,2,4}$. The structure of this S-ring is getting more clear from the lattice of the subgroups belonging to $\mathcal{S}^*(\mathcal{A})$ given at Fig. 1. If $p_1 = p_2$ and $p_3 = p_4$, then this lattice contains all subgroups of $\mathbb{Z}_{p_1 p_2 p_3 p_4}$. Also one can see that $\mathcal{A}_{p_1 p_2} = \mathcal{A}_1 \wr \mathcal{A}_2$ and $\mathcal{A}_{p_3 p_4} = \mathcal{A}_3 \wr \mathcal{A}_4$.

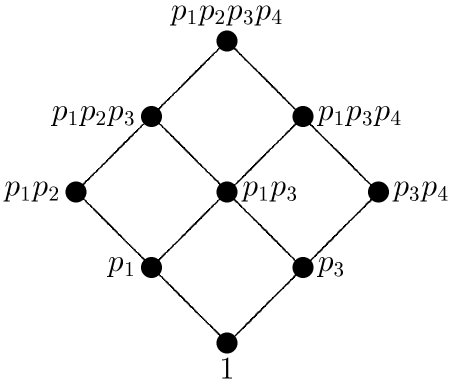


Figure 1: Subgroup lattice of $\mathcal{A}$

Below we will study the properties of the S-ring $\mathcal{A}$ depending on the choice of the parameters $p_i$, $d$ and $f_{i,j}$. If the parameters are fixed, then given a divisor $d'$ of $d$ we will denote by $\mathcal{A}(d')$ the S-ring obtained in the same way as $\mathcal{A}$ with $d$ replaced by $d'$ and $f_{i,j}$ by its restriction to the subgroup of $K_i$ of order $d'$. Thus $\mathcal{A} = \mathcal{A}(d)$ and $\mathcal{A}(d') \supset \mathcal{A}$ for all $d'$.

**Lemma 4.1** *In the above notation let $f = f_{1,3} \circ f_{2,3}^{-1} \circ f_{2,4} \circ f_{1,4}^{-1}$. Then the following statements hold:*

(1) *if $f \neq id_{K_1}$, then $\mathcal{A}$ is a non-Schurian S-ring,*

(2) *if, additionally, $f$ is identical on the subgroup of order $d'$ and the factorgroup modulo it for some $d'$ dividing $d$, then $\mathcal{A}(d')$ is a non-separable S-ring.*

**Proof**. Let us first consider an arbitrary automorphism $g \in \mathrm{Aut}(\mathcal{A})$. Set

$$g_{1,3} = (g_{p_1p_2p_3})_{p_1p_3}, \quad g_{2,3} = (g_{p_1p_2p_3})^{p_2p_3}, \quad g_{1,4} = (g^{p_1p_2p_4})_{p_1p_4}, \quad g_{2,4} = (g^{p_1p_2p_4})^{p_2p_4},$$

$$g_1 = (g_{1,3})_{p_1} = (g_{1,4})_{p_1},\ g_2 = (g_{2,3})_{p_2} = (g_{2,4})_{p_2},\ g_3 = (g_{1,3})_{p_3} = (g_{2,3})_{p_3},\ g_4 = (g_{1,4})_{p_4} = (g_{2,4})_{p_4}.$$

Then $g_{i,j} \in \mathrm{Aut}(\mathcal{A}_{i,j})$, $i = 1,2$, $j = 3,4$, and $g_k \in \mathrm{Aut}(\mathcal{A}_k)$, $k = 1, \ldots, 4$ (see Subsection 2.3). By [5, Theorem 3.4,(1),(b$_2$)] any automorphism of the S-ring $\mathcal{A}_{i,j}$ over $\mathbb{Z}_{p_ip_j}$ is the multiplication by some element of $\mathbb{Z}^*_{p_ip_j}$ (we recall that $p_i \neq p_j$). So there exist $m_{i,j} \in \mathbb{Z}^*_{p_ip_j}$ such that $x^{g_{i,j}} = m_{i,j}x$, $x \in \mathbb{Z}_{p_ip_j}$, for all $i,j$. Moreover, the automorphism $g_k$ being induced by $g_{k,3}$ as well as by $g_{k,4}$ for $k = 1,2$, and by $g_{1,k}$ as well as by $g_{2,k}$ for $k = 3,4$ defines $m_k \in \mathbb{Z}^*_{p_k}$ such that $x^{g_k} = m_kx$, $x \in \mathbb{Z}_{p_k}$, and also $m_{i,j} = (m_i, m_j)$ for all $i,j$. Let us check that

$$f_{i,j}(m_i) = m_j, \quad i = 1,2,\ j = 3,4. \tag{13}$$

Indeed, it follows from the definition of $m_{i,j}$ that $1^{g_{i,j}} = m_{i,j}$ and hence $m_{i,j} \in K_{i,j}$. So $(m_i, m_j) = m_{i,j} = (m_i, f_{i,j}(m_i))$ (see (12)). This proves (13). Now taking into account the definition of $m_i$'s we have

$$f_{i,j}(y^{g_i}) = f_{i,j}(m_iy) = f_{i,j}(m_i)f_{i,j}(y) = m_jf_{i,j}(y) = f_{i,j}(y)^{g_j} \tag{14}$$

for all $y \in K_i$, $i = 1, 2$, $j = 3, 4$.

Let us prove statement (1). It follows from (14) that $f(y^{g_1}) = f(y)^{g_1}$ for all $g \in \mathrm{Aut}(\mathcal{A})$ and $y \in K_1$. So $f(1^{g_1}) = f(1)^{g_1} = 1^{g_1}$. Thus

$$1^{g_1} \notin X \quad \text{for all} \quad g \in \mathrm{Aut}(\mathcal{A}). \tag{15}$$

where $X = \{x \in K_1 : \ f(x) \neq x\}$. By the assumption of statement (1) the set $X$ is not empty. Let $x^* \in X$. It follows from (15) that $x^*$ and 1 as elements of the subgroup $\mathbb{Z}_{p_1}$ belong to different orbits of $\mathrm{Aut}(\mathcal{A})$. Since these elements are in the same basis set of $\mathcal{A}$, the S-ring $\mathcal{A}$ is not Schurian.

Let us prove statement (2). Set

$$x_3 = f_{1,3}(x^*),\ x_2 = f_{2,3}^{-1}(x_3),\ x_4 = f_{2,4}(x_2),\ x_1 = f_{1,4}^{-1}(x_4), \tag{16}$$

where $x^*$ is define as above. Then $(x_i, x_j) \in K_{i,j}$ for all $i = 1, 2$ and $j = 3, 4$. Denote by $\varphi_{i,j}$ the weak isomorphism of the S-ring $\mathcal{A}'_{i,j} = (\mathcal{A}(d'))_{i,j}$ given by the multiplication by $m'_{i,j} = (x_i, x_j)$. Since obviously $(\varphi_{1,3})^{p_3} = (\varphi_{2,3})_{p_3}$ and $(\varphi_{1,4})^{p_4} = (\varphi_{2,4})_{p_4}$, by Theorem 3.3 there exist uniquely determined weak isomorphisms $\varphi_{1,2,3}$ and $\varphi_{1,2,4}$ of the S-rings $\mathcal{A}'_{1,2,3} = \mathcal{A}'_{1,3} \wr_{p_3} \mathcal{A}'_{2,3}$ and $\mathcal{A}'_{1,2,4} = \mathcal{A}'_{1,4} \wr_{p_4} \mathcal{A}'_{2,4}$ such that $(\varphi_{1,2,3})_{p_1p_3} = \varphi_{1,3}$, $(\varphi_{1,2,3})^{p_2p_3} = \varphi_{2,3}$ and $(\varphi_{1,2,4})_{p_1p_4} = \varphi_{1,4}$, $(\varphi_{1,2,4})^{p_2p_4} = \varphi_{2,4}$ respectively. Let us check that

$$(\varphi_{1,2,3})^{p_1p_2} = (\varphi_{1,2,4})_{p_1p_2}. \tag{17}$$

Since $(\mathcal{A}'_{1,2,3})^{p_1p_2} = (\mathcal{A}'_{1,2,4})_{p_1p_2} = \mathcal{A}'_1 \wr \mathcal{A}'_2$ where $\mathcal{A}'_i = (\mathcal{A}(d'))_i$, $i = 1, 2$, it suffices to prove that $(\varphi_{1,3})_{p_1} = (\varphi_{1,4})_{p_1}$ and $(\varphi_{2,3})_{p_2} = (\varphi_{2,4})_{p_2}$. The latter is straightforward from the definition of $\varphi_{i,j}$'s. Since the weak isomorphisms $(\varphi_{1,3})_{p_1}$ and $(\varphi_{1,4})_{p_1}$ correspond to the multiplications by $x^*$ and $x_1$ respectively, to prove the former one we only have to verify that $(x^*)^{-1}x_1 \in K'_1$ where $K'_1$ is the subgroup of $K_1$ of the order $d'$. However, it follows from the assumption of statement (2) that $f(K'_1x) = K'_1x$ for all $x \in K_1$. So $(x^*)^{-1}x_1 = (x^*)^{-1}f(x^*) \in K'_1$ because $x_1 = f(x^*)$ due to (16).

It follows from Theorem 3.3 and (17) that $\varphi_{1,2,3} = \varphi_{p_1p_2p_3}$ and $\varphi_{1,2,4} = \varphi_{p_1p_2p_4}$ for a uniquely determined weak isomorphism $\varphi : \mathcal{A}' \to \mathcal{A}'$. The definitions imply that $\varphi$ is identical on $\mathcal{A}$ and $\varphi_{p_1}$ takes the basic set of $\mathcal{A}'_1$ containing 1 to the basic set of $\mathcal{A}'_1$ containing $x^*$. Suppose that $\varphi$ is induced by a strong isomorphism $g$ of $\mathcal{A}'$. Then $g$ is an automorphism of $\mathcal{A}$ and $1^{g_1} \in K'_1x^*$. On the other hand, it follows from the assumption of statement (2) that $f(h) = h$ for all $h \in K'_1$. Since $f(x^*) \neq x^*$, this implies that $K'_1x^* \subset X$ where $X$ is as in (15). Thus $1^{g_1} \notin K'_1x^*$ by (15). This contradiction shows that $\varphi$ is not induced by any strong isomorphism of $\mathcal{A}'$ and hence $\mathcal{A}'$ is not separable.■

**Proof of Theorem 1.1.** Let $n = p_1p_2p_3p_4n'$ and $D$ be as in the theorem. Let further $\mathcal{A} = \mathcal{A}(\{p_i\}, d, \{f_{i,j}\})$ be the S-ring over $\mathbb{Z}_{p_1p_2p_3p_4}$ constructed in the beginning of this section. To prove statement (1) set $d$ to be any divisor of $D$ greater than 2 (e.g., $D$). Then the order of the group $\mathrm{Aut}(K_1)$ is at least 2. So the isomorphisms $f_{i,j}$ can be chosen satisfying the condition of statement (1) of Lemma 4.1 (e.g., $f_{1,3}$, $f_{2,3}$, $f_{2,4}$ are arbitrary and $f_{1,4} \neq f_{1,3} \circ f_{2,3}^{-1} \circ f_{2,4}$). Thus this lemma implies that the S-ring $\mathcal{A}$ is not Schurian. Now if $\mathcal{B}$ is an arbitrary S-ring over $\mathbb{Z}_{n'}$, then since $\mathcal{A} = (\mathcal{A} \wr \mathcal{B})_{p_1p_2p_3p_4}$ the S-ring $\mathcal{A} \wr \mathcal{B}$ is also not Schurian, and statement (1) follows.

To prove statement (2) set $d$ to be any non-squarefree divisor of $D$ (e.g., $D$). Let further $d' > 1$ be a positive integer such that $(d')^2$ divides $d$ (e.g., any prime $l$ such that $l^2$ divides $d$). Set $f$ to be any automorphism of $K_1$ taking a generator of it to a different generator belonging to the same coset modulo the subgroup $K_1'$ of $K_1$ of order $d'$. Then $f$ acts trivially on $K_1'$ and $K/K_1'$. Choose the automorphisms $f_{i,j}$ so that $f_{1,3} \circ f_{2,3}^{-1} \circ f_{2,4} \circ f_{1,4}^{-1} = f$. Then obviously they satisfy the condition of statement (2) of Lemma 4.1. By this lemma the S-ring $\mathcal{A}' = \mathcal{A}(d')$ is not separable. Let $\varphi$ be a weak isomorphism from $\mathcal{A}'$ to $\mathcal{A}'$ such that $\mathrm{Iso}(\mathcal{A}', \mathcal{A}', \varphi) = \emptyset$. Now if $\mathcal{B}$ is an arbitrary S-ring over $\mathbb{Z}_{n'}$, then the weak isomorphism from the S-ring $\mathcal{A}' \wr \mathcal{B}$ to itself corresponding by Theorem 3.3 to the pair $(\varphi, \mathrm{id}_{\mathcal{B}})$ is not induced by any strong isomorphism. So the S-ring $\mathcal{A}' \wr \mathcal{B}$ is not separable and statement (2) follows.■

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