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THE IHARA–SELBERG ZETA FUNCTION OF A FINITE GRAPH AND SYMBOLIC DYNAMICS*

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Abstract.

In this paper we find a simple combinatorial treatment of the Bass' evaluation of the Ihara–Selberg zeta function for a finite graph in terms of a natural symbolic dynamics attached to a graph. A generalization of this result for a graph with labeling is given.

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Introduction

The study of zeta functions of finite graphs is an interesting problem as from the calculational side as from the point of view of their relationship with network problems, dynamical systems, etc. In this paper we will deal with zeta function of finite connected (not necessary regular) graph attached to trivial representation of the fundamental group of such a graph and depending on a labeling of the edges of the graph.

Zeta function of such kind is known to be closely related to the combinatorial laplacian acting on the space of the 0- (or: 1-) automorphic forms on a graph. (The possibility to pass from 0-forms to 1-forms reflects the combinatorial duality on a graph). Zeta function of a finite connected graph (and, in fact, its more abstract version: zeta function of a tree lattice) has been studying intensively during last period in the series of papers [3, 6–8] where the rationality theorems for it (and its generalizations) were proven. The rationality results for the Ihara–Selberg zeta function in the case of a finite connected *regular* graph (see [9, 1]) can be regarded as a special case of the abovementioned more general ratonality theorems. Recently Foata and Zeilberger [4] developed a new general combinatorial framework basing on the notion of the algebra of Lyndon words. As one of the consequences of the Foata and Zeilberger results was the Bass' evaluation of the Ihara–Selberg zeta function for graph.

In this paper we give a simple combinatorial treatment of the Bass' evaluation for concrete choice of the tree lattice $\Gamma := \pi_1(X)$ by means of symbolic dynamics attached to a finite connected graph (see Theorems 1 and 2). The approach to zeta function of finite graphs seems to be very natural from the point of view of Ruelle's theory [17–19].

This paper is organized as follows.

In Sec. 1 we recall some basic facts relating the geometry of finite graphs: free group action on trees, fundamental group of a graph, the notion of primitive cycles of a graph including a natural bijection between a set of primitive Γ-conjugacy classes and a set of primitive cycles.

In Sec. 2 we give a symbolic dynamics approach to zeta function of a finite connected graph without labeling. This result is closely related to the well known fact in Ruelle's theory: zeta function is considered as a counting function of the (weighted) number of fixed points of the homeomorphism of a shift and its powers.

In Sec. 3 we give a generalization of this result to the case of finite connected graphs with labeling. We also notice that the random walk approach to Alexander polynomial of [14] could be used to produce a random walk approach to determining the top row of the Kontsevich integral of a knot (cf. [12]) and to a construction of universal invariants for some class of 3-manifolds [5].

In Sec. 4 the result of Sec. 3 is commented from the point of view of the usual symbolic dynamics approach where the inverse value of zeta function of a finite connected graph with labeling could be regarded as the determinant of the weighted transfer operator.

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1. Background on Geometry of Graphs

Let X be a finite connected (not necessary regular) graph. Consider a covering of such a graph by a universal tree $\widetilde{X}$, which will be locally-finite infinite tree, equipped with a *free group action* of the fundamental group $\Gamma := \pi_1(X)$ of a graph X.

Recall the notion of the *fundamental group* of a graph. Fix a vertex $v_0 \in VX$. For a graph X its *fundamental group* $\pi_1(X)$ is defined to be a group of homotopy classes $< C >$ of a closed paths

C in X, such that $o(C) = t(C) = v_0$, and a multiplication in $\pi_1(X)$ is induced by a compositon of paths. Clearly, this definition (up to a conjugation) is independent on the choice of a vertex $v_0 \in VX$.

The fundamental group $\pi_1(X)$ acts on the covering tree $\widetilde{X}$ *freely* (see [20]). This means that:

(1) group $\Gamma := \pi_1(X)$ acts on the covering tree $\widetilde{X}$ without inversions, i.e.:

$$\Gamma{\cdot}e{\neq}\Gamma{\cdot}\overline{e}, \quad \text{for any} \quad e \in EX;$$

(2) stabilizer of any vertex $v \in VX$ in a group Γ cosists from the unit element only (for all vertices of a graph X):

$$\Gamma_v := \{\gamma \in \Gamma : \gamma v = v\} = \{\mathbf{id}\},$$

for any $v \in VX$.

The quotient of the action of the group Γ on the tree $\widetilde{X}$, $\Gamma \setminus X$ is defined correctly, and it can be canonically identified with the finite connected graph X. Orientation of the tree $\widetilde{X}$ is induced by the corresponding orientation of the graph X. This example of the free group action on $\widetilde{X}$ is a special case of the action of the uniform tree lattices (see [3]).

Recall for further description some basic facts relating the correspondence between the fundamental group of a graph and the cycles of the graph. We say that a sequence of the edges $C := (e_1, \dots, e_m)$, $m \in \mathbf{N}$, defines an (edge) path C from $v := \partial_0(e_1)$ to $v' := \partial_1(e_m)$ of the length m, if $\partial_0(e_i) = \partial_1(e_{i-1})$, for any $1{<}i{\leq}m$. A path $C := (e_1, \dots, e_m)$ is called to have *no backtracking* (in terminology of [6, 7]) if $e_i{\neq}\overline{e}_{i+1}$ for any $1{\leq}i{\leq}m-1$.

Definition 1 (see [6]). *A path $C := (e_1, \dots, e_m)$ is called to be closed if $\partial_1(e_m) = \partial_0(y_1)$.*

The closed path without backtracking $C = (e_1, \dots, e_m)$ is called to be reduced, if either $m = 1$ or $e_1 \neq \overline{e}_m$. In other words, the closed path C without backtracking is called to be reduced, if and only if C and $C{\circ}C = C^2$ – are both have no backtracking.

For any $m \in \mathbf{N}$ we define two sets:

$$\mathfrak{C}_m(X) := (C : C = \text{closed path having no backtracking; length}(C) = m), \tag{1.1}$$

$$\mathfrak{C}_m^{\text{red}}(X) := (C : C \in \mathfrak{C}_m(X) : C = \text{reduced.}) \tag{1.2}$$

To any closed path without backtracking $C \in \mathfrak{C}_m(X)$ we associate a unique reduced path $C^\star \in \mathfrak{C}^{\text{red}}_{n-2k}(X)$, obtaining by removing the edges (e_1, e_m), $(e_2, e_{n-1}), \dots, (e_k, e_{n-2k})$, where $\overline{e}_i = e_{n-i+1}$, $(1{\leq}i{\leq}k)$ and $e_{k+1}{\neq}\overline{e}_{n-k}$. This passage is illustrated in Figure 1. (Figure 1 is given after [6]).

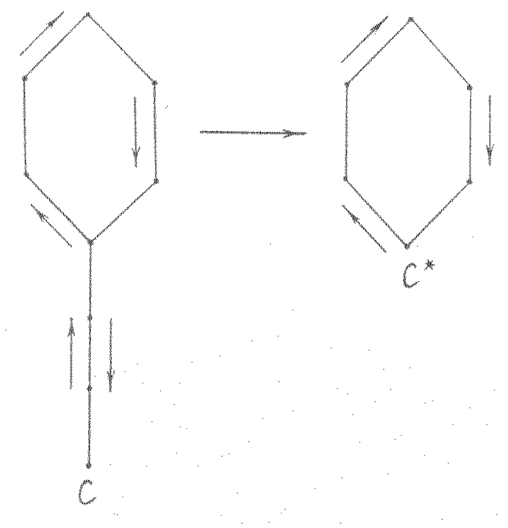

Figure 1.

If $C \in \mathfrak{C}_m^{\mathrm{red}}(X)$ is a reduced path, then for any $k(1\leq k\leq m-1)$, closed path

$$\sigma^k(C) := C_k := (e_{k+1}, e_{k+2}, \ldots, e_n, \ldots, e_k), \tag{1.3}$$

derived from C by shifting the origin by k steps is also reduced path. The closed path C_k is called to be conjugate to C; this will be denoted as $C\simeq C_k$. This defines an equivalence relaton on the set $\mathfrak{C}_m^{\mathrm{red}}(X)$ of the reduced closed paths of the length m. Class of equavalence C is called a *cycle* of X (of the length m); this class will be denoted by $[C]$.

In other words, formula (1.3) defines an action of a cyclic group of order m on $\mathfrak{C}_m^{\mathrm{red}}(X)$, and the orbits of such action is called the *cycles* in graph X of the length $m \in \mathbf{N}$. The set of the cycles of X of a length $m \in \mathbf{N}$ is denoted by $\mathbf{Cycl}_m(X)$. The reduced closed path $C \in \mathfrak{C}_m^{\mathrm{red}}(X)$ (or: the cycle $[C] \in \mathbf{Cycl}_m(X)$) is called to be a *primitive*, if it is not of the form $C = D^l$ with $l>1$ for some reduced closed path $D \in \mathfrak{C}_d^{\mathrm{red}}(X), d|m$ (see [6]).

Denote the set of all closed reduced primitive paths in X of a length $m \in \mathbf{N}$ by $\mathfrak{C}_m^{\mathrm{red,prim}}(X)$. It is not easy to see that there is a decomposition (see [6]):

$$\mathfrak{C}_m^{\mathrm{red}}(X) = \bigcup_{d|n} (\mathfrak{C}_d^{\mathrm{red,prim}}(X))^{n/d} \tag{1.4}$$

(disjoint union).

It is clear from the Figure 1 that any closed path C in X defines a cycle $[C^\star]$ of a reduced (closed) path $C^\star$, which is homotopic to C. By a constructon of the fundamental group $\pi_1(X)$, a map $< C > \longrightarrow [C^\star]$ induces a natural bijection (see [6]):

$$(\text{conjugacy classes in} \quad \pi_1(X, v_0) \quad \simeq \quad (\text{cycles in} \quad X). \tag{1.5}$$

If a set of the primitive conjugacy classes of $\Gamma := \pi_1(X)$ is denoted by $\mathfrak{P}_\Gamma$, then, in these on notatons, the bijection (1.5) induces the bijection (see [6]):

$$\mathfrak{P}_\Gamma \simeq \quad (\text{primitive cycles in} \quad X) \tag{1.6}$$

(1–1 correspondence) and

$$\mathfrak{P}_\Gamma(d) \simeq \mathrm{Prim}_X(d), d \in \mathbf{N}, \tag{1.7}$$

(1–1 correspondence), where $\mathfrak{P}_\Gamma(d) := \{\{P\} \in \mathfrak{P}_\Gamma : \deg P = d\}$, and

$$\mathrm{Prim}_X(d) := (\text{primitive cycles in} \quad X \quad \text{of length} \quad d), \quad d \in \mathbf{N}.$$

(Recall (see [20]) that a degree deg of a hyperbolic Γ-conjugacy class is defined uniquely by its geometric action on the covering tree $\widetilde{X}$.)

Now we are in a position to define a rank of a graph X after [3]. An *orientation* of X is, by defintion, is subset E^+X in EX such that $EX := E^+X \coprod \overline{E^+X}$. Then it follows that $|EX| = 2|E^+X|$. If we put $C_1^+ := \mathbf{C}^{(E^+X)}$, then the folloiwng exact sequence holds:

$$0 \longrightarrow \mathbf{H}_1(X, \mathbf{C}) \longrightarrow C_1^+ \overset{\partial_0 - \partial_1}{\longrightarrow} C_0 \longrightarrow \mathbf{H}_0(X, \mathbf{C}) \longrightarrow 0.$$

Betti numbers, assigned to the graph X, is defined by the usual procedure:

$$h_j(X) := \dim_{\mathbf{C}} \mathbf{H}_j(X, \mathbf{C}) \quad \text{be} \quad j^{th} \quad \text{Betti number},$$

and for the *Euler characteristic* of X we have:

$$\chi(X) := h_0(X) - h_1(X) = |VX| - \frac{1}{2}|EX|. \tag{1.8}$$

The *rank* of X is defined via the formula (see [3]):

$$\mathrm{rank}\,(X) := h_1(X) = h_0(X) - \chi(X). \tag{1.9}$$

It is easy to see that $h_0(X)$ is equal to the number of connected components of the graph X. Consequently, if X is a connected graph, then

$$\mathrm{rank}\,(X) = 1 - \chi(X) :=$$

$$\text{rank of the free group} \quad \pi_1(X) := \dim{}_{\mathbf{C}}\mathbf{H}_1(X, \mathbf{C}) \tag{1.10}$$

(where we omit the dependence $\pi_1(X)$ from the basic point). By defintion, the formula for rank of X (or, for the rank of the corresponding fundamental group of X) can be rewriten as follows:

$$\mathrm{rank}\,(X) = |GEX| - |VX| + 1,$$

where by $|GEX|$ is denoting the number of geometric edges in X. For example, for connected $(q+1)$-regular graph X, the number of geometric edges in X can be calculated as follows:

$$\frac{1}{2}|EX| = |GEX| = \frac{(q+1)}{2}|VX|,$$

and rank of the free group $\pi_1(X)$ is equal to

$$r = \frac{q-1}{2}|VX| + 1.$$

Fix (once again) the concrete situation under consideration. Let $\widetilde{X}$ be a covering tree of a graph X, equipped with a free action of a free group $\Gamma := \pi_1(X)$, such that the quotient of this action is canonically identified with X. The set of the primitive conjugacy classes of the group Γ (distinct from unit class) will be denoted by $\mathfrak{P}_\Gamma$. This set bijectively corresponds to the set of primitive cycles of the graph X. $\mathfrak{P}_\Gamma$ is called to be a *geometric spectrum* of the graph $X \simeq \Gamma \backslash \widetilde{X}$.

2. Zeta Function of a Finite Graph and Symbolic Dynamics

As shown in [3] the geometric spectrum of a graph $X := \Gamma \setminus \widetilde{X}$ (i.e., the spectrum of its primitive cycles) is closely related to the spectrum (in the operator theory sense) of *automorphic Laplacian* acting in the space of automorphic forms on X. In mathematical terms, this relationship can be written as an exact determinant expression of L-function of the graph via corresponding automorphic Laplacian. By definition, L-function attached to the graph X and a finite-dimensional $\mathbf{C}$-representation $\rho : \Gamma \longrightarrow GL(V_\rho)$ is defined by means of the infinite (formal) product which is taken over the set of the primitive Γ-conjugacy classes (or, in equivalent terms, over the set of the Γ-classes of Γ-hyperbolic ends of the covering tree $\widetilde{X}$):

$$L(\Gamma, u, \rho) := \prod_{\{P\} \in \mathfrak{P}_\Gamma} \det\,(I_V - u^{\deg P}\rho(P))^{-1}. \tag{2.1}$$

As we have seen in Sec. 1, primitive Γ-conjugacy classes bijectively correspond to the primitive cycles of X. This allows to rewrite (2.1) as the (formal) product over the set of primitive cycles in X:

$$L(\Gamma, u, \rho) := \prod_{[C]:primitive} \det\left(I_V - u^{l([C])}\rho(< C >)\right)^{-1}, \tag{2.2}$$

where by $< C >$ is denoting the Γ-conjugacy class, corresponding to the cycle $[C]$. We are going to compare this L-function with determinants of certain combinatorial operators acting in the spaces of automorphic forms on X. Define the following vector spaces of Γ-automorphic 0- and 1-cochains on $\widetilde{X}$:

$$H_\rho(V\widetilde{X}) := \{\Gamma - \text{equivariant functions} \quad f : V\widetilde{X} \to V_\rho\},$$

i. e., $f(\gamma v) = \rho(\gamma) f(v)$, for any $\gamma \in \Gamma, v \in V\widetilde{X}$, and

$$H_\rho(E\widetilde{X}) := \{\Gamma - \text{equvariant functions} \quad f : E\widetilde{X} \to V_\rho\},$$

i. e., $f(\gamma e) = \rho(\gamma) f(e)$, for any $\gamma \in \Gamma$, $e \in E\widetilde{X}$.

One can easy to see that the values of Γ-automorphic 0- (resp., 1-) cochain on $\widetilde{X}$ are determined completely by the values on VX (resp., EX). It follows that these spaces are finite-dimensional:

$$\chi_\rho(r_0) := \dim_{\mathbf{C}} H_\rho(V\widetilde{X}) = |VX| \dim_{\mathbf{C}} V_\rho,$$

$$2\chi_\rho(r_1) := \dim_{\mathbf{C}} H_\rho(E\widetilde{X}) = |EX| \dim_{\mathbf{C}} V_\rho.$$

Consider the following operators acting on $H_\rho(\widetilde{X})$:

$$(Q_\rho f)(v) := q(v) f(v), \quad \text{for any} \quad v \in V\widetilde{X},$$

where $q(v) + 1 := d(v)$ is the degree of the vertex $v \in V\widetilde{X}$ (the number of the edges joining to v);

$$(\delta_\rho f)(v) := \sum_{e \in E_0(v)} f(\partial_1 e), \quad \text{for any} \quad v \in V\widetilde{X},$$

where $E_0(v) := \{e \in E\widetilde{X} : \partial_0 e = v\}$;

$$\Delta_\rho(u) := I - \delta_\rho u + Q_\rho u^2,$$

with $I := Id_{V_\rho}$.

Define the opeartor T_ρ acting on $H_\rho(E\widetilde{X})$:

$$(T_\rho f)(e) := \sum_{(e, e_1)_{\text{red}}} f(e_1), \quad \text{for any} \quad e \in E\widetilde{X},$$

where summation is taken over all reduced paths (e, e_1) in $\widetilde{X}$.

Finally, define the number $\hat{\chi}_\rho(\Gamma)$ (the equvariant Euler characteristic) by:

$$\hat{\chi}_\rho(\Gamma) := \chi_\rho(r_0) - \chi_\rho(r_1) = \chi(X) \dim_{\mathbf{C}} V_\rho.$$

(The reader could find in [3] more general expression of $\hat{\chi}_\rho(\Gamma)$ for more general lattices.)
With these notations the following rationality formulas for $L(\Gamma, u, \rho)$ holds [1]:

$$L(\Gamma, u, \rho) = \det (I - uT_\rho)^{-1} = (1-u^2)^{\hat{\chi}_\rho(\Gamma)} \det (\Delta_\rho(u))^{-1}. \tag{2.3}$$

As a consequence of the first determinant expression we have the following equalities for the traces of powers of T_ρ:

$$Tr(T_\rho^n) = \sum_{d|n} d \sum_{\{P\}\in \mathfrak{P}_\Gamma, \deg P = d} Tr(\rho(P)^{n/d}), \tag{2.4}$$

for any $n \in \mathbf{N}$. As the simplest case we consider $\rho := \mathbf{1}$ (the trivial representation of the group Γ). In this case we return back to the notion of *zeta function* of a finite graph $X \simeq \Gamma \setminus \widetilde{X}$ (considering as the infinite (formal) product):

$$Z_\Gamma(u) = \prod_{\{P\}\in \mathfrak{P}_\Gamma} (1 - u^{\deg P})^{-1}. \tag{2.5}$$

As follows from (2.3) the rationality formula for $Z_\Gamma(u)$ is

$$Z_\Gamma(u) = \det (I - uT_{\mathbf{1}})^{-1}. \tag{2.6}$$

This equality means that the following equalities (which can be viewed as the special case of (2.4)) hold:

$$Tr(T_{\mathbf{1}}^n) = \sum_{d|n} d|\mathfrak{P}_\Gamma(d)|. \tag{2.7}$$

The rationality result for $Z_\Gamma(u)$ can be interpreted via symbolic dynamics. To do that we associate the *Markov toplogical chain* to finite connected graph $X = \Gamma \setminus \widetilde{X}$ as follows. Consider a finite alphabet generated by the set of (oriented) edges EX of X. A transition matrix T is exactly a matrix of the operator $T_{\mathbf{1}}$ in a natural basis attached to EX, i.e.:

$$t(e, e^{'}) := 1, \quad \text{if} \quad (e, e^{'}) \quad \text{is reduced path in} \quad X, \quad \text{and} \quad t(e, e^{'}) := 0, \quad \text{otherwise.}$$

Denote: $T := \{t(e, e^{'})\}_{e,e^{'}\in EX}$. The set of the states of the Markov topological chain (Σ_T, σ) with the transition matrix T consists of all sequences of symbols from EX:

$$\Sigma_A \ni \Psi := (\dots, e_k, e_{k+1}, \dots)$$

such that $t(e_k,\, e_{k+1}) = 1, and k \in \mathbf{Z}$. Symbolic dynamics on (Σ_T, σ) is generated by the homeomorhism of a shift:

$$\sigma : \Sigma_T \longrightarrow \Sigma_T, \quad \Psi^{'} = \sigma(\Psi), \quad (\Psi^{'})_k = (\Psi)_{k+1}, \quad k \in \mathbf{Z},$$

where k-component for Ψ is denoted by $(\Psi)_k$, $k \in \mathbf{Z}$. Let $\mathrm{Fix}\, \sigma^n (n \in \mathbf{N})$ be a set of the fixed points of nth iteration of the homeomorphism of the shift:

$$\Psi \in \mathrm{Fix}\, \sigma^n \Longleftrightarrow \sigma^n(\Psi) = \Psi$$

[1] These formulas can be viewed as the part of Bass' theory [3] for the Ihara–Selberg zeta function of a tree lattice, see also [1, 6, 7].

As usual, define a zeta function $Z_{\rm top}^{(\Sigma_T,\sigma)}(u)$, attached to the Markov topological chain (Σ_T,σ) by the formula:

$$Z_{\rm top}^{(\Sigma_T,\sigma)}(u)=\exp\left\{\sum_{n=1}^{\infty}\frac{|\mathrm{Fix}\,\sigma^n|}{n}u^n\right\}. \tag{2.8}$$

As well known (see [17]), zeta function $Z_{\rm top}^{(\Sigma_T,\sigma)}(u)$ attached to the Markov topological chain with finite number of states (recall that X is assumed to be finite and connected) is a rational function on whole u-plane:

$$Z_{\rm top}^{(\Sigma_T,\sigma)}(u)=\det{(I-uT)^{-1}}. \tag{2.9}$$

This equality is a consequence of the infinite series of trace formulas:

$$|\mathrm{Fix}\,\sigma^n|=TrT^n,\ n\in\mathbf{N}. \tag{2.10}$$

As noted above the matrix T is the matrix of the operator $T_{\mathbf{1}}$ in the basis attached to EX [2]. Hence we have the following theorem.

Theorem 1.

$$Z_{\Gamma}(u)=Z_{\rm top}^{(\Sigma_T,\sigma)}(u). \tag{2.11}$$

Proof. Now we will give a combinatorial proof of Theorem 1 basing on the bijective correspondence between Γ-conjugacy classes and cycles in X. Really, we can note that the statement of the Theorem 1 can be reformulated on the language of the primitive cycles of the graph X. Let Prim_X be the set of the primitive cycles of the graph $X{\simeq}\Gamma\backslash\widetilde{X}$. Since the primitive Γ-conjugacy classes are in the bijective correspondence with the primitive cycles of the graph $X{\simeq}\Gamma\backslash\widetilde{X}$, and $\deg\,P=l([C])$ if $P{\leftrightarrow}[C]$, then to verify (2.11) is the same as to verify:

$$u\frac{d}{du}\log Z_X(u)=u\frac{u}{du}\log Z_{\rm top}^{(\Sigma_T,\sigma)}(u),$$

where

$$Z_X(u):=\prod_{[C]\in\mathrm{Prim}_X}(1-u^{l([C])})^{-1},$$

and the product is taken over the set of all primitive cycles of the graph X. Consider:

$$\log Z_X(u)=-\sum_{[C]\in\mathrm{Prim}_X}\log{(1-u^{l[C]})}=\sum_{[C]\in\mathrm{Prim}_X}\sum_{k=1}^{\infty}\frac{u^{kl([C])}}{k}=$$

$$\sum_{k=1}^{\infty}\sum_{d=1}^{\infty}\sum_{[C]\in\mathrm{Prim}_X,l([C])=d}\frac{u^{kd}}{k}. \tag{2.12}$$

Applying the Euler operator to the right hand side of (2.12), we have:

$$u\frac{d}{du}\log Z_X(u)=\sum_{k=1}^{\infty}\sum_{d=1}^{\infty}d\sum_{[C]\in\mathrm{Prim}_X(d)}u^{kd}, \tag{2.13}$$

[2] The action of $T_{\mathbf{1}}$ on $H_{\mathbf{1}}(E\widetilde{X})$ is defined by Γ-equivariantness.

since, by defintion: $\mathrm{Prim}_X(d) := \{[C] \in \mathrm{Prim}_X(d) : l([C]) = d\}, d \in \mathbf{N}$. Take a substitution $n := kd$, making use that from the definition of the cycles follows that the natural map $C \mapsto [C]$ from $\mathfrak{C}_d^{\mathrm{red,prim}}(X)$ to $\mathrm{Prim}_X(d)$ is d-to-one. Therefore one can rewrite (2.13) as follows:

$$u\frac{d}{du}\log Z_X(u) = \sum_{n=1}^{\infty}\sum_{d|n}\sum_{C\in\mathfrak{C}_d^{\mathrm{red,prim}}(X)} u^n =$$

$$\sum_{n=1}^{\infty}\sum_{C\in\mathfrak{C}_n^{\mathrm{red}}(X)} u^n = \sum_{n=1}^{\infty}|\mathfrak{C}_n^{\mathrm{red}}(X)|u^n.$$

But $|\mathfrak{C}_n^{\mathrm{red}}(X)| = |\mathrm{Fix}\,\sigma^n|$ for any $n \in \mathbf{N}$. Really, if $\Psi \in \mathrm{Fix}\,\sigma^n$, for some $n \in \mathbf{N}$, then $\Psi = (\ldots, C, C, C, \ldots)$, for some $C \in \mathfrak{C}_n^{\mathrm{red}}(X)$. Conversly, reduced path $C \in \mathfrak{C}_n^{\mathrm{red}}(X)$ generates the fixed point Ψ of σ^n by a rule $\Psi := (\ldots, C, C, C, \ldots)$. Therefore, $C_n^{\mathrm{red}}(X)$ is in the bijective correspondence with $\mathrm{Fix}\,\sigma^n$ for any $n \in \mathbf{N}$. It follows that:

$$u\frac{d}{du}Z_X(u) = u\frac{d}{du}\sum_{n=1}^{\infty}\frac{|\mathrm{Fix}\,\sigma^n|}{n}u^n = u\frac{d}{du}Z_{\mathrm{top}}^{(\Sigma_T,\sigma)}(u).$$

This completes the proof of Theorem 1. ♣

Note once agian that the main tool in verifying (2.10) is a passage to verifying more simple equality basing on the correspondence between the Γ-conjugacy classes and the cycles in X. Therefore, as shown above, the exact determinant expression for zeta function can produced via theory of the Markov topological chains. Consequently, this zeta function can be considerded as the generating function for the number of the fixed points of the natural symbolic dynamics attached to a finite graph.

Remark (see [2]). Generalizing slightly the previous discussion one can consider for a finite $0-1$-transition matrix T and for a topological Markov chain (Σ_T, σ) *weighted* matrix $T_{i,j}^g$, defined by the formula:

$$T_{i,j}^g := t_{i,j}g_{i,j}, \tag{2.14}$$

where $g : \Sigma_T \longrightarrow \mathbf{C}$ is the function, such that $g(\Psi_1, \Psi_2, \ldots)$ depends on Ψ_1, Ψ_2 only. A *weighted dynamical zeta function* is defined by the formula:

$$\zeta_g(z) := \exp\left\{\sum_{m=1}^{\infty}\frac{z^m}{m}\sum_{\sigma^m(\Psi)=\Psi}\prod_{k=0}^{m-1}g(\sigma^k(\Psi))\right\}, \tag{2.15}$$

and *weighted transfer-operator* acting on functions $\Phi : \Sigma_T \longrightarrow \mathbf{C}$ is defined by the formula:

$$L_g(\Phi) := \sum_{\Psi^{'}\in\Sigma_T,\sigma(\Psi^{'})=\Psi}\Phi(\Psi^{'})g(\Psi^{'}). \tag{2.16}$$

The matrix T^g can be viewed as the matrix of the operator L_g acting on the finite-dimensional vector space of the functons depending on Ψ_1 only. One can deduce the following trace formula generalizing the trace formula (2.10):

$$Tr(T^g)^m = \sum_{\Psi\in\Sigma_T,\sigma^m(\Psi)=\Psi}\prod_{k=0}^{m-1}g(\sigma^k(\Psi)). \tag{2.17}$$

Therefore, we have the following exact determinant representation for $\zeta_g(z)$:

$$\frac{1}{\zeta_g(z)} = \exp\left\{-\sum_{m=1}^{\infty}\frac{z^m}{m}Tr(T^g)^m\right\} = \det(I - zT^g). \tag{2.18}$$

We refer the reader to [17–19] for more detailed studying this topic. The reader could find in [2] a more information on the modern progress in the theory of dynamical zeta functions as well as on their applications to the problems of symbolic dynamics (see also [16]).

3. Zeta Function for Graphs with Labeling via Symbolic Dynamics

It is not easy to see how to generalize the previous result for *finite connected graphs with labeling*. Recall that a labeling for a finite connected graph X means an assignment:

$$e \mapsto \underline{\mathbf{u}}(e) \in \mathbf{U},$$

for any $e \in EX$. The set $\mathbf{U} = \{\underline{\mathbf{u}}(e)\}_{e\in EX}$ is the set of independent commuting variables. In other words, by a labeling of a graph $X(\simeq\Gamma\backslash\widetilde{X}$, where $\widetilde{X},\Gamma$ are the same as before), we mean Γ-equivariant mapping

$$\underline{\mathbf{u}} : E\widetilde{X} \longrightarrow \mathbf{U} \quad \text{such that} \quad \underline{\mathbf{u}}(\gamma e) = \underline{\mathbf{u}}(e), \quad \text{for any} \quad e \in EX, \quad \gamma \in \Gamma.$$

For any reduced path $C \in \mathfrak{C}_k^{\mathrm{red}}(X)$ we will write

$$\underline{\mathbf{u}}^C := \underline{\mathbf{u}}(e_1)\underline{\mathbf{u}}(e_2)\ldots\underline{\mathbf{u}}(e_k), \quad \text{if} \quad C = \{e_1, e_2, \ldots, e_k\}.$$

It is easy to see that $\underline{\mathbf{u}}^{C_1} = \underline{\mathbf{u}}^{C_2}$ if C_1 and C_2 belongs to the class of the cycle $[C] \in \mathbf{Cycl}_k(X)$. For sake of simplicity we will asuume that:

$$\underline{\mathbf{u}}(e) = \underline{\mathbf{u}}(\overline{e}),$$

for any edge $e \in EX$ (as in [6]). As in [6] we will denote a polynomial ring of $\underline{\mathbf{u}}$ over $\mathbf{C}$ by $\mathbf{C}[\underline{\mathbf{u}}]$. Then zeta function of a finite connected graph $X := \Gamma\backslash\widetilde{X}$ with labeling (considering as the element of the polynomial ring $\mathbf{C}[\underline{\mathbf{u}}]$), is defined as the infinite (formal) product (see [6]):

$$Z_\Gamma(\underline{\mathbf{u}}) = \prod_{[C]\in \mathrm{Prim}(X)} (I - \underline{\mathbf{u}}^C)^{-1}, \tag{3.1}$$

where $< C >$ be a primitive Γ-conjugacy class corresponding to a cycle $[C] \in \mathrm{Prim}(X)$. Let (Σ_T, σ) be the Markov toplogical chain attahced to the finite connected graph $X(\simeq\Gamma\backslash\widetilde{X})$ as described in Sec. 2. For any $\Psi \in \Sigma_T$ we deifne an operator:

$$B_{\underline{\mathbf{u}}}(\Psi) = \underline{\mathbf{u}}(e_0)$$

if $\Psi = (\ldots, e_0, e_1, \ldots) \in \Sigma_T$. Then a *topological zeta function* (considering as the element of the ring $\mathbf{C}[\underline{\mathbf{u}}]$), attached to (Σ_T, σ) is defined as the infinite (formal) power series:

$$Z_{\mathrm{top}}^{(\Sigma_T,\sigma)}(\underline{\mathbf{u}}) = \exp\left\{\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\Psi\in \mathrm{Fix}\,\sigma^n}\prod_{l=0}^{n-1} B_{\underline{\mathbf{u}}}(\sigma^l(\Psi))\right\}. \tag{3.2}$$

Now we are in a position to compare $Z_\Gamma(\underline{\mathbf{u}})$ with $Z_{\mathrm{top}}^{(\Sigma_T,\sigma)}(\underline{\mathbf{u}})$: corresponding theorem will be a special case of the more general version considering in [15].

Theorem 2. *The following equality holds in $\mathbf{C}[\underline{\mathbf{u}}]$:*

$$Z_{\text{top}}^{(\Sigma_T,\sigma)}(\underline{\mathbf{u}}) = Z_\Gamma(\underline{\mathbf{u}}). \tag{3.3}$$

Proof. Since the constant terms in both zeta functions are equal to unity, then we have to verify that:

$$\mathbf{D}\log Z_{\text{top}}^{(\Sigma_T,\sigma)}(\underline{\mathbf{u}}) = \mathbf{D}\log Z_\Gamma(\underline{\mathbf{u}}),$$

$$\text{where}\quad \mathbf{D} := \sum_{e\in GEX} \underline{\mathbf{u}}(y)\frac{\partial}{\partial \underline{\mathbf{u}}}$$

be the (reduced) Euler operator. $\mathbf{D}$ acts on homogeneous elements of the ring $\mathbf{C}[\underline{\mathbf{u}}]$ of degree d as the multiplication by d. As in [6] one can see that:

$$\mathbf{D}\log Z_\Gamma(\underline{\mathbf{u}}) = \sum_{k=1}^{\infty}\sum_{d=1}^{\infty} d \sum_{[C]\in \text{Prim}_Y(d)} \underline{\mathbf{u}}^{C^k}. \tag{3.4}$$

It follows from the definition of the cycles that the natural map $C\mapsto[C]$ from $\mathfrak{C}_d^{\text{red,prim}}(X)$ to $\text{Prim}_X(d)$ is d-to-one. Take a substitution $kd = n$ and rewrite (3.4) as follows:

$$\mathbf{D}\log Z_\Gamma(\underline{\mathbf{u}}) = \sum_{n=1}^{\infty}\sum_{d|n}\sum_{C\in\mathfrak{C}_d^{\text{red,prim}}(X)} \underline{\mathbf{u}}^{C^{n/d}} =$$

$$\sum_{n=1}^{\infty}\sum_{C\in\mathfrak{C}_n^{\text{red}}(X)} \underline{\mathbf{u}}^{C} = \sum_{n=1}^{\infty}\sum_{\Psi\in\text{Fix}\,\sigma^n}\prod_{l=0}^{n-1} B_{\underline{\mathbf{u}}}(\sigma^l(\Psi)) =$$

$$\mathbf{D}\left\{\sum_{n=1}^{\infty}\frac{1}{n}\sum_{\Psi\in\text{Fix}\,\sigma^n}\prod_{l=0}^{n-1} B_{\underline{\mathbf{u}}}(\sigma^l(\Psi))\right\} = \mathbf{D}\log Z_{\text{top}}^{(\Sigma_T,\sigma)}(\underline{\mathbf{u}}),$$

since we can use the same arguments as in the proof of Theorem 1 and the formula (1.4) reresenting $\mathfrak{C}_n^{\text{red}}(X)$ as a disjont union. This completes the proof of Theorem 2. ♣

We note that a zeta function described above is expressed via an exact rationalty formula, which is the special case of the formulas described in [3] (see also [4, 6]). Namely, the determinant representation for $Z_\Gamma^{-1}(\underline{\mathbf{u}})$ is:

$$Z_\Gamma^{-1}(\underline{\mathbf{u}}) = \det(I - \mathfrak{B}_{\underline{\mathbf{u}}}), \tag{3.5}$$

where $\mathfrak{B}_{\underline{\mathbf{u}}}$ is the Bass operator for labelled graph [3] acting in a space of Γ-equivariant 1-cochains ans I is an identity operator acting in this space. Recalling the expression of determinant of $N{\times}N$-matrix $I - \mathfrak{B}_{\underline{\mathbf{u}}}$ in terms of supertrace

$$\det(I - \mathfrak{B}_{\underline{\mathbf{u}}}) = str\hat{\mathfrak{B}}_{\underline{\mathbf{u}}},$$

(where $\hat{\mathfrak{B}}_{\underline{\mathbf{u}}}$ is an extension of $\mathfrak{B}_{\underline{\mathbf{u}}}$ to the whole exterior algebra $\Lambda^\star V$ and a vector space V is spanned on the set EX) one can see that $Z_\Gamma^{-1}(\underline{\mathbf{u}})$ has a similarity to the definition of the

[3] Subscript $\underline{\mathbf{u}}$ corresponds to the fixed labelling of a graph.

Alexander–Conway polynomial (see [11]). As was stated in [14] this not only similarity. Namely, it was found in [14] that det $(I - \mathfrak{B}_{\underline{\mathbf{u}}})$ (up to the factor of the power of t, depending on a knot [4]) coincides with the symmetric Alexander polynomial (or, in equivalent terms, with the Alexander–Conway polynomial [5]) of a knot for special choice of the Bass' matrix $\mathfrak{B}_{\underline{\mathbf{u}}}$ regarding as a function of t. Lin–Wang approach to the Alexander–Conway polynomial of a knot K [6] is based on a model of random walk on knot diagrams and uses the Wirtinger presentation of a knot group and Fox' free differential calculus. The calculation combining well known state sum expressions for colored Jones polynomial of a knot with this presentation of the Alexander–Conway polynomial of a knot leads to a new proof of the Melvin–Morton conjecture [14].

Therefore due to Lin–Wang model developing in [14], we provide with a random walk approach to the Alexander–Conway polynomial. The Alexander–Conway polynomial of a knot, as well known (see [12] and references therein), determines the top row of the Kontsevich integral of a knot. Recall breifly the basic formulas relating to the question. Consider the Alexander–Conway polynomial as a formal power series in variable h defined by the skein relations:

$$C_{K^+}(h) - C_{K^-}(h) = (\exp(h/2) - \exp(-h/2))C_{K_s}(h), \quad C_U(h) = 1,$$

where K^+, K^-, and K^s are three knots which differ only in a ball and where they are represented as an overcrossing, an undercrossing and orientation-preserving smooothing of the crossing, and U is an unknot.

Rational knot invariants $d_{2n}(K)$ are determined by the (normalized version of) the Alexander–Conway polynomial:

$$C_K(h) = \exp(-2\sum_{n=1}^{\infty} d_{2n}(K)h^{2n}).$$

Then the top row of the Kontsevich integral of a knot is determined:

$$Z(K) = \exp(\frac{f(K)}{2}\theta + \sum_{n=1}^{\infty} d_{2n}(K)\omega_{2n} + X),$$

where $Z(K)$ is the framed version of the Kontsevich integral [13] of a knot, f(K) is the framing of a knot, θ is the web diagram with a single chord, ω_{2n} is the wheel diagrams ("with 2n spokes") and X is a series of diagrams with connected dashed graphs of negative Euler characteristic.

The similar things could be done for some class of 3-manifolds. Here we mean an example of 3-manifold $M := S^3_K$ which is obtained by surgery of zero-framed knot K in S^3. In this case the universal Le–Murakami–Ohtsuki (LMO) invariant $Z^{LMO}(S^3_K)$ is determined by the (normalized) Alexander–Conway polynomial $A(K)$ (see [5]) which in the same time has a random walk interpretation. Therefore, as a byproduct, we obtain a possibility of a random walk approach to the construction of universal invariants for this class of 3-manifolds.

4. From Weighted Transfer Operator to Ratonality Formula

Here we are going to relate a rationality formula for zeta function (3.1) of a finite connected graph with labeling to a weighted transfer operator attached to such graph. To solve a problem one can identify the (equivariant version of) Bass' operator with certain weighted transfer operator which attached to a graph.

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[4] In this case the labeling of a graph is a function of variable t.
[5] In fact, the Alexander and Conway polynomials are related to each other (see [10], Prop. 9.3).
[6] The knot K is the closure of 0-framed link T which can be chosen appropriately.

Recall that the inverse value of the zeta function of a finite connected graph with labelling defined as (3.1) has the following determinant representation:

$$Z_\Gamma^{-1}(\underline{\mathbf{u}}) = \det(I - \mathfrak{B}_{\underline{\mathbf{u}}}), \tag{4.1}$$

where $\mathfrak{B}_{\underline{\mathbf{u}}}$ is Bass' operator defining by its matrix $\mathfrak{B}_{\underline{\mathbf{u}}}(e, e^{'})$: [7]

$$\mathfrak{B}_{\underline{\mathbf{u}}}(e, e^{'}) := \underline{\mathbf{u}}(e) \quad \text{if} \quad (e, e^{'}) \quad \text{is reduced path in} \quad X, \quad \text{and} \quad \mathfrak{B}_{\underline{\mathbf{u}}} := 0, \quad \text{otherwise}.$$

(This $\mathfrak{B}_{\underline{\mathbf{u}}}$-matrix can be considered as a generalization of a transition T-matrix from the Sec. 2.)

The rationality result for $Z_\Gamma(\underline{\mathbf{u}})$ (where group Γ acts freely on a tree with finite quotient) is the special case of Bass' rationality theorem for more abstract zeta function. Applying the Euler derivative to (4.1) one can see that the rationality result for $Z_\Gamma(\underline{\mathbf{u}})$ follows from the trace formulas for the powers of $T_{\underline{\mathbf{u}}}$:

$$Tr\mathfrak{B}_{\underline{\mathbf{u}}}^n = \sum_{d|n} d \sum_{[C]\in \mathrm{Prim}_X(d)} \underline{\mathbf{u}}^{C^k}, \quad n \in \mathbf{N}. \tag{4.2}$$

These formulas (after the substitution $kd = n$) can be viewed as the special case of Bass' result in [3] deriving in more general ground (see also [19]). From the another point of view (4.2) can be regarded as the generalization of (2.7). Then, like as in the proof of Theorem 2, we have:

$$\sum_{d|n} d \sum_{[C]\in \mathrm{Prim}_X(d)} \underline{\mathbf{u}}^{C^k} = \sum_{C\in \mathfrak{C}_n^{\mathrm{red}}(X)} \underline{\mathbf{u}}^C = \sum_{\Psi\in \mathrm{Fix}\,\sigma^n} \prod_{l=0}^{n-1} B_{\underline{\mathbf{u}}}(\sigma^l(\Psi)), \quad n \in \mathbf{N}. \tag{4.3}$$

Now, we attach the "2-variable" weighted dynamical zeta function (depending on labelling $\underline{\mathbf{u}}$ of a graph X and variable z) $\zeta_{\underline{\mathbf{u}}}(z)$ to the weighted matrix $\mathfrak{B}_{\underline{\mathbf{u}}}$ as follows:

$$\zeta_{\underline{\mathbf{u}}}(z) := \exp\left\{ \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{\Psi\in\Sigma_T, \sigma^n(\Psi)=\Psi} \prod_{l=0}^{n-1} B_{\underline{\mathbf{u}}}(\sigma^l(\Psi)) \right\}. \tag{4.4}$$

Combining (4.2) and (4.3), we have

$$Tr\mathfrak{B}_{\underline{\mathbf{u}}}^n = \sum_{\Psi\in \mathrm{Fix}\,\sigma^n} \prod_{l=0}^{n-1} B_{\underline{\mathbf{u}}}(\sigma^l(\Psi)), \quad n \in \mathbf{N}.$$

This relation implies the following rationality result for $\zeta_{\underline{\mathbf{u}}}(z)$. Denote the polynomial ring of variables $\underline{\mathbf{u}}$ and z over $\mathbf{C}$ by $\mathbf{C}[\underline{\mathbf{u}}, z]$.

Corollary. *The following equality holds in* $\mathbf{C}[\underline{\mathbf{u}}, z]$*:*

$$\zeta_{\underline{\mathbf{u}}}^{-1}(z) = \det(I - z\mathfrak{B}_{\underline{\mathbf{u}}}). \tag{4.5}$$

Specializing at $z = 1$ gives (once again) the rationality result (3.5) for $Z_\Gamma^{-1}(\underline{\mathbf{u}}) = \zeta_{\underline{\mathbf{u}}}^{-1}(1)$.

[7] Such matrix could be considered as the weighted matrix in a sense of a remark in Sec. 2.

Conclusion

As we have seen in this paper the Ihara–Selberg zeta function of a finite graph can be considered as Ruelle type zeta function attached to a natural symbolic dynamics defined on a graph. Generally, this method is applied to a finite graph with labeling. In this case the corresponding Bass' operator can be viewed as a transfer matrix of a (generalized) symbolic dynamics.

The intriguing case is considered in [14] in studying of a model of a random walk on the knot diagrams. In this case the inverse of the Alexander polynomial of a knot is the Ihara–Selberg zeta function of a graph (corresponding to a knot diagram) with labeling. Combining this result with well known state sum expressions for colored Jones polynomial of a knot gives a new graph-theoretic proof of the Melvin–Morton conjecture for a knot. This can be viewed as an example of the effective application of the Ihara–Selberg theory in three–dimensional topology.

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