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ERGODIC THEORY ON SL(n) AND ANOMALIES IN THE LATTICE POINT PROBLEM: RELATED QUESTIONS AND OPEN PROBLEMS

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ABSTRACT

The present paper contains a short survey of the results on anomalously small, particularly logarithmically small, errors in the lattice point problem for polyhedra in $\mathbb{R}^n$. Relevant open problems are also briefly discussed.

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Let $\mathcal{O} \subset \mathbb{R}^n$ be a compact region with a piecewise smooth boundary of volume vol $\mathcal{O}$ and $\Gamma \subset \mathbb{R}^n$ an unimodular lattice. We write $t\mathcal{O}$ for the dilation $\mathcal{O}$ by a parameter $t > 0$ and $t\mathcal{O} + X$ for the translation $t\mathcal{O}$ by a vector $X \in \mathbb{R}^n$.

Consider the number $N(t\mathcal{O} + X, \Gamma)$ of lattice points $\gamma \in \Gamma$ which fall into the region $t\mathcal{O} + X$, and put

$$N(t\mathcal{O} + X, \Gamma) = \operatorname{vol} \mathcal{O} t^n + \mathcal{R}(t\mathcal{O} + X, \Gamma),$$

$$R(t\mathcal{O}, \Gamma) = \max_{X \in \mathbb{R}^n} \mathcal{R}(t\mathcal{O} + X, \Gamma).$$

The classical lattice point problem deals with the behavior of the error $R(t\mathcal{O}, \Gamma)$ as $t \to \infty$ (see, for example, [Kr] and [G-L]). Typically, say, for regions like circles in $\mathbb{R}^2$ or balls in $\mathbb{R}^n$, the error is of order $O(t^q)$ with an exponent $q = q(\mathcal{O}, \Gamma) > 0$.

Definition. *For given $\mathcal{O}$ and Γ the error $R(t\mathcal{O}, \Gamma)$ is called anomalously small (resp. logarithmically small) if $R(t\mathcal{O}, \Gamma) = O(t^\varepsilon)$ with arbitrarilu small $\varepsilon > 0$ (resp. if $R(t\mathcal{O}, \Gamma) = O(\log^p t)$ with some $p = p(\mathcal{O}, \Gamma) > 0$).*

We conjecture that for all regions $\mathcal{O}$ and lattices Γ one has the following omega-bound $R(t\mathcal{O}, \Gamma) = \Omega(t^{\frac{1}{2}m})$ where $0 \leq m \leq n-1$ is a maximal number of nonvanishing principal curvatures on the boundary $\partial\mathcal{O}$. By this means even a minor distortion of the boundary prevents the appearence of anomalously small errors. Thereby such errors may arise only for polyhedra. In dimansions $n > 2$ the conjecture seems to be rather difficult.

Now we wish to state known results on anomalously small errors. We begin with two dimensions. Let $P \subset \mathbb{R}^2$ be a compact polygon with h sides, and let $\theta_1, \dots, \theta_h$ be angles between the sides and one of the coordinate axes and $\xi_1 = \tan\theta_1, \dots \xi_h = \tan\theta_h$. We write UP for the rotation of P by a matrix $U \in SO(2)$.

Theorem 1 ([H-L], [Kh], [Skr1], [Skr2]). *With the above notation one has the following:*

(i) *If each of $\xi_1, \dots, \xi_h$ is badly approximable, then $R(tP, \mathbb{Z}^2) = O(\log t)$.*

(ii) *For almost all $U \in SO(2)$ (in the sense of the Haar measure on $SO(2)$) $R(tUP, \mathbb{Z}^2) = O(\log^{1+\varepsilon} t)$ with arbitrarily small $\varepsilon > 0$.*

(iii) *If each of $\xi_1, \dots, \xi_h$ is algebraic irrationality, then $R(tP, \mathbb{Z}^2) = O(t^\varepsilon)$ with arbitrarily small $\varepsilon > 0$.*

Statements (i) and (ii) were, respectively, given in [H-L] and [Kh] on the base of continued fractions (see also [Skr1]). Statement (iii) was given in [Skr2] as a consequence of Roth's theorem on approximation of algebraic irrationalities (cf. [Sch.]).

In arbitrary dimensions $n \geq 2$ we have the following results. Let

$$Nm\Gamma = \inf\{|Nm\gamma| : \gamma \in \Gamma \setminus \{o\}\}$$

be a homogeneous minimum of a lattice $\Gamma \subset \mathbb{R}^n$ with respect to the norm form $NmX = x_1 \dots x_n$, $X = (x_1, \dots, x_n) \in \mathbb{R}^n$. We recall that a lattice $\Gamma \subset \mathbb{R}^n$ is called admissible if $Nm\Gamma > 0$. For each n admissible lattices in $\mathbb{R}^n$ can be easily constructed explicitly in terms of totally reall algebraic number fields of degree n (see [G-L], [Skr3], [Skr5]).

Theorem 2 ([Skr1], [Skr3], [Skr5]). *Let $\Pi \subset \mathbb{R}^n$ be a parallelepiped with edges parallel to the coordinate axes and Γ be an admissible lattice, then $R(t\Pi, \Gamma) = O(\log^{n-1} t)$.*

Theorem 2 was a first result on anomalously small errors in arbitrary dimensions. Applications of Theorem 2 to the theory of algebraic number fields, uniformly distributed point sets and spectral theory can be found in [Skr1], [Skr3], [Skr4], [Skr5], [N-Skr1], [N-Skr2] (see also [Kl-Ma], [M], [St] for related topics).

The set of unimodular lattices in $\mathbb{R}^n$ can be naturally identified with the following homogeneous space $\mathcal{L}_n = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ and the space $\mathcal{L}_n$ admits a unique $SL(n,\mathbb{R})$-invariant probability measure $\mu_{\mathcal{L}}$ (see, for example, [Ra]).

It is worth noting that admissible lattices form a dense subset in $\mathcal{L}_n$ of zero measure $\mu_{\mathcal{L}}$ (see Theorem 6 below). What is a typical behavior of errors on the space $\mathcal{L}_n$?

Theorem 3 ([Skr5]). *Let $P \subset \mathbb{R}^n$ be an arbitrary compact polyhedral, then for almost all lattices $\Gamma \in \mathcal{L}_n$ (in the sense of invariant measure $\mu_{\mathcal{L}}$) one has $R(tP,\Gamma) = O(\log^{n-1+\varepsilon} t)$ with arbitrarily small $\varepsilon > 0$.*

Theorem 3 shows that the appearance of logarithmically small errors for polyhedra is quite typical from the metric point of view.

One can also consider submanifolds of lattices in $\mathcal{L}_n$ leading to logarithmically small errors. The existence of such submanifolds follows at once from Theorem 3 and Fubini's theorem. However, we can give much more concrete result.

Theorem 4 ([Skr5], [Skr-St]). *Given an arbitrary compact polyhedron $P \subset \mathbb{R}^n$ and a lattice $\Gamma \subset \mathbb{R}^n$, then for almost all $U \in SO(n)$ (in the sense of the Haar measure on $SO(n)$) one has $R(tUP,\Gamma) = O(\log^{n-1+\varepsilon} t)$.*

Notice that a similar result with a bound of the order $O(\log^{2(n-1)+\varepsilon} t)$ was earlier given in [Skr5]. The improvement given in Theorem 4 is based on an analysis of hidden symmetries of the problem generated by the Weyl group for $SL(n,\mathbb{R})$ (cf. [Skr-St]).

In dimensions $n > 2$ there is known only one result, stated above in Theorem 2, on explicit constructions of regions and lattices with logarithmically small errors. However, for anomalously small errors the following explicit constructions can be given.

Let $P_k \subset \mathbb{R}^n$ be a compact polyhedron, such that all its vertices have coordinates in a real algebraic number field k, $[k:\mathbb{Q}] < \infty$. Notice that each combinatorial type of polyhedra in $\mathbb{R}^n$ can be realized as a polyhedron P_k over a suitable field k.

Let $k = k_0 \subset k_1 \subset \ldots \subset k_n$ be a tower of real algebraic fields of finite degrees with $[k_{j+1} : k_j] \geq n+1$, $0 \leq j \leq n-1$. By these conditions, in each field k_j, $1 \leq j \leq n$, we can choose n numbers ξ_{ij}, $1 \leq i \leq n$, such that $1, \xi_{1j}, \ldots, \xi_{nj}$ are linear independent over subfield $k_{j-1} \subset k_j$. Introduce the following nonsingular matrix $A_k = c(\xi_{ij})$, where constant c is given by $\det A_k = 1$, and define the following lattice $\Omega_k = A_k\mathbb{Z}^n \in \mathcal{L}_n$.

Theorem 5 ([Skr5]). *With the above notation one has $R(tP_k,\Omega_k) = O(t^{\varepsilon})$ with arbitrarily small $\varepsilon > 0$.*

Proofs of the above results are based on a new general bound for the number of lattice points in polyhedra (see [Skr5], Theorem 6.1).

Roughly our approach to the study of the behavior of errors $R(tP,\Gamma)$ can be outlined as follows. With the set of faces of the polyhedron P and the lattice Γ we associate a finite set of lattices $\Gamma_f^{\perp} \in \mathcal{L}_n$. For example, in the simplest case where the polyhedron is a rectangular parallelepiped Π, this set consists of a single lattice $\Gamma^{\perp}$ dual to Γ. Next we consider orbits $\{\delta\Gamma_f^{\perp} : \delta \in \Delta\}$ of lattices $\Gamma_f^{\perp}$ under the action of a discrete subgroup $\Delta \subset D$ of the group of unimodular diagonal matrices $D \subset SL(n,\mathbb{R})$. It turns out that the rate of approach of these orbits to "infinity" (the cusp of the homogeneous space $\mathcal{L}_n$) determines the behavior of the corresponding error $R(tP,\Gamma)$ as $t \to \infty$.

For example, if all lattices $\Gamma_f^{\perp}$ are admissible, then their orbits $\{\delta\Gamma_f^{\perp} : \delta \in \Delta\}$ are relatively compact by Mahler's reterion (see [G-L]), and we obtain a minimal bound for the error given in Theorem 2.

We show in [Skr4] that from metric point of view for almost all lattices $\Lambda \in \mathcal{L}_n$ the orbits $\{\delta\Lambda : \delta \in \Delta\}$ tend to infinity very slowly. This circumstance leads to Theorems 3 and 4. Similarly, Theorem 5 follows from the fact that orbits $\{\delta\Omega_k^{\perp} : \delta \in \Delta\}$ for special algebraic lattices Ω_k also tend to infinity very slowly.

It is worth noting that the study of the structure of orbits $\{\delta\Lambda : \delta \in \Delta\}$, $\Lambda \in \mathcal{L}_n$, is of much interest for geometry of numbers. We state here only one result in this direction. Introduce the following quantities

$$Nm_q\Gamma = \inf\big\{(\log(2 + \|\gamma\|))^q |Nm\gamma| : \gamma \in \Gamma \setminus \{o\}\big\}, \quad q \geq 0,$$

where $\|\cdot\|$, is the euclidian metric on $\mathbb{R}^n$. Notice that $Nm_0\Gamma = Nm\Gamma$.

Theorem 6 ([Skr5], [Kl-Ma]). *For almost all lattices $\Gamma \in \mathcal{L}_n$ one has the following:*

(i) $Nm_q\Gamma > 0$ *for* $q > n-1$,

(ii) $Nm_q\Gamma = 0$ *for* $q \leq n-1$.

Statement (i) was proved in [Skr5], while statement (ii) was given in [Kl-Ma].

In conclusion we wish to outline some of main open problems related to anomalies in the lattice point problem:

(i) How to construct explicitly lattices leading to logarithmically small errors for an arbitrary polyhedron? Even for specific algebraic polyhedra $P_k \subset \mathbb{R}^n$ this question is close related with an open problem in diophantine approximations on logarithmically small bounds in the classical Schmidt's subspace theorem (cf. [Sch], [La]).

(ii) What is a minimal possible order $\varphi_{\min}(t)$ in logarithmically small bounds in the lattice point problem? One can conjecture that $\varphi_{\min}(t) = \log^{n-1} t$ in $\mathbb{R}^n$. In two dimensions the conjecture is true at least for parallelograms, that follows at once from known Schmidt's lower bound for the L_∞-discrepancy (cf. [B-C], [M]).

(iii) Suppose that $\varphi_{\min}(t) = \log^{n-1} t$ and introduce the quantity

$$\psi(\mathcal{O}, \Gamma) = \sup \lim_{t \to \infty} \frac{R(t\mathcal{O}, \Gamma)}{\log^{n-1} t}$$

for given $\mathcal{O} \subset \mathbb{R}^n$ and $\Gamma \subset \mathbb{R}^n$.

Theorem 2 implies that $\psi(\Pi, \Gamma) < \infty$ for rectangular parallelepipeds Π and admissible lattices Γ. One can conjecture that in dimensions $n > 2$ the quantity $\psi(\mathcal{O}, \Gamma) = \infty$ for all lattices Γ if a convex region $\mathcal{O}$ does not coinside with a parallelepiped (cf. Theorem 1 and 2). Notice that the conjecture has parallels with the well known conjecture of Littlewood (cf. [C-S-D], [L-G]). In particular, in the present context the following question should be of much interest: Given an admissible lattice $\Gamma \subset \mathbb{R}^n$. For what rotations $U \subset SO(n)$ lattices $U\Gamma$ are also admissible?

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